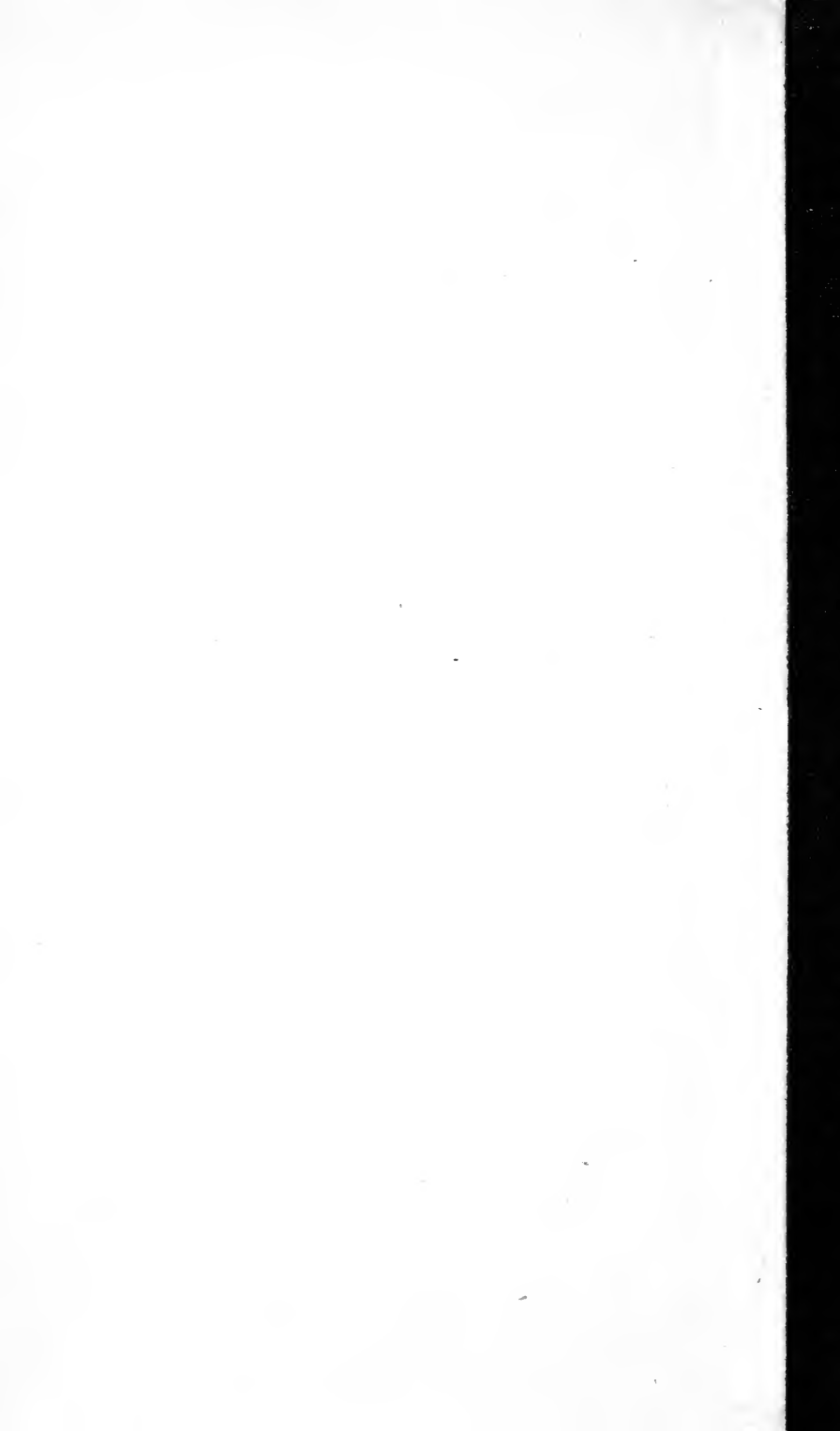




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CRYSTALLOGRAPHY

AND

PRACTICAL CRYSTAL MEASUREMENT



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CRYSTALLOGRAPHY

AND

PRACTICAL CRYSTAL MEASUREMENT

BY

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PREFACE

IN this book an endeavour has been made to present at the same time both a guide to practical work in crystallography and all the essential theory of the subject, not only as regards crystal morphology but also with respect to the physical properties of crystals. The aim has been to give the main facts unencumbered with obsolete nomenclature, notation, and methods, which confuse and discourage the student or enquiring reader, and occupy time that could with greater advantage be spent in acquiring familiarity with the goniometer and a knowledge of facts of real use in the practical experimental investigation of crystals.

For this reason the notation of Naumann for the labelling of crystal faces is discarded for the more scientifically founded method of Miller, which is in harmony with the important recent development and completion of the geometrical theory of crystal structure, and regards crystals as homogeneous structures built up by the arrangement of their chemical molecules, and the atoms of which they are composed, according to definite schemes of symmetry. For the same reason the conceptions of hemihedrism and tetartohedrism are eliminated in favour of the now well-established principle, that the various classes of the same system of symmetry are different entirely because they possess in a definite manner more or less of the elements of symmetry (planes and axes of symmetry) possible to the system, in various stages and modes of combination, from the class possessing the minimum essential elements of the system to that endowed with the maximum number.

Another feature of the book is that the forbiddingly mathematical aspect of so many works on crystallography has been altogether avoided; the student is shown that there are really no mathematical difficulties involved, but that, on the contrary, all the necessary calculations of crystal angles and elements, as well as of the physical constants, are both simple and straightforward, requiring only an ordinary acquaintance with simple trigonometry and the use of a table of logarithms. Four pages only of mathematical instructions and formulæ, given in Chapter VII., form the key to all the morphological calculations required.

The practical use of the goniometer, which at once clears away all preconceived difficulties, is taught from the very beginning of the book, immediately after the necessary instructions for selecting or preparing

crystals for goniometrical purposes have been given. A typical crystal is measured as early as Chapter IV., and from the actual observations made with it all the salient fundamental facts of crystal morphology are discovered in their natural sequence, thus leading up to their systematic consideration in subsequent chapters.

The method adopted in this succeeding part of the book has then been to give first a chapter on the symmetry of the crystal system under consideration, and to follow it immediately with another in which a characteristic well-formed crystal of a substance crystallising in that system is actually worked through on the goniometer; indeed in the cases of several of the systems two such crystals of substances belonging to different classes of the system are thus employed as practical typical examples. In all cases these are chosen either from readily procurable minerals and such as form small well-developed crystals suitable for goniometry, or from easily prepared chemical salts known to afford good crystals without difficulty. From these practical measurements in each case the symmetry is deduced, the stereographic and clinographic projections of the crystal are accurately drawn, the crystal elements and angles are calculated from the best measured basal angles, and the results are expressed in the approved tabular form, precisely as if for publication.

In one of these cases—the measurement of copper sulphate as an example of triclinic symmetry in Chapter XX.—the whole chapter is really a record of an original investigation now published for the first time; for the existing data concerning the crystallography of copper sulphate are so confused that a reinvestigation was imperative. The student is thus enabled to follow an actual piece of research through all its stages.

The fact that no two crystals are ever alike entirely saves this method from being a cut-and-dried one, affording no opportunity for original thought and treatment. For the crystals of suitable small size and perfection, purchased from the mineral dealer, or grown in the laboratory by the student personally, with the object of repeating the measurements here detailed and working through the chapter practically with the goniometer, while being similar can never be copies, and will generally afford some additional or different faces, or be deficient in others, compared with those exhibited by the crystal described in this book as typical of the substance.

A special endeavour has been made in Chapter IX. to give a full but concise account of the important work on the geometrical theory of homogeneous crystal structures, and in Chapter XXXI. of the development of the fruitful idea of molecular distance ratios, the dimensions along the three axial directions in space of the elementary cell (the habitat of a molecule) of the space-lattice of the crystal structure, our nearest approach to the determination of the molecular dimensions. Practical guidance is also given in the succeeding chapter for the determination of the density of crystals, now in consequence rendered so very important.

The illustrations are almost entirely original and of two kinds, namely (1), direct reproductions of the author's own drawings, which include all the figures of crystals, and (2) wood-cut engravings; the latter are used to illustrate every instrument and piece of apparatus referred to in the text, a large number of which are in the author's own laboratory. The drawings of crystals have all been constructed to scale and according to the axial ratios and angular elements of actual substances, the very few copies of crystal forms being taken with full references and acknowledgments from the memoirs of original authors. Besides the essential instruction in clinographic projection given in Chapter XXV., the great utility of the stereographic projection, on the lines indicated by Penfield, as an important aid to the drawing of crystals, has been emphasised. The illustrations of interference figures afforded by crystal plates in convergent polarised light are almost exclusively reproductions of actual photographs taken by the author.

An attempt has been made to render the optical portion of the book a special feature. So little practical aid has hitherto been forthcoming for the student in this domain of crystal optics, that it is no wonder the optical details are usually so meagre in the published descriptions of the crystals of new substances. The optical part of a crystallographic investigation is frequently of exceeding interest and often of prime importance, and it is hoped that a really practical guide to this branch of the work will be found in the book. Moreover, the science of optics has been undergoing such remarkable changes of recent years, owing to the rapid succession of discoveries of the first rank, that an introductory chapter (XXXIV.) has been given in which the present position of the theory of light is briefly presented, as a definite basis on which to rest the subsequent chapters applied to crystal optics.

The chapters on the microscope include a full account of the important new methods introduced by Becke and von Fedorow, and it is hoped that they will be a help in presenting the difficultly accessible work of these investigators to English readers. The chapters on the thermal expansion and elasticity of crystals are also of a fully practical character, and the instruments employed, which are mostly original, illustrated.

The last chapter (LV.) includes a brief account of the liquid crystals discovered by Lehmann, with instructions for their study by observation or screen projection.

The author desires to record his indebtedness to Principal Miers, F.R.S., for much kindly help and criticism, which has materially enhanced the value of the book, and for his permission to illustrate by new wood engravings the whole of the instruments designed by him. Also to his successor at Oxford, Professor Bowman, for similar facilities concerning his additions to the Oxford instruments. To Dr. Herbert Smith of the Mineralogical Department of the British Museum (Natural History) at South Kensington the author's best thanks are due, for like privileges in connection with the two three-circle goniometers with

which he has enriched the science, as well as with respect to the total refractometer and other attributes of goniometrical-optical research on crystals which he has devised, and for the permission to redraw certain illustrations of his memoir concerning the gnomonic projection of crystals. To Dr. Hutchinson of Cambridge the author is also indebted for the kind loan of the beautifully prepared copper engraving of his stereographic net, and of the blocks illustrating his stereographic protractor and his universal apparatus. Also the author's sincere thanks are due to Mr. T. V. Barker for most kind assistance in presenting the work of Professor von Fedorow, with whom he studied in St. Petersburg, adequately before the readers of this book. For much kindly help and many valuable suggestions in connection with the geometrical theory of crystal structure (Chapter IX.), and with the valency theory of Pope and Barlow (Chapter XXXIII.), the author is deeply indebted to Mr. W. Barlow, F.R.S. The author also desires to record his hearty thanks to the firm of Carl Zeiss, for kindly placing at his disposal for experimental demonstration their beautiful apparatus for the study of liquid crystals, together with facilities for preparing a wood engraving of the heating microscope employed. Similarly, to Messrs. Swift for the loan of electros of certain attributes of the Dick microscope, and for facilities for illustrating this beautiful instrument in its absolutely latest form with a new wood engraving. Also the author is particularly desirous of recording his thanks to the firm of Steeg and Reuter of Homburg, for very kind help in obtaining suitable sections of mineral crystals for the preparation of the photographs of interference figures, and for the loan of electros of their well-known sectioning apparatus.

The author cannot conclude without paying a tribute to the careful work of the engraver, Mr. Frank Butterworth, to whose personal skill and draughtsmanship it is due that the wood engravings of this book are such faithful representations to the last details of the most recent form of crystallographic instruments. The author also wishes to express in conclusion his gratitude to Messrs. Macmillan for their unflinching kindness and consideration during the production of the book, and particularly for their public-spirited expenditure on a work of this kind, which can scarcely hope for a very wide circulation in the early days of this growing science of crystallography, when its value is only just beginning to be realised. That there is a great future before this subject, the science of the organised and perfect solid, is assured, however, and that the book may help forward the progress of crystallography by enabling it to be more widely studied, and its value to be more fully appreciated, and especially that there may be attracted to it an increasing number of earnest spirits seeking for a field of fascinating and richly rewarding research, is the author's chief aim and desire concerning it.

A. E. H. TUTTON.

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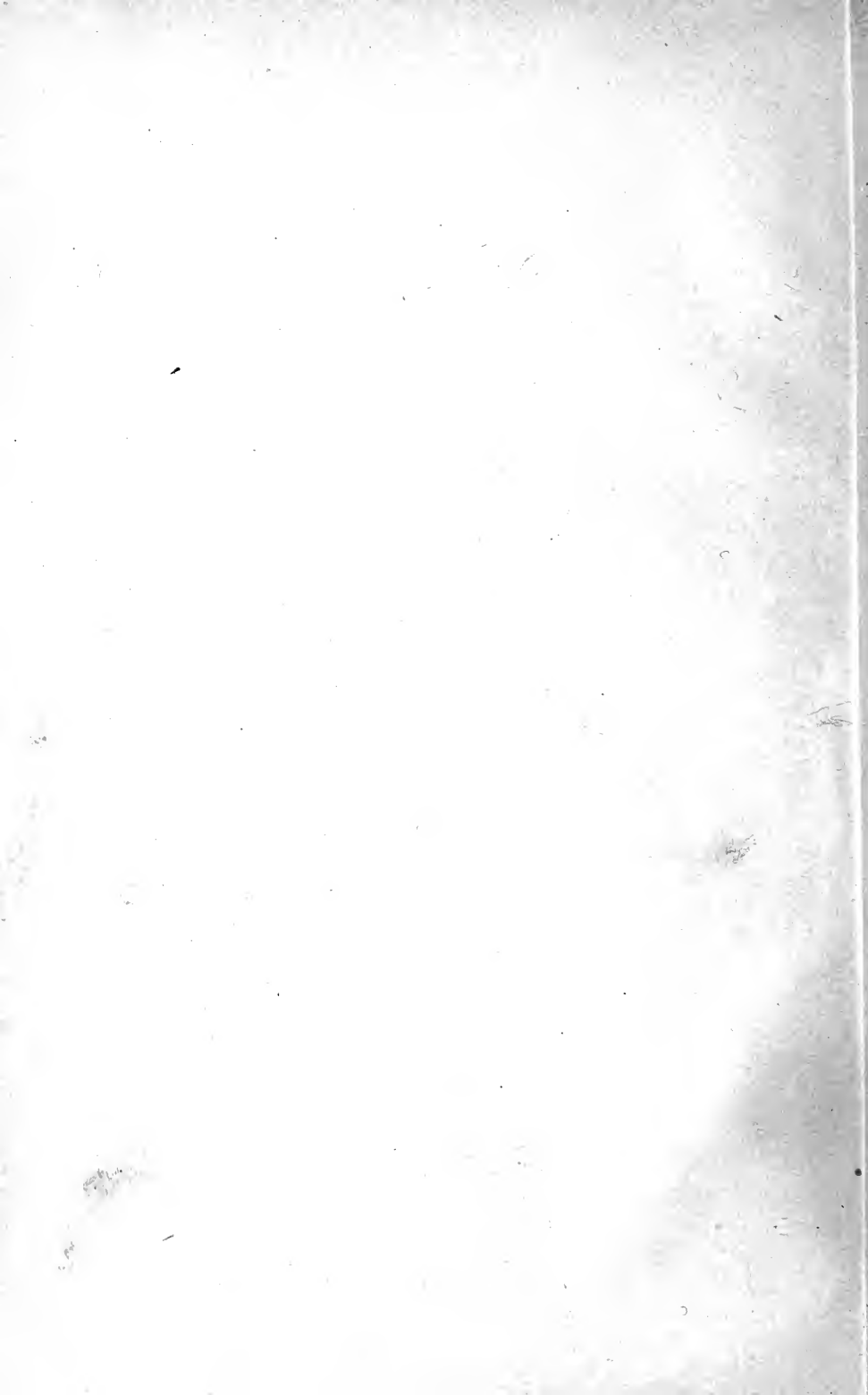
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PART I

MORPHOLOGICAL



CHAPTER I

THE NATURE OF CRYSTALS.

A CRYSTAL is solid matter in its most perfectly developed and naturally organised condition. It is characterised by an exterior form of extraordinarily regular geometrical design, carried out much more accurately than can be attained by artificial construction, except by great expenditure of time and rarely attained skill; also by an internal structure of a nature likewise so regular that the arrangement of the ultimate particles or structural units, which are in general the chemical molecules, about any one point in the structure is precisely the same as about every other point. If the growth of the crystal has been slow, undisturbed, and unrestricted in all directions, its external shape is that of a closed solid the surface of which is entirely made up of numerous truly plane facets, technically termed "faces," meeting in straight edges, brilliantly smooth as if highly polished, and the arrangement of which with respect to each other, as measured by their mutual inclinations, is characteristic of the particular substance of which the crystal is composed. The appearance of such a perfect crystal is shown in Fig. 1, which represents a crystal of the common substance, sulphate of potash, K_2SO_4 .

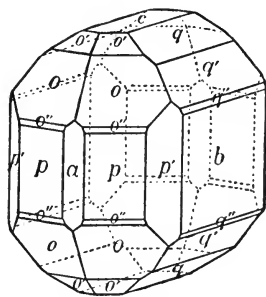


FIG. 1.—Potassium Sulphate Crystal.

It will be observed that in the above definition a crystal is regarded as a solid, and the crystals employed for the purposes of measurement, both of angles and of optical and other physical constants, must obviously be such, although they may vary in hardness from that of the diamond down to that of gypsum and of the softer chemical salts and preparations. A most interesting borderland between the softest of crystals, as we ordinarily know them, and liquids, has been brought to our notice during the last fifteen years, however, by the brilliant investigations of Lehmann concerning certain viscous organic liquid substances and even mobile oils, which he has shown possess the property of double refraction and the power of definite orientation of their molecules in organised

assemblages which he terms "liquid crystals." Keen controversy has been excited as to the suitability or otherwise of this term, and over the theoretical deductions of Lehmann. A brief account of these remarkable substances will be given in Chapter LV., together with an indication of the manner in which the definition of a crystal requires to be modified if these so-called liquid crystals are to be included in the scope of the definition.

Faces of Crystals perfectly True Planes.—In referring to the faces as being truly plane, it is not merely meant that they are as flat and highly polished as the facets imparted to a precious stone by the jeweller's lapidary, but that they are optically true planes, as perfectly even as those surfaces which the most highly skilled opticians can only produce by days and often weeks of most patient labour. The result is that perfect natural crystal faces reflect light like a mirror, and the reflections which they afford of surrounding objects resemble those objects as truly as in the case of the optically worked reflecting surfaces just referred to. Moreover, this is true whether the substance of the crystal is opaque or transparent. If the latter, then part of the light only is reflected, the amount depending on the angle of incidence, being least for perpendicular incidence, but anyhow ample to afford excellent images. The other part penetrates the substance of the crystal, and suffers dispersive refraction in accordance with the well-known laws of refraction. Further, on reaching the faces on the other side of the crystal the dispersed rays may partially pass out, and partially be reflected back again in accordance with the ordinary law of reflection at a transparent surface, or they may be internally totally-reflected, provided the angle of incidence on one or more of the back faces is greater than the so-called critical angle, the whole of the light incident on that face or those faces being then totally-reflected, in accordance with the optical law of total-reflection which governs what occurs on an attempted passage from a more highly (in this case the crystal) to a less highly (the air) refractive medium.

It is to this latter fact that is due one of the most beautiful properties of transparent crystals, namely, the scintillation of spectrum-coloured rays when the crystal is viewed in sunshine or a bright artificial light, which causes so many of the harder and more durable of naturally occurring crystals, such as the diamond, sapphire, ruby and emerald, to be so highly valued as jewels and gem-stones. A diamond, provided with numerous cut and polished facets for use as a brilliant (as unfortunately the natural faces of the diamond have usually been rounded by corrosion) exhibits the phenomenon to perfection; for owing to the very high refractive power of diamond, and its exceptionally great dispersion (long spectrum), almost all the light which penetrates the crystal, namely, all which internally meets any face or facet at any angle greater than $24^{\circ} 13'$, suffers internal reflection before being permitted eventually to emerge in brilliant spectrum colours from the crystal.

The high order of reflection from crystal faces is the foundation of

practical crystallography, for upon it is based the use of the chief instrument of the crystallographer's laboratory, the reflecting goniometer, with the aid of which the angles between the crystal faces are measured. The crystals found in nature, or prepared by the usual artificial means (deposition from a saturated solution of the substance in a solvent, cooling of the fused substance, or condensation of the vaporised substance), only more or less rarely exhibit the state of perfection described in the opening paragraphs. But sooner or later such perfect crystals are met with naturally, or are obtained by careful preparation, in the case of crystallisation from solution by taking precautions to avoid rapid changes of temperature, concussion, or other brusque disturbance of the solution during the period of deposition of the crystals. It amply repays to take such precautions, and to select carefully only the very best crystals for measurement on the goniometer, for the results are far more valuable than the mean results of a very large number of measurements with inferior crystals.

The Constancy of Crystal Angles.—The fact that the crystalline form is the most distinctive and characteristic property of a solid substance, offering the most infallible evidence of its identity, would alone suffice to render crystal measurement of the highest importance. "By crystalline form," however, is not meant the relative extent of development of the different faces, but the arrangement of those faces, as defined by the geometrical angles between them. To render this clearer, illustrations of two other crystals of the same substance which

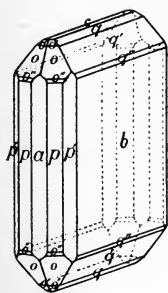


FIG. 2.

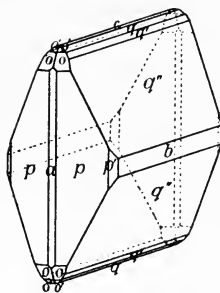


FIG. 3.

Two further Types of Potassium Sulphate Crystals.

is represented in Fig. 1, potassium sulphate, are given in Figs. 2 and 3. Precisely the same faces are present in all three cases, but their relative development is so different that the crystals would appear at first sight to have no single point of similarity. It was this great diversity of appearance, consequent on great variety displayed in the development of the faces of the crystals of one and the same substance, that for so many years delayed the acceptance of the conclusion published by the Abbé Haüy so long ago as the year 1784, that every substance of definite chemical composition is distinguished by its own particular crystalline form. In the case of potassium sulphate actual

measurements of the angles between any analogous well-developed pair of adjacent faces on the three crystals represented in the figures, which were particularly good crystals employed in the author's research on the alkali sulphates, showed that in every case they were identical to within two minutes of arc. This constancy of the interfacial angles, so excellently illustrated in the specific case of potassium sulphate, is a general law of nature, as regards the crystals of any one definite substance, and the law forms the corner-stone of the science of crystallography.

Final Proof of Angular Constancy and Individuality even in Cases of Extreme Similarity.—Yet the perfect truth of this fundamental law has only quite recently been fully proved. For the principle of "isomorphism," discovered in the year 1819 by Mitscherlich, implied, as indicated by the term used to designate it, that certain series of similarly constituted substances,—such for instance as the numerous well-known series of chemical salts in which the metal present may be replaced by any other metal of the same family group,—possessed identical crystalline forms. Although Mitscherlich himself was subsequently of opinion that only similarity and not identity of form obtained in these cases, the published measurements of the crystals of the members of such series were so conflicting until so late as the year 1890 that it was impossible on existing data definitely to decide the question. Since the year just mentioned, however, the author has carried out an organised series of investigations, having for their main object the settlement of this fundamental question, employing highly perfect crystals, prepared with very special care, of over forty salts belonging to four of the more important of these series. The result has been to establish definitely the fact that relatively small but very real differences of angle do occur between the crystals of different members of any one series, and that each member of the series is thus characterised by its own distinctive angles. Further, the differences have been shown to be subject to a definite and very interesting law, namely, that they follow the order of the differences between the atomic weights of the metals or other dominating interchangeable chemical elements which give rise to the series, so that the angles are functions of the atomic weights. Indeed, the law is much more far-reaching than this, for it embraces in its scope not only the relations of the angles between the exterior faces, but also the whole of the internal as well as external, including the optical and thermal, properties of the crystals, all of which are also functions of the atomic weights of the interchangeable elements. It has been stated by the author as follows: **The difference in the nature of the elements of the same family group, which is manifested in their regularly varying atomic weights, is also expressed in a similarly regular variation of the characters of the crystals of an isomorphous series of salts of which these elements are the interchangeable constituents.**

This generalisation, which is based on the indubitable result of many thousands of accurate measurements, finally disposes of all doubt

as to the constancy of the crystal angles of any one and the same substance; it defines the limited sense in which Mitscherlich's law of isomorphism is true, thus reconciling the views, formerly considered antagonistic, of Haüy and Mitscherlich, and leaves as the expression of a fundamental truth the statement that: **To every solid crystallisable substance of definite chemical composition there corresponds a crystalline form which is proper and special to that substance, and by which the latter can be recognised and identified.**

The small interfacial angular differences observed in the cases of isomorphous series, which have been shown to be subject to the law of progression with the atomic weight of the interchangeable chemical elements of the same family group which produce the series, increase with diminution of the symmetry, and diminish as the symmetry of the crystal increases, until, when the perfect symmetry of the cubic system is attained, they disappear altogether. For the angles of a cubic crystal are fixed by the symmetry, and are therefore invariable. In the case even of a cubic crystal, however, the optical, thermal, and other properties continue to obey the law. The differences of interfacial angle in these strictly isomorphous series rarely attain 3° , even in the lowest systems of symmetry.

CHAPTER II

THE PREPARATION AND SELECTION OF MEASURABLE CRYSTALS, AND THE NATURE OF THE PROCESS OF CRYSTALLISATION FROM SOLUTION.

The Size of Crystals for Measurement.—The most suitable crystals for goniometrical measurement are usually of small size, somewhere between that of a pin's head and that of a pellet of small shot. For the minute faces of such crystals as a rule exhibit less distortion than larger ones. There are cases, however, of substances the faces of which habitually show such perfection of planeness that somewhat larger crystals, as much as three or four millimetres in diameter, or, in the case of elongated prismatic crystals, across their narrowest part, may be safely employed, but such instances are rare. On the other hand, discrimination requires to be used as to the acceptance of angular measurements from crystals of less than a millimetre diameter; for the images of the reference signal-slit of the goniometer, reflected by the smaller faces, which must obviously be very minute on such crystals, are relatively so much less brilliant that it is difficult to distinguish a simple perfect image, derived from a truly plane face, from the brightest of a bundle of images derived from a distorted face, owing to the feeble illumination of the weaker subordinate images of the bundle, which may cause them to be overlooked. Hence, while very small crystals are generally preferable for measurement, discretion must be employed. As a rule, a crystal of an intermediate size between the dimensions just referred to will prove a happy choice.

The Selection of Crystals of Minerals.—In the case of mineral crystals it is only necessary to examine all the material available, and to select, with the aid of a pocket lens, those crystals which are of suitable size, in accordance with the above considerations, and which are obviously the best formed and endowed with faces apparently free from distortion or striation. It may be that several such will require to be tested on the goniometer itself, for every one that is finally chosen as fulfilling the *desiderata*.

The Preparation of Crystals from Solutions.—When the substance under investigation is a chemical compound which is crystallised, as is most frequently the case, from a solvent (usually water, but in cases of carbon compounds probably alcohol, ether, acetone, benzene or other

organic liquid), a number of different crops of crystals should be prepared. The solutions are preferably placed in small flat-bottomed glass beakers or thin glass crystallising dishes, all scrupulously clean, and every possible precaution must be taken against draughts of air or other cause of rapid change of temperature, against mechanical disturbance of any description whatsoever, and particularly against the intrusion of dust. The crystallising vessels should rest on felt, and the latter on wood, the wooden top of the working bench or table being quite suitable, provided all metal or other good conductors are removed, and non-conducting screens are placed so as to surround the felt-covered part. The room should be a quiet one, free from dust and not liable to sudden change of temperature. For the obvious reason that a large room can rarely be given up for the purpose, it is preferable to set apart a small room for the crystallisation, or at any rate to reserve a quiet corner for this all-important work.

It is usually unwise to crystallise by cooling from strongly super-saturated solutions, as the crystals then form so rapidly that they are generally distorted, or acicular, or otherwise unsuitable for measurement. The best course is to prepare a hot solution not quite saturated. If there be the slightest deposit in the solution it should be filtered, and the clear liquid allowed to fall directly into the several small crystallising dishes or beakers to be employed, and to the extent of not more than an inch deep in each. The beakers ought to be first warmed, as a very cold beaker may start crystallisation at once. During cooling to the ordinary temperature there will be some evaporation of the solvent, with the production of a solution which will be more or less super-saturated by the time the atmospheric temperature is reached, according to the amount of evaporation and to the difference in solubility of the substance at the boiling temperature of the solvent and at the atmospheric temperature. As a rule crystallisation will supervene during the succeeding night, and next morning a crop of apparently well-developed crystals will be found, at any rate in one or more of the beakers. A little experience with the substance under investigation will soon teach one the best conditions for crystallisation, and they will vary for different substances as considerably as do the solubilities of those substances. The rules just given, however, will be found to be of very general application.

Removal and Storage of the Crystals.—Having thus secured one or several such crops the crystals should be taken out, as early in the morning as is convenient, as otherwise the normal rise of the atmospheric temperature with the altitude of the sun will warm the mother liquor sufficiently to round the edges of the crystals, owing to the solubility being usually greater the higher the temperature. The removal is best effected by decanting off the mother liquor (the technical term used to designate the solution standing over deposited crystals) into another clean beaker, which had better be first warmed. The clear liquid in this second dish may then be advantageously brought just to boiling over a very small flame of the Bunsen gas burner, and the dish set aside

for the production of a second crop on the succeeding night. The crystals left in the first beaker may then be gently removed and placed on a double sheet of clean white blotting or filter paper to drain. They should next be moved about over the sheet by several small strips of blotting-paper in succession, until free from traces of mother liquor, and be eventually transferred to a fresh sheet of blotting-paper and allowed to dry completely in ordinary dry air, assuming that the substance is not hygroscopic. When, however, the substance is a deliquescent one (absorbing moisture from the air), the paper and crystals must be placed in a desiccator containing concentrated sulphuric acid, calcium chloride, phosphoric anhydride, or other powerful absorbent of water vapour, to dry.

Each crop of crystals thus obtained should be stored in a separate small bottle, labelled with the name or chemical formula of the substance and the date of collection of the crop. The contents of each can then be separately examined for crystals of suitable size and the required perfection of faces for use in the measurements, and any peculiarities of habit on the part of the crystals of that particular crop studied.

Procedure in the Case of very Deliquescent Crystals.—In the case of the crystallisation of very deliquescent substances during damp weather, it is frequently advisable, and sometimes necessary, to place the crystallising vessels inside the receiver of an air-pump, along with the usual desiccating dish of sulphuric acid, or, in extreme cases, of phosphoric oxide, and to reduce the pressure of the air within the receiver, by means of the pump, to something less than one-third of the outside atmospheric pressure. The progress of the crystallisation requires in this case to be carefully observed every few hours, as otherwise the crystals are apt to grow to too large a size.

A very convenient form of desiccator for use either at the ordinary or under reduced pressure, is that of which a battery of six is shown in Fig. 4. It is simple, and never gets out of order or collapses at a low interior pressure, as some of the more elaborate vacuum receivers are apt to do. A strong glass bell-jar, with ground tubular open neck and ground lower rim, rests by the latter on a slab of plate glass also ground. The neck is closed by a good indiarubber stopper, of such size and taper that the exterior atmospheric pressure is incapable of forcing it through the neck, and perforated by a single hole through which there just passes a short glass tube bent at right angles, having a well-fitting stopcock blown on the horizontal limb. The fitting of the truly plane rim of the bell-jar on the slab, of the caoutchouc stopper in the neck, and of the stopcock in its tubulure on the short tube, is in each case rendered air-tight by a thin film of pure vaseline. It is a great mistake to use cerate or other wax, as this is no substitute for well-fitting glass grinding, and it is not only a poorer closing medium but also, owing to its hardening, gives infinite trouble on again opening the receiver, generally causing upsetting of the crystallising vessels or derangement of the crystals exposed on filter-paper for drying. If the crystallisation is to proceed under ordinary

pressure, and the receiver to be thus used as an ordinary desiccator, it is only necessary to close the stopcock. If the pressure is to be reduced, the stopcock is opened and the tube connected to the air-pump, water-pump, or, in case the laboratory is supplied with automatic vacuum, to the bench vacuum connection tap. As soon as the pressure has been reduced to the desired amount, the stopcock is closed and the receiver thus isolated. It is often convenient, if a large number of crops of crystals are to be prepared, to possess a battery of half a dozen such desiccators, as shown in Fig. 4, and to connect them to the vacuum

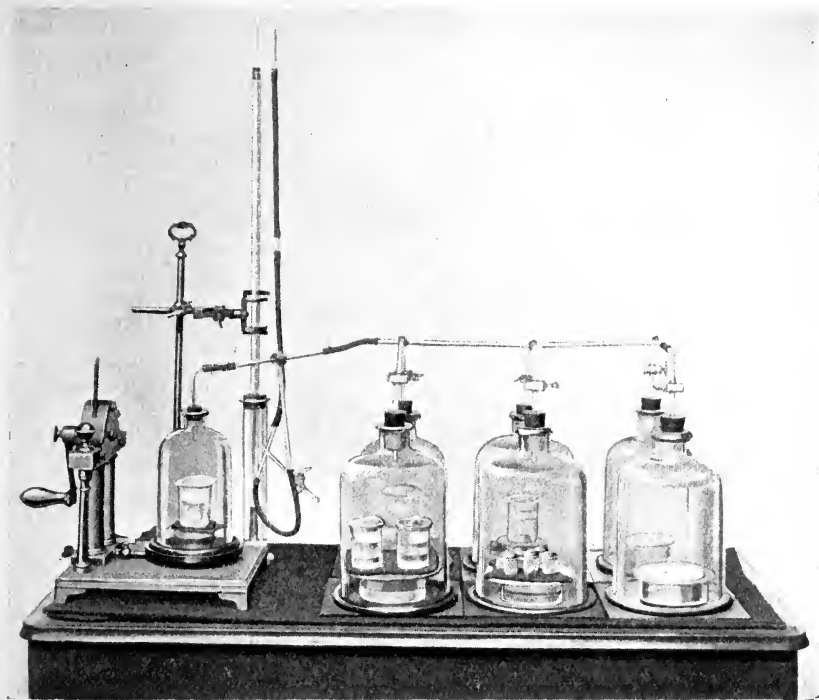


FIG. 4.—Apparatus for Crystallisation under Reduced Pressure.

producer through the intervention of a glass main tube directly connected with the latter, and having the corresponding number of little side tubes blown on it, whereby to make the connections to the stopcock tubes of the receivers. All six can then be exhausted at once. If, however, the pressure is to be different in the different cases, which is rare, the receivers can just as readily be separately exhausted. It is an advantage that the main tube should be attached to a simple form of manometer, for indicating immediately the pressure in the receivers or any particular receiver before shutting off the taps. It is only necessary to use a long narrow glass tube for this purpose, carried vertically upwards at first on leaving the connection, then bent over and

brought closely parallel to itself vertically downwards into a cup of mercury. But it is more convenient to have a partially caoutchouc (thick walled) connection, as shown in Fig. 4, which can be broken for the insertion of an open T-piece with stopcock, for convenience in re-admitting the air. The height of the manometer tube should obviously be somewhat over thirty inches, and a simple scale of millimetres or inches should be provided to indicate the height of the mercury in the tube above the level of that in the cup.

Larger Crystals for Use in Optical Investigations.—Before leaving the subject of the preparation of crystals suitable for the purposes of investigation, it may be remarked that crystals of somewhat larger size than those best suited for goniometrical work are an advantage in studying the optical properties, provided the internal structure is uniform throughout, as indicated by their perfect transparency. For these optical purposes it is necessary to prepare, by cutting or grinding, parallel-sided section-plates and 60° -prisms out of some of the crystals. The reduced pressure method just described may often be employed with success, but an even better method is to effect the crystallisation at the ordinary pressure, but in a large confined chamber containing as usual a dish of sulphuric acid, so as to permit evaporation to proceed with little restraint. A glass case 28 inches long, 18 inches wide and 16 inches high, mounted on a cloth-covered wooden base with a fairly tightly fitting plinth, has served the author's purpose well. The dish of vitriol is set in the centre, and the crystallising beakers at regular intervals around. As fast as evaporation of the water of the solutions occurs, the vapour is absorbed by the sulphuric acid. Excellent crystals of many inorganic salts have been obtained by this method, of extraordinary perfection as regards development, planeness of the faces, and internal homogeneity; and of adequately large size not only for use in the optical investigation, but also to enable sufficiently large section-plates to be cut from them for use in demonstrations with the lantern projection polariscope.

The Conditions for Crystallisation from Solution.—In order that we may apply the preceding instructions with advantage, it is advisable that we should study in some detail the nature of the process of crystallisation, so far as it is yet understood. The elementary fact that every soluble substance dissolves in the solvent to a definite extent for every particular temperature, and that in general the extent or degree of solubility differs with the temperature, and is usually greater the higher the latter, has been assumed in the preceding sections. The inter-dependent relations of solubility and temperature are represented by the well-known "solubility curve," which is obtained by plotting a curve in the usual manner on squared paper, taking temperature for abscissæ and concentration (relative amount of the salt in solution as compared with the amount of the solvent present) for ordinates. Any point on the curve indicates by its co-ordinates the amount of the substance which the solvent can hold in solution at a specific temperature. For instance, Fig. 5 shows the solubility curve of ammonium selenate,

$(\text{NH}_4)_2\text{SeO}_4$, constructed from a series of actual determinations by the author of the solubility of the salt at different temperatures. It is nearly a straight diagonal line, the dotted line being a truly straight one also drawn for comparison, and it indicates that at 10°C . 121 parts by weight of ammonium selenate dissolve in 100 parts of water, while 197 parts of the salt are dissolved by 100 parts of water at 100° , for intermediate temperatures the amount being nearly proportionately intermediate, as for instance at 60° when 165 parts dissolve in 100 of water.

Although it is a general rule that the higher the temperature the

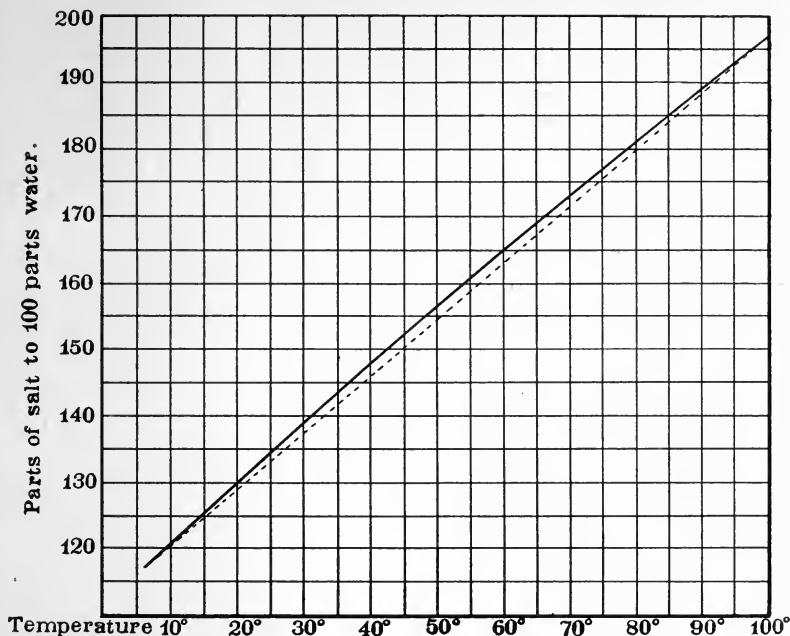


FIG. 5.—Solubility Curve of Ammonium Selenate.

greater is the quantity of salt which can be held in solution, and that the difference in solubility for the neighbourhood of the ordinary temperature and for 100° , the boiling temperature of the common solvent water, is considerable, still there are many exceptions to the rule, and many cases where solubility increases up to a definite temperature and then diminishes again, particularly where the salt is one which crystallises with water of crystallisation. Also there is one notable case, that of common table salt, sodium chloride, NaCl , in which the solubility is almost the same at all temperatures of the solvent, water; thus 100 grams of water at 14° dissolve 35.87 grams of salt, whereas at the boiling temperature, 100° , only the slightly greater amount 39.61 grams are dissolved. Hence

the divergence in solubility of different salts or other soluble chemical compounds is very great, and each case has to be most carefully determined experimentally, and its curve constructed, if we wish to have trustworthy data regarding the solubility. For this reason the author always includes determinations of solubility in his crystallographic investigations of soluble salts.

But bearing these facts in mind, it is, broadly speaking, the general increase in solubility with rise of temperature which, as we have seen, gives rise to the regular deposition of the dissolved substance out of the solvent in the form of crystals, on the cooling of a solution which had been saturated at a temperature higher than the atmospheric, down to the latter. The amount of the solid substance thus deposited in crystals corresponds to the excess over and above the amount of the solid which the solvent can hold at the lower temperature of equilibrium with that of the atmosphere.

While this fact is roughly true, there are many interesting points of slight variation which the crystallographer is bound to observe in the course of a lengthy experience and to take account of. For instance, it is frequently observed that when a solution has been brought to the condition in which, on cooling down to the atmospheric temperature, it ought to crystallise if it immediately obeyed the rule in accordance with its solubility curve, it declines to do so; and it may still refrain from crystallising, if the room be quite still, for some days. The solution is then obviously in a supersaturated condition. Eventually, however, under conditions which will presently be explained, crystallisation does occur, generally accompanied by the disengagement of heat, and proceeds very rapidly until the excess of the solid corresponding to the degree of supersaturation has been deposited in the crystalline form. Moreover, it is an interesting fact that the crystallisation frequently goes too far, that is, further than just sufficient to bring about the theoretical equilibrium corresponding to the degree of solubility at the temperature of the atmosphere and solution. It is as if the molecules of the dissolved substance are carried down with such impetus by the crystallising forces, when once set in action to produce the regular solid, that the liquid is positively brought into a state of slight under-saturation. The result is that if crystals are left in the solution for any considerable time after they have formed (assuming no appreciable evaporation to occur so as to reduce the solvent and render the liquid again saturated), their edges will become rounded owing to this excess of crystalline matter redissolving again when the solution and crystals have time to establish full equilibrium. For true equilibrium is only attained when the solid phase (the crystals) and the liquid phase (the solution) have been allowed to remain for some time in contact. This rounding of the solid angles is also increased by the rise of temperature which accompanies rapid crystallisation, although the two causes are quite distinct, and the interesting phenomenon is particularly noticed in the cases of rapid crystallisations from strongly supersaturated solutions, and in those cases more especially when large

rapidly formed individual crystals are the result. These are not conditions, however, which have been recommended in the foregoing pages as suitable for the growth of measurable crystals; for, as we have seen, solutions only slightly supersaturated are infinitely preferable for our purpose.

The phenomenon of over-crystallisation, and the subsequent rounding of the crystals produced if allowed to remain in the mother liquor, has been mentioned here because it is one that has not hitherto attracted the attention which it merits, and one with which the practical crystallographer has to reckon, and most of all to avoid, by eliminating all conditions conducive to rapid crystallisation, which is always, moreover, provocative of distortion.

Supersolubility.—The phenomenon of “supersaturation,” the converse of “over-crystallisation,” is not confined to a few substances, such as the well-known instance of sulphate of soda, but is a much more general one than was formerly thought to be the case. The main conditions in general are that the cooling solution shall remain tranquil, protected from all disturbance (although this is not essential in the case of sulphate of soda, which does not spontaneously crystallise at temperatures above -8°), and particularly, in all cases, including that of sulphate of soda, that it must be protected from the intrusion of dust particles from the air.

The common lecture experiment, with a strongly supersaturated solution of sodium sulphate, illustrating supersaturation, may just be recalled in passing. As much of the crystallised salt, $\text{Na}_2\text{SO}_4 \cdot 10\text{H}_2\text{O}$, as corresponds to nearly the correct amount for saturation at the temperature of 32.5° , at which temperature sodium sulphate exhibits its maximum degree of solubility, is dissolved in water in a large flask. One hundred parts by weight of water dissolve as much of the crystallised salt $\text{Na}_2\text{SO}_4 \cdot 10\text{H}_2\text{O}$ as corresponds to 50 parts of Na_2SO_4 at 32.5° and less than this amount at any other temperature, so that the strongest supersaturated solution can be made by dissolving as much of the salt as possible at 32.5° . The reason for this maximum in the curve is that above 32.5° the decahydrate decomposes, losing its water and then dissolving as the anhydrous salt Na_2SO_4 , which is abnormal in exhibiting decreasing solubility as the temperature rises. These interesting facts are graphically expressed by the curves given in Fig. 6, which are combined in one diagram from a paper by Hartley, Jones and Hutchinson (*Journ. Chem. Soc.*, 1908, vol. 93, p. 828). The neck of the flask should, immediately after completion of the solution of the powdered crystals, be plugged with cotton wool, to filter out any dust particles which might otherwise intrude themselves, and be set aside to cool; it may then be preserved for an indefinite time without crystallising. But if the plug be removed and a small crystal of the salt be dropped in, crystallisation instantly occurs, and in fact the whole contents of the flask become a semi-solid white mass of crystals. A considerable rise of temperature accompanies the act of crystallisation, for the energy expended when a solid is converted into the liquid condition by the action of a solvent is given out again in the form of heat on the liquid resuming its solid condition. The supersaturated solution may also be caused to crystallise spontaneously by cooling it down to -8°C ., even when the plug of cotton wool is maintained in the neck. The experiment is still more effective if the warm concentrated solution of sodium sulphate is placed in a tube closed at one end and drawn out near the other, in such a manner that the solution may be boiled; while steam is issuing from the drawn-out portion the latter can be fused and sealed off at the blowpipe. If the sealed tube is

then left to cool to the ordinary temperature, or even to 0° , no crystallisation occurs, although crystallisation would occur if it were further cooled in a freezing mixture to -8° . But if the end of the tube be broken off and air admitted, crystallisation almost at once begins at the small surface at the top of the liquid column, where the air impinges, and rapidly spreads down the tube till all is apparently a solid mass of white interlacing crystals, the tube feeling distinctly warm to the hand. If the tube and solution are cooled in powdered ice to 0° before the point is broken off, the temperature at once rises, on crystallisation, to 18° .

Löwel¹ was the first to investigate systematically the reasons for the crystallisation or non-crystallisation of such supersaturated solutions of Glauber's salt (as the decahydrated sodium sulphate is commonly called); and he discovered the interesting fact that the air in contact with the solution had something to do with it, and eventually attributed it to "catalytic" action of some constituent of the air. He found that

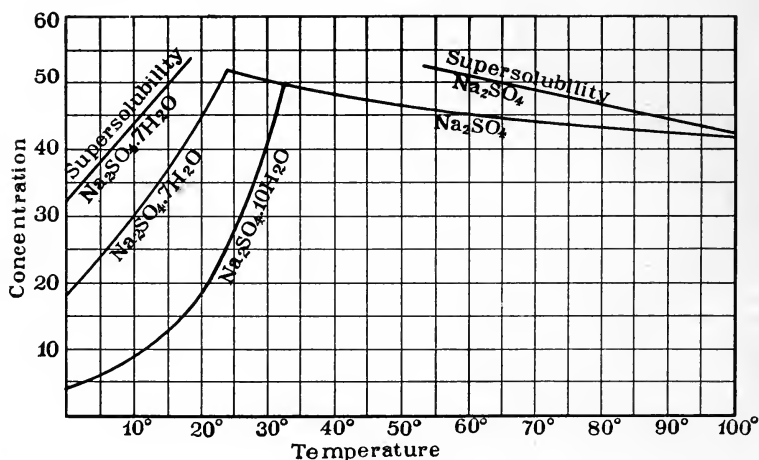


FIG. 6.—Curves of Solubility of Sodium Sulphate.

if the air were washed by passage through water, crystallisation was delayed, but also that the converse process, passing the air through drying agents such as sulphuric acid or caustic alkalies, had the same effect, so that it was not a question of moisture but of something removed mechanically by the wash-liquid. Moreover, even passage of the air through a series of empty flasks delayed the crystallisation of the sodium sulphate solutions left in contact with the air so treated, and, best of all, filtration of the contact-air through cotton wool was most effective in arresting crystallisation.

Shortly afterwards, at one and the same sitting of the French Academy, a remarkable coincidence occurred, of a character which has not been rare in the history of science, for two different experimenters, Violette² and Gernez,³ communicated papers in which practically identical

¹ *Ann. chim. phys.*, 1850 (3), 29, 62.

² *Comptes rendus*, 1865, 60, 831.

³ *Ibid.*, 1865, 60, 833.

conclusions were arrived at, fully explaining the interesting phenomenon in question. The facts presented were briefly these :

(1) The crystallisation of supersaturated solutions of Glauber's salt is brought about by the introduction of a solid substance from the air. For the crystallisation occurs the more rapidly the greater the extent of the surface of the solution which is exposed to the air. Thus it occurs more rapidly in an open dish than in a flask, and in the latter the crystallisation is the slower the narrower the neck. Air from the country is more sterile than the air of towns, and when performing the experiment in the country crystallisation occurred more slowly if the experimenter changed his clothes for others which had not travelled with him from the town.

(2) The substance thus causing crystallisation is soluble in water, and it loses its property of inducing crystallisation when it is heated.

(3) The air which causes crystallisation contains sodium sulphate. The air of towns notoriously contains traces of the salt, owing to the combustion of coal in domestic and industrial fires, which send small but easily detectable quantities of sulphurous acid into the atmosphere, where it becomes oxidised to sulphuric acid, which then (or possibly the sulphurous acid before oxidation) combines with the traces of soda which are well known to be universally present in the air, to form sodium sulphate (or sulphite which becomes oxidised to sulphate).

The conclusion is obvious, namely, that the substance in the air causing the crystallisation of supersaturated solutions of sodium sulphate is **that substance itself**, disseminated in the air in excessively minute crystals.

Gernez afterwards¹ extended his observations to other substances, and discriminated clearly between those supersaturated solutions which do not spontaneously crystallise, but require the introduction of a germ-crystal of the same substance, and those which crystallise on shaking, on rubbing the sides of the vessel with a glass rod, or on employing other modes of disturbance which bring about spontaneous crystallisation, the temperature at which the latter occurs (except in the case of sodium sulphate, where it is -8°) being generally readily observable. But Gernez does not appear to have recognised that the same substance shows both kinds of supersaturated solutions, and considered that the class to which a substance belonged was a property of the substance.

A further step was next made by Lecocq de Boisbandran,² who showed that not only were crystal-germs of the substance itself able to provoke the crystallisation of supersaturated solutions, but that also germs of such substances as were isomorphous with the substance were able to do so. Gernez³ about the same time made the particularly interesting discovery, that not only the crystal system required to be the same in order to call forth crystallisation, but also, in the case of crystals belonging to other than the holohedral class of any system,

¹ *Comptes rendus*, 1865, 60, 1027.

² *Ann. chim. phys.*, 1866 (4), 9, 173.

³ *Comptes rendus*, 1866, 63, 843.

that is, crystals showing less than the full symmetry of the system, the particular variety must be the same. Thus for instance, a solution of sodium ammonium racemate (racemic acid being a molecular compound of ordinary dextro-tartaric acid and of lævo-tartaric acid) which on crystallisation had previously been found by Pasteur to yield an equal mixture of right-handed and left-handed crystals of sodium ammonium dextro- and lævo-tartrates, on contact with a crystal of the right-handed variety of sodium ammonium tartrate only deposited right-handed crystals of the dextro-salt, whilst a left-handed crystal of complementary enantiomorphous symmetry (fully described in Chapter XVII.) only produced a crop of similar left-handed crystals of the lævo-tartrate; the right-handed salt has no power of compelling the crystallisation of the left-handed variety, and *vice versa*.

A still further interesting step was then made by J. M. Thomson,¹ who showed that strict isomorphism was essential in the crystal-germ, that for instance the crystallisation of an alum solution, although the simple forms of the cubic system (the cube, octahedron, and rhombic dodecahedron) in which the alums crystallise are alike for all substances, was not provoked by cubic crystals of common salt or by octahedral or rhombic dodecahedral crystals of magnetic oxide of iron. It will be shown in Chapter IX. that this means that not only must the form but also the internal structure of the crystalline substance be the same, an exceedingly important point. It will subsequently be shown that the index to such structural similarity is practical identity of the distances of separation of the centres of gravity of adjacent chemical molecules along the crystallographic axial directions, which distances are expressed by the constants termed "molecular distance ratios," a full account of which will be given in Chapter XXXI.

Other chemists and physicists have from time to time added further data concerning supersaturated solutions, notably de Coppet, Lefebvre, Roozeboom, and Ostwald.²

All agree that among supersaturated solutions there are those which, when crystal-germs are excluded, may be preserved under certain conditions apparently for an unlimited length of time without spontaneously depositing crystals (forming the solid phase, as it is termed by Ostwald, in conformity with the nomenclature introduced by the acceptance by physical chemists of the "phase rule" of Willard Gibbs). Such solutions were called by Ostwald "metastable."

Others there are which after a short time, assuming germ-crystals also excluded, spontaneously develop the solid phase. These Ostwald termed "labile."

Metastable solutions always exhibit a lower concentration compared with the labile solutions of the same substance. By increasing the concentration the metastable solution passes consequently into the labile condition, and the particular concentration for which the change occurs was termed by Ostwald the "metastable limit."

¹ *Zeitschr. für Kryst.*, 1881, 6, 94.

² *Lehrbuch der allg. Chemie*, vol. ii. part ii. p. 780.

The metastable limit depends chiefly on the nature of the substance, the temperature and the pressure. It is also influenced by certain other conditions to be defined later.

The state of supersaturation can be removed, not only by the agency of solid germs (excessively minute crystals) of the dissolved substance itself, but also by the action of solid germ-crystals of a substance isomorphous with it, and possessing close similarity of molecular volume and of the molecular distance ratios to be described in Chapter XXXI. Indeed some substances may effect the same object which are not strictly isomorphous, but which form mixed crystals with the substance dissolved, the formation of mixed crystals having been shown to be due to the possession of almost identical molecular distance ratios.

The quantity of the solid substance which, in the form of a crystal-germ, can exert this influence, is very minute, but not undeterminably so, direct experiments having indicated an order of from 10^{-9} to 10^{-12} of a gramme of the solid substance for the weight of the germ-crystal.

The means of defining the metastable limit and of constructing the supersolubility curve graphically representing it have now been discovered by H. A. Miers,¹ who has also added considerably to our further knowledge of the behaviour respectively of both metastable and labile supersaturated solutions. When, instead of allowing the cooling supersaturated solution to remain at rest, it is continuously stirred in an open vessel, only a slight shower of crystals appears when the temperature of saturation, indicated by the ordinary solubility curve and corresponding to the amount of salt present, is attained, and the liquid then continues to cool without depositing the main bulk of the excess—which one might expect would crystallise out if the ordinary solubility curve represented the whole truth—until a temperature about 10° lower than the saturation temperature is reached, when a second and much more copious shower of crystals falls, corresponding to arrival at the metastable limit. When, however, a closed vessel is used, a sealed tube for instance, the first shower never falls at all, nothing whatever occurring until the temperature 10° lower than the saturation temperature is reached, when the whole excess corresponding to the two showers added together is suddenly and rapidly precipitated in the crystalline form. The tube may be shaken for any length of time above this temperature without any deposition of crystals occurring. This latter observation is a confirmation of the experiments of the previous observers already referred to, but the novelty of Miers' work consists in the isolation of two distinct showers, corresponding to the ordinary and the supersolubility curves, by the continuous-stirring method in open vessels. The deposition of the first slight shower is due to crystallisation being started by inoculation of the solution with crystal-germs of the same or an isomorphous or structurally similar substance introduced from the air, while the second shower at the lower temperature is due to the spontaneous crystallisation of the excess of solid. The first shower

¹ *Phil. Trans.*, 1903, A, 202, 459; *Journ. Chem. Soc.*, 1906, 89, 413; *Proc. Roy. Soc.*, 1907, A, 79, 322.

thus corresponds to the ordinary solubility curve, which may be said to be the curve of crystallisation by inoculation, and the second shower to the supersolubility curve of spontaneous crystallisation, the "metastable limit" of Ostwald. The interval between the two curves represents the "metastable condition" of the supersaturated solution, and the condition for spontaneous crystallisation corresponding to the area of the curve-diagram beyond the supersolubility curve corresponds to Ostwald's "labile condition." Fig. 7 on page 22 will assist in rendering this clear.

Miers has carried the subject an important stage further by his discovery of an admirable method of experimentally tracing the supersolubility curve. A glass prism of known refractive index is mounted on the crystal-holder of an inverted goniometer, and immersed in the solution, which is contained in a glass trough the front part of which facing the observing telescope is formed by a parallel-faced truly plane plate of glass, and the refractive index of the solution is determined by the method of total reflection within the prism. The mode of procedure, illustrated by a figure showing the actual instrument (Fig. 443), will be fully described in Chapter XLVI.—on refractive index determinations. Such determinations of refractive index were found to be possible during the whole course of the cooling and stirring of the solution, and to afford an exact means of determining when the labile shower of crystals commences to fall. For the instant crystals begin to be deposited the refractive index shows a corresponding marked diminution, and a thermometer immersed in the solution indicates the temperature. The change in refractive index is very slight at the temperature corresponding to the ordinary solubility curve, but much more considerable and very sudden at the temperature of the supersolubility curve, the two changes corresponding to the slight and the copious nature of the two respective crops of crystals. As the ordinary solubility curve is determined accurately by the usual well-known methods, based upon the estimation of the actual amount of dissolved substance contained in a given amount of the solution, which is always decanted for the purpose from above crystals with which it has been lying in equilibrium-contact for some time previously, attention need only be concentrated on the determination of the temperature and concentration corresponding to the deposition of the copious crop at the metastable limit, and the estimation is a very accurate one at this stage, owing to the suddenness of the change and copiousness of the crop. By making a large number of such observations with solutions of various degrees of concentration, observing the change with each as it fell from a temperature of about 50° until the crystal cloud appeared, and then combining the results, the supersolubility curve was accurately traced, for it followed the maxima of all the individual curves, and it was found to lie generally more or less parallel to the ordinary solubility curve at a distance indicating an almost constant difference of about 10° .

The explanation is as follows. As the refractive index of a soluble solid substance is greater than that of pure water, the refractive index of the solution is higher than that of water, and is the higher the greater

the amount of salt present, that is, it rises with the concentration. As, moreover, the refractive index is generally reduced by raising the temperature of the refracting solid or liquid, it follows that the refractive index of a solution cooling slowly for 50° without evaporation (a layer of oil being placed on the top to prevent this) shows a rising refractive index until the labile cloud of crystals appears (the slight metastable cloud scarcely affecting it), when the sudden removal of solid causes a correspondingly sudden fall in the refractive index, the effect being greater than merely just enough to arrest the growth of the index. Hence, when the concentration is accurately known, it is only a matter of ascertaining the temperature at the exact moment when the maximum refractive index is observed. But the slight metastable crystallisation, corresponding to the ordinary solubility curve, alters the concentration from that originally started with, and somewhat complicates matters. The actual maximum refractive index at the instant the labile cloud forms is itself, however, an infallible indication of the true strength of the solution at the moment, provided, as was actually done, a preliminary series of determinations of the refractive indices of clear solutions of known strengths at different temperatures had been made. Hence the method is really very simple and complete in itself.

The advantage of this over any method based on mere observation of the formation of the cloud of crystals lies in its precision, for the liquid is already usually turbid from the formation of the first slight shower and an amount of slow growth of crystals which follows it, so that it is not possible to see clearly when the labile cloud first begins to form. The idea of making the solution record its own strength at the critical moment by its refractive index, and the moment afterwards to indicate that same critical point by its sudden jump in refraction, is an ingenious one.

In Fig. 7 are given a typical pair of solubility and supersolubility curves. The lower one marked S is the ordinary solubility curve, and the upper one lettered SS is the supersolubility curve. The dotted curve ABCD shows the normal course of the crystallisation of a solution made by adding the salt to hot water, and which is stirred while being allowed to cool. The solution cools from A to B without anything being observed. But at B crystals make their appearance, forming from germ-crystals introduced from the air. From B to C the liquid is still cooling and crystals are growing slowly, but at C the labile condition is reached at contact with the supersolubility curve, and a cloud of crystals is deposited, the concentration falling rapidly to D on the solubility curve, in most cases with a rise of temperature. It will be noted that crystallisation does not necessarily begin immediately on entering the solubility curve, but usually well inside the metastable region between the two curves. For it will probably happen that a germ-crystal does not enter until after the condition corresponding to first contact with the solubility curve has been attained. But the spontaneous crystallisation of the heavy cloud occurs quite sharply at C, the point of contact with the supersolubility curve, when the

solution is agitated by stirring. If the solution were allowed to cool at perfect rest, however, crystallisation might not occur until the labile condition had been well entered, at some point beyond the supersolubility curve, if it so happened that a germ-crystal did not fall into the solution until this degree of concentration had been attained.

In the important case of a not too strong solution crystallising at rest, from the metastable condition,—the case which the author has recommended in the previous pages as the most suitable for growing good crystals for the purposes of measurement,—the crystals are first started by inoculation from the air, say at B, while the solution is only

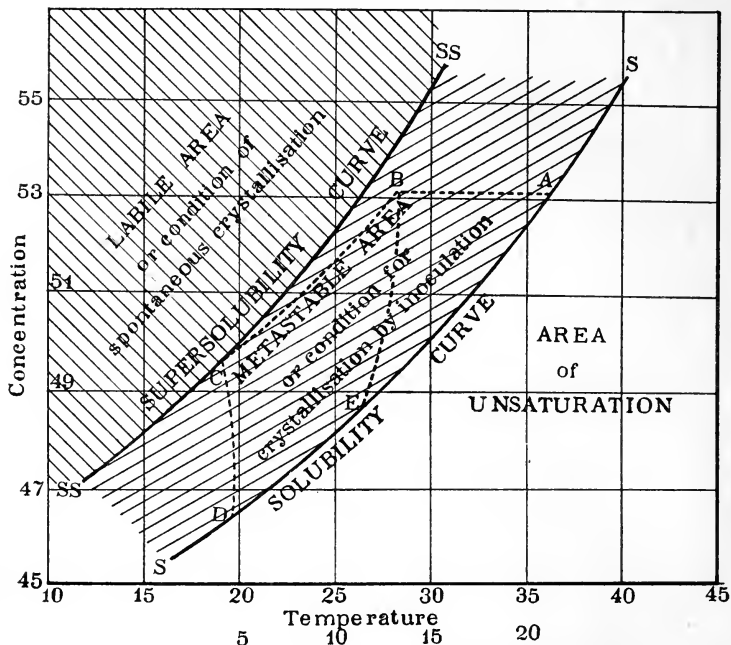


Fig. 7.—Diagram explanatory of Labile and Metastable Conditions of Supersaturated Solutions.

very slowly cooling from A to B. The concentration then diminishes by withdrawal of material from the solution, and the change of temperature being so slow, the labile condition is never reached, the course of the crystallisation being indicated by the dotted curve BE, for which the temperature abscissæ given below the word temperature are more suitable. Thus no cloud of crystals is produced, but only the further steady growth of the crystals first started, the very conditions desired.

In the previous Fig. 6 (page 16) the supersolubility curves are also drawn of anhydrous sodium sulphate, Na_2SO_4 , and the heptahydrated salt $\text{Na}_2\text{SO}_4 \cdot 7\text{H}_2\text{O}$, as given by Hartley, Jones and Hutchinson in the memoir already alluded to. The decahydrated salt does not appear to possess a definite limit of supersolubility. The crystals obtained by

these authors by spontaneous crystallisation, as distinct from those induced in the lecture experiment by the access of germ-crystals, consisted of the heptahydrated salt in the great majority of cases, the decahydrated salt being of comparatively rare and irregular occurrence, except when the cooling occurred very swiftly to -10° .

These experiments have been described at some length because of their importance to the crystallographer, in teaching him the exact conditions under which crystallisation can and does occur. Usually, the conditions recommended in the previous section of this chapter will correspond to the crystallisation of solutions only slightly supersaturated, most probably in the metastable state, the conditions being those of the region between the solubility and supersolubility curves. The solutions will be at as perfect rest as is attainable, but will be open to the air, thus permitting access of crystal-germs and also slow evaporation; hence crystallisation will probably be started by a crystal-germ of the substance itself, or of a substance isomorphous or isostructural with it, and it will proceed slowly and steadily, and as there is no agitation there will be no rapid acceleration of crystallisation if or when the labile state is attained, but a steady further accretion of crystalline material on the lines slowly and deliberately (and therefore perfectly orderly without distortion) laid down during the previous stage of metastability, that is, as a further growth on the same crystals.

In all probability the evaporation and crystallisation will eventually nearly balance each other, the former just keeping very slightly in advance and maintaining the solution very slightly supersaturated, that is, in the condition of metastability. It will also be understood how important is the instruction which has been given, to remove crystals from the mother liquor (when it is desired to collect them for use) early in the morning, before the rising temperature of the room and solution renders the latter less saturated and possibly slightly unsaturated, resulting in the crystals having their edges rounded, and their faces etched.

CHAPTER III

THE GONIOMETER.

It is astonishing how soon the apparent difficulties of crystallography disappear after a short acquaintance with the practical use of the goniometer. Owing to the more or less geometrical and mathematical foundation of the subject, and to the fact that this aspect of crystallography has usually been the one first brought to the notice of students, and is especially prominent in most of the text-books, many students who might otherwise have been attracted to an essentially fascinating and eminently practical study have been discouraged and repelled. If, instead, a few lessons on the practical use of the goniometer can be substituted, the fundamental principles underlying the symmetry of crystals, and the geometrical and numerical relationships of faces, unfold themselves so naturally and obviously that the learner is charmed as with a series of most interesting original discoveries. It will be the writer's object in this and the following chapters thus experimentally to lead the student onwards, introducing only sufficient theory at a time to render the practical steps already taken intelligible, and explaining it more fully when the actual phenomena have been personally observed.

Our first duty is to understand thoroughly the reflecting goniometer, which has been vastly improved since its original invention by Wollaston in the year 1809. After a brief description of the contact goniometer, an account will first be given of the most accurate and at the same time handy form of horizontal-circle goniometer in use in the best laboratories to-day, and subsequently of a simpler and cheaper form of instrument, with a vertical circle, eminently suitable for student use.

The Contact Goniometer, as constructed to-day by Messrs. Steeg and Reuter, is shown in Fig. 8. It was invented by Carangeot of Paris in the year 1780, and was used in pretty much the same form as that shown in Fig. 8 by the earlier crystallographers Romé de l'Isle (for whom Carangeot also made a large number of crystal models) and the Abbé Haüy, previous to the invention of the reflecting goniometer by Wollaston. It is still of use in measuring the interfacial angles of large crystals, either because their surfaces are too dull to afford reflections on the reflecting goniometer, or because the crystals are too large to be mounted on the latter. It consists of a divided semicircular arc, to the

centre of which a pair of slotted bars provided with straight edges are pivoted, in such a manner that they can be fixed at any angle by means of the pivot-screw. The crystal requires to be firmly supported while the two straight edges are laid closely in contact with the two faces in question, the angle between which is to be measured. A sheet of white cardboard is placed well behind so as to afford a brilliantly white background, and the observer then looks along the direction of the edge between the two faces, and, while maintaining the plane of the instrument perpendicular to this edge, carefully adjusts the straight edges so that absolutely no trace of the white background is seen between them and the faces against which they are laid. The milled-headed nut of the pivot-screw is then tightened.

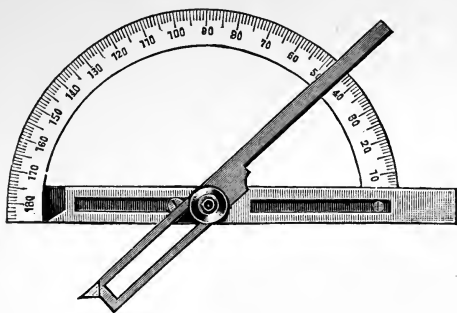


FIG. 8.—The Contact Goniometer.

One of the straight edges is set so that it is parallel to the diameter corresponding to the zero division of the semicircular arc. The central line of the bar is then over this zero mark. Two slots are cut in this bar, on both the right and left side of the pivot, as shown in the figure. The other inclined bar is cut away altogether on the right down to a bevelled edge, which forms the central line of the bar and thus indicates the correct angle between the straight edges on the arc. The measuring edge of the bar to be employed as straight edge is in each case indicated by the bar being made to terminate at a point lying in the edge in question. This particular edge of each bar has been made an absolutely straight line, as accurately as experimental skill can achieve it. The point is also useful when the instrument is used with mineral crystals occupying a cavity in a matrix; the slots on the left also enable the bars to be shortened to any extent to facilitate the use of the instrument even when the cavity is quite shallow.

The Reflecting Goniometer.—The principle of this all-important instrument will be fully discussed with the aid of an explanatory diagram (Fig. 20) at the opening of the next chapter, in connection with the mode of use of the apparatus. It will suffice to say here, before passing to the description of the instrument itself, that its purpose is to measure the angle between any two faces on a crystal, by obtaining a reflection of the light from a collimator (a tube with a lens at one end and an illuminated slit at its focus at the other end) from each of the two faces in succession, and receiving the reflected rays in a telescope, where an image of the slit, which latter is known as the "signal" or "signal-slit," is produced, and can be adjusted to the intersection of the cross-wires as reference spot. The angle through which the crystal requires to be rotated, in order thus to bring the two images

one after the other to the reference spot, is measured on a divided circle, and is the angle between the normals to the two faces, the supplement of the real internal dihedral angle between the faces.

A telescope and collimator are not indispensable, a slit in a distant otherwise darkened window serving admirably as a signal, and a mirror carried by the goniometer and in which the light from the window slit is also reflected, affording an image of the slit which serves equally well as reference line to which the images reflected from the crystal faces can be allocated. But the most accurate goniometers are always

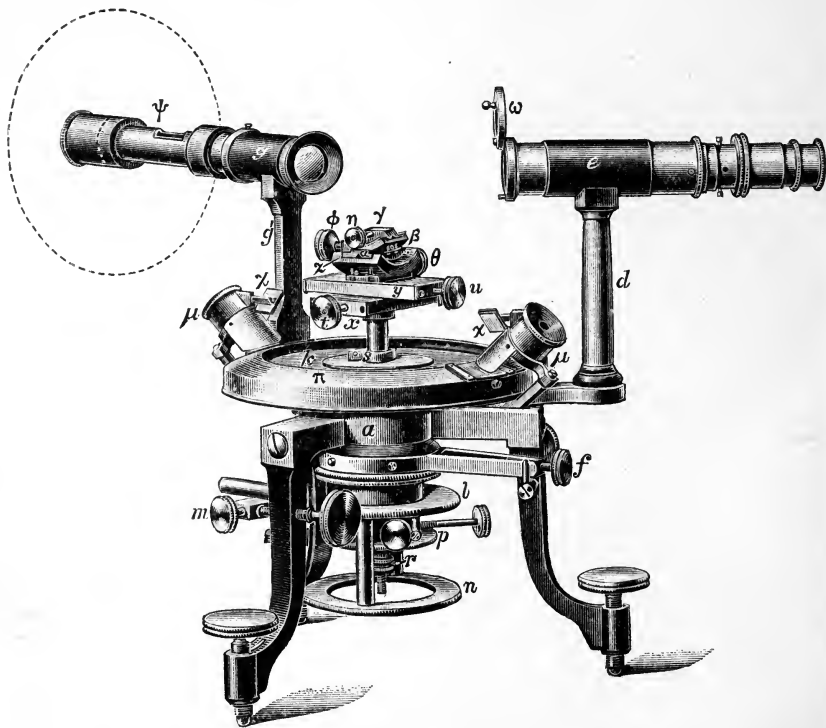


FIG. 9.—No. 2a Fuess Reflecting Goniometer.

supplied with collimator and telescope, like a spectrometer or spectro-scope, the crystal replacing the prism. The best of these reflecting goniometers will be described first.

The Fuess Horizontal-Circle Goniometer.—One of the best forms of reflecting goniometer, with which many original investigations have been carried out, including those of the author, is that constructed by the firm of R. Fuess of Berlin, No. 2 model, and due to the recommendations of such well-known crystallographers as von Lang, Websky, von Groth, Liebisch, and others. It possesses a circle which reads with the aid of a vernier accurately to half-minutes. Its construction will be rendered

clear by Figs. 9 and 10, the former representing the general appearance of the instrument, and the latter a vertical section on the scale of one quarter the actual size. It consists of three essential parts, (1) the rotating horizontal divided circle for the measurement of the angles, (2) a telescope and collimator, the latter provided with the signal-slit, for the observation of the reflections of this signal from the crystal faces; and (3) a delicate but easily and rapidly manipulated apparatus for adjusting the crystal.

The whole is carried by a stout circular table of brass, *a*, supported by three feet provided with levelling screws, and having a hollow cone bored in its centre, where it is thickened. Within this boring rests, and is rotatable, the conical axis *b* which

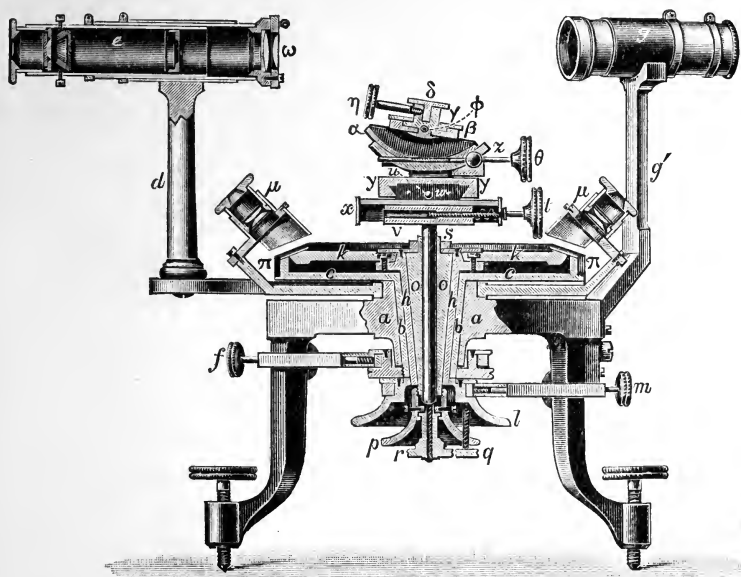


FIG. 10.—Section of No. 2 Fuess Goniometer.

carries a circular plate *c*, provided at two diametrically opposite positions on inlaid silver arcs with vernier divisions, each of 30". To this vernier-circle there is rigidly attached, below, a horizontal arm, which carries at its outer end a vertical column *d* on which is supported the observing telescope *e*. Hence the latter moves with the vernier-circle, and both may be fixed at any position by the clamping screw *f*, which presses a ring against the lower flange of the rotating cone *b*. The collimator *g* is carried by a column *g'*, which is definitely fixed to the table *a*, so that when the clamping screw *f* is tightened the collimator and telescope are fixed relatively to each other, and this may be achieved at the angular position with respect to each other which is found most convenient for the observation of the reflections from crystal faces, which is usually somewhat over 90°.

In the hollow rotatable axis *b* a second similar one *h* is capable of rotation, which carries at its head the divided circle *k*, the silver scale of which is divided directly to 15', and at its lower extremity a hollowed milled disc *l* for the purpose of rotating the

divided circle by the hand, together with all that its hollow cone bears within it, including the crystal on its holder, when the telescope and vernier have been fixed by f . A fixing screw m , working against a clamping ring as in the case of f , is provided for fixing the circle and all that it carries, quite independently of the fixation of the vernier-circle and telescope, so that either the divided circle with the crystal, or the verniers with the telescope, can be separately either moved or fixed. Moreover, the fixing by f and m does not occur directly to the fixed base or feet of the instrument, but to the ring in each case, and this ring is continued horizontally into a narrow V-shaped arm in order to provide a fine adjustment, by a screw working through the basal foot at right angles to the direction of f or m . The arm carries a strong spring alongside it, free at the outer end but screwed to the ring at the inner end, and when the adjusting screw pushes the arm, the free end of the spring is pressed against a little bracket carried by the foot, and thus fine adjustment is afforded until the spring and arm are forced into contact, and on releasing the screw the arm retrogresses, along with it owing to the pressure of the spring. Hence, the divided circle and the crystal carried in rigid attachment with it may be fixed by m , and the telescope and the pair of verniers which move with it rotated about them; they may then be fixed by f at any desired point, and the fine adjustment carried out by use of the adjusting screw. Equally well, the telescope and verniers may be fixed by f , and the divided circle and the crystal which it carries may be rotated instead, by means of the disc l , or more readily still by a ring n attached below l by two vertical rods, and eventually fixed in the desired position by m , and finally adjusted by the adjusting screw.

Within the conical axis h there rotates still a third one o , manipulated by the milled disc p attached to its lower end. This axis is primarily intended for the support of the crystal, which is thus carried separately and not directly by h , in order that the rotation of the delicately divided circle may be avoided during all the preliminary adjustment of the crystal, and unnecessary wear and tear of that all-important axis saved. Moreover, the further advantage is secured of rendering the crystal axis very free of movement, not having the weight of the circle to carry except during measurements, and thus enabling the preliminary adjustments to be very rapidly performed. The crystal axis o and the circle axis h can be locked together for the measurements either by a simple screw q as shown in Fig. 10 (No. 2 model), or by a fixing screw and a fine adjustment screw similar to f and m , as shown in Fig. 9, the latter being the arrangement on the latest form of this admirable Fuess instrument, model No. 2a, which has been employed by the author throughout his researches, and which is also shown with its accessory fittings in Fig. 11.

The crystal is still not carried directly by the conical axis o , but at the head of an innermost cylindrical axis, which is keyed to prevent rotation, and the lower end portion of which is narrowed, and provided with a screw thread which gears with a flanged vertically immovable nut terminating below for the purpose of manipulation in a small milled disc r , the rotation of which consequently causes vertical motion of the crystal axis. The object of this inner axis is thus to provide the crystal with an adjustment for height, so as to bring it to the level of the plane of the optic axis of the telescope and collimator. It may be fixed at this height, if required, by tightening the collar s by means of a key provided.

The innermost axis just referred to bears the adjusting apparatus at the head of which the crystal itself is carried. This adjusting apparatus consists of two mutually rectangular horizontal movements for centring the crystal, and above them two circular movements in vertical planes also at right angles to each other, for adjusting the tilt of the crystal so as to bring the intersecting edge of any two faces, or the axis of any zone of faces, truly vertical and parallel to the goniometer axis. The two combined pairs of movements enable the edge or zone axis to be brought exactly into the axis of the divided circle of the goniometer. The lower centring movement is manipulated by the traversing screw t , and the upper one at right angles to the lower by the similar

screw *u*. Each of these two screws is surrounded by a spiral spring confined between the fixed piece *v* or *w* (the nut of the screw) and the traversing box *x* or *y* to prevent "backlash." The fixed piece *w* of the upper movement is only fixed as far as regards movement by *u*, being rigidly attached on the top of and at right angles to the traversing

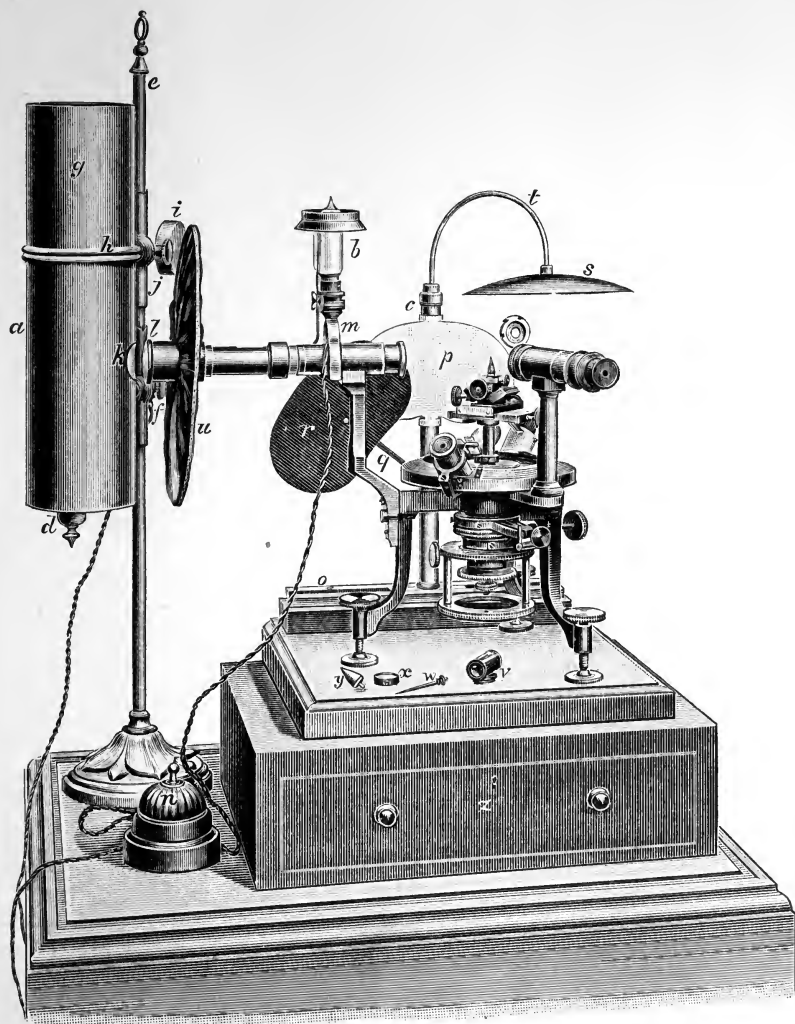


FIG. 11.—No. 2a Fuess Goniometer in Actual Use.

box *x* of the lower movement, with which it therefore moves. On the traversing box *y* of this upper centring movement is fixed the cylindrical bed *z* of the lower rotatory adjusting segment *a*, and on the latter the similar but smaller bed *β* of the upper segment *γ*, the circular movement of the two sliding cylinder-segments being effected by the two tangent screws *θ* and *φ*, provided with spring arrangements seen under the

segments in Fig. 9, for ensuring perfect contact and no free play. The two segments have a common centre of movement at a height somewhat above the small tabular crystal holder δ , which fits, by a little peg in a central vertical boring in the upper segment, in which position it may be fixed by the screw η ; this enables the crystal to be supported at approximately the centre of movement by and at the apex of a little cone of goniometer wax built up on the tabular holder-top, and thus the amount of centring required by the centring screws t and u is minimised. The angle of movement of each segment is about 40° on each side of the centre, which is ample to allow of several adjacent zones of faces being measured without re-setting the crystal on the wax. The wax employed is a judicious mixture, generally about two parts to one part, of beeswax and pitch, melted up together and thoroughly blended; when cold it is sufficiently adhesive and soft to permit of ready attachment of the crystal, while being adequately solid not to allow slipping after attachment.

The divisions of the divided circle k and of the vernier-circle c are protected by a brass cover-cap, π , attached below to c , and the scales themselves, which are engraved on silver surfaces inclined (the circle plate being bevelled) towards the observer for convenience of reading, are read through two diametrically opposite glass-covered windows, at the places where the vernier divisions of c are engraved, by means of a pair of microscopes μ carried on suitable arms from a ring fitting loosely round the boss of the fixed table α . This method of support enables the microscope in either case to follow the vernier divisions; the latter are evenly illuminated by light reflected from a little adjustable mirror, χ , and diffused through a thin screen of horn, and each microscope is furnished with an adjustment to the proper focal position for clear reading of the scale and vernier divisions.

The collimator g , carried by the column g' in rigid connection with the fixed table of the instrument, is provided with an achromatic lens at the end nearest the crystal, and at the outer end, at the focus of the lens, with one of the four following interchangeable signals, each of which is fitted in a separate draw-tube sliding in the outer collimator tube. (1) An ordinary rectilinear slit, provided with an adjustment for one of the jaws, in order to be able to vary the width of the opening by means of

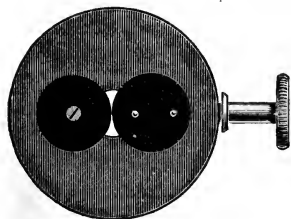


FIG. 12.—The Websky Signal-Slit.

a little traversing screw. (2) A "Websky" slit, shown in Fig. 12, formed by two circular discs arranged in front of a round aperture, and the separation of which can be varied by the movement of one of them by means of a traversing screw, the milled head of which is shown to the right in the figure. This is the form of signal-slit which is practically always employed in the measurement of crystal angles, for it combines the advantages of a narrow central part, the image of which may be readily adjusted exactly to the vertical spider-line of the telescope eyepiece, with broad ends which

transmit much more light than a rectangular slit narrow enough for accurate adjustment, and consequently affords bright images even from poor or minute faces which would scarcely show the image of a rectangular slit at all. (3) A round pinhole, which is sometimes useful to determine a small deviation of a face out of a zone, or for the study of groups of vicinal faces (adjacent faces only very slightly inclined, that is, nearly parallel, and affording signal-images very close together). (4) A "Schrauf" signal which is sometimes used in preference to (3) in the cases of very brilliant faces, and which consists of two broad rectangular slits arranged cross-wise like the diagonals of a square, and with a horizontal and a vertical cross-wire at their intersection in the centre of the field.

It is an advantage to employ in front of the slit an "illumination" tube ψ as shown in Figs. 9 and 11, consisting of a brass tube slipping over the end of the signal tube

and widening towards the outer end, where it carries a condensing lens somewhat larger than the collimator and telescope-objective lenses and the focus of which is the slit, so that the light from the illuminating lamp is concentrated on the latter. The tube has an open rectangular window in its narrower part near the slit, for convenience of rotating a Nicol prism which is provided to slide, when required, in the tube at this part, when it is desired to polarise the light entering the collimator, as during the determination of the refractive indices of doubly-refracting crystals. Longitudinal marks on the Nicol and on the window edge enable the former to be set with its polarising plane in the desired direction, vertical, horizontal, or at 45° .

The observing telescope *e* is provided with four interchangeable eyepieces, three of which are positive achromatic combinations magnifying respectively about seven, five, and three times, the last of which is almost always used as being the best combination for affording good lighting of the signal reflections from crystal faces, while still yielding adequate definition and delicacy of measurement. The fourth is used in connection with a lens supported, when required, by a suitable adapter about an inch in front of the objective; this combination furnishes an image which is actually less than the real size of the signal-slit, and is designed for use in those cases where the image afforded by the ordinary eyepiece is very feeble, owing either to the very minute size of the crystal or the particular face, or to the dulness of the face, the concentration of light in such a small image rendering it visible and adjustable to the intersection of the crossed spider-lines when it would be almost invisible and certainly not accurately adjustable if the ordinary third eyepiece were used, which far more than counterbalances the loss of delicacy in the measurement. Each eyepiece carries at the focus of the lens combination, which is adjustable in a draw-tube, a pair of rectangular crossed spider-lines, mounted in an inner ring adjustable for its position in the tube by four screws; the outer eyepiece tube in its turn slides in the main objective tube so that the telescope can be accurately adjusted for parallel rays, and when this is achieved the position can be made recoverable at any time by means of a collar, which can be fixed by a tightening screw and which has a V-shaped projection fitting in a corresponding notch in the fixed objective tube, and which is so adjusted that when the eyepiece is pushed home not only is adjustment for parallel rays effected but also the crossed spider-lines are accurately horizontal and vertical.

In front of the objective, which is a similar achromatic lens to that carried by the collimator, there is also attached, by a marginal pin so that it can be rotated out of the way when the telescope is to be used as such for parallel rays, an additional lens *w* which converts the telescope into a low-power microscope the focus of which is in the axis of the goniometer, and which, therefore, affords the observer a magnified view of the crystal and thereby greatly facilitates the adjustment of the latter, and also enables the reflecting face to be identified by actual observation, the face appearing brilliantly illuminated while the rest of the crystal is relatively dark.

A convenient mode of mounting the instrument, on a mahogany base forming the plinth for a glass case to protect the whole when not in use, is shown in Fig. 11, which exhibits also three essential accessories. These are: (1) the goniometer lamp *a*, (2) an intermittent light *b* for reading the verniers, and (3) a simple apparatus *c* for furnishing a white or black background.

The goniometer lamp is a metallic filament electric glow-lamp of 32-candle power with clouded glass, mounted on an arm *d* capable of sliding up and down a tall standard *e* and of being fixed at the convenient height opposite the lens of the illuminating tube by means of a milled-headed screw *f*. The observer is shielded from the light by means of an enveloping vertical copper cylinder *g*, supported in a ring *h* carried by an adjustable arm similar to that which supports the lamp, and which also is continued on the other side of the standard into a counterpoise *i* to

facilitate the sliding, and is capable of fixation at the convenient height by means of a screw *j*. A circular aperture *k* $1\frac{1}{4}$ inch in diameter is cut out of the cylinder at a little more than one-third of its height, and the arm is adjusted so as to bring this aperture exactly opposite the lens *l* of the illumination tube, so that the light may fill the latter. The vernier light *b* is a small cylindrical-bulb electric glow-lamp of 8-candle power, mounted on a little fitting *m* above the outer fixed tube which carries the collimator, and shaded downwards on to the verniers, the green shade being removed in the figure in order to reveal the lamp. A switch *n* is provided on the mahogany base, to the observer's left, for this lamp requires to be switched off after each reading of the verniers, as the placing of the signal-image reflected from a crystal face, to the cross-wires of the telescope, requires to be carried out in the dark, with no light, that is, except that from the goniometer lamp. For this reason goniometrical observations are either carried out at night, or in a room provided with means of darkening.

The construction of the simple background apparatus will be clear from Fig. 11. For the accurate adjustment, to the cross-wires, of the image of the signal reflected from a crystal face, a dark background is required, while the preliminary adjustment and centring of the crystal by the two circular and two rectangular movements provided for the purpose is best carried out in front of a white background, until the operation is approximately achieved. In order to supply either background as desired, the little arrangement shown at *c* in the illustration was devised. It consists of a brass pillar screwed into a bevelled foot-plate, which is adjustable to the most convenient position by sliding in a dove-tailed groove in a heavy metal base *o*. The pillar carries the fixed white xylonite background *p* in front, and is pierced by an axle at about a third of its height, below the xylonite plate; to the front end of the axle is attached an arm *q* which carries the black ebonite background *r*. The amount of rotation is so limited by a pin and a slot that at one extremity of movement the black background is in position perpendicular to the axis of, and opposite to, the telescope, while at the other it is rotated over so as to leave the white background exposed instead. A thin brass canopy *s* to protect the crystal from scattered overhead light is also loosely attached to the top of the pillar by a curved rod *t* rotatable in a socket in the pillar-head, so that it may be turned out of the way when reading the verniers. The other accessories shown in Fig. 11 are the screen *u* of thick folded green silk over a circular wire frame, for screening off the light issuing from the aperture *k* from the observer's eye; the Becker fitting *v*, the adjusting needle *w* and glass-plate mirror *x*, the use of all three of which will be explained in the next section; and a second crystal holder with its wax cone *y*. There is also a drawer *z* in which accessories, note-book, crystals in their miniature bottles, or other articles required may be stored.

Adjustment of the Goniometer.—Before proceeding to any measurements of crystal angles with the instrument it is essential that it should be carefully adjusted. For this purpose the following series of operations should be carried out in the order named, as originally given by Websky.

(1) The eyepiece to be employed, preferably the one magnifying about three times, is first adjusted, as regards the position of the draw-tube carrying the positive achromatic lens combination within the outer tube carrying the crossed spider-lines, until the latter are clearly focussed. The eyepiece is then inserted in the telescope tube and its position varied until the whole telescope is adjusted for parallel rays, as indicated by the clear focussing of some very distant object.

(2) A special holder, *w* in Fig. 11, resembling the crystal holders in

its mode of attachment to the goniometer, but carrying a vertical needle instead of a tabular head for the attachment of a wax cone, is inserted by its peg in the central hole in the top of the crystal-adjusting movements, and adjusted by the latter vertically in the axis of the goniometer. The eyepiece is then rotated, if necessary, until one of the spider-lines is parallel to the needle as viewed through the micro-telescope, the lens ω (Fig. 9) in front of the objective having been rotated into position. It may now happen, after full use has been made of the centring movements and the needle remains apparently immovable as the goniometer axis carrying it is rotated, that the images of the needle and the vertical spider-line are not coincident, but only parallel. They must now be brought into coincidence, and this can be done either by moving the spider-lines horizontally or by an adjustment of the setting of the pillar which supports the telescope. The crossed spider-lines are mounted in an annular frame which slides with some little play in the main eyepiece tube, and is adjustable from outside by four diametral screws, a horizontal pair and a vertical pair; the two former enable the required adjustment to be made. But when the adjustment required is more than a very minute amount, and involves displacing the intersection of the spider-lines from the optic axis of the telescope, the author prefers to effect the adjustment of the whole telescope, which is really also the more correct thing to do. For this purpose an addition has been made to the instrument as furnished by Fuess, of three little adjusting screws passing from below through the arm which carries the pillar, and pressing up against the circular base of the pillar near its periphery. Starting with these screws not in action, and the base firmly screwed to the arm by the stout central fixing screw, on observing that the vertical spider-line lies to one side of the needle, it is easy to bring the little screw on the opposite side, or two of the three screws, into action while slightly releasing the fixing screw, until the needle and spider-line are coincident, and subsequently to tighten up the fixing screw again. Probably the latter operation will have very slightly again displaced the spider-line from coincidence with the needle image, and one or two closer and closer approximations may be necessary before final adjustment is achieved.

(3) A small circular plate of glass provided with the instrument, with surfaces ground truly plane and parallel, and mounted in a circular frame, x in Fig. 11, with peg for insertion in the hole in the top of the adjusting movements, is next adjusted approximately parallel to the axis of the goniometer, and then left as nearly as can be judged perpendicular to the axis of the telescope. A small fitting carrying another small plate of glass arranged at, or adjustable to, 45° to the axis of the telescope, is now fixed in front of the eyepiece, and a lamp arranged to reflect light from the inclined glass plate through the telescope to the circular glass plate on the goniometer axis, so as to obtain both a direct view of the crossed spider-lines by scattered light and their reflection in the parallel glass plate. An excellent

little fitting for the purpose is now supplied by Messrs. Becker of Hatton Garden. It is shown at *v* in Fig. 11, and more clearly in Fig. 13. It screws on to the end of the cap-tube in front of the eyepiece instead

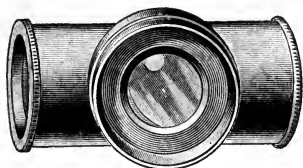


FIG. 13.—The Becker Transparent Eyepiece Mirror.

of the usual cap with eye-hole, and the inclined plate is carried in a little tube fixed transversely to the very short tube bearing the screw thread just referred to; this transverse tube is closed at one end, at the other carries a lens to concentrate the light from the lamp on the glass plate, and is pierced with a hole in front of the centre, through which the observer looks instead of through the usual eyehole. The illumination

of the spider-lines afforded by this fitting is excellent, and enables the adjustment to be very rapidly carried out. The parallel glass plate should now be rotated in order to see if it is exactly parallel to the axis of the goniometer, as indicated by the images of the spider-lines reflected from the two surfaces of the plate occupying in turn exactly the same position in the field, and if this is not the case it should be adjusted so that it is so. If now the intersection of the spider-lines as reflected is not quite coincident with the intersection as seen directly, it should be made so either by raising or lowering the spider-line fitting by its adjusting screws, or by using the adjusting screws at the base of the telescope pillar as already described for adjustment (2). The axis of the telescope will then be truly normal to the plate and perpendicular to the goniometer axis. The horizontal spider-line should also be exactly perpendicular to the goniometer axis and coincident with its own reflected image, and should appear to remain so, although both spider-lines and their intersection will traverse the field (moving at right angles to the goniometer axis), when either the plate or the telescope is slightly rotated about the goniometer axis. When finally satisfied that the adjustment of the spider-lines, eyepiece, and the whole telescope is perfect, the collar of the eyepiece should be brought along so that its V fits the notch in the outer telescope tube, and fixed there by the clamping screw, so that the adjustment may be at once recovered whenever this eyepiece is again replaced after temporary removal. The fitting in front of the eyepiece is now removed, and the ordinary cap with hole substituted.

(4) The small circular-hole signal is next inserted in the collimator, and the telescope rotated into the same straight line with the collimator so as to observe the hole; the latter is then brought to sharp focus by movement of the signal tube in the outer collimator tube, and the latter adjusted, by means of the adjusting and fixing screws by which its supporting column is attached to one of the legs of the tripod, so that the centre of the image of the signal-hole lies on the horizontal spider-line. The Websky and the straight-edged signal-slits may next be inserted in turn in the collimator tube, and moved about therein until focussed clearly by the telescope in the truly vertical position and

symmetrical to the crossed spider-lines of the telescope. After this adjustment is achieved, in the case of each of the three signals, the sliding collar should be moved up so that its projecting V fits into the notch of the collimator tube, and fixed there by the clamping screw, so that when any one of the signals is required it can at once be pushed into position. The Schrauf diagonal slit and cross-wire signal should next be inserted in the collimator tube, and the intersection of the cross-wires brought into coincidence with that of the spider-lines of the telescope by the four traversing screws of the signal itself, provided for the purpose, after which in this case also the sliding collar may be fixed. This signal may be advantageously left in for the adjustment of the other eyepieces than the one magnifying three diameters already adjusted, after which it should be replaced by the Websky signal, when the goniometer will be quite ready for use in crystal measurement.

For the measurement of crystal angles at higher temperatures than the ordinary and the determination of the minute change of angle brought about by rise of temperature, which may amount to only a few seconds, a larger horizontal goniometer reading to seconds is constructed by Fuess, the No. 1a. This instrument, however, and its use in higher temperature work, will be described in Chapter XXVIII. It is also eminently suitable for the determination of refractive indices. It is not so handy, however, for crystal-angle measurements at the ordinary temperature, in which the extra refinement beyond half-minutes is of no advantage; for analogous angles on the very best crystals of the same substance differ by as much as a whole minute, owing to slight disturbance of the conditions during deposition. The instrument No. 2a is undoubtedly the goniometer *par excellence* for crystal-angle work at the ordinary temperature.

A simpler and much cheaper form of horizontal-circle goniometer suitable for student use is also now furnished by Fuess, which is quite an efficient instrument although not so delicate as the No. 2 or 2a models, and not fitted with such excellent means of ready and accurate adjustment.

Troughton and Simms Goniometer.—An accurate form of horizontal-circle goniometer constructed by Messrs. Troughton and Simms is shown in Fig. 14.

Its principle is the same as that of the Fuess instrument.

The lowest *a* of the three milled heads shown under the circle-cone is rotated to raise or lower the crystal *b*, and its holder *c* and adjusting apparatus *d*; the middle one *e* rotates the latter with the crystal, during adjustment, and the large upper milled head *f* is used to rotate the circle. A novel feature, due to Dr. H. A. Miers, concerns the lens in front of the objective of the telescope; for instead of the usual one rotatable merely in or out of the field, in order to convert the telescope into a microscope while observing the crystal itself, this lens *g* and a second one *h* are mounted at the extremities of a short cross-piece carried by a horizontal bar *i*, the whole forming a T-piece which is capable of swinging about a bearing *j* carried by a short tube *k* sliding along the outer telescope tube *l*. When arranged as shown in the figure, the first lens *g* is in the usual position for crystal observing purposes, but when the T-piece is swung back the second lens is brought in front of the eyepiece,

and serves to follow up the signal-image reflected from any particular face, until it becomes replaced by the illuminated face itself, as the lens is drawn away from its

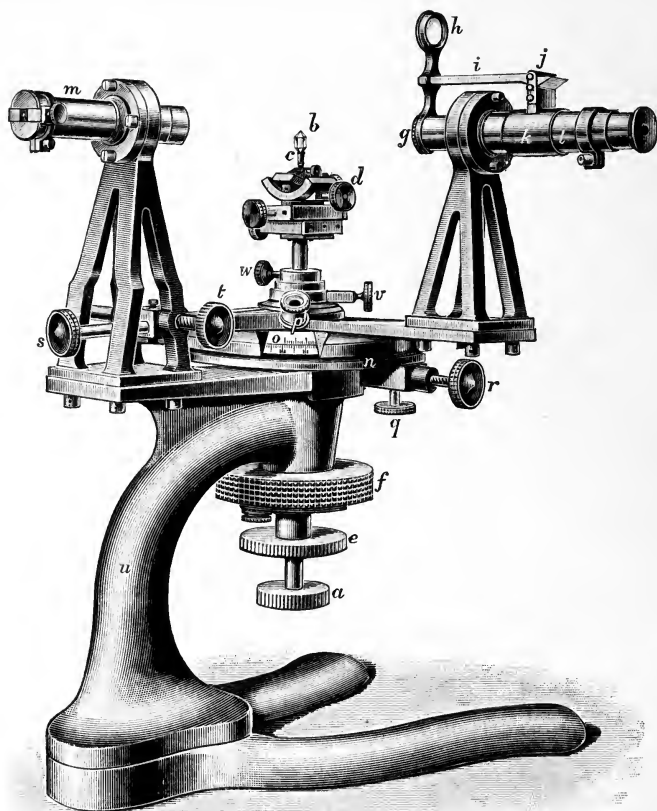


FIG. 14.—Troughton and Simms Horizontal-circle Goniometer.

position close to the eyepiece towards the eye. This device is of use when it is doubtful from which of several illuminated and very closely adjacent faces an image is proceeding, and was invented for the study of such vicinal faces.

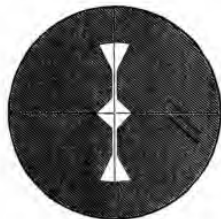


FIG. 15.—Miers Signal-slit.

In Fig. 14 the collimator is marked with the letter *m*. The circle is indicated by *n* and its vernier by *o*; *p* is the reading microscope, *q* the fixing screw and *r* the fine adjustment of the circle. The fixing screw for the telescope is marked *s* and the fine adjustment *t*. The whole is supported on a convenient form of rigid stand *u*. The crystal-holding and adjusting axis may be fixed to the circle by the screw *v*, and *w* is a clamping screw for fixing the innermost steel axis after it has been raised or lowered by the milled head *a*. A modification of the Websky slit was also introduced by Miers, the nature of which will be clear from Fig. 15.

Relative Advantages of Horizontal- and Vertical-Circle Goniometers.

—The chief advantages which the horizontal-circle form of goniometer possesses over the form with vertical circle are the following three: (1) the crystal is less liable to displacement during the measurements, by the yielding of the wax cone of support; (2) larger crystals can be used with greater convenience, largely owing to the greater rigidity of the wax support; (3) the large amount of possible movement of the telescope, nearly 360° , enables the instrument to be employed for the determination of refractive indices, as well as for the measurement of crystal angles. At the same time the vertical-circle instrument possesses the following three relative advantages, namely: (1) the inclined telescope (if one is employed) is easier to observe down than the horizontal one; (2) it is easier to read the circle, the graduations being on the edge of the vertical disc; (3) the milled heads for rotation of the various axes, and possibly also those of the adjusting and centring screws, are somewhat more accessible. On the whole, however, the author vastly prefers the horizontal-circle instrument for refined work. But for teaching purposes a relatively much cheaper instrument can be constructed on the vertical-circle principle, and the best, a very efficient, form of this goniometer will next be described.

The Miers Student's Goniometer.—This is a much less costly form of goniometer constructed on the vertical-circle principle by Messrs. Troughton and Simms to the designs of Dr. H. A. Miers, and is eminently suited for the use of students. The observation by telescope and collimator is replaced, in the simple form of the instrument recommended, by the use of a mirror to reflect the image of a distant signal-slit, placed in an otherwise darkened window, and which is likewise caused to be simultaneously reflected from the crystal face, the reading of the circle being taken when the two reflections are brought into coincidence to an observer looking vertically downwards on to crystal and mirror from above. A general view of the instrument is given in Fig. 16, and Fig. 17 will assist in explaining its principle.

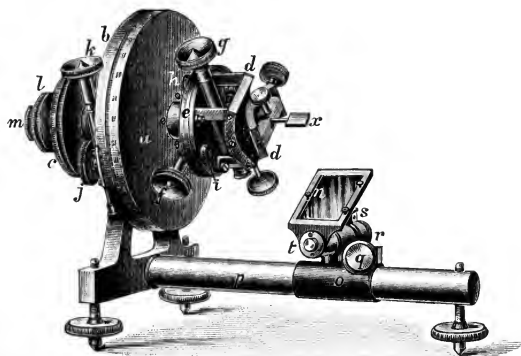


FIG. 16.—Miers Student's Goniometer.

In the latter illustration M represents the fixed mirror, I the incident light originating from the signal-slit cut in a window shutter at not less than 20 feet from the mirror, ab and bc two adjacent faces of the crystal, the edge b between them being arranged so as to be parallel to the mirror and the face ab so as to afford a reflected image of the

signal-slit. This image is then brought, by rotation of the crystal, to coincide exactly with the similar image of the signal-slit reflected in the mirror, and the crystal being so small compared with the size of the mirror there is no difficulty in seeing both images at the same time. When coincidence is attained a reading of the circle is taken. The crystal is then further rotated until the image reflected from *bc* is similarly brought to coincidence with the mirror image, and another reading of the circle made. The difference of the readings is the angle required, namely θ , that between the normals *nn* and *mm* to the two faces. For if we draw these normals to the faces *ab* and *bc* respectively

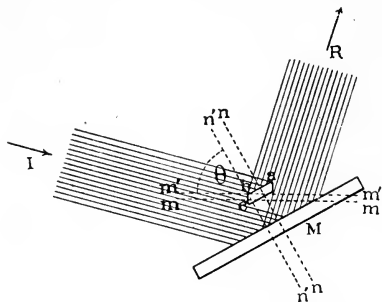


Fig. 17.—Principle of Student's Goniometer.

tively at their centres, and then move them parallel to themselves until they occupy the positions *n'n'* and *m'm'* in which they intersect at the adjusted edge *b*, the centre of rotation, then on rotation of the crystal from the position shown in the figure, when the image of the signal reflected by the face *ab* is adjusted, to that when the image from *bc* is similarly adjusted, the normal *m'm'* moves through the angle θ , that is, until it becomes identical with that occupied under the conditions represented in the figure by *n'n'*. This angle θ is thus the required crystallographic angle between the normals to the two faces *ab* and *bc*.

The actual construction of the instrument will be clear from Fig. 16. The circle *a* is vertical and divided directly on the periphery to half-degrees, while a vernier on the fixed disc *b* against which the circle rotates enables the latter to be read to one minute; it is rotated by the large milled head *c*, and is kept in close contact with the fixed disc by means of a circular steel spring inserted between *c* and the bearing of the axle. The crystal is carried at the end of an adjusting *d* and centring *e* apparatus on the same principle as that of von Lang on the Fuess instrument, namely, two mutually rectangular centring movements and two circular adjusting movements likewise arranged at right angles. The centring movements are of different construction from those of Fuess, each depending on the movement of a circular plate by a screw *f* or *g*, about a pin *h* or *i* passing through the disc near the periphery into a second similar disc, and with the aid of a boss which is kept pressed up against the screw by a spring piston on its other side in the same straight line as the screw. The crystal is thus moved along two circular arcs crossing each other at right angles, and to which the directions of the two screws *f* and *g* are tangential, which arrangement serves the same purpose of centring as if the arcs were straight lines. The discs are kept in close contact by a pin working in a circular arc-shaped slot at the opposite side of the disc to the axle pin, a spring being confined between the broad head of the pin and the outer disc to effect the essential close contact. The cylindrical adjusting segments are similar to those of the Fuess goniometer.

The circle is provided with a fine adjustment, the screw *j* clamping the divided rotatable circle to the fixed circle *b*, and *k* being the fine adjustment screw which is subsequently brought into operation. The milled head *l* turns the centring and adjusting piece with the crystal, and the smaller outer milled head *m* locks that piece and all that it carries to the divided circle.

The mirror n is carried above a tube o sliding on the basal cylindrical bar p , in such a manner as to be capable of adjustment in two vertical and mutually rectangular planes. The tube o can be clamped anywhere along the bar p by the fixing screw q of a tightening collar. A short column rises from o carrying a similar but somewhat narrower tube r at right angles to o , and provided with a similar tightening collar at s . Within this a cylinder slides, but is keyed so as to be incapable of rotation, and may be fixed at any position by s ; to one end of the cylinder the mirror is attached by means of the capstan-headed screw t , which can be loosened so as to allow the mirror to be adjusted by rotation and then screwed home. The mirror is thus capable of four movements, namely, rotation about t and also of o about p , these two rotations being in planes at right angles; also two perpendicular rectilinear movements by the sliding of o along p and of the cylinder in r . The adjustment of the mirror is achieved once for all by setting it parallel to the axis of the goniometer, while the screw t is loosened and with the aid of the signal-slit in the window and a small disc of truly plane and parallel-surfaced glass, mounted instead of a crystal at the end of the crystal-adjusting movements, as described for the adjustment of the Fuess goniometer. The screw t is then finally fixed, and the other movements employed until the mirror is in the best position for seeing the simultaneous reflections from a crystal and the mirror, when the clamping screws should be tightened.

A similar instrument with an equally accurate circle but with a simpler crystal-adjusting apparatus of the type originally used by Wollaston, the inventor in the year 1809 of the reflecting goniometer, is also constructed by Messrs. Troughton and Simms for the small sum of five pounds, and is shown in Fig. 18. The construction of the two simple adjusting movements d' will be quite clear from the illustration; two rods u and v at right angles to each other, one of them v carrying the crystal holder x , can be rotated in bearings by milled heads y and z , and serve for the adjustment of the crystal edge, that is, of the pair of faces intersecting in the edge. The centring is achieved by means of the slotted bar e' , which is capable of being clamped to the circle when the crystal is centred, by the milled head a shown in the figure, and which carries the adjusting movements at the end of an elbow β springing from its inner end.

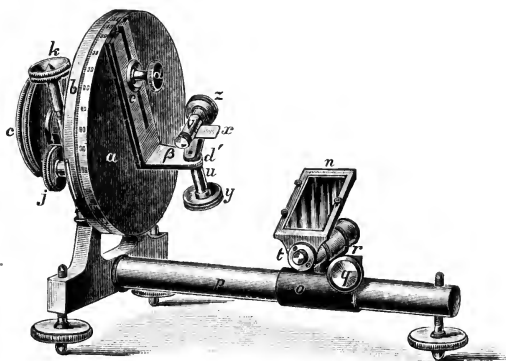


FIG. 18.—Simple Form of Student's Goniometer.

In using either form of this goniometer the observer looks down from above on to the crystal and mirror, but with the eye accommodated for distance, so as to focus sharply the images of the window-slit reflected from the crystal face under observation and from the mirror. In order to adjust the crystal so that the two images exactly coincide, it is convenient to cut off half of the light falling on the

mirror from the signal-slit, by means of a little cardboard screen placed on the table in front of the goniometer, of the right height to perform this function while allowing the light to fall freely on the crystal. By moving the screen about over the table part or all of the signal reflection from the mirror can be cut off, so as to distinguish it plainly from the crystal reflection, and when half only is cut off it is easy to make the signal-image from the crystal form an exact prolongation of the remaining half, and in this way obtain a better coincidence than by complete overlapping, as it is difficult to distinguish the separate edges when near coincidence. When this coincidence has been thus attained the reading of the circle is taken.

If it is found inconvenient to employ a window-signal, for instance because the room is very small, a collimating arrangement of a small slit in an opaque disc placed at the focus behind a large lens of about five inches focal length, the latter being brought close to the goniometer and the former illuminated by a table lamp, serves admirably.

The author finds it better to have for use with this arrangement a. separate mirror-fitting with the mirror raised higher, nearer to the crystal, than the ordinary one, as the two images are then better covered and illuminated by light from the central part of the lens.

By the addition of a collimator γ and telescope δ to this instrument,

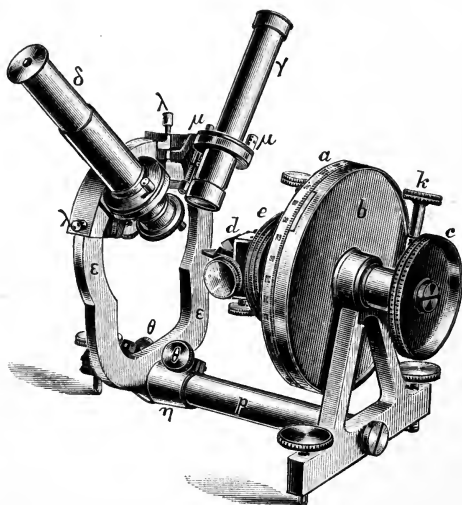


FIG. 19.—Miers Student's Goniometer with Telescope and Collimator.

as shown in Fig. 19, it can be converted into an efficient goniometer more or less comparable with the Fuess instrument at less than half the cost. The levelling screw at the end of the bar p is temporarily removed, while the mirror-fitting is replaced by one ϵ carrying the telescope and collimator, after which the levelling screw is again fixed in position. The sliding tube η which carries this fitting may be clamped at the proper position by two screws θ , which tighten a pair of split collars. The telescope δ and collimator γ move over a vertical circular arc of rather more than a semi-

circle, and they may be fixed at the suitable positions by means of screw clamps. Adjustment screws λ and μ are provided which enable the optical tubes to be adjusted exactly in the vertical plane and directed truly to the axis of the instrument.

In Chapter XXIX. some further forms of goniometer having two and three circles will be described.

CHAPTER IV

THE MEASUREMENT OF CRYSTAL ANGLES, PRACTICALLY EXEMPLIFIED ON A CRYSTAL OF POTASSIUM SULPHATE.

THE most practical method of illustrating the mode of measuring the angles of crystals will be to take an actual example, from among the numerous crystals of high perfection measured by the author. During the process the regular plan of symmetry on which the crystal is built up will be elucidated, and the mode of graphically representing the results in a "spherical" or "stereographic projection" explained. The example chosen is a crystal of potassium sulphate, a salt already referred to in the first chapter and illustrated in Figs. 1, 2, and 3, and so common and cheap, while exceptionally pure, as to be within the reach of every one. In preparing similar suitable crystals for measurement, in order to repeat these sample measurements as a first practical exercise in goniometry, the general rules given in Chapter II. must be followed, especial care being taken that the solutions, when cooled to the ordinary temperature, shall not be more than the slightest amount supersaturated. The normal sulphate of potassium, K_2SO_4 , is only sparingly soluble in water, 10 grammes dissolving in 100 cubic centimetres of water at the ordinary atmospheric temperature, so that solutions stronger than ten per cent cannot be made at this temperature, and even at 100° C. potassium sulphate is only soluble to the extent of 26 grammes *per* 100 c.c. There is great risk of obtaining long irregular prisms unsuitable for measurement by cooling from a solution saturated at 100° , but a solution just slightly supersaturated at the ordinary temperature, say by solution of about 12 grammes of the salt in 100 c.c. of boiling distilled water and slow and spontaneous cooling, will generally be found to deposit at least one or two well-developed crystals of the type shown in Fig. 20 (which is a repetition of Fig. 1, made here for the reader's convenience), on leaving the solution to continue to cool and deposit its excess of salt during the night, under the protected conditions described in Chapter II. One or two crystallisations may be necessary to secure really excellent crystals, and sometimes such solutions remain thus slightly supersaturated for several days before crystallisation ensues, but with a little perseverance good crystals will eventually be obtained. The solution will probably be in the metastable condition

and require to have its crystallisation started by means of a minute crystal of the salt falling into it from the air, as explained in Chapter II. For both potassium sulphate and its isomorphous analogue, ammonium sulphate, which would equally well start the crystallisation, are very common salts, and germ crystals of them are probably floating in the air of every laboratory.

Having thus secured a satisfactory crop of small crystals and stored them in a miniature bottle, properly labelled as described in Chapter II, one of the best may be selected, of the small size there recommended and which may be as much as three millimetres across the widest part in the case of the salt under consideration. Such a crystal will afford very similar measurements to those obtained by the author with the example now to be dealt with.

Example of General Procedure for Measurement of a Crystal Angle.—It will be evident from the first cursory inspection that the

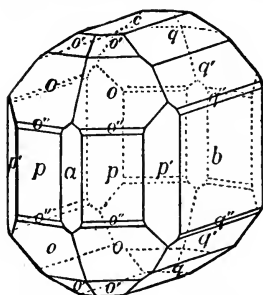


FIG. 20.—Potassium Sulphate Crystal.

broad face labelled in Fig. 20 with the letter *b* is an important plane of the crystal, and that the smaller faces *a* and *c* are approximately perpendicular to this face *b* and to each other. It will, therefore, be advisable to commence operations by measuring the angles between the faces *a* and *b*, between *a* and *c*, and between *b* and *c*, in order to ascertain whether these angles are really exactly 90° , in which case the three planes will probably be planes of symmetry. Beginning with *a* and *b*, the crystal is attached to the little wax cone of the crystal holder of the goniometer by one of the *c* faces, taking care not to soil by contact with the wax more than a small part of that face. There is scarcely any soiling if the wax is properly made and not too new. The intersection of *a* and *b*, imagining the faces extended so as to meet, instead of the edge being blunted by the intermediate faces, will thus be parallel to the goniometer axis, that is, vertical if the Fuess or other horizontal-circle goniometer is being employed, or horizontal when the Miers student's goniometer is used. The crystal holder should then be fixed in its socket in such a position that the broad *b* face is parallel to the tangent screw of one of the two adjusting movements. The telescope should be fixed at somewhat more than a right angle from the collimator if the former instrument is being used, and the inner axis carrying the crystal should be left free to move easily and independently, by unclamping it if necessary from the circle. The crystal should first be centred, with the micro-lens in position, by means of the two centring movements and the milled head which effects the raising or lowering of the axis. The inner axis is then rotated until one of the faces, say *b*, is brought into the position from which a reflection of the Websky signal-slit of the collimator may be expected to be visible in the telescope, as indicated by the brilliant illumination of the face itself as seen

through the micro-telescope. If the setting on the wax has been fairly true to that intended, the *b* faces having appeared perfectly vertical when the crystal has been rotated so as to bring the plane of those faces parallel to the axis of the micro-telescope, that is, so as to enable them to be seen in profile, the image of the signal will probably be visible, either at once or on slight rotation of the axis, although most likely it will not be symmetrical to the horizontal spider-line. If this be so, the image can be brought into such symmetrical adjustment by rotation of the tangent screw of the adjusting movement at right angles to the face. If it be not so, the image is probably just vertically out of the field, and the micro-lens had better be temporarily placed in position again while the tangent screw is worked so as to see which direction of rotation brings about a better general illumination of the face, and then adjusted for the maximum illumination; on again removing the micro-lens the signal image will be visible and can be readily adjusted further to the required symmetrical position with respect to the horizontal spider-line.

Having thus adjusted the face *b*, a precisely similar adjustment must be carried out for the face *a*, this time with the aid of the other tangent screw.

If the Miers goniometer without telescope is being used, it is only a question of adjusting the ends of the elongated oblong image of the window-slit, reflected from the crystal face, to coincide exactly with the ends of the image reflected from the mirror. The object is even better attained if a little transverse bar be laid across the centre of the slit, as a mark by which to adjust the image into coincidence.

On reviewing the image from *b*, if the crystal has been arranged so that this face is strictly parallel to the adjusting movement, its adjustment will have remained unimpaired. But this is only attainable by a rare chance, and in general the image will have been slightly displaced from the adjusted position. A partial turn of the tangent screw should suffice to readjust it, and then the image from *a* should be again examined, any necessary slight correction of its adjustment made in the same way, and thus after one or two closer and closer approximations the two faces should be obtained in simultaneously perfect adjustment. The operation will be the simpler, the readjustments being the slighter, the closer the setting of the face *b* to parallelism with the tangent screw has been. After thus accomplishing the "adjustment of an edge," as the operation of adjusting two faces is technically called, a finishing touch should be given to the centring of the crystal, swinging the micro-lens of the telescope into position for the purpose; after this is achieved everything is ready for the actual measurement.

The result of these preliminary operations is that the crystal is truly centred to the common intersection of the three axes of the goniometer-circle, collimator, and telescope, and that the faces *a* and *b* are strictly parallel to the goniometer axis, while also they are precisely perpendicular to the plane of the axes of the collimator and telescope and therefore

also to the plane of the circle. On rotation of the circle, which carries the crystal with it whether the inner crystal axis is locked to it by the fixing screw or not, it will be obvious that when each of the two faces in turn is brought into the position at which the reflected image of the signal-slit is symmetrical to the crossed spider-lines of the telescope, it is then situated perpendicular to the imaginary straight line bisecting the angle between the axes of the collimator and telescope and lying in the same horizontal plane with those axes, in accordance with the optical law of reflection that the angles of incidence and of reflection are equal. In other words, the normal to each face in turn is brought into coincidence with the bisectrix in question, and thus the angle through which the crystal has been rotated is the angle between the normals to the two faces.

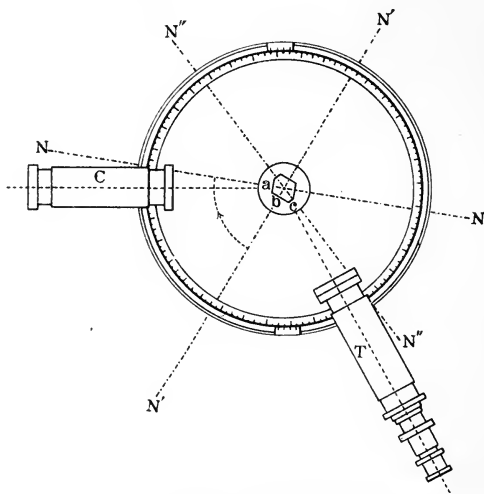


FIG. 21.—Principle of Reflecting Goniometer.

The conditions will be rendered clear by Fig. 21, in which T represents the telescope and C the collimator; *a* and *b* are the two crystal faces, the angle between which is first to be measured, *c* is a third face, and N, N', and N'' are their respective normals. N' is also the direction of the bisectrix of the angle between the telescope and collimator, the face *b* being adjusted. The interfacial angles as drawn are other than 90° in order to render the illustration as general as possible, although in the case of potassium sulphate the angle $ab = 90^\circ$. It will be obvious that the angle through which rotation has been effected during the successive placing to the eyepiece spider-line of the images of the signal reflected from the faces *a* and *b*, is the angle between N and N', indicated in the diagram by the dotted arc. This is the angle required to be measured, for it is much more convenient to consider as the crystal angles the angles between the normals to the faces, rather than the actual angles of intersection of the faces them-

selves; for they are the supplements of the latter angles, and besides being in general the smaller, not exceeding 90° , and therefore the more conveniently recorded of the two supplementary angles, they are the angles directly employed in the stereographic projection and graphical representation of the crystals, as well as in the calculation of the crystal elements. Similarly the goniometrical angle between the faces b and c is that between the normals N' and N'' , for the latter in its turn has to be brought into the position now occupied by N' . The size of the crystal relative to that of the goniometer is much exaggerated in the diagram for the sake of clearness.

Before making the actual measurement of the angle between any two faces thus adjusted, the circle and crystal axes should be clamped together by the fixing screw provided, so that no relative movement may occur during rotation. It is naturally convenient to take the circle readings in descending order, and therefore to bring first the left-hand face, a , into the position shown for b in the figure, perpendicular to the bisectrix N' . Supposing the Fuess goniometer (or similar horizontal goniometer of Troughton and Simms) to be used, as in the case of the author's measurements of potassium sulphate, in placing the image of the Websky signal to the vertical spider-line the dark background should be employed and the vernier light switched off, leaving the room in darkness except for the goniometer lamp in front of the illuminating tube, and which is closely shaded except opposite the condenser of the latter by the copper cylinder. The narrow central part of the signal-image is then to be made coincident with the vertical spider-line, the preliminary adjustment having already arranged the image symmetrical to the horizontal spider-line. The signal-slit should have been once for all previously adjusted so that the narrow central part is just sufficiently wide to enable the vertical spider-line to be clearly defined within it, the curved edges of the image just overlapping the spider-line to an equal minute amount on each side.

The circle reading for the face a may then be taken, the vernier lamp being temporarily switched on for the purpose. A difference of one minute of vernier reading will be so clearly perceptible as to bring the vertical spider-line into contact with, or even slightly to pass, one of the edges of the image. The vernier lamp should then be switched off again, and the circle and all that it carries, including the crystal, rotated by the large milled head until the image of the signal-slit from the second face b is similarly adjusted, as actually shown in the figure, symmetrically to both spider-lines, when another reading may be taken in the same manner. The difference of the two readings is the required angle between the normals to the two faces. During the rotation from the position for the first image to that for the second, it is usually advisable to slip the micro-lens into position in front of the objective, in order that one can observe the crystal throughout and see when the approximate position for the second image is arrived at by the brilliant illumination of the face which then occurs, and at the same time to

verify that it is really the desired face which is furnishing the signal-image. The micro-lens is then again slipped off during the observation and final placing of the image itself.

Actual Observations with the Example chosen, Potassium Sulphate.—The result of the measurement of the angle between the faces *a* and *b* of our potassium sulphate crystal is to prove that the angle is, as surmised, exactly $90^{\circ} 0'$. The next fact that is observed is the very important one that the adjustment of these two faces has automatically adjusted quite a number of other faces. During the rotation from *a* to *b*, if the micro-lens were in position, the intermediate faces to the right of *a* marked in Fig. 20 with the letters *p* and *p'* would be seen to be brightly illuminated in turn, and if the micro-lens were pushed aside the corresponding image of the signal-slit in each case would be seen to be symmetrical to the horizontal spider-line, and a reading of the circle could be taken when each was also brought by the rotation into the position symmetrical also to the vertical spider-line, just as accurately as in the cases of *a* and *b*. Moreover, after passing the image from *b*, and thus completing the first quadrant, it would be found that two similar faces *p'* and *p* would be met with in the second quadrant in the reverse order, that is, symmetrically placed with respect to *b*, and that at the position exactly 180° from *a* and 90° from *b* we should find a face parallel to our original *a* face. Still proceeding through the third quadrant, we should discover two more intermediate faces *p* and *p'*, then a face parallel to our original *b* face, at exactly 180° from that face and at the $270^{\circ} 0'$ graduation of the circle. Finally, rotating on through the fourth quadrant, we should discover a fourth pair of *p'* and *p* faces, and eventually arrive at our original *a* face itself again, at precisely the original circle reading of $0^{\circ} 0'$ (which is the same as $360^{\circ} 0'$) if there had been no slipping of the crystal on the wax.

The following actual measurements were obtained by the author for the crystal of potassium sulphate shown in Fig. 20. The faces marked with an asterisk are those of the right-hand front quadrant in the figure, the asterisked faces *a* and *b* being those actually there labelled:—

Face.	Circle Reading.	Interfacial Angle.
<i>b</i>	$360^{\circ} 0'$	$bp' = 30^{\circ} 10'$
<i>p'</i>	329 50	$p'p = 29 57$
<i>p</i>	299 53	$pa = 29 52$
* <i>a</i>	270 1	$ap = 29 48$
* <i>p</i>	240 13	$pp' = 30 0$
* <i>p'</i>	210 13	$p'b = 30 12$
* <i>b</i>	180 1	$bp' = 30 14$
<i>p'</i>	149 47	$p'p = 29 58$
<i>p</i>	119 49	$pa = 29 49$
<i>a</i>	90 0	$ap = 29 47$
<i>p</i>	60 13	$pp' = 30 0$
<i>p'</i>	30 13	$p'b = 30 13$
<i>b</i>	0 0	

From an inspection of this table of angles four most interesting and important facts are to be derived, namely :—

(1) Twelve faces have been adjusted parallel to the goniometer axis by the mere act of adjusting two of them. These twelve faces may be said to constitute a “zone.” The direction of their edges of intersection, which are all parallel to the goniometer axis and to each other, will be the “zone axis”; and all the faces, their edges of intersection, and the zone axis are perpendicular to an imaginary plane which may be termed the “zone plane.”

(2) The arrangement of the twelve faces in the zone is obviously a regular one, based on a definite plan of symmetry; that is to say, there is a regular repetition of similar faces at the same angular intervals.

(3) Every face has a parallel face on the other side of the crystal, at truly 180° from it, hence the imaginary centre of the crystal is a centre of symmetry.

(4) The two pairs of faces aa and bb , each pair consisting of two parallel faces, are arranged at right angles to each other, and the intermediate faces p and p' are distributed symmetrically on each side of them, that is to say, in such a regular manner that the planes a and b are planes of symmetry.

An analysis of the table of angles shows that the latter fall into three groups of equal angles, the maximum difference from equality of the individual values for the three groups being only $4'$, $3'$, and $5'$ respectively, limits within which the same angle on different crystals of the same substance and of fairly high perfection may vary, so that they can be equally well accepted as indicating intended equality in the case of symmetrical angles of the same zone on the same crystal. The three groups, together with their mean values, are given below. But it cannot be too prominently emphasised that the mean value must never under any circumstances whatever be taken until the question of the symmetry has been definitely settled from a consideration of the individual angular values themselves.

Angles between b and p' Faces.	Angles between p' and p Faces.	Angles between p and a Faces.
30° 10'	29° 57'	29° 52'
30 12	30 0	29 48
30 14	29 58	29 49
30 13	30 0	29 47
Mean value } 30° 12'	Mean value } 29° 59'	Mean value } 29° 49'
for bp'	for $p'p$	for pa

The symmetry will be at once apparent if these results are expressed graphically, as in Fig. 22, in which the positions of the intersections of the normals to the various faces with a circle of reference, which represents the goniometer circle, are indicated by circular dots.

The shading will render the symmetry clear, for the angles shaded

with continuous lines are each $30^{\circ} 12'$, those shaded with dotted lines are $29^{\circ} 49'$, and those not shaded are $29^{\circ} 59'$. The fact that the planes *aa* and *bb* are planes of symmetry will also be quite obvious.

Before passing to the consideration of other faces on the crystal, it may be inquired, how do these mean values for the three angles on this crystal compare with similar results derived from other crystals of potassium sulphate? From eleven excellent crystals, yielding respectively 32, 30, and 29 different and independent values of *bp'*, *p'p*, and *pa*, the

final mean angles $30^{\circ} 12'$, $30^{\circ} 0'$, and $29^{\circ} 48'$ were obtained, and these are the values for these three angles which were published by the author. This closeness, the difference from the values for the crystal under consideration (one of the eleven) being *nil* in one case and only one minute in the other two, will afford some idea of the extraordinary precision with which nature achieves her aim when left undisturbed, and the truly wonderful degree of refinement to which the law of constancy and individuality of crystal angle is carried.

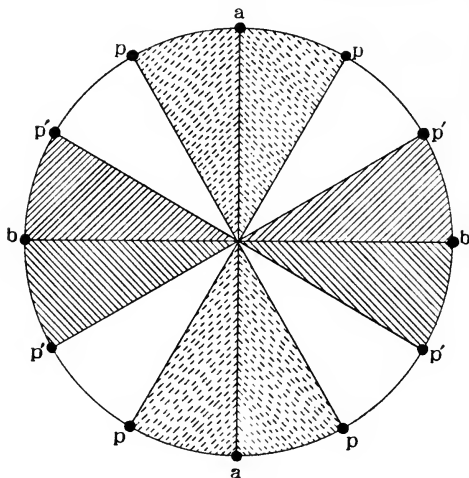


FIG. 22.

Another fact which is evident from the table of angles and Fig. 22 is that the symmetry is almost hexagonal, for the angles are all within a few minutes of 30° . It is not strictly hexagonal, however, for the difference of $12'$ is well beyond the limits of possible error. For instance, among all the eleven crystals and thirty-two measurements of different *bp'* angles, the nearest value to 30° was $30^{\circ} 8'$, while the farthest from 30° was $30^{\circ} 15'$, the mean being $30^{\circ} 12'$ as already given. Hence there can be no possible doubt that the crystal angle *bp'* is larger than 30° . It is in cases like this that a novice in crystal measurement might be inclined to jump to the conclusion that hexagonal symmetry was really intended, and might take the mean of the whole of the angles in the zone, which would then be bound to work out to exactly $30^{\circ} 0'$, a twelfth aliquot part of 360° . In so doing he would fall into the worst possible form of error. Such symmetry as we have here is called pseudo-hexagonal, but it is really of the type termed orthorhombic, which will subsequently be fully described in Chapter XV. Before anything can be decided, however, as to the symmetry the other faces of the crystal must be studied, and attention will now be directed to them.

The crystal holder should be removed from its position at the head of the adjusting movements, and the crystal reset on the wax cone in

such a manner that it stands on an a face, with the faces b and c vertical. The holder should then be replaced in position and rotated until the broad b face is approximately parallel to the tangent screw of one of the adjusting movements. The faces b and c are next to be adjusted and centred exactly as in the former case of a and b . It will then be found that not only are these two faces and their two parallel faces adjusted, but also a whole series of intermediate faces, q , q' , and q'' , repeated in each quadrant, are equally accurately adjusted. In short, there is again presented the phenomenon of a "zone" of faces. Their disposition will be rendered clear by Fig. 23, and the actual values found

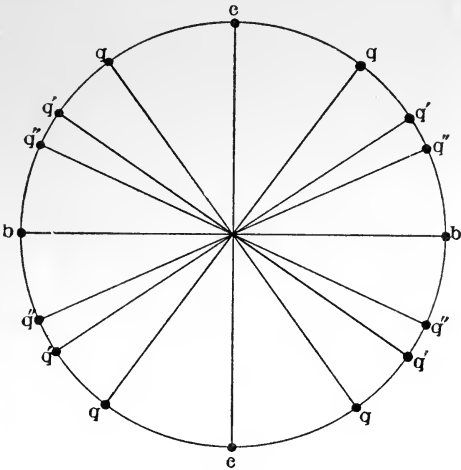


FIG. 23.

for the various angles are given in the next table. Similar angles are collected together in the further short tables, and, after being thoroughly satisfied that only such values are grouped together as are really symmetrically equivalent, their means taken. These latter values are very close to the mean values for the whole eleven crystals, which were respectively: $bq' = 33^\circ 59'$, $q'q = 19^\circ 26'$, $qc = 36^\circ 36'$, $q'q'' = 9^\circ 44'$, and $q''b = 24^\circ 13'$. The concordance is again to one minute of arc, and in one case absolute.

Face.	Circle Reading.	Interfacial Angle.
b	$360^\circ 0'A$	$bq' = 34^\circ 2'$
q'	$325 58$	$q'q = 19 20$
q	$306 38 A$	$qc = 36 36 A$
c	$270 2 A$	$cq = 36 37 A$
q	$233 25 A$	$qq' = 19 23$
q'	$214 2$	$q'b = 34 2$
b	$180 0$	$bq' = 33 52$
q'	$146 8 A$	$q'q = 19 26 A$
q	$126 42 A$	$qc = 36 38 A$
c	$90 4 A$	$cq = 36 35 A$
q	$53 29 A$	$qq' = 19 31$
q'	$33 58$	$q'q'' = 9 44$
q''	$24 14$	$q''b = 24 12$
b	$0 2 A$	

TABULATION OF SIMILAR ANGLES.

bq' .	$q'q$.	qc .	$q'q''$.	$q''b$.
34° 2'	19° 20'	36° 36' A	9° 44'	24° 12'
34 2	19 23	36 37 A		
33 52	19 26 A	36 38 A		
33 56	19 31	36 35 A		
Mean 33° 58'	Mean 19° 25'	Mean 36° 37'		

It will be again clear from the measurement of this zone, and especially from the contemplation of the graphical representation in Fig. 23, that bb and cc are planes of symmetry, about which the intermediate faces q and q' are symmetrically arranged. The face q'' was only found developed once on this crystal, but it was found twice on several other crystals of the eleven measured, and on one crystal three times repeated in the zone, so that there can be no doubt that, like q and q' , as well as p and p' on the zone already considered, it belongs to a set of four faces, symmetrically situated two on each side of the plane b . Such a set of similarly symmetrically situated faces is termed a "form." It is for this reason that the same symbolic letter is given to each of the four such faces, a letter being assigned to each form.

The forms on the two zones now measured are eight in number, namely: (1) the two faces a , (2) the two faces b , (3) the two faces c , (4) the four faces p , (5) the four faces p' , (6) the four faces q , (7) the four faces q' , and (8) the four faces q'' .

A form consisting of two parallel faces only, such as a , b , and c , is conveniently termed a "pinakoid," from $\pi\acute{\iota}\nu\alpha\kappa\acute{\iota}$, "a slab." Each of the forms p , p' , q , q' and q'' is of "prismatic" character, for the four faces of each make up, if considered alone, an open four-faced prism, in section a parallelogram, a rhombus, the axis of the prism being vertical (normal to the c faces) in the cases of p and p' , and front-to-back horizontal (normal to the a faces) in the cases of q , q' and q'' .

With respect to the character of the signal-images afforded by the faces of the two zones now measured on the crystal under consideration, those yielded by the faces of the first zone were all of the most excellent quality, being brilliant and single, with two exceptions in which they were fainter owing to the faces being very small (narrow); in the two cases in question, however, they were sharp and single, so that the relative faintness was no drawback. The faces of the second zone were not so uniformly excellent, only about half the images being brilliant and single. Hence the degree of closeness of the values for the similar angles is not so high as in the case of the first zone, but it is yet good and the mean values are highly trustworthy, as is proved by their agreement with those for the whole eleven crystals. It is of considerable importance that some indicating mark should be placed against circle readings corresponding to perfect single images, and the author employs the letter A for this purpose, and also to indicate angular

values obtained by taking the difference between two A readings. In the tables of readings and angles which follow, and for the zone just given, this has been done. All the images for the zone ab were of A quality.

The next move should be to determine the relative position of the two zones, by measurement of the angle between the faces a and c , and thus to ascertain definitely that they are at right angles, as indicated by a cursory inspection; and also at the same time to ascertain the presence or absence of intermediate faces in the zone ac . On adjusting a and c , the crystal being set up on the wax cone on the broad face b , it was found that the four angles ac , ca , ac and ca were respectively $90^\circ 0'$, $90^\circ 1'$, $90^\circ 2'$, and $90^\circ 0'$; and there were no intermediate faces present. Evidently therefore, the three "zone planes" of the three zones of faces now investigated are at right angles to each other (a "zone plane" being the plane at right angles to the faces of a zone), the axes of the zones being respectively the normals to the faces c , a , and b , themselves also at right angles to each other. These three faces are obviously of prime importance as regards the symmetry of the crystal. If they were alone present they would form a rectangular solid block, the edges of which would be parallel to the three normals.

We may therefore at once take a further step in theoretical crystallography by stating that the directions of these three principal normals, in cases like the present when the three planes are mutually perpendicular, are taken as the "axes" of the crystal, c being the vertical axis, a the axis passing from back to front towards the observer, and b the lateral axis. In the most general (triclinic) type of crystal the angles between these principal normals would not be 90° , in which case the three axes chosen would be parallel to the intersecting edges of the three principal facial planes, and would be mutually obliquely inclined. But in the case before us the axes are rectangular. These three rectangular axes a , b , c , of potassium sulphate may be considered, equally with the general case of oblique axes, to intersect, by moving them parallel to themselves till they do, at the centre of the crystal, and we have next to consider how their relative lengths are determined. The data are already partially at our hand, in the measurements of the positions of the intermediate faces in the two zones ab and bc .

If a "form" had been discovered, consisting of well-developed faces, on each of those two zones inclined at exactly 45° to the rectangular faces, we should probably have been dealing with a cubic crystal of three equal rectangular axes. But we have seen that the intermediate faces are not arranged at 45° , but at angles nearly but not quite 30° in the case of the zone ab , and at 36° and 34° respectively in the case of the zone bc . The symmetry has already been foreshadowed as being that of the orthorhombic system, of three rectangular but unequal axes, and these angles are quite in accordance with such a supposition. It will be necessary to choose one "form" in each of these two zones as the fundamental one, determining the lengths of the axes by its intersection with them, and this will be p in the zone ab and q in the zone bc , for reasons which will be fully entered into later, but chiefly because it is

agreed by convention that of the two lateral axes b shall be the longer. Before we are quite in a position to make this choice, however, it is essential that we should study the other faces on the crystal.

It will be obvious that the faces marked o in Fig. 20 are important ones, and that the smaller faces of similar octahedral or pyramidal character, o' and o'' , are apparently arranged in the same zone with them, which zone also includes the c and p faces. This zone, moreover, occurs in duplicate, there being one such zone on each side of the plane of the axes a and c . The crystal was left in position on the goniometer with the zone ac adjusted, and it should now be tilted by the adjusting movement the tangent screw of which is parallel to c so as to bring one pair of faces p into adjustment with the pair of c faces. After completing the adjustment with the aid of the other tangent screw it will be found that the faces o , o' , and o'' are, as appeared likely, situated truly in the cp zone, and are consequently in perfect adjustment also. The zone should then be measured, and when this is completed the same tangent screw as before should be employed (working the segment over to the other side of the centre) to bring the other pair of p faces into adjustment with the c faces, so as to measure the duplicate zone cp . When the adjustment is complete, with the aid of the final touches from both tangent screws as usual, and the crystal properly centred, the o , o' , and o'' faces of this zone will also be found to be in perfect adjustment, and the zone can at once be measured. The measurements obtained with both the zones are set out side by side in the next table.

Zone to right of a .			Zone to left of a .		
Face.	Circle Reading.	Interfacial Angle.	Face.	Circle Reading.	Interfacial Angle.
c	360° 0'A	co' 36° 42'A	c	360° 0'A	co' 36° 45'A
o'	323 18 A	$o'o$ 19 30 A	o'	323 15 A	$o'o$ 19 32 A
o	303 48 A	op 33 47 A	o	303 43 A	op 33 43 A
p	270 1 A	po 33 48 A	p	270 0 A	po 33 47 A
o	236 13 A	oo' 19 26 A	o	236 13 A	oo' 19 31 A
o'	216 47 A	$o'e$ 36 47	o'	216 42 A	$o'e$ 36 42
c	180 0	co' 36 44	c	180 0	co' 36 46
o'	143 16	$o'o$ 19 25	o'	143 14 A	$o'o$ 19 25
o	123 51 A	op 33 49 A	o	123 49	oo'' 9 49
p	90 2 A	po 33 46 A	o''	114 0	$o'p$ 24 0
o	56 16 A	oo' 19 28 A	p	90 0 A	po 33 51 A
o'	36 48 A	$o'e$ 36 46 A	o	56 9 A	oo' 19 18
c	0 2 A		o'	36 51	$o'e$ 36 50
			c	0 1 A	

Values of co' .	Values of $o'o$.	Values of op .	Values of co' .	Values of $o'o$.	Values of op .
36° 42'A	19° 30'A	33° 47'A	36° 45'A	19° 32'A	33° 43'A
47	26 A	48 A	42	31 A	47 A
44	25	49 A	46	25	49
46 A	28 A	46 A	50	18	51 A
36° 45'	19° 27'	33° 48'	36° 46'	19° 27'	33° 47'

It will be immediately obvious that the angles of the right zone are equal to those of the left zone, and that, therefore, the plane between them, that of the axes a and c , is a plane of symmetry with respect to these zones, just as truly as it was found to be as regards the p and p' faces of the ab zone and the q and q' faces of the bc zone. The same is equally true of the relations of these two zones, and the individual angles of which they are made up, to the other two principal planes of the axes, ab and bc . So that these three axial planes, parallel to the faces a , b and c respectively, are undoubtedly planes of symmetry with respect to all the faces developed on the crystal, the inclinations of those faces being the same on either side of them.

It will be further observed that the faces o and o' are repeated eight times on the crystal, four on the right-hand zone and four on the similar zone to the left of the fundamental plane of the normals to a and c . The form o'' was only represented by a single measurable face, but traces of several of the other seven possible faces were observed, the signal-images derived from them being, however, too faint for accurate allocation to the cross-wires. Each of these forms o , o' , and o'' is of "pyramidal" character, for each of them, if alone present, would build up a doubly terminated pyramid, that is, two four-faced pyramids base to base.

Stereographic Projection.—We have now arrived at a stage in the measurement of the crystal of potassium sulphate when the need for a mode of graphically representing the whole of the faces of the crystal in a single drawing, showing at a glance the mutual arrangement of the various zones, is irresistibly borne in upon us. We have hitherto represented the zones separately, in Figs. 22 and 23, but these figures, while showing the symmetry of the faces in the one zone represented in each case, give no indication of the relationship of the zone drawn to any other zone. By thus discovering the want of such a condensed and complete graphical representation of the symmetry of the crystal for ourselves, we can at once appreciate the immense service which the stereographic projection, first introduced by Neumann in the year 1823, has been to crystallography.

The essence of the method of the stereographic projection is that the normals to the crystal faces are considered as the radii of a sphere. It is hence sometimes called the "spherical projection," although strictly speaking this term applies to the actual projection on the surface of a sphere, while "stereographic projection" refers to the projection of the latter on to the plane of the paper, as actually used. It was clearly shown in Chapter I. that the relative amount of development of crystal faces is quite a secondary matter, without influence on the symmetry, and that the direction of the faces, as indicated by their mutual inclinations, is what is of real moment in determining the symmetry and in offering, by the constancy of the angles, an infallible guide to the individuality of the crystallised substance. Our stereographic projection, therefore, takes cognisance only of the directions of the faces, as represented most conveniently by their normals.

An arbitrary point within the crystal is taken as the centre of the sphere of projection, or origin, and radii are drawn from it perpendicular to the crystal faces, or to their planes produced. Each of these radii will meet the sphere in a point which is termed a "pole," and from the disposition of these poles, which are taken as representing the faces, the symmetry of the crystal can be seen at a glance. It requires to be clearly understood at the outset, however, that the normal drawn from the centre will not necessarily pass actually through the face itself, because it may happen that the face is very small and thrown out of the line of the normal to the indefinitely produced and extended face, owing to the specially large development of other faces; the faces are, however, considered as planes of indefinite extent, to which the normals are drawn perpendicular, from the centre. The spherical projection thus gets rid of the confusion due to the fortuitous development of particular faces, and presents a graphical expression of the arrangement of the faces, indicative immediately of the symmetry, and showing the facial poles at their correct angular distances, as determined directly by the measurements. For, as has been clearly shown already, it is the angles between the normals which are directly determined on the goniometer.

In order to render the spherical projection really useful, the sphere and its poles are projected on to the plane of the paper, and in such a manner that the poles of faces of the same zone will be represented in the projection as lying either on the same circular arc or on the same straight line. For all the zone circles, formed by the intersection of the various zone planes with the sphere, are projected as circles and not as ellipses. Their two limiting cases are: (1) the equatorial plane itself, which is a complete circle, termed the "primitive circle," and within the circumference of which the whole figure falls; and (2) those zone circles the planes of which are perpendicular to the primitive zone circle, which project into straight lines (diameters of the primitive circle) intersecting at the centre of the primitive circle and of the sphere which it represents in plan. The centre is also the projection of the "pole of projection," which lies on the surface of the sphere. The actual mode of constructing the "stereographic projection" will now be described.

The zone plane of one of the most prominent zones of faces on the crystal is chosen as the plane of projection. This plane is supposed to pass through the centre of the sphere, and the axis of the zone is of course perpendicular to it and is supposed to be projected in the centre. The sphere will be cut by the plane in a circle, the primitive circle, and a circle is consequently described round the centre, of convenient radius, to represent this primitive circle. The sphere is thus divided by this circle into two hemispheres, and each hemisphere is projected upon the plane of projection by imagining the eye placed at the pole of the other hemisphere. For there are really two poles of projection, one for each hemisphere, and each is naturally removed 90° from every point on the primitive circle. If straight lines be drawn from this pole of the one hemisphere to all the poles of faces (the points where the face-normals cut the sphere) on the other hemisphere, these straight lines will cut the

plane of the primitive circle (the plane of projection) in points which are the required projections of those face-poles. The same plane of projection can also be used on which to project, from the eye situated at the pole of the other (opposite) hemisphere, the poles of faces in the second hemisphere, in exactly the same manner. If the plane of projection is, as is true in the cases of all but triclinic crystals, a plane of symmetry, the projection points of the poles of the two hemispheres will coincide on the plane of projection, and such pairs of poles are usually represented by a dot with a miniature ring around it. They may be regarded as joined by imaginary straight lines perpendicular to the plane of projection, and of equal lengths above and below it.

Construction of Stereographic Projection for Typical Crystal of Potassium Sulphate.—We will now proceed to construct the stereographic projection of potassium sulphate, and in so doing the main rules for the construction of such projections in general will be elucidated.

The directions of the edges of intersection of the faces a , b , and c have already been chosen as the crystallographic axes, and as the three faces are all perpendicular to each other, these directions are obviously identical with those of the normals to the three faces. It has also been decided that a shall run between back and front, that b shall be the lateral axis (running right and left), and that c shall be the vertical axis. We take, therefore, the zone plane of the zone of faces $app'b$ as the plane of projection, and describe our primitive circle, Fig. 24, with any convenient centre and radius, to represent it. The correct positions of the intermediate faces p and p' are then to be set off, as was already done for Fig. 22, in accordance with the measurements, and these positions must be indicated by circular dots to represent the poles, two pairs of similar dots being also placed at the positions of the fundamental faces a and b . The centre is the pole of the pair of c faces, and should, therefore, also be marked with a dot. Opposite

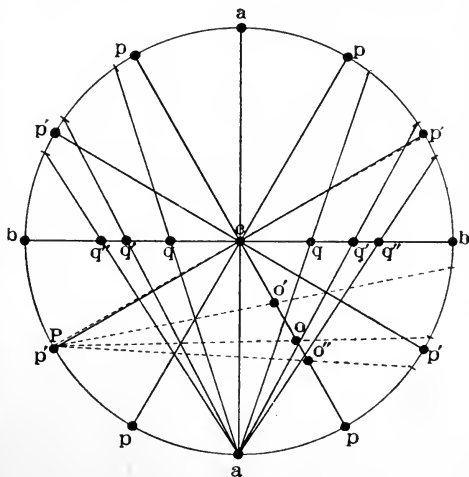


FIG. 24.

pairs of parallel faces should then be joined by diameters, and as all these diameters will naturally pass through the centre c , they will be the projections of possible zones, of which c will furnish a pair of faces.

No faces were observed between the a and c faces in the zone ac , but in the zone cb there were three forms of intermediate faces found,

namely, q , q' , and q'' , and their positions are next to be found on the projection. In order to achieve this it is only necessary to set off from b along the primitive circle arcs equal to bq'' , bq' , and bq , and to join these points on the circle to the pole (in the opposite hemisphere) of the zone circle (projected as diameter) beb , this pole being the point a . The points in which these straight lines will cut the diameter beb , representing the zone circle, will be the projections of the required poles of the faces q , q' , and q'' . This should be done for the three such poles on each side of the centre, as shown in Fig. 24, in which the lower a pole is taken as the pole of the zone $[bc\bar{b}]$, the markings off of the angles bq , bq' , and bq'' having been carried out along the upper semicircle.

The positions of the poles of the faces o , o' , and o'' in the two similar zones cp may next be found in the same way. For the projections of these zones are the diameters pcp . It is necessary first to mark off the position of one of the poles of these zones, say P , at 90° from p on the primitive circle; then to set off along the primitive circle from p , on its other side away from P , the angles po , po' , and po'' corresponding to the intermediate faces o , o' , and o'' , and to join these points to the pole P just found, as shown in Fig. 24 in dotted lines. Where they intersect the radius cp are the positions of the facial poles o , o' , and o'' . This only need be done on one side of c , for the angles are the same on the other side and can be simply pricked off at the like distances by the compasses. The poles of the similar zone cp , on the other side of the plane of symmetry of which the diameter aca is the projection, can also be marked off with the compasses at the same time, and in this manner all the eight poles of the forms o , o' , and o'' may be found, and the positions of all the forms observed on the crystal will then have been located. The stereographic projection thus completed, and to which Figs. 22, 23, and 24 have led up, is shown in Fig. 25. The poles of three other forms, $q''' = \{012\}$, $m = \{102\}$, and $s = \{132\}$, are included on it, as these forms have been discovered by the author on other isomorphous salts of the alkaline sulphate and selenate series. It will be remembered that the poles o , o' , o'' , q , q' and q'' each represent two faces, one in the hemisphere above the plane of projection and one in the hemisphere below. The dots may therefore each be ringed to mark this fact, although in general practice the fact is taken as understood, and the rings omitted.

On studying the projection thus obtained, a further interesting fact is apparent, namely, that the zones indicated by the primitive circle itself, and the six diameters, representing great circles the planes of which are perpendicular to the primitive circle and plane of projection, are not the only zones into which the faces fall, but that they also arrange themselves into eight other zones, along which the angles ought to be measured. If circular arcs, for instance, are drawn through the points a and o , to either the right or left, the arc will be found in both cases to pass also through the pole q ; and if arcs be also drawn through the poles p and o' of adjacent quadrants they will be similarly found in both cases to pass

also through q . In Fig. 25 arcs have been drawn to connect the poles of all the chief zones, thus finally completing the stereographic projection of the example chosen, potassium sulphate. This mode of projection is thus exceedingly valuable from another point of view, namely, that when the few main zones have been discovered experimentally and drawn into the projection, the latter at once shows that the faces are also arranged along other zones, and so suggests the course of the further measurements, which should be carried out along those additional zones. In the next tables are given the measurements along eight such further zones in the case of the crystal before us, namely, those along which the circular arcs are drawn.

The two pairs of zones aoq and $ao'o'$ are most conveniently measured by setting the crystal on the wax and arranging the holder as if

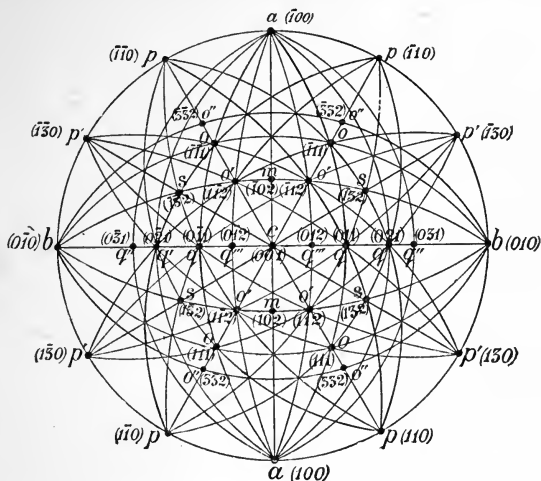


FIG. 25.—Stereographic Projection of Alkali Sulphates and Selenates.

the zone ac were to be again adjusted and measured; then leaving the a faces so adjusted but rotating the other tangent screw so as to bring first the q face on one side of c into adjustment instead of c , and after measuring that zone, then the q face on the other side of c . Similarly, with the same setting, the two zones comprised by the a and o' faces may be adjusted and measured. In the case of the pair of zones bo and bo' the crystal should be set again as if to measure the zone ab , and then, while keeping the b face as adjusted, the crystal should be rotated with the other tangent screw, on each side of a in turn, so as to bring also into adjustment the desired o or o' face, parallel to the goniometer axis. In order to measure the four pairs of zones poq , $po'q$, $p'o'o'q$, and $p'o'q$, it is necessary to be able to recognise the various faces at sight, and as these more difficult zones are left to the last, by this time the observer will have become thoroughly familiar with the peculiarities of the faces, and can readily recognise any particular face. If any difficulty should,

however, be experienced in doing so, it is a great help if, while measuring the earlier fundamental zones, little rough sketches of the shapes or marking of the different faces have been made. When thus at home among the various faces no difficulty will be experienced in adjusting any, even the most complicated, zone to the required position for measurement, namely, with its zone axis parallel to the goniometer axis. The tables referred to, of the results of the measurements of these zones, now follow :

Zone aoq .	Zone bo .	Zone poq' .	Zone $p'oo'$.
$ao = 43^\circ 52'$ $oq = 40^\circ 8'$	$bo = 65^\circ 36'$ $oo = 48 47$	$po = 65^\circ 9'$ $oq' = 49 11$ $qp = 65 40$	$p'o = 43^\circ 59'$ $oo' = 45 47$ $o'q' = 46 0$ $q'p' = 44 13$
Zone ao' .	Zone bo' .	Zone $po'q$.	Zone $p'o'q$.
$ao' = 58^\circ 44'$ $o'o' = 62 34$	$bo' = 72^\circ 42'$ $o'o' = 34 36$	$po' = 72^\circ 23'$ $o'q = 34 52$ $qp = 72 46$	$p'o' = 58^\circ 48'$ $o'q = 62 14$ $qp' = 58 58$

The actual circle readings are not recorded here, as the method of arranging them in the note-book has been sufficiently indicated in connection with the earlier zones. It will be enough to state that in passing round the complete zone aoq , the angles ao and oq are each met with four times, and as there are two such zones the crystal furnishes these angles eight times each, and the values given in the above table are the mean of the eight in each case. Similarly, ao' occurs eight times, and $o'o'$ four times; bo and bo' each occur eight times and oo and $o'o'$ four times, and the mean values are given in the table. In the cases of the remaining four cross zones, each angle is repeated (occurs in duplicate) in each zone, and as each zone is itself repeated (the pair of zone circles being symmetrical to the diametral plane pc or $p'c$, although the poles themselves are only arranged on them symmetrically to the centre), four values are afforded in all for each angle, and the mean of the four is given in the table.

Having now completed the measurements of the interfacial angles of the typical crystal of potassium sulphate, it will be well to take stock of the facts which have been learnt during the process. The mode of utilising the measurements for the calculation of the elements of the crystal, and the checking of the measurements by calculation, will be dealt with in Chapter VIII., after the essential principles have been elucidated on which descriptive crystallography is based, and to which this chapter has afforded an introduction.

Summary of Facts revealed by Measurement of a Typical Crystal.—(1) Each face developed on the crystal has been found to be accompanied by a parallel face on the opposite side of the crystal, so that the centre of the crystal is a "centre of symmetry."

(2) The faces have been found to lie in "zones," which are such that when any two faces, other than a parallel pair, have been adjusted

parallel to the axis of the goniometer, the whole of the remaining faces of the zone to which these two belong are also found to be equally accurately adjusted parallel to the goniometer axis, so that by rotation of the latter the signal images reflected from all the faces in turn are brought into the field of view and in perfect adjustment symmetrical to the horizontal spider-line.

(3) The faces of any and every zone have been found to occur in a regular order of repetition, dependent on a definite plan of symmetry according to which the whole scheme of faces is arranged.

(4) The crystal exhibits three pairs of fundamentally important faces, a , b , and c , inclined at exactly 90° to each other, and all the other faces, as also their zones, are arranged symmetrically as regards their inclination to these three planes, which may thus be termed "planes of symmetry." Their intersections may be taken as the three rectangular "axes" of the crystal.

(5) The faces not only fall naturally into zones, but into sets (each set consisting of different faces) of two, four, or eight similarly orientated faces with respect to the symmetry, which are termed "forms." The faces a , b , and c occur in pairs, termed "pinakoids"; the faces p , p' , q , q' , and q'' occur in fours, and are of "prismatic" character; and the faces o and o' occur in eights, and are of double "pyramidal" character, consisting of two four-faced pyramids base to base.

(6) The "stereographic projection" represents the arrangement of the faces of a crystal in a highly convenient manner, the confusion due to fortuitously large or small relative development of particular faces being eliminated, while the order of symmetry is prominently displayed. It has the further great convenience that the interfacial angles directly read off on the goniometer, the angles between the normals to the faces, are those actually graphically expressed in the projection, which is that of an imaginary sphere enveloping the crystal, projected on to the plane of the paper, the circular dots or "poles" representing the crystal faces being the projections of the points of intersection of their normals with the sphere. The projection shows all the zones either as circular arcs (a complete circle in the case of the primitive circle representing the plane of projection) or diametral straight lines, so that it can be readily constructed with the aid of a pair of compasses and a ruler. It is a most valuable aid in discovering the more complicated cross-zones, other than the fundamental ones; and, moreover, it lends itself immediately to the methods of calculation of spherical trigonometry.

CHAPTER V

CRYSTAL AXES, INDICES OF FACES, AND LAW OF RATIONALITY.

Crystal Axes.—It was shown in Chapter IV. that three important faces, parallel to planes of symmetry if the crystal be endowed with such, are chosen as the axial planes, and their intersections as the fundamental crystal “axes,” to which for descriptive purposes the whole morphology of the crystal is referred. If, as was the case with the typical crystal of potassium sulphate studied in that chapter, there were three planes inclined at 90° to each other developed on the crystal, these planes would naturally be chosen as the axial planes, and the axes of the crystal would then be mutually perpendicular to each other. In the stereographic projection (Fig. 25) these three axes are represented by the diameters *aa* (running back and front) and *bb* (running right and left), and by the central point *c*, which represents in plan the third axis perpendicular to the plane of projection. As will be shown in detail in subsequent chapters, in considering the various possible types of crystal symmetry, the three axes are by no means necessarily at right angles to each other, and it is best to consider the general case, which does actually occasionally occur in the crystals of a few substances, namely that of the so-called triclinic crystals, in which the three axes are inclined to each other at unequal angles and are of unequal lengths. In Fig. 26 *XOX'*, *YOY'*, *ZOZ'* represent the three crystal axes intersecting at a centre *O*, and the directions of which are parallel to the intersections of three principal faces of the crystal, *ABO*, *ACO*, *BCO*. Each of these three facial planes may be and generally is represented on the crystal by two parallel faces at opposite sides of the crystal, that is, on opposite sides of the centre *O*, as shown in Fig. 27, where the faces *a*, *b*, and *c* each have a parallel fellow indicated by dotted lines, as they would be invisible if the crystal were opaque. The actual edges of intersection formed by these three pairs of faces on the crystal are twelve in number, consisting of three sets of four parallel and equal edges. The three different edges meeting in any one of the eight corners of the solid are those which are taken as the directions of the three axes, and they can be imagined without any impropriety to be moved parallel to themselves so as to bring the corner to the centre *O* of the crystal, when they will correspond to *OX*, *OY*, and *OZ* of Fig. 26. Each of the

three axes will then radiate from O through the middle of a parallel pair of faces, as shown in Fig. 27, A, A', B, B', C and C' being the six points of emergence. For the amount of development of the crystal in any direction is a purely fortuitous circumstance, and, as has been amply proved, is without influence on the angles of mutual inclination of the faces, which angles are the real expression of the crystal structure. The planes XOY, YOZ, and XOZ are termed the "axial planes," and the angles between the axes, which are the same as the angles between the crystal edges, are termed the "axial angles," and are labelled α , β , γ , in the order shown in Fig. 26. These angles are not to be confounded with those between the crystal faces chosen as axial

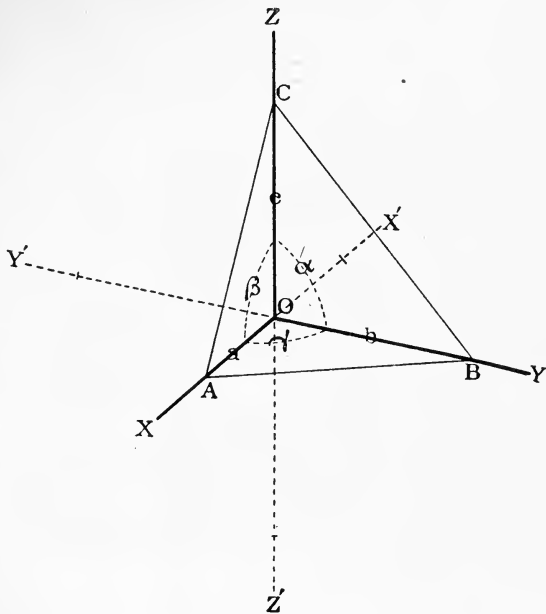


FIG. 26.

planes, with which they are only identical when at least one plane of symmetry is developed, as in crystals belonging to the system termed monoclinic; in triclinic crystals they are never identical. The axes themselves are generally labelled a , b , c , and the angle α is opposite to the axis a , the angle β is opposite to the axis b , and the angle γ is opposite to the axis c . The three fundamental faces themselves are also generally labelled a , b , and c , as well as their parallel fellows, according to the axis which emerges through them. Hence the interfacial edge cb is parallel to axis a , the edge ac to axis b , and the edge ab to axis c , in each case the edge being parallel to that axis the letter expressing which does not occur in the two letters representing the faces intersecting in the edge. This will be quite clear from Fig. 27. The three planes containing the axes divide up the space within the crystal into eight parts, which are referred to as "octants."

When a fourth face ABC (Fig. 26) inclined to all three axes is present on the crystal, such as the face o in Fig. 27, in addition to the three fundamental faces (and their parallel fellows) chosen as the directions of the axial planes, it will cut the axes in lengths OA, OB, OC which are termed the "parameters" of the face. These parametral axial lengths cut off by a selected fourth face serve the purpose of defining

the position not only of the fourth face in question, but also, by making use of them as standard or unit lengths, that of any other face on the crystal, with respect to the three axes, just as in solid geometry any plane is defined by reference to three axes in space. In the case of the fourth or parametral plane itself the ratio of the three parametral lengths at once gives the position of the face, whereas for all other faces inclined to all three axes the lengths are expressed relatively to these fundamental lengths. In order that the definition of the position of a face may be complete, however, it is necessary to know also the

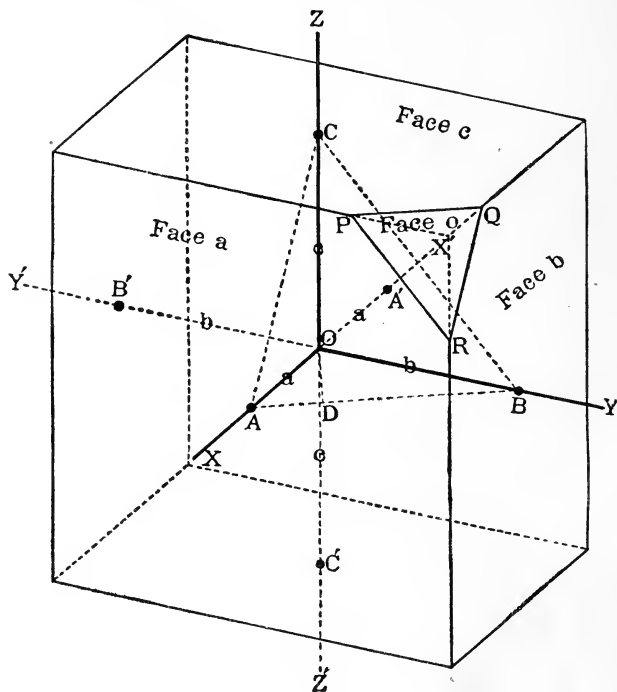


FIG. 27.

particular octant in which the face produces, by its intersection with the axes, the triangle corresponding to ABC; this closed figure formed by the plane ABC and the axial planes then defines the position, even although the face spreads over into other octants after intersection with the axes. As there are eight octants, there will be eight faces possible having similar parameters, the whole making up an octahedral solid having the shape of two four-faced pyramids base to base; and to distinguish the octant referred to by any particular set of parameters, intercepts falling in the back and left-hand halves OX' and OY' of the lateral axes XOX' and YOY' , and on the lower half OZ' of the vertical axis ZOZ' , are marked with the negative sign -.

It will be obvious that the parameters of any face may be multiplied or divided by any number whatsoever without altering their ratio, and it is the last alone which determines the angular position of the face; mere extent of development has been shown to be of no moment whatever as regards defining the elements of the crystal structure, merely resulting in certain faces being pushed outwards parallel to themselves and certain others withdrawn parallel to themselves. Consequently, it is quite permissible to represent the parameters by the simplest numbers which will exhibit their ratio correctly, and it is therefore customary to express the primary parameters, those of the selected fourth face, so that one parameter = 1, the right and left lateral axis b being now universally chosen as this axis of unit length.

If, therefore, we choose the face ABC as representing the "primary" or "unit" form,—because it cuts all three axes at lengths not far removed from equality, or by reason of its prominent development, or because cleavage is developed parallel to it,—its parameters $OA = a$, $OB = b$, $OC = c$ will be the "**lengths of the axes**," and their ratios $a:b$ and $c:b$ (as $b = 1$) will be the "**axial ratios**" of the crystal. When referring to the crystallographic axes it is customary to restrict the use of the three small letters a , b , c , to the designation of these relative parametral lengths of the three axes, OA , OB , OC , and, as already mentioned, to make $b = 1$. Thus a , b , c do not only signify the axial directions, but definite lengths along those directions. The ratios thus generally (now practically universally) take the form $a:b:c$, that is, $a:1:c$. As the axis b is generally chosen as the longer horizontal axis the value of a will be less than unity. The value of c is in general something between 0.5 and 1.5. The actual figures are expressed in each case to four decimal places. For instance, in the case of triclinic copper sulphate, it will be shown in Chapter XX. that $a:b:c = 0.5715:1:0.5575$.

Elements of a Crystal.—The two ratios $\frac{a}{b}$ and $\frac{c}{b}$, together with the three interaxial angles α , β , γ , are termed the "**elements**" of the crystal, and the four primary planes which determine the elements, namely, the three axial planes and the primary parametral plane inclined to them, are termed the "**elementary faces**" of the crystal.

In the case of highest symmetry, which will subsequently be shown to be found in the cubic system, the three axes are of equal lengths and are arranged at right angles to each other; moreover, their planes are planes of symmetry. In this case the ratio of the axes, as determined by the primary inclined form, the regular octahedron, will simplify to $1:1:1$, and the interaxial angles are each 90° . Between this case of perfect symmetry and the general case of triclinic crystals, in which the three axes are of unequal lengths and are all inclined at angles other than 90° to each other, and in which also the crystal is without any planes of symmetry, it will be shown in subsequent chapters that there are several stages of intermediate symmetry. In the case of the orthorhombic potassium sulphate crystal investigated in the last chapter, the axes are arranged rectangularly, so that all the interaxial angles are 90° ,

but the three axes are of unequal lengths; their three planes are, however, planes of symmetry. The actual ratio of the axes found was $a:b:c=0.5727:1:0.7418$, and the mode of calculating it will be dealt with in Chapter VIII.

Indices of a Face.—Having decided which faces to choose as the four elementary faces and consequently settled the “elements” of the crystal, any other face H K L, Fig. 28, can at once be defined as regards its position and direction by the “intercepts” which it makes, or

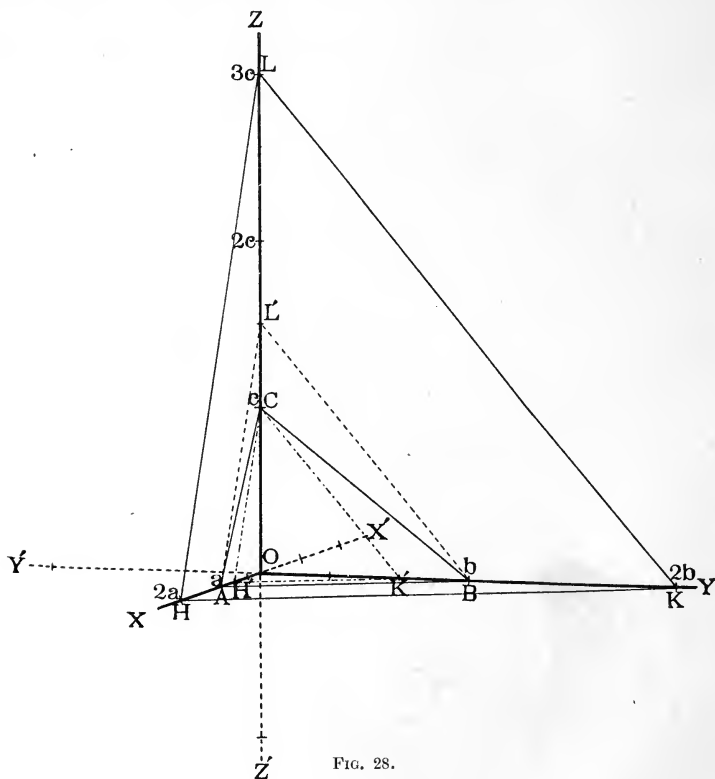


FIG. 28.

lengths (relative to the axial lengths a, b, c) which by its intersection it cuts off, along the three axes. We can express those intercepts in terms of the axial lengths, the primary intercepts or parameters, a, b, c , by stating them as $ma:nb:rc$. For instance, the face o'' on the potassium sulphate crystal dealt with in the last chapter—and which, as well as the rectangular axes a, b, c and the parametral face ABC, is drawn as HKL in Fig. 28 correctly to scale according to the customary convention to be described in Chapter XXV.—makes the intercepts on the three respective axes: $2a:2b:3c$; this expression, therefore, may be taken as the symbol of the face. This is the **Weiss notation**, or method of

distinguishing crystal faces, having been proposed by Weiss in the year 1818.

According to the simplest form of this notation, in the case of a face having different intercepts on the three axes, the longer ones are expressed as multiples of the shortest, which latter is taken as unity. For instance, in the case just quoted the relationship is equally expressed by the symbol $a:b:\frac{3}{2}c$, and the dotted lines ABL' indicate the position of the face in Fig. 28, just as well as the parallel plane HKL . The faces other than the primary are thus considered as lying outside the latter, but in contact at one point or along a line. In this case there is contact along the line AB , which would thus be the direction of intersection of the primary parametral face and the face in question.

To take another example, suppose the intercept on the axis a to be the same as that of the primary form, that on b being $1\frac{1}{2}$ times that of the primary form and that on c 3 times the primary value; the symbol will then be $a:\frac{3}{2}b:3c$. Here there is contact at the point a only with the primary plane.

For the parallel face in the lower, back, left octant, all the signs would be negative, and written thus: $-a:-\frac{3}{2}b:-3c$. And for the faces in the other octants, the signs of such of the intercepts are negative as occur along the axes on the negative side of the centre.

If a face be parallel to any axis, its intercept on that axis is infinitely long, and its coefficient is ∞ . For instance, if the face were parallel to the vertical axis the above symbol would become $a:\frac{3}{2}b:\infty c$.

These symbols of Weiss are of truly scientific character, and an accurate expression of the relative intercepts cut off from the axes by the various faces, but they are too long to be convenient symbols for the marking of crystal faces, either in a drawing of the crystal or its stereographic projection.

Millerian Notation.—It is much more convenient not to label the face directly with the intercepts themselves, but to divide the latter by the highest number among the three, so that the other two will become fractional parts, and then to take the denominators of these fractions as the indices of the face, when the numerators are a , b , and c respectively. In other words, the unit intercept becomes the longest according to this method of notation, instead of the shortest as in the case of the Weiss method. It is equivalent to moving the face parallel to itself towards the centre until one of the intercepts becomes identical with that of the primary parametral face, and the other two intercepts lie within the unit values, as shown by the broken-and-dotted lines $H'K'C$ in Fig. 28; it will be remembered that in this diagram ABC represents the unit face the intercepts of which are a , b , c , that HKL represents the face o'' of the potassium sulphate crystal investigated in the last chapter, the intercepts of which are $2a:2b:3c$, and that ABL' represents the same face but moved up parallel to itself towards the centre until the intercepts are expressed in their lowest terms $a:b:\frac{3}{2}c$ and the first two become

1a and 1b. Moving it now still further towards the centre until the largest intercept, that on the c axis, has become 1c and the face cuts the axis at C itself, so that L (of the general facial cognomen HKL) is coincident with C, the other two intercepts H' and K' will obviously be fractional parts of OA and OB respectively, namely, dividing $2a : 2b : 3c$ by 3, the highest figure among them :

$$OH' = \frac{2}{3}a, OK' = \frac{2}{3}b, OC = c;$$

that is, the symbol becomes :

$$\frac{2}{3}a : \frac{2}{3}b : c, \text{ or } \frac{a}{\frac{3}{2}} : \frac{b}{\frac{3}{2}} : \frac{c}{1}.$$

The symbols taken to represent the face are these denominators $\frac{3}{2}$, $\frac{3}{2}$, 1, or multiplying them by 2 to get rid of the fractions : 3, 3, 2. These latter three numbers are the Millerian "indices" of the face, and they are written as a symbol within brackets, thus : (332). The brevity and general convenience of such a symbol will be at once apparent.

The "symbol" of a face is therefore an expression written within brackets thus :—(hkl), where h , k , l are the "indices," these latter being three numbers such that

$$\frac{a}{h} \frac{b}{k} \frac{c}{l}$$

are the intercepts made by the face on the axes.

Suppose we take another example, in which all three indices are different, say a face the intercepts of which are $a : 2b : 4c$. Dividing by the highest number present, 4, we get $\frac{a}{4}$, $\frac{b}{2}$, $\frac{c}{1}$, and the indices will consequently be (421). This is the **Millerian system of crystallographic notation**, so called because it was adopted by W. H. Miller, Professor of Mineralogy at Cambridge, in the year 1839, in his historic *Treatise on Crystallography*. It had already been previously suggested by the Rev. Dr. Whewell, but the great work of Professor Miller first brought it into prominence. It is the notation that will be employed throughout this book.

Symbol of a Form and of its Individual Faces.—The indices of the other faces of the same form will be the same, except that whenever the intercept is measured along the negative side of the origin (OX', OY', or OZ' in Figs. 26-28) a negative sign is placed over the number. Thus, if the o'' face referred to be that of the lower-left-front octant of the crystal of potassium sulphate, its indices will be ($\overline{3}\overline{3}2$), as the intercepts occur along the negative parts of the b and c axes, but along the positive part of the a axis.

When the symbol is written in the ordinary kind of simple brackets given above, the indices are intended to be those of a face, and if there are no negative signs above any of the numbers, it means the face in the top-right-front octant. In order to indicate the whole form of eight, four or two faces, the symbol of this same face is used, but is placed within a pair of brackets of the type { thus : {332}.

General Relationship of Crystal Axes, Parameters, Intercepts, and Indices.—These relationships may now be summarised as follows. The italic letters a , b , c represent the axes of the crystal, of definite relative lengths, and these lengths or parameters are determined by the intercepts made along the directions of the axes by the inclined face which is chosen as the unit, parametral, or primary face ABC (one of the eight possible faces of the primary octahedral or bi-pyramidal form), so that $OA = a$, $OB = b$, $OC = c$. The letters a , b , c thus represent at the same time the directions of the axes, their relative lengths (expressed by the axial ratio $a : b = 1 : c$), and the parametral intercepts (intercepts made on the axes by the parametral or primary inclined plane). These same italic letters are also used in crystallography to represent the three fundamental faces (and their parallel faces) chosen for axial planes, the two a faces being those intersected by the back-to-front a axis, the pair of b faces by the lateral b axis, and the two c faces (generally termed the basal pinakoid) being cut by the c axis; the incidence of the axis on the face is rectangular if the face is a plane of symmetry. This double use of these three first letters of the alphabet is not really confusing, but rather a convenience, provided it is always clearly indicated whether a fundamental face or an axis is being referred to; if a face, the letter is generally accompanied by its indices, thus: $a(100)$, or $a = (100)$, so that no ambiguity is possible.

No general agreement has yet been arrived at as to the letter by which the faces of the primary parametral form $\{111\}$ shall be labelled. The author generally employs the letter o , as in Fig. 27, which reminds one that the form is of octahedral character.

Any other face than the four primary faces is generally referred to in diagrams (Fig. 28) as HKL, and is distinguished in the drawing, description, and stereographic projection of the crystal by an italic labelling letter, common to the form, and by the symbol $\{hkl\}$, its form of eight, four or two faces being $\{hkl\}$, in which :

$$h = \frac{OA}{OH} = \frac{a}{OH}, \quad k = \frac{OB}{OK} = \frac{b}{OK}, \quad l = \frac{OC}{OL} = \frac{c}{OL},$$

or as it is more usually written, to give the values of the intercepts OH, OK, OL :

$$OH = \frac{OA}{h} = \frac{a}{h}, \quad OK = \frac{OB}{k} = \frac{b}{k}, \quad OL = \frac{OC}{l} = \frac{c}{l}.$$

The three indices h , k , l consequently define the position of the face completely, when the parameters OA , OB , OC (a , b , c) are known. The values can be reduced to their simplest form, in which one of them = 1 or some other small number (2 in the case of the o'' face of the potassium sulphate crystal), by multiplying or dividing all three indices by the same positive whole number; for, as already shown, this is only equivalent to a translation of the face parallel to itself, which has no effect on the elements of the structure.

Indices of the three Axial Planes.—Each face of the three

fundamental pairs of faces parallel to the axial planes, that is to say, those faces which were chosen for axial planes, will obviously be cut by one axis only, as shown in Fig. 27, and will be parallel to the other two axes, for these latter are parallel to its edges of intersection with the two other axial planes; hence, to use the ordinary geometrical convention, it will only meet those two other axes at an infinite distance. When, however, the intercept becomes ∞ , the index becomes 0, thus, if $OH = \infty = \frac{a}{h}$, then $h = \frac{a}{\infty} = 0$.

Hence the symbol of a face parallel to two axes will have in it two ciphers, and the other index will be 1. The symbol of the face a will be (100), because it cuts the forward axis a only, that of the face b will be (010) because it cuts the lateral axis b only; and that of the face c , the basal plane, will be (001) because it cuts the vertical axis c only. The faces parallel to these will be ($\bar{1}$ 00) (0 $\bar{1}$ 0), and (00 $\bar{1}$).

Indices of the Primary Inclined (Parametral) Faces.—The face chosen as the fourth fundamental face ABC, which defines by its intersection with, or intercepts on, the three axes the unit lengths of those axes, will obviously have the symbol (111), its position being in the top, right-hand-front octant. The other seven faces of the form (octahedral in character) will be ($\bar{1}$ $\bar{1}$ 1) for the face in the top-front-left octant, ($\bar{1}$ 1 $\bar{1}$) for that in the top-back-left octant, and (1 $\bar{1}$ 1) for the face in the top-back-right octant. There will be four similar faces of like indices in the four lower octants, the symbols only differing by having also a negative sign over the third 1, to mark the fact that the vertical c axis is cut in the lower negative half.

Relationship and Mutual Conversion of the Symbols of Weiss and Miller.—The elegance of Miller's method, the brevity of the symbol, rendering it eminently suitable for the labelling of crystal faces either on large specimens or on drawings of the crystals, and its direct application to the stereographic projection and the methods of calculating by spherical trigonometry, have all combined to bring it into general use, and it is now the almost universally accepted method of notation, and is employed by Professor von Groth as the notation of the unique publication of the subject, the *Zeitschrift für Krystallographie*. The symbols of Weiss, however, are the direct expression of the intercepts, and it is of some consequence to be able to convert the one kind of symbol into the other, even if only for the purpose of finding the intercepts themselves, from the Millerian indices, when it is desired to draw the crystals to scale according to the conventional method to be hereafter described in Chapter XXV.

In order to convert the Millerian symbol (hkl) into Weiss notation, we multiply each term of the expression for the intercepts

$$\frac{a}{h} : \frac{b}{k} : \frac{c}{l}$$

(in which h, k, l are the Millerian indices and a, b, c are the axial inter-

cepts of the primary face, that is, the unit lengths of the three axes) by the product of $h.k.l$; the ratios then become

$$kla : hlb : hkc.$$

That is, we multiply together the two other Millerian indices in order to get the Weiss coefficient of any one axis, reducing, if necessary, the coefficients thus found to their simplest relation.

For example, the Weiss symbol for the face represented in Millerian notation by (123) is $6a : 3b : 2c$, or simpler, $3a : \frac{3}{2}b : c$.

It is well to remember that Weiss's intercept-coefficients are directly proportional, and Miller's indices inversely proportional, to the lengths of the intercepts made by the face under consideration on the crystal axes. The highest intercept-coefficient consequently corresponds to the lowest index, and *vice versa*.

As an example of the converse process, suppose the Weiss expression for a face to be $a : \frac{3}{2}b : \frac{1}{2}c$. Reducing this to the form $\frac{a}{h} : \frac{b}{k} : \frac{c}{l}$, by dividing all through by 3, we get $\frac{a}{3} : \frac{b}{2} : \frac{c}{6}$. The Millerian symbol is therefore (326). Or taking once more the face o'' of potassium sulphate, which has the Weiss intercept-coefficients $2a : 2b : 3c$, we proceed to reduce the latter expression to the form $\frac{a}{h} : \frac{b}{k} : \frac{c}{l}$ by dividing all through by 6, when we get $\frac{a}{3} : \frac{b}{3} : \frac{c}{2}$. The Millerian symbol is consequently (332). A safe rule for the reduction of the Weiss expression $ma : nb : rc$ to the Millerian expression $\frac{a}{h} : \frac{b}{k} : \frac{c}{l}$ is to divide the former by the

product mnr , that is, to get the Millerian index for any axis we multiply together the intercepts on the other two; then if the resulting values of h, k, l admit of further simplification by means of a common divisor, we so simplify them. Using this rule in the case of the o'' face just referred to, $m.n.r = 12$, and h, k, l work out respectively to 6, 6, 4. These will simplify by dividing by 2, which thus reduces them to 3, 3, 2.

The Facial Form-Symbols of Naumann.—A modified and more or less arbitrary form of Weiss notation was for many years very popular for the description of crystal faces, which was devised by C. F. Naumann, and propagated in his popular *Elemente der Mineralogie*. It combines the Weiss parametral values with certain arbitrary signs, which differ for the various crystal systems. The general symbol corresponding to the Millerian hkl , or to the Weiss expression written in the form $ma : b : nc$, is mPn in the tetragonal and rhombic systems, where the axes are rectangular, the P standing for primary pyramid. In the monoclinic and triclinic systems in which the axes are oblique the P is marked on its stem with an oblique stroke, thus P , or a rectangular one for a face of the orthozone of the monoclinic system, thus P . In the rhombic system the P is marked over the top with a straight stroke, thus \bar{P} , or with a little arched (inverted) accent, thus \acute{P} , according as the face is parallel to the macro- or brachy-diagonal, the longer b axis or the shorter a axis of

the crystal. In the cubic system, where the primary pyramid P of Millerian indices $\{111\}$ is replaced by the octahedron, the letter O is used instead of P , so that the general form in this system is mOn . When the face is parallel to a crystallographic axis, instead of intercepting it, m or n is replaced by the sign of infinity ∞ . Thus the cube faces are all marked $\infty O\infty$. The primary prism is ∞P , or ∞O in the cubic system, this latter symbol being that of the rhombic dodecahedron. The primary pyramid itself is O or P , and the basal plane is always oP . A face such as a brachydomal prism of the rhombic system is $m\bar{P}\infty$, for instance.

These symbols of Naumann are now quite obsolete for the purposes of accurate crystallography. They are interesting, however, historically, and are still often seen in the older text-books and sometimes in museums marked on the faces of large mineral crystals. They had, indeed, a certain value for descriptive purposes, when once the arbitrary signs and symbols had been thoroughly mastered, being short, and one symbol covering the whole of the faces of a form, that is, the whole of the faces present of equal value as regards the symmetry.

In the event of the necessity arising to study any crystals thus marked, or to unravel the symmetry of crystals described in the older memoirs or treatises, a capital and very complete list of the symbols of all the principal forms of all seven crystal systems in Naumannian notation will be found in the *Physikalische Krystallographie* of P. von Groth.

The Law of Rationality of Intercepts or Indices.—We now come to one of the most remarkable and important facts in crystallography. After having fixed the elements of the crystal and determined by reference to the primary form the symbols of all other faces developed on the crystal, from the data afforded by the angular measurements, we find that, provided our measurements have been accurate and the crystals are of the desirable high state of perfection, **the whole of the symbols thus obtained consist of small whole numbers**, generally not larger than 3 or 4, and very rarely indeed amounting to 6 or more, which latter only happens on crystals such as those of natural minerals which are extraordinarily rich in faces. In other words, taking the now generally accepted Millerian notation for greater convenience, the values of the indices h, k, l can always be expressed as small whole numbers, chiefly 1, 2, 3, or 4. This is the law of "rationality of indices," and it can be briefly expressed in the statement: "The indices of any and every face on a crystal are three small whole numbers."

This important law is equally true, however, of intercepts as of indices, as will be obvious from the close relationship between them. Expressing the intercepts by the Weiss notation $ma:nb:rc$, the law requires that m, n , and r shall be whole numbers.

The law in its complete form may be expressed in the following two statements, the first referring to intercepts and the second to indices:—

The intercepts on the crystal axes, made by any face of the crystal, are such as can be expressed as rational multiples of the

parametral unit lengths of those axes, as determined by the intersection of those axes with the primary face which has been chosen as the parametral plane.

If OA, OB, OC be the crystal axes intersecting at O in the centre of the crystal, and parallel to the edges of the three faces chosen as the fundamental faces and directions of the axial planes, and if their respective lengths be $a = OA$, $b = OB$, $c = OC$, as determined by the intercepts made upon them by a fourth face chosen as the primary or parametral face, and if, further, H, K, L be the points at which the axes are met by any other facial plane developed on the crystal, then in the expressions

$$OH = \frac{a}{h}, \quad OK = \frac{b}{k}, \quad OL = \frac{c}{l}$$

h , k , and l are small whole numbers.

This law is strictly true whichever three faces meeting in three edges are chosen as the axial planes, but for obvious reasons, when symmetry is developed in the crystal, such faces as are parallel to the planes of symmetry are chosen for the axial planes. Whichever three faces are chosen, they form with their parallel faces the six faces of a closed figure having twelve edges, which is the cube in the case of highest symmetry, a rectangular block in the case of an intermediate type of symmetry such as the orthorhombic crystal of potassium sulphate already considered, or a similar block but without any right angles in the general case of triclinic crystals in which no planes of symmetry are developed, as shown in Fig. 27.

The subject of the proper choice of axial planes and of the proper arrangement in space of the three planes chosen, that is, the correct "setting up" of the crystal, is so important from the point of view of the internal structure of crystals, that a special section will be devoted to it in a later chapter (XXXIII.), after the various types of symmetry and the cleavage of crystals have been discussed.

CHAPTER VI

ZONES, THEIR TEACHING, AND REPRESENTATION IN STEREOGRAPHIC PROJECTION.

Symbol of a Zone.—The fact that the faces of a crystal are arranged in zones, each of such a character that the successive pairs of adjacent faces composing it meet in edges which are all parallel to each other and to the zone axis, was elicited in Chapter IV. by the practical investigation on the goniometer of the typical crystal of potassium sulphate. Such faces belonging to the same zone are said to be “cozonal” or “tautozonal.” It was also seen that the adjustment to the axis of the goniometer of any two faces of the zone other than parallel faces brought about the automatic adjustment of the whole zone of faces. Two faces are consequently adequate to determine a zone, and it is found convenient, and of great help in ascertaining the indices of other faces, both of the same and of other zones, that the indices of these two determining faces should be combined so as to afford us a symbol of the whole zone. To distinguish a zone symbol from that of a face, it is always written in square brackets, thus: [431]. The indices of the zone are found as follows: Let the symbols of the two determining faces be (hkl) and $(h'k'l')$. We write the indices of each symbol twice over in the same horizontal line, arrange those of one symbol below those of the other, and cross-multiply all except the two end pairs, thus:

$$\begin{array}{c|ccccc|c} h & k & l & h & k & l \\ & \times & \times & \times & & \\ h' & k' & l' & h' & k' & l' \end{array}$$

Then, if $[uvw]$ be the symbol of the zone, the three indices u , v , w , are found from the above cross-multiplication by taking the difference of the products of each adjacent pair of cross-multiplied facial indices. That is,

$$u = kl' - lk'; \quad v = lh' - hl'; \quad w = hk' - kh'.$$

The values thus found for the indices of a zone may be reduced to their simplest relationship by dividing them all by any common divisor, or they may all be multiplied by any one and the same number, or all their signs may even be changed, without altering their nature as indices of

the zone. This last fact regarding the sign will be obvious, for the sign simply depends on which of the two facial indices is written above and which below the other.

Use of Zone Symbols in finding Facial Indices.—The task of finding the indices of a subsidiary face on a crystal is often greatly facilitated by the fact that the face is situated at the intersection of two zones, and is thus common to the two zones. For the symbol of such a face is at once given by the cross-multiplication of the symbols of the two zones, in the same manner as that just employed for the finding of a zone symbol. For instance, in a crystal of ammonium selenate belonging to the monoclinic system measured by the author, a certain face *n* was situated at the intersection of two zones, as will be clear from the stereographic projection¹ of the salt given in Fig. 29,

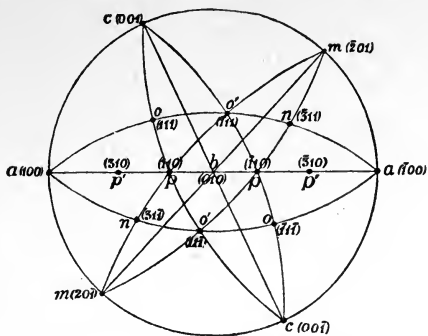


FIG. 29.—Stereographic Projection of Ammonium Selenate.

the particular *n* face in question being that in the upper-right quadrant of the projection; the two faces nearest to it on one of the zones were $(\bar{1}00)$ and $(\bar{1}11)$, and the two nearest faces on the other zone were $(\bar{2}01)$ and $(\bar{1}10)$. Cross-multiplying, first, the pair of faces on each of the two zones, we get the two zone symbols:

$$\begin{array}{cccccc} 1 & 0 & 0 & \bar{1} & 0 & 0 \\ & \times & \times & \times & & \\ \hline \bar{1} & 1 & 1 & 1 & 1 & 1 \end{array} = [01\bar{1}]$$

$$\begin{array}{cccccc} \bar{2} & 0 & 1 & \bar{2} & 0 & 1 \\ & \times & \times & \times & & \\ \hline \bar{1} & 1 & 0 & 1 & 1 & 0 \end{array} = [\bar{1}\bar{1}2]$$

Then by cross-multiplication of the zone symbols we obtain the symbol of the face, as follows:

$$\begin{array}{cccccc} 0 & 1 & \bar{1} & 0 & 1 & \bar{1} \\ & \times & \times & \times & & \\ \hline \bar{1} & \bar{1} & 2 & \bar{1} & \bar{1} & 2 \end{array} = (\bar{3}11)$$

The face *n* has, therefore, the indices $(\bar{3}11)$.

That the signs of the indices are merely a function of the order in which the two zone symbols are taken will be at once apparent, for if we place the $[\bar{1}\bar{1}2]$ zone on the top line instead of the bottom line, and the $[01\bar{1}]$ zone below, we shall obtain $(3\bar{1}\bar{1})$ for the indices of the face *n*. Hence, the signs may be changed (provided all three in the symbol are so changed) without causing the indices to cease to represent the plane

¹ Taken from the author's memoir, *Journ. Chem. Soc.*, 1906, 89, 1065.

of the face. As a matter of fact, the two sets of signs correspond to the two parallel faces on opposite sides of the crystal, the two zones meeting twice on opposite sides of the centre. The ambiguity as to sign disappears when the position of the face as regards the octant in which it occurs is considered, for it is then obvious which of the axes will be cut by the face on the positive side and which on the negative side of the centre. There is, however, a safe rule for fixing the signs, dependent again on the occurrence of three faces in a zone. It is as follows:—

If the known indices of two faces in a zone be added together, each index of the one to that for the same axis of the other, the sum is the symbol of a face lying between the two faces in question, and in the same zone. In other words, it affords the indices of a third face in the zone, replacing the edge in which the two faces under consideration meet.

For instance, in the case of the example just given, and illustrated in Fig. 29, if we add together the indices of the faces m and p we get:

$$\begin{array}{r} \bar{2}01 \\ \bar{1}10 \\ \hline 3\bar{1}1 \end{array}$$

and $(\bar{3}11)$ is the symbol of the face n with its correct sign.

We cannot adopt this rule to give us with certainty the indices themselves, however, but only in order to fix the sign; because there may be several faces intermediate between the two faces the symbols of which are known, and the result may not be the particular one required. The cross-multiplication of the two zone-symbols must be resorted to, in order indubitably to obtain the indices, as has just been described. The case before us (Fig. 29) is an excellent example, for if we add the indices of the two faces o' and a of the second zone, we get the symbol $(\bar{2}11)$, which is not that of the face n , but of another face in the zone between n and o' , its pole being in fact at the point where the diameter mm crosses the zonal arc. Indeed, if we add together the indices of $m = (\bar{2}01)$ and of $b = (010)$ the pole at the centre (that of the undeveloped clino-pinakoid), we get $(\bar{2}11)$.

One more use of the zone symbol $[uvw]$ is to verify whether a face (hkl) belongs to a particular zone or not. For when it does,

$$hu + kv + lw = 0.$$

Taking again the same example, the symbol of the zone $o'a$ is $[\bar{1}\bar{1}2]$, and that of the face n is $(\bar{3}11)$, and

$$hu + kv + lw = \bar{3} \times \bar{1} + 1 \times \bar{1} + 1 \times \bar{2} = 3 - 1 - 2 = 0.$$

Also the symbol of the zone mp is $[01\bar{1}]$, and

$$hu + kv + lw = \bar{3} \times 0 + 1 \times 1 + 1 \times \bar{1} = 0 + 1 - 1 = 0.$$

Thus the face n does lie in the two zones mp and $o'a$.

Two further facts follow from the above considerations, namely:

A plane which is common to two zones is always a possible face of the crystal; and all the faces of even the most highly developed crystal (rich in faces) are connected together in zones.

Moreover, as all the zone symbols are got by cross-multiplication of rational facial indices, the zone symbols themselves are rational; consequently, the indices of any other face, obtained by the cross-multiplication of the zone-symbols of two zones to which the face is common, must also be rational. Hence: All the faces which are indicated as possible by constructing all possible zone circles on a stereographic projection, and marking their intersections by dots to represent facial poles, must be identical with the faces indicated as possible by taking all the possible combinations of rational indices.

The law of rational indices is, therefore, implied in the statement that all the faces of a crystal lie in zones constructed from four of its faces, no three of which lie in the same zone; for all the faces of a crystal have simple rational indices referred to its axes, which are determined by those four faces, three determining the directions of the axes and the fourth their lengths. In such a form it is known as the "zone law."

We can now take one further concluding step in the extremely valuable teaching of zone relationships, and one which is of untold value in the calculation of crystal angles and crystal elements. It is given in the succeeding paragraphs.

Law of Rational Double-Ratios, or Anharmonic Ratio of Four Poles in a Zone.—If *a*, *b*, *c*, *d* be four faces of a crystal lying in one and the same zone, the double-ratio (anharmonic ratio) of the sines of the angles between them is a rational number, which is expressed by the corresponding double-ratio of the indices of the four faces. This far-reaching law has one immediate consequence, namely, that there can only be three faces in a zone inclined at any angles whatsoever; for any fourth face also making any angle whatsoever would make the value of the double-ratio irrational. Consequently, this fourth angle must have a definite value, dependent on the values assumed by the other three angles, that is, on the position taken up by the three faces first considered. The formula in which the law is mathematically expressed, and which is an exceedingly simple one, enables us to determine either the fourth angle, given the indices of all four faces and the angles between three of them, or the indices of the fourth face, given the four angles and the indices of three faces. This formula may be written:

$$\frac{\sin(a:b)}{\sin(a:c)} \cdot \frac{\sin(d:c)}{\sin(d:b)} = \frac{(hkl) \text{ of } a \times (hkl) \text{ of } b}{(hkl)_a \times (hkl)_c} \cdot \frac{(hkl)_a \times (hkl)_c}{(hkl)_d \times (hkl)_b}$$

By "multiplication" of the indices on the right-hand side of the equation is to be understood "cross-multiplication," in a sense to be presently fully explained.

This principle of the anharmonic ratio of four poles in a zone affords us the indispensable connection between the indices of the faces (on the right in the equation) and the measured interfacial angles (on the left),

which latter are the practical means open to us of defining the relative positions of the facial planes of crystals. We arrive at the principle from the following considerations.

It will be remembered from the first principles of geometry that if A, B, C, D be four points on a straight line, then any one of the ratios

$$\frac{AB}{AC} \cdot \frac{DC}{DB} \text{ or } \frac{AC}{AD} \cdot \frac{BD}{BC} \text{ or } \frac{AB}{AD} \cdot \frac{CD}{CB},$$

or of the same ratios inverted, is termed an anharmonic ratio of the four points. It may be any one of the six ratios in which each point occurs once in the numerator and once in the denominator.

In the same manner, if A, B, C, D be four straight lines meeting at a common point O, as in Fig. 30, the anharmonic ratio of the sines of the angles is any one of six ratios, as before, of the type

$$\frac{\sin AOB}{\sin AOC} \cdot \frac{\sin DOC}{\sin DOB}.$$

If, now, we combine these two geometrical and trigonometrical principles, by allowing the straight line first considered to intersect the four straight lines forming the angles by their meeting at O, say in the points A, B, C, D (Fig. 30), then it is readily capable of proof that the anharmonic

ratio of the four points is equal both in magnitude and sign to the corresponding anharmonic ratio of the sines of the angles, that is :

$$\frac{AB}{AC} \cdot \frac{DC}{DB} = \frac{\sin AOB}{\sin AOC} \cdot \frac{\sin DOC}{\sin DOB}.$$

Care must be taken that the order of the letters is the same on each side of the equation, and that the signs are correct. For $BA = -AB$, and $\sin BA = \sin(-AB) = -\sin AB$.

Now, a precisely similar relationship holds between the indices of the faces on a crystal and the angles between those faces ; and if a, b, c, d be the four faces of the crystal lying in the same zone, then :

$$\frac{\sin ab}{\sin ac} \cdot \frac{\sin dc}{\sin db} = \frac{ab}{ac} \cdot \frac{dc}{db},$$

provided ab, ac, dc, db on the right-hand side of the equation are taken to mean the products (by cross-multiplication) of the indices of the respective faces, and on the left-hand side the angles between the faces. With these significations accepted for the letters, this equation is identical with that given on the previous page as the general formula expressing the anharmonic ratio of four poles in a zone. Hence, the anharmonic ratio of the sines of the angles of four tauto-

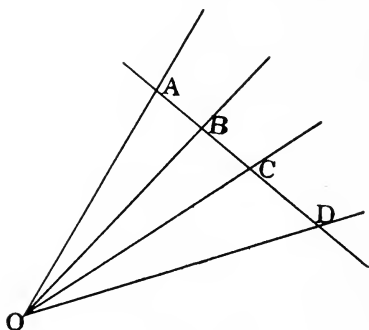


FIG. 30.

zonal (cozonal) planes is equal to the anharmonic ratio of their indices.

The immense practical utility of this formula will be at once apparent when it is remembered that the value of the anharmonic ratio of the indices is, as stated at the head of this section, a rational simple number, or simple fraction the numerator and denominator of which are low rational numbers, not generally higher than 4, and often only 2 or 3. This must be so, because of the rationality of the indices themselves. Consequently, if we know the positions of three poles in a zone, any other pole which makes the anharmonic ratio rational is that of a possible face of the crystal, and the angle made by this possible face with the other faces may be found for any rational number, corresponding to any desired or suggested indices. The angle thus obtained by calculation can then be compared with the measured angle between the actual fourth face and one of the other three faces, and the suggested indices of the fourth face thus either verified or shown to be untenable, when another likely rational number could be tried. Generally there are only two possibilities, and one or other of the two corresponding symbols will be found to yield a calculated angle which will agree with the measured angle. Or the converse use may be made of the formula, namely, knowing with certainty the indices of the fourth face, from the application of the zone rules, we may find the angle, that is, the position of the fourth face in the zone. For it may happen that this particular angle has not been measured, on account of bad reflections of the signal-image from the face in question; or if it has been measured, it may be desirable to verify the value obtained for the angle, as the reflections may have been imperfect, or at any rate of a lower order of perfection than those afforded by certain other faces, the angles between which are used as the basis of the calculations.

To illustrate the use of this important principle in practice, an example will be taken from the calculations made by the author from his measurements of the monoclinic crystals of ammonium selenate.

In the zone $[c\ o\ p\ o']$, shown in the stereographic projection, Fig. 29, and in shortened diagrammatic form in Fig. 31, there are the poles of four faces of different forms, the fifth pole being that of the c face parallel to the first and belonging to the same form. The four poles to be considered, therefore, are those of $c = (001)$, $o = (111)$, $p = (110)$, and $o' = (1\bar{1}\bar{1})$. The angles cp and po' were known with certainty to be respectively $77^\circ 26'$ and $43^\circ 16'$ from the measurements, which had been made on numerous excellent crystals, on all of which faces of the forms $c = \{001\}$, $p = \{110\}$, and $o' = \{1\bar{1}\bar{1}\}$ had been developed. The form $o = \{111\}$ had been of rarer occurrence, and it was desired to check the values obtained for the angles co and op by calculation from the indisputable values of cp and po' .

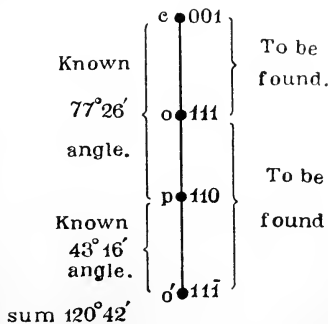


FIG. 31.

Setting down the anharmonic ratios of the sines of the angles on one side of the

equation, in the order convenient for our purpose, and then the indices on the other side, over each other in pairs (so as to be conveniently placed for cross-multiplication) and corresponding to the pairs of faces forming the respective angles, all in the same order as the angles are taken, we obtain :

$$\frac{\sin co}{\sin cp} \cdot \frac{\sin o'p}{\sin o'o} = \frac{001}{110} \cdot \frac{11\bar{1}}{110}$$

When the values of the known angles are inserted the left-hand side of the equation becomes :

$$\frac{\sin co}{\sin oo'} \cdot \frac{\sin 43^\circ 16'}{\sin 77^\circ 26'}$$

The right-hand side is to be cross-multiplied by taking the second and third pairs each time, rather than the first and second ; for the latter procedure would make two ciphers $\frac{0}{0}$ the result in the case of the first fraction, and thus render the final result indeterminate and valueless. Cross-multiplying, therefore, the second and third pairs, we obtain :

$$\begin{array}{r} 001 \quad 11\bar{1} \\ \times \quad \times \\ 111 \quad 110 \\ 001 \quad 11\bar{1} \end{array} = \frac{0 \times 1 - 1 \times 1}{0 \times 0 - 1 \times 1} \cdot \frac{1 \times 0 - \bar{1} \times 1}{1 \times 1 - \bar{1} \times 1}$$

$$\begin{array}{r} \times \quad \times \\ 110 \quad 111 \end{array} = \frac{0-1}{0-1} \cdot \frac{0+1}{1+1} = \frac{1}{2}$$

Hence, the whole equation simplifies to :

$$\frac{\sin co}{\sin oo'} = \frac{\sin 77^\circ 26'}{2 \sin 43^\circ 16'}$$

The right-hand part of the equation works out to a number, and this may be considered as $\tan \theta$, in accordance with a familiar trigonometrical device ; that is :

$$(a) \quad \frac{\sin co}{\sin oo'} = \frac{\sin 77^\circ 26'}{2 \sin 43^\circ 16'} = \tan \theta$$

Working out the value of θ with the aid of logarithms :

Log. 2	0.30103	Log. $\sin 77^\circ 26'$	$\bar{1}.98947$
Log. $\sin 43^\circ 16'$	$\bar{1}.83594$	Log. $2 \sin 43^\circ 16'$	0.13697
	<u>0.13697</u>		<u>$\bar{1}.85250$</u> Log. $\tan 35^\circ 27'$

The value of θ is thus $35^\circ 27'$. The object of considering the numerical value of the right-hand part of the equation as the tangent of an angle is in order that we may convert $\frac{\sin co}{\sin oo'}$ into a form suitable for logarithmic calculation, and in which the two unknown angles co and oo' may be reduced to one unknown by taking advantage of our knowledge of their sum, for this is $= cp + po' = 120^\circ 42'$. Such a convenient form is obtained by employing the well-known formula for $\tan (45^\circ - \theta)$, namely, $\frac{1 - \tan \theta}{1 + \tan \theta}$. For in the equation :

$$(b) \quad \frac{1 - \tan \theta}{1 + \tan \theta} = \tan (45^\circ - \theta)$$

we can substitute for $\tan \theta$ its equivalent value $\frac{\sin co}{\sin oo'}$ taken from equation (a) ; we then get :

$$(c) \frac{1 - \frac{\sin co}{\sin oo'}}{\frac{\sin co}{\sin oo'}} = \frac{\frac{\sin oo' - \sin co}{\sin oo'}}{\frac{\sin oo' + \sin co}{\sin oo'}} = \frac{\sin oo' - \sin co}{\sin oo' + \sin co} = \tan (45^\circ - \theta).$$

We can now transform the third fraction in (c) into the form of the quotient of the tangent of half the difference by the tangent of half the sum of the two angles, a result well known in trigonometry, and which calls for no proof here. Thus,

$$(d) \frac{\sin oo' - \sin co}{\sin oo' + \sin co} = \frac{\tan \frac{oo' - co}{2}}{\tan \frac{oo' + co}{2}} = \tan (45^\circ - \theta).$$

It now only remains to transpose, thus :

$$(e) \tan \frac{oo' - co}{2} = \tan \frac{oo' + co}{2} \cdot \tan (45^\circ - \theta).$$

The terms on the right-hand side of this final equation (e) are both known, the first being half the known sum of the angles ($120^\circ 42'$), namely, $60^\circ 21'$, and ($45^\circ - \theta$) being $9^\circ 33'$. Hence, we have

$$\tan \frac{oo' - co}{2} = \tan 60^\circ 21' \cdot \tan 9^\circ 33'.$$

Working this out logarithmically,

$$\text{Log. tan } 60^\circ 21' \quad 0.24471$$

$$\text{Log. tan } 9^\circ 33' \quad 1.22593$$

$$\hline 1.47064 = \text{Log. tan } 16^\circ 28',$$

we find that half the difference of the two angles is $16^\circ 28'$. We now possess, therefore, the knowledge that

$$oo' + co = 120^\circ 42',$$

and

$$oo' - co = 32^\circ 56'.$$

By addition, therefore,

$$2oo' = 153^\circ 38', \text{ from which } oo' = 76^\circ 49';$$

and by subtraction

$$2co = 87^\circ 46', \text{ from which } co = 43^\circ 53'.$$

The remaining angle op is at once afforded by the difference of the given value of cp and the value now found for co , or of the found value of oo' and the given value of po' . That is,

$$op = \left\{ \begin{array}{l} cp - co = 77^\circ 26' - 43^\circ 53' \\ \text{or} \\ oo' - po' = 76^\circ 49' - 43^\circ 16' \end{array} \right\} = 33^\circ 33'.$$

Thus the final result of the calculation is that :

$$co = 43^\circ 53',$$

$$op = 33^\circ 33'.$$

It may be interesting to remark that the angles found by actual measurement in this particular case were $co = 43^\circ 51'$ and $op = 33^\circ 30'$, which affords both a satisfactory proof of the accuracy of the calculations, and an example of the usual degree of agreement between calculated and measured angular values, in such cases as the present, when the degree of reliability of the measured value is not quite so high as in the cases of the more frequently measured angles used as the basis of calculation.

The above example having been worked through from first principles, it may have appeared more complicated than it really is. For we have only to work out in practice the value of $\tan \theta$ in equation (a), and then at once to set down the equation (e) and work out from it the value of the difference of the two angles required; this, by addition to and subtraction from the known sum of the two angles, at once enables us to find the two required angles, either of which defines the position in question of the fourth pole in the zone.

When one of the angles is a right angle, the formula becomes simplified to the ratio of two tangents. For example, in the case of the same crystal of monoclinic ammonium selenate the question arose of finding the magnitude of the angle ap' in the zone $[ap'pb]$ shown in the stereographic projection, Fig. 29, and diagrammatically to represent the conditions in Fig. 32. The angle ap had been thoroughly well measured (36 times) and shown to be $59^\circ 38'$, but ap' had only been measurable 4 times, and so it was desired to calculate its value for the sake of verification, from the more certain value of ap . The angle ab was a right angle, the plane b being the symmetry plane and a one of the fundamental faces, that is, chosen as an axial plane, lying in the zone perpendicular to the symmetry plane.

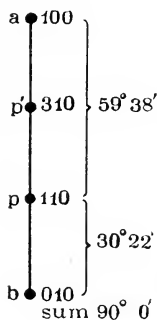


FIG. 32.

Writing down the anharmonic ratio, starting as usual from the two ends, we get :

$$\frac{\sin ap'}{\sin ap} \cdot \frac{\sin bp}{\sin bp'} = \frac{ap'}{ap} \cdot \frac{bp}{bp'}$$

Putting in the values of the indices on the right-hand side, and the values of the known angles on the left-hand side, and instead of $\sin bp$ ($bp = 30^\circ 22'$) and $\sin bp'$ substituting their equivalents (being a right-angled triangle) $\cos ap$ and $\cos ap'$, that is, the cosines of their complements, we obtain :

$$\frac{\sin ap'}{\sin 59^\circ 38'} \cdot \frac{\cos 59^\circ 38'}{\cos ap'} = \frac{3 \ 1 \ 0}{1 \ 0 \ 0} \cdot \frac{0 \ 1 \ 0}{0 \ 1 \ 0}$$

$$\begin{array}{cc} \times & \times \\ 1 \ 1 \ 0 & 3 \ 1 \ 0 \end{array}$$

Cross-multiplying the first two pairs of indices in each case, as the second and third pairs would not yield a finite result, we obtain :

$$\tan ap' \cdot \cot 59^\circ 38' = \frac{1}{1} \cdot \frac{\bar{1}}{3}$$

or

$$\tan ap' \cdot \cot 59^\circ 38' = \frac{1}{3}$$

Transposing, and replacing $\frac{1}{\cot 59^\circ 38'}$ by $\tan 59^\circ 38'$, we obtain the simple expression :

$$\tan ap' = \frac{1}{3} \tan 59^\circ 38'$$

Working this out :

$$\text{Log. } \tan 59^\circ 38' \ 0.23217$$

$$\text{Log. } 3 \ 0.47712$$

$$\text{Log. } \tan ap' \ \bar{1}.75505 = \text{Log. } \tan 29^\circ 38'$$

we find that :

$$ap' = 29^\circ 38'$$

The value of the angle obtained by taking the mean of the four measurements was $29^{\circ} 36'$, the two values thus satisfactorily confirming one another.

This last is also an excellent example of the converse use of the anharmonic ratio of four poles in a zone for determining the symbol of a face. For there was some doubt when the crystals of ammonium selenate were being investigated as to whether the proper symbol of the face p' were (210) or (310). For the crystals are not isomorphous with those of orthorhombic ammonium and potassium sulphates, but belong to the monoclinic system, and are thus quite different from the latter salts, so that we cannot compare the indices. Now, if the symbol be (310), the angle is, as we have seen, $29^{\circ} 38'$. But if it were (210), and we substituted these indices for (310) on the right-hand side of the expression for the anharmonic ratio, we should get:

$$\tan ap' = \frac{1}{2} \tan 59^{\circ} 38',$$

and ap' would work out to be $40^{\circ} 29'$. Now, the position of the face was found by direct measurement four times, and although there was not such certainty about the mean result as regards the last minute (it has been shown to differ $2'$ from the calculated value) as in the case of the basal angle ap , which had been measured as often as thirty-six times on different crystals and on different parts of the same crystal (the angle being repeated four times on going round the zone), still it was absolutely certain that the true value was quite near to $29\frac{1}{2}^{\circ}$ and consequently very far removed from $40\frac{1}{2}^{\circ}$. Hence the indices were certainly (310) and not (210).

So much space has been devoted to this principle of four poles in a zone because of its unique practical importance. For with its aid, and that of the principles which have been elucidated for finding the indices of faces by zonal intersections, together with such of the ordinary formulæ for the solution of spherical triangles as are suitable for logarithmic calculation (these triangles being shown directly on the stereographic projection), the whole of the calculations of practical morphological crystallography can be carried out. It is essential, therefore, that the few principles discussed in this chapter should be thoroughly mastered, for when that desirable consummation is achieved all difficulty with the subject will disappear.

The invaluable nature of the stereographic projection will have impressed itself more and more during the elucidation of these principles. For it not only furnishes in one complete diagram a review of the symmetry of the crystal, but also indicates, by showing at once all the possible triangles formed by the numerous intersecting zone circles, how we can proceed to make the required calculations. That is, it enables us to see at a glance when to employ the anharmonic ratio for the determination of the position of any fourth pole in a zone, and when to determine the angle between any pair of faces whatsoever with the aid of one or other (and which) of the usual formulæ of spherical trigonometry above referred to, the more practically convenient of which

will be given at the commencement of the next chapter. It affords, moreover, as will subsequently be shown, the diagram for the calculation of the relative lengths of the crystal axes and the axial angles, in the cases where these latter are not 90° . It is advisable, therefore, that this chapter should close with a concise account of the principles which govern the construction and use of the stereographic projection, including one or two important propositions not already elucidated during the measurement of the typical crystal of potassium sulphate.

The Stereographic Projection of the Arrangement of Crystal Faces.—The elementary rules concerning this invaluable mode of representing graphically the symmetry of a crystal have already been elicited in Chapter IV., during the process of measuring the typical crystal of potassium sulphate. It is necessary now, however, after having thus fully considered the subject of indices and that of zonal relationships, to summarise them, and to extend them so as to meet the cases of crystals in general.

The following are the three chief principles and facts on which the practical application of the stereographic projection depends.

(1) Every circle on the sphere appears in the projection as a circle (the primitive circle) or circular arc, or as a diameter of the primitive circle; and if the circle in question be a great circle (the section of a sphere by a plane passing through the centre of the sphere), it appears in the projection as a diameter, or as a circular arc which cuts the primitive circle at the ends of a diameter. Moreover, all zone circles perpendicular to the primitive circle have their poles on the latter, and appear in the projection as diameters.

(2) The pole of a zone circle is the projection of that point on the sphere which is situated 90° from all points on the zone circle; in other words, it is the pole of the zone plane, or of a face at right angles to the faces of the zone.

(3) If the poles of any two faces belonging to a zone be joined to the pole of the zone circle, and if the two straight lines thus obtained be produced until they cut the primitive circle, then the arc thus cut off and enclosed on the primitive circle will be equal to the angle between the faces.

The following practical rules for the application of these principles will be found to cover the cases of crystals in general, including the more difficult ones of triclinic or asymmetric crystals.

In order to find on the projection the positions of the poles of the different faces constituting a zone, after having determined the angles between the normals to the faces on the goniometer, the zone circle must first be constructed, and its pole found. To do this we proceed on the lines indicated by paragraphs (2) and (3).

Let $ABA'B'$ in Fig. 33 be the primitive circle, and A and P be the poles of two faces, the former lying on the primitive circle zone. The positions of the poles on the primitive circle are given at once, as already explained in Chapter IV., by merely setting off along it

the measured interfacial angles of the zone, chosen as the primitive zone of the crystal, and marking their positions by dots, and the dot A represents one of the faces of this zone. The position of the pole P will here be supposed to be given; the mode of finding it will be described on the next page but one. As a matter of fact A and P are the poles $b = (010)$ and $o' = (\bar{1}11)$ of an actual triclinic substance measured by the author, methyl triphenylpyrrholone, the whole stereographic projection of which is given in Fig. 35. The face A' parallel to A at the other end of the diameter AOA' will also belong to the zone AP, and so we can describe a circular arc through the three points APA', by the usual geometrical process, and this arc will be the projection of the zone circle. In order actually to describe the arc we require to find its centre, which will lie on the diameter BOB', at right angles to AA', or its prolongation. In general the usual process referred to would be to draw straight lines perpendicular to and bisecting the two straight lines joining A and P, and A' and P, the point of intersection of these perpendiculars being the required centre. It suffices here, however, to draw one such perpendicular, CD, as the diameter acts as one and its intersection C with either perpendicular, or both, is the required centre.

The projection of the pole of this zone circle APA' thus described will

lie on the diameter BOB', for this latter is the projection of the zone circle the plane of which is perpendicular to that of the zone circle APA' and to that of the primitive circle, and which contains all points situated 90° from A or A'. But the desired pole of the zone circle APA' must not only be 90° removed from A or A' but also from all other points on the zone circle, Q, for instance, being such a point. Now, as A is the pole of the zone BQB', if we draw the straight line joining A and Q and produce it to R on the primitive circle, and if we then mark off the point S at 90° on that circle, to the left of R, and join AS, then the projection of the desired pole will be at T, where AS cuts the diameter BQOB'. For QT, when projected on to the primitive circle from the pole A of the zone circle represented in the projection by the diameter BQTB', on which zone circle the required pole must lie, projects into an angle of 90° on the primitive circle.

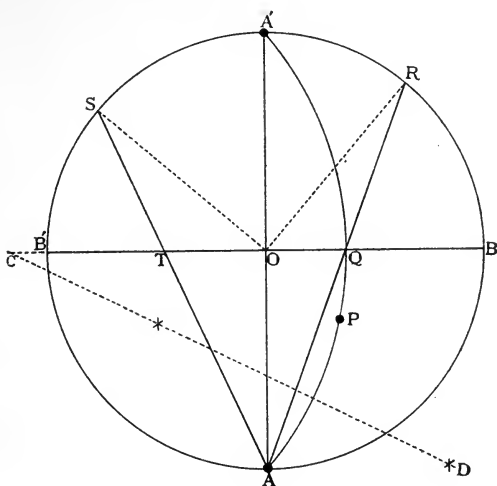


FIG. 33.

by construction, so that, by principle (3) *T* is the projection of a point situated 90° from *Q*, as desired.

It is interesting to observe that as the radius of the zone circle becomes greater, and the curve consequently becomes flatter, the pole recedes along the perpendicular diameter and approaches the primitive circle, as instanced by the pole *P* of the zone circle *bqcq'b* in Fig. 35, until, in the limiting case, the zone circle also becomes a diameter and the pole then lies actually on the primitive circle itself, at the extremity of the perpendicular diameter, in accordance with the last sentence of principle (1).

It will generally happen, except in the few cases met with of entire absence of symmetry, or of all but centro-symmetry, in the crystals of a substance, that the stereographic projection can be completed without having actually to construct (from a couple of angular measurements only, defining the position of a face not on the primitive zone with respect to faces on that zone,) a zone circle which is not a diameter in the projection. The presence of symmetry, as in the typical case of potassium sulphate considered in Chapter IV., generally ensures that, after having drawn in the poles on the primitive circle, representing the faces on the principal zone, and having joined the poles of opposite parallel faces by diameters, the poles of most of the remaining faces will lie on those diameters; and even the more complicated, rare, or subsidiary faces will lie on such curves, representing other zones, as can be readily drawn in; for three points will already have been found on them, namely, two on the primitive circle at the ends of a diameter, and the third on another diameter. It is only a matter of finding the centre of the curve and actually constructing it, which can be done either by trial, or by the usual procedure of plane geometry as given on the last page.

Having thus constructed the zone circle and found its pole, it only remains to apply rule (3) in order to be able to find the positions of the poles of the faces of the zone, namely, by setting off the measured angles along the primitive circle and on the further side of it from the pole of the zone circle, starting from one of the two facial poles common to the primitive and zone circles, and joining the points thus obtained on that circle to the pole of the zone circle; the points of intersection of those straight lines with the zone circle will be the required poles, exactly as described on page 55, Chapter IV., for potassium sulphate and graphically shown in Fig. 24.

The procedure is not quite so simple, however, in the absence of symmetry, or where there is merely centro-symmetry, as in the crystals of the triclinic system. In such a case we can only proceed as already indicated as far as choosing the best-developed zone for that which is to be considered as perpendicular to the plane of projection, and drawing in the primitive circle with its poles at the angular distances measured between the normals to the faces. An example is given in Fig. 35, which represents the stereographic projection of a typical triclinic substance methyl triphenylpyrrholone.

One of the actual crystals of this substance investigated by the author is shown in Fig. 34, drawn to scale by the conventional method to be described in Chapter XXV. The construction of the triclinic axes corresponding to the crystal is fully described in that chapter, and illustrated in Fig. 390. The basal plane $c=(001)$ is now no longer perpendicular to this zone of faces chosen to afford the primitive circle, and parallel to the plane of projection, as was the case with the orthorhombic crystal of potassium sulphate, so it cannot be represented by the usual dot at the centre of the primitive circle. We know from the measurements, however, the inclination of the important face c , which we choose as basal plane (001) , to the two other primary faces $a=(100)$ and $b=(010)$. If we let θ and ϕ represent these two angles $a : c = (100) : (001)$ and

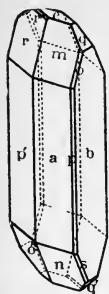


FIG. 34.—Crystal of Methyl Triphenylpyrrolone.

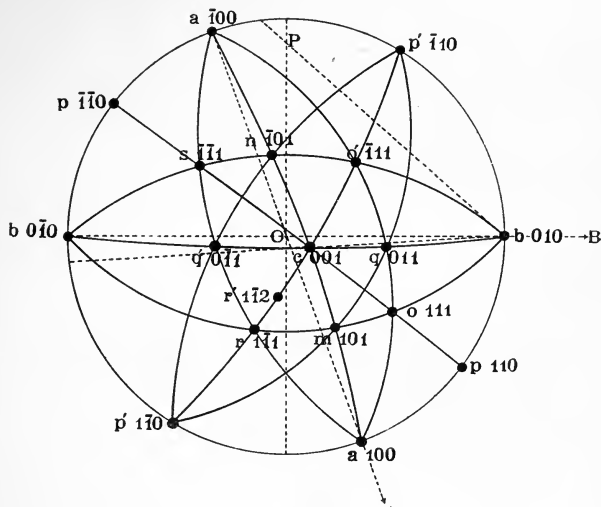


FIG. 35.—Stereographic Projection of Methyl Triphenylpyrrolone.

$b : c = (010) : (001)$ respectively, we can then make the following simple construction and calculation in order to arrive at the position of the pole of the basal plane c on the projection. We produce the diameters (dotted in Fig. 35) aa' and bb' , the former forwards and the latter to the right, and then r being the radius of the primitive circle, we take lengths OA and OB along them from the centre O such that :

$$OA = \frac{r}{\cos \theta}, \text{ and } OB = \frac{r}{\cos \phi}.$$

Then from each of the points A and B we draw a circular arc the radius of which is, in the respective cases,

$$Ac = r \tan \theta, \text{ and } Bc = r \tan \phi.$$

These two arcs will intersect each other within the primitive circle at the pole c of the basal plane (001) . We have then three points on

each of the zones *aca* and *bc b*, and can therefore at once construct the two zone circles.

In the case of methyl triphenylpyrrolone, the radius r of the primitive circle in the original drawing, of which Fig. 35 is an approximate half-size reproduction, was 55 millimetres; θ , the angle *ac* as measured on the crystal, was $82^\circ 2'$, and ϕ , the measured angle *bc*, was $77^\circ 24'$. The calculations for OA and OB, and for Ac and Bc, were then as under:

Log. r	1.74036	Log. r	1.74036		
Log. $\cos \theta$	1.14175	Log. $\cos \phi$	1.33874		
Log. OA	2.59861	OA = 396.8 mm.	Log. OB	2.40162	OB = 252.1 mm.
Log. r	1.74036	Log. r	1.74036		
Log. $\tan \theta$	0.85403	Log. $\tan \phi$	0.65067		
Log. Ac	2.59439	Ac = 393.0 mm.	Log. Bc	2.39103	Bc = 246.1 mm.

As the points A and B are thus respectively about seven and five times as far from the centre O as the points *a* and *b*, they are not shown in Fig. 35, the arrows indicating their directions. They were, of course, actually found on the large sheet of cardboard on which the original drawing was made, and employed as the centres for the arcs Ac and Bc, but for reproduction purposes it was necessary to eliminate these two very long straggling construction lines. In the case of copper sulphate they were not nearly so long, so they are shown in the Fig. 252 representing the stereographic projection of that substance in Chapter XX.

The poles of the remaining faces can be found in a similar manner if necessary, but usually it is only necessary to find *c*, or, in the absence of a basal plane, some one other pole. Occasionally, as in the case of copper sulphate crystals, described in Chapter XX., the angle *ac* or *bc* (in the case of copper sulphate it is *bc*) is slightly more than 90° , and it is then more convenient to choose another face than *c*, the angles of which with *a* and *b* are less than 90° , for convenience of calculation by use of the formulæ just given. The face $q = (011)$ is convenient in the case of copper sulphate. Other pairs of poles of parallel faces on the primitive circle will form zones with the pole just found, and the construction of these two or three zone circles enables us at once to determine the positions along them, by the method already described, of the poles of the other faces belonging to these zones, when, as is likely, others are present. Having thus found the positions of quite a number of facial poles, other zones will be apparent passing through fresh combinations of them, and their zone circles can be similarly readily constructed. Consequently the whole stereographic projection can as a rule be constructed without having recourse to calculation for the position of more than one pole, and, as we have already seen, if symmetry is developed even this one calculation can generally be dispensed with.

It will thus be apparent in how emphatic a manner the stereographic projection brings out the important fact that the faces of crystals are arranged in zones, and how clearly it renders apparent any symmetry that may exist, as regards both the disposition of the individual faces and that of the zones into which they fall.

Aids to the Construction of Stereographic Projections; Wulff and von Fedorow's Ruler for Circles of large Radius; Hutchinson's Stereographic Protractor and Stereographic Net.—The drawing of a circular arc of large radius, that is, of a flat curve, in the cases of great circles on a stereographic projection which are nearly straight lines, is often a somewhat difficult matter with even a large pair of compasses, and the exact attainment of the required accuracy with which the arc passes through the ends of the diameter on the primitive circle, where diameter, primitive circle, and the large arc should intersect, as well as through any other poles already found, is not easy. For such cases a **curved ruler**, of which the curvature is capable of variation while preserving the circular nature of the arc, was suggested by both Wulff and von Fedorow, and is now constructed, in accordance with the designs of the latter, by Fuess of Berlin. It is shown in Fig. 36. It consists essentially of a strip of steel *a*, sufficiently thin to be readily bent into a bow by means of a screw *b*, applied like an arrow except for two important differences. Firstly, it is applied to the bow itself and not to the equivalent of the string, which is a rigid bar *c* moved parallel to

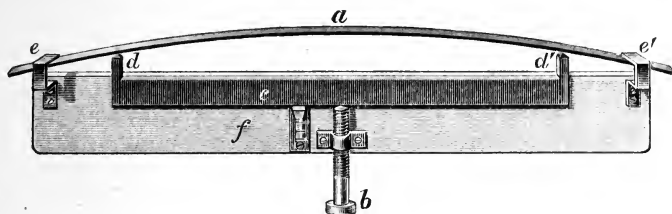


FIG. 36.—Von Fedorow's Ruler for Circular Arcs.

itself by means of the screw; and secondly, the force is not applied in one place at the middle of the bow but at two places *d* and *d'* equidistant and not very far from each end. The rigid bar terminates at these positions in a pair of wedges or slightly rounded knife-edges parallel to the surface of the curved strip, which is slightly oblique in order to afford greater facility for the use of the ruling pen; the wedges push the strip circularly outwards so that the maximum movement is at the centre, while it is prevented from moving outwards at all at the ends, but is left free to move in the direction of its length, by means of and between a pair of guides *ee'*, which act as the fulcrum supports; it can readily adjust itself, therefore, to any circular arc which it is constrained to form by the action of the screw and the pressure of the wedges which the latter actuates. The mother-nut of the screw is fixed rigidly to the basal plate *f*, by which the apparatus rests on the drawing in the position required for the construction of the circular arc.

The steel strip acts in this instrument like the bar supported on two knife-edges near its ends, in the well-known elasticity experiment, and which is loaded above at two points symmetrically situated with respect to the two supports and within the latter nearer to the centre. The curve produced by such a bar is nearly a circular arc, and in the case of

the steel strip the approximation to a circle, when the screw is only used to the extent required to produce fairly flat curves for which the compasses are not readily available, is so close that the difference is not perceptible on the scale of the drawing. A full discussion of the theory of the instrument will be found in von Fedorow's memoir.¹

The **protractor** devised by Dr. Hutchinson to facilitate the construction of stereographic projections is constructed in boxwood by Harling (47 Finsbury Pavement, London). It is shown in Fig. 37, and its method of use in Fig. 38. It is intended for a projection the primitive circle of which is of $2\frac{1}{2}$ inches radius. It is about a foot long and is exactly 2.5 inches wide. A circle of 5 inches diameter is very convenient in size for a stereographic projection intended for publication, by photo-process reproduction on a reduced scale. The necessary circles to represent the various zones on a stereographic projection of such a size may be readily drawn with the large compasses of any good box of mathematical instruments, or, when the radius is very large, by means of the Wulff-Fedorow circular ruler just described.

The protractor has a zero line, OZ in Fig. 38, drawn across it at right angles to its length, and at a distance from one end rather greater than its width. Taking Z as centre, where the zero line intersects one of the longer edges, the opposite long edge ST is graduated by means of a circular dividing engine. The portion of the scale from O to T is divided into degrees, just like an ordinary protractor, and every tenth division is numbered in duplicate, the numbers running both ways, as shown in the figures. The divisions along OS are the same as those from 0° to 45° along OT, but are numbered so that each division represents 2° instead of 1° . The finer divisions are omitted in the figures for the sake of clearness, but they are very clearly engraved on the protractor itself.

The distance from O of any division lying between O and T is obviously numerically equal to the tangent of the corresponding angle, read on the upper row of numbers, when measured with a scale for which the radius of the primitive circle is taken as the unit. Such a scale is provided on the back of the protractor, and in order that the tangent may be expressed as usual in decimals, its numbered unit is a quarter of an inch, one tenth of 2.5 inches; each numbered unit is further subdivided into tenths, so that the actual engraved unit-mark of the scale is one-fortieth of an inch.

The distance from O of any division along OS is numerically equal to the tangent of half the angle as read on the lower row of numbers, and the divisions themselves correspond to degrees stereographically projected on a diameter. This enables us at once to perform the following operations.

(1) The position of a facial pole P, situated on any diameter and the angular distance of which from the centre of the primitive circle (or its complement, from the primitive circle itself) is known, may at once be marked off, by placing the protractor along the diameter in question,

¹ *Zeitschr. für Kryst.*, 1893, 21, 618.

of the compass passes through P in constructing the circular arc BPZ. This construction depends on the fact that if r be the radius of the primitive circle and θ the angle OP, then pP the radius of any great circle BPZ is numerically equal to $r \operatorname{cosec} \theta$.

(3) The pole P' of a great circle such as BPZ may also be easily found with the protractor still in the same position. We take the reading of P on the scale OS, using the upper numbers, divide this reading by two, and find on the upper number scale of OT the reading corresponding, which will be that of the required pole P' of the great circle BPZ. Thus in Fig. 38 the upper row reading for P along OS is 50° , half this is 25° , and at 25° along OT, using the upper row of numbers, we have the position of P'. The proof of this is, that the stereographic position of P on OS being 40° and that of P' being (stereographically also) $25^\circ \times 2$, or 50° , the sum of the two stereographic positions $40^\circ + 50^\circ = 90^\circ$, and by definition the pole of a great circle is 90° removed from all points upon it. (It will be remembered that a "great circle" is the section of a sphere by a plane which passes through the centre of the sphere, a "small circle" being a section by any other plane.)

A very valuable property of the stereographic projection is its angular truth, for the angles between great circles on the sphere are preserved in the projection. We can, therefore, determine the angles between zones which intersect at a certain point by drawing tangents to the zone circles at the point where they meet, and measuring the angle between these tangents. This, of course, can readily be done by using the ordinary protractor scale along OT.

It will often be required, also, to measure the angle between two crystal faces of which the projected positions have been found. This requires the construction of the great circle passing through the points and the determination of the pole of this great circle. This can always be done by means of the protractor, using operations (2) and (3), and making use of other information which will be available to enable the terminations of the great circular arc on the primitive circle to be fixed, that is, the ends of a diameter which are also points on the great circle in question.

Many other problems can also be solved with equal facility with the aid of this stereographic protractor, such, for instance, as the construction of a small circle, of given angular radius, round any point either on or within the primitive circle, and an account of them will be found in the original memoir of Dr. Hutchinson.¹

The last of the problems above referred to, and many others relating to stereographic projection, may be more rapidly solved by means of a **stereographic net**. The use of such nets has been pointed out by von Fedorow, Wulff, Penfield, and other recent investigators, and a very convenient one has been prepared by Dr. Hutchinson for a primitive circle of 2.5 inches radius, corresponding to the stereographic protractor. It is shown in Fig. 39. The great circles and small circles

¹ *Min. Mag.*, 1908, 15, 93-112.

constructed on this net are only 2° apart, so that the positions of poles on the projection can be found with great precision. A copy of the net on tracing paper is exceedingly useful, as it enables the positions found on it to be directly pricked through on to the drawing which is being made on Bristol board for the purposes of reproduction, or if publication is not intended, on to the drawing which is being made on drawing paper or in the note-book, without injuring the original net, a new tracing being taken whenever the one in use becomes spoilt with frequent use. If reproduction be intended, another useful method of procedure is to make a preliminary drawing on tracing paper, and to place it above the net (either the one in the book or the loose one on tracing paper), when the particular problem can readily be solved. For instance, in the case of the second one above referred to, the measurement of the angle between two points, the tracing-paper drawing is placed over the net so that the two centres are coincident, and with a needle stuck through them; the angular distance between the two points in question may then be determined by rotating the drawing above the net until the two points lie along the same great circle, or evenly between two adjacent ones. The angular distance between the points may then be read off immediately by means of the small circles. The problem can also be solved without the use of a tracing by means of a three-legged compass; two of its legs are arranged so as to be placed on the two points when the third is on the centre of the primitive circle; the compass points are then transferred to the net, the same point being placed on the centre, and the compass is rotated about this leg until the other two legs lie on the same great circle.

An exceedingly interesting account of the history of the stereographic projection will be found in Dr. Hutchinson's memoir, going

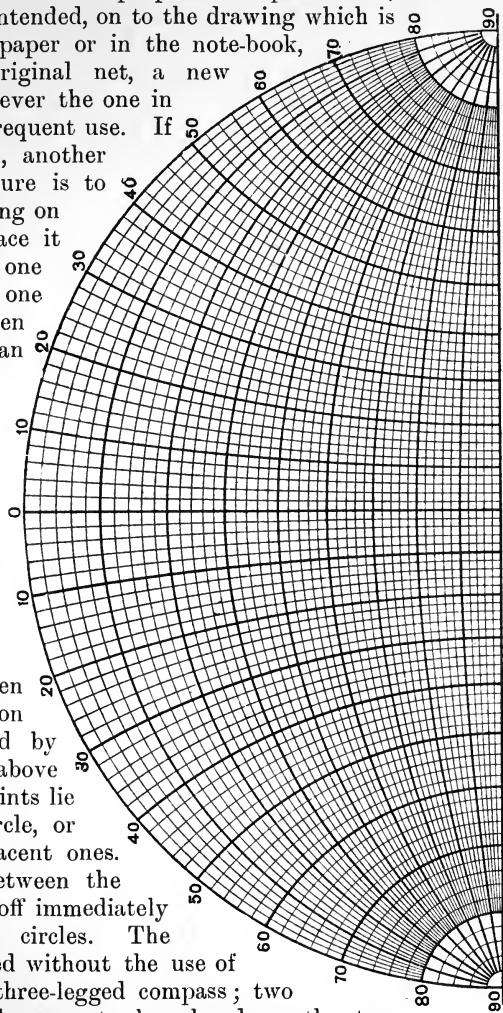


FIG. 39.—Hutchinson's Stereographic Net.

back to Ptolemy, and to his greater predecessor Hipparchus, whose writings concerning the projection are unfortunately lost to us, except what we know through the preserved works of Ptolemy. It has already been stated in Chapter IV. that its use in crystallography was first pointed out by Neumann, in his *Beiträge zur Krystallonomie*, dated 1823. Its use in England was taken up by Professor W. H. Miller of Cambridge, and forms the basis of his *Treatise on Crystallography* published in 1839 already referred to in the last chapter, as the graphical expression of the system of facial indices which he introduced, and which has ever since borne his name and is now practically universally employed. Des Cloizeaux employed it later in his *Manuel de minéralogie*, the first volume of which was published in 1862. It has been much further developed, however, of late years by von Fedorow¹ and Wulff² who both published stereographic nets for a primitive circle of 10 centimetres radius, and indicated the use of the triangular compass with them. Benno Hecht also published a net for a circle of 5 c.m. radius in his *Anleitung zur Krystallberechnung* dated 1893, and Sommerfeldt³ has since extended this beyond the boundary of the primitive circle. Goldschmidt⁴ has also discussed the stereographic projection as regards its use for dealing with the data obtained by the use of the two-circle goniometer.

The name of Penfield stands out predominately, however, as regards recent work on the stereographic projection. In a series of papers to the *American Journal of Science*, beginning with two in 1901 (ser. 4, vol. xi. pages 1 and 115), and including two important ones in vols. xiii. and xiv., Penfield has given a most exhaustive treatment of the subject, and published stereographic nets printed on transparent celluloid. Penfield also introduced sheets of paper on which are printed a circle of 7 c.m. radius divided into degrees, together with four scales; the first two give the radii of great and small circles respectively, the third is a scale of stereographically projected degrees, and the fourth is one the unit of which is the one-hundredth part of the radius.

All these constructions of Penfield's can now be carried out with the stereographic protractor of Hutchinson, and if the latter be constructed for a 7 c.m. radius it will be found more convenient with Penfield's circle than the four scales given by Penfield himself. But, as already stated, the net of 2.5 inches (6.3 c.m.) radius given by Hutchinson is one of a particularly convenient size, and the 2.5 inch protractor and net together afford the best aids to the construction of stereographic projections which have yet been described.

¹ *Zeitschr. für Kryst.*, 1893, 21, 617, and 1903, 37, 138.

² *Ibid.*, 1893, 21, 253, and 1902, 36, 14.

³ *Ibid.*, 1906, 41, 164.

⁴ *Ibid.*, 1899, 30, 260.

CHAPTER VII

THE FEW INDISPENSABLE FORMULÆ REQUIRED FOR THE CALCULATION OF CRYSTAL ANGLES AND ELEMENTS.

HAVING in the last chapter elucidated the main lines on which the general structure of crystals is conceived, and brought into prominence the important principle of rationality, as regards both intercepts and indices, and also developed the interesting laws regarding the arrangement of faces in zones, we are now in a position to work out to a logical conclusion the typical experimental results obtained with the orthorhombic crystal of potassium sulphate the goniometry of which was described in Chapter IV., and we shall proceed to do so in the next chapter. Before so doing, the chief formulæ employed in the calculation of crystal angles from the necessary given data—one, two, three, or five thoroughly well measured basal angles, according as the symmetry is tetragonal or hexagonal, orthorhombic, monoclinic, or triclinic respectively, as will be fully proved and explained in subsequent chapters—and in the derivation of the crystal elements, will be dealt with in this chapter and rendered simple and straightforward.

Students of crystallography have so frequently been discouraged by the formidable array of mathematical formulæ set out in books on the subject, particularly with respect to the calculation of crystal angles and elements, that it is one of the author's chief aims to dispel the cloud of misconception which has arisen as to the difficulty of the science. The only mathematical endowment required is the ability to employ the ordinary formulæ for the solution of spherical triangles, that is, an elementary knowledge of plane trigonometry and the ability thereby conferred to use the very similar formulæ of spherical trigonometry. An advanced study of the latter subject is unnecessary, although, of course, the writer would be the last person to deny that a superior knowledge of higher mathematics is an equipment of the utmost value to an investigator in this branch of science. It will, however, be the author's special task to make the calculation of goniometrical results simple and clear. The few essential formulæ will first be stated in this chapter, and then with this endowment a typical series of crystals of all the degrees of symmetry will be worked out from first principles in subsequent chapters, beginning with the calculation of the results

for potassium sulphate, as the first of these typical crystals, in a general manner which should always be employed with regard to every crystal investigated.

Chief Formulæ used in Calculation of Crystal Angles from the Basal Angles.—The few fundamental formulæ in question, which are of constant use in all such calculations, will now be stated.

(1) The first of these is the all-important **anharmonic ratio of four poles in a zone**, which has been fully set out and explained in the last chapter.

(2) Next come **Napier's rules for the solution of right-angled spherical triangles**. A spherical triangle has six parts or elements, namely, three sides and three angles, the sides as well as the angles being, unlike plane triangles, circular arcs and therefore stated in angular measure just like the angles. In a right-angled spherical triangle, one of the six parts is known to be 90° , and only the other five parts appear in Napier's rules. If we draw a vertical line AB, and from

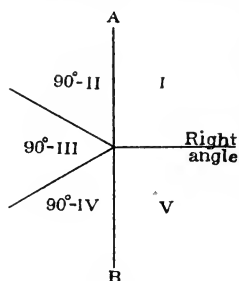


FIG. 40.

its centre draw on the right hand another line at right angles to it, and on the left-hand side two lines radiating from the same central point on AB, at about 60° to each other and to AB, as shown in Fig. 40, then the five parts of a right-angled spherical triangle other than the right angle may be considered as represented diagrammatically by the five angles or compartments of the figure. The sixth part, the right angle itself, is supposed to be represented by the horizontal line to the right of AB, and the compartments of the figure above and below it represent

the two parts of the spherical triangle adjoining the right angle; these two parts are, of course, not right angles as actually shown in the figure, the line AB being drawn as a vertical straight one for the express purpose of dividing the five parts into two sets, one set consisting of the two parts adjoining the right angle and the other set consisting of the other three parts, for a reason which will be apparent directly. In some text-books the five radiating lines separating the five parts are drawn at equal angular distances, but the relation which it is desirable to emphasise is not so well brought out on such a figure. The right angle of a spherical triangle may, it must be remembered, be either a side or an angle, for the reason already mentioned that the sides are, equally with the angles, measured in degrees and minutes. Hence, in giving a general statement of Napier's rules it will be clearer not to letter the parts, as we should do in the case of any specific triangle; for the right angle which disappears may be any one of the six parts a, b, c (the sides), A, B, C (the angles). Instead, we shall number them I to V, I being either of the two parts (angle or side) adjoining the right angle and V being the other part adjoining the right angle. Then we place the other three parts in the left-hand compartments of the diagram, in the order in which we meet with them in going round the triangle. Thus, if we elect to start by

placing the part I above the rectangular horizontal line, that is, in the top right-hand compartment, II will come in the top left-hand compartment, III in the middle compartment on the left-hand side of AB, IV in the lower compartment of the same side, and V finally in the lower right-hand compartment. The horizontal line itself, representing the sixth 90° part, does not enter into the formulæ.

Napier's rules are then as follows :

$$\text{The sine of the middle part} = \begin{cases} \text{The product of the tangents of the} \\ \text{adjacent parts, or} \\ \text{The product of the cosines of the} \\ \text{opposite parts.} \end{cases}$$

Provided : (a) That by "the middle part" we mean any one of the five parts which it may be convenient to choose as such either because it is the part the value of which it is desired to calculate, the two adjacent or the two opposite parts being known, or because it is one of the two known parts, the other of which together with the part required form the two opposite or adjacent parts ; that all parts are equally eligible for such choice, whichever side of the line AB they may be ; that by "the adjacent parts" we mean the two parts lying next to and on each side of the part chosen as middle part ; and that by "the opposite parts" we mean the two parts opposite to the part chosen as middle part. (b) That if any part—be it the middle part, an adjacent part, or an opposite part—lie to the left of the line AB, the angle " 90° - the angular value of the part" shall be understood, instead of the angular value of the part itself. Now, as $\sin(90^\circ - \theta) = \cos \theta$, $\cos(90^\circ - \theta) = \sin \theta$, and $\tan(90^\circ - \theta) = \cot \theta$, this simply means that the sine of the middle part becomes the cosine of that part when the particular compartment in which the part occurs lies to the left of AB, that the tangent of an adjacent part becomes the cotangent of that part when it lies in a compartment to the left of AB, and that the cosine of an opposite part similarly becomes the sine of that part when it lies left of AB. The reason for making the line of division AB prominent by making it vertical, and the left side at once visibly apparent, will now be appreciated.

When once these conditions have been learnt, Napier's rules will be found to render the solution of the numerous right-angled triangles met with in the calculation of crystal angles a matter of the utmost simplicity, and the method is eminently suitable for ready logarithmic calculation. Numerous examples will be worked out in the next, and later chapters.

(3) When right-angled triangles are not available we have to make use of the ordinary formulæ for the solution of any required part of any oblique-angled spherical triangle. If the three sides be a, b, c and the three angles A, B, C, of which A is opposite to the side a , B opposite b , and C opposite c , then when any three of the six parts are known we can always find the other three by the usual formulæ of spherical trigonometry. In lettering the sides and angles it is convenient that a should be a greater side than b , and consequently that the angle A should be larger than B, so that $a - b$ and $A - B$ may be positive and not

negative, as these expressions occur in the formulæ given below under (c).

In every case only such formulæ should be chosen as are adapted to logarithmic calculation. The chief and most useful of these will now be enumerated.

(a) The first series of formulæ of practical use are those for the calculation of the three angles A, B, C, having given the three sides a, b, c . These are as follows, when s = half the sum of the sides :

$$\sin^2 \frac{A}{2} = \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c},$$

$$\sin^2 \frac{B}{2} = \frac{\sin (s-a) \sin (s-c)}{\sin a \sin c},$$

$$\sin^2 \frac{C}{2} = \frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}.$$

These formulæ are very easy to remember, for in each case the square of the sine of half the angle to be calculated is equal to an expression which is of similar type in all three cases, in which besides s the sides adjacent to the angle under calculation alone occur, and the side opposite to the angle in question does not occur. That is, if the angle A for instance is being calculated, only b and c appear in addition to s on the right-hand side; in case of B, only a and c ; and in case of C, only a and b .

(b) To find the third side, which we may call c , knowing two sides, a and b , and the included angle, C.

We assume : $\tan \theta = \tan b \cos C$.

Then : $\cos c = \frac{\cos b \cos (a - \theta)}{\cos \theta}.$

Both these are very short calculations suitable for the use of logarithms.

(c) To find the two sides (a and b), given the two angles (A and B) and the included side c .

We first calculate $a + b$ and $a - b$ from the formulæ :

$$\tan \frac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{1}{2}c,$$

$$\tan \frac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{1}{2}c.$$

By addition of $a + b$ to $a - b$ we obtain $2a$, and finally its half a . Similarly, by subtracting $a - b$ from $a + b$ we obtain $2b$, and hence b .

It will usually be found that the formulæ which have now been enumerated in this chapter are all that are required for the calculation of the whole of the interfacial angles and crystal elements for all crystals up to and including those of the monoclinic system; in the case

of this latter system, from the three best measured angles as basis of calculation. Indeed, in the cases of crystals of higher symmetry, including the orthorhombic, the formulæ for four poles in a zone and Napier's rules generally suffice; only two basal angles are required in cases of orthorhombic symmetry, and these are further reduced to one basal angle in the cases of hexagonal and tetragonal crystals. In the rarer cases of triclinic crystals the formulæ of section (3) are chiefly required, and five basal angles must be known, belonging to at least two different triangles, the three sides of the fundamental triangle being conveniently three of the five known angles, for treatment by formulæ (a) of this section (3). That is to say, it is convenient to have measured at the outset the three sides a , b , c of the fundamental triangle, as a basis for all the calculations; for they are the three angles between the fundamental faces of the crystal parallel to the three axial planes, and the three angles A , B , C which can be calculated from them by formulæ (a) of section (3) will be the elementary axial angles α , β , γ (see Fig. 37), as will be shown in the next section of this chapter when we come to consider the mode of calculating the crystal elements. Having commenced by making these fundamental calculations of α , β , γ with the aid of formulæ (a), we shall be able, with the further aid of the other two basal angles, to calculate not only the axial ratios, but also, by stepping from one triangle to another of the stereographic projection in such order as always to find three known parts from which to calculate the other three, the whole of the interfacial angles of the crystal. We can thus find, even in the most difficult case, the "general" case of a triclinic crystal, the value of every arc between every pair of poles on the stereographic projection. Occasionally in the case of triclinic crystals, other of the well-known formulæ for the solution of oblique-angled spherical triangles may be required than are given in section (3), but when such rarer cases occur it will be easy to obtain the formulæ from any standard text-book of spherical trigonometry. In Chapter XX. a practical example of a fairly complex triclinic crystal will be worked through in detail.

General Mode of Calculation of the Crystal Elements.—Fig. 41 represents the upper-right-front octant of the stereographic projection of a triclinic crystal,¹ the most general case. The axial angles α , β , γ have the situations shown in the figure; for the arc ac represents the edge between the faces $a = (100)$ and $c = (001)$, the arc ab represents the edge between the faces $a = (100)$ and $b = (010)$, and the arc bc represents the edge between the faces $b = (010)$ and $c = (001)$, and as the axes have been chosen parallel to these edges, the angles between the edges will be equal to the axial angles. Therefore, the angle of intersection of the arcs ac (axis b) and ab (axis c) is the axial angle α , that at the intersection of ab and bc (axis a) is β , and that between bc and ac is γ (see Fig. 26). Hence, the three angles can at once be calculated when the three arcs ab , bc , and ca are known, by the formulæ (a) of section

¹ Fig. 41 is taken from the stereographic projection of methyl triphenylpyrrolone, a triclinic substance described by the author, *Journ. Chem. Soc.*, 1890, 57, 724.

(3), just given. The arcs in question will generally be three of the five well-measured basal angles, but if any or all of them are not included in the basal angles, they can be calculated from any such five angles as are taken for the given data.

As regards the calculation of the **axial ratios**, the positions of at least

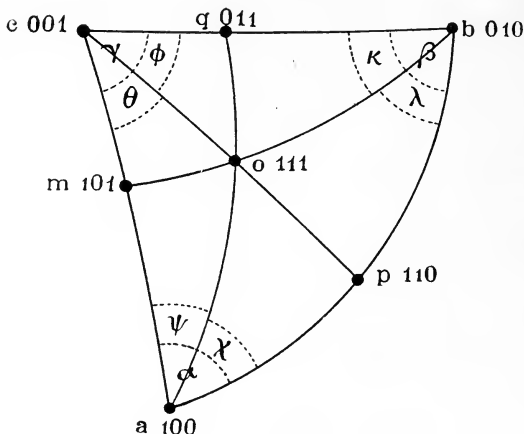


FIG. 41.

two of the poles $p = (110)$, $m = (101)$, $q = (011)$, and $o = (111)$ will probably have been included in the five basal angles, and those of the other two can be calculated with the aid of the formulæ of section (3), or, in cases where none are included in the given data, all can be so calculated. We shall then know the angular values of the whole of the arcs between the various poles shown in Fig. 41, and can, therefore, calculate the angles θ and ϕ , χ and ψ , κ and λ , shown on the figure, with the aid of the same formulæ, any five of the six angles ab, bc, ca, oa, ob, oc being adequate for the purpose of such calculations. Then:

$$\frac{a}{b} = \frac{\sin \theta}{\sin \phi}; \quad \frac{c}{b} = \frac{\sin \psi}{\sin \chi}; \quad \frac{a}{c} = \frac{\sin \lambda}{\sin \kappa}.$$

The first two of these ratios suffice to give us the whole of the axial ratios $a:b:c$, for b is always taken as unity; so that the quotient of the two sines, in the cases of the first two equations just given, expresses the required ratio of a or c to b directly. In the case of tetragonal crystals either the second equation for the ratio $c:b$ or the third equation for $a:c$ is alone sufficient, as in this system $a = b$.

That this simple calculation is correct can readily be shown. Fig. 27, which is reproduced in Fig. 42, represents the essential faces of a triclinic crystal, the three faces a, b, c being parallel to the axial planes and the fourth face o being the parametral plane; while OX, OY, OZ are the axes. Further, the dotted triangle A, B, C represents a plane parallel to the parametral plane intercepting the axes a, b, c at A, B, C , and if a perpendicular to AB be dropped from O , namely, OD , then in the triangle OBA :

$$\frac{a}{b} = \frac{OA}{OB}; \quad \text{and} \quad \frac{\sin OBD}{\sin OAD} = \frac{OD}{OD} = \frac{OB}{OA} = \frac{a}{b}.$$

Now OBD is the angle between the edges ac (parallel to the axis b) and co , which is the angle θ in Fig. 41, and OAD is the angle between the edges bc (parallel to the axis a) and co , which is the angle ϕ , hence :

$$\frac{a}{b} = \frac{\sin \theta}{\sin \phi}.$$

In the same manner it can readily be shown that :

$$\frac{c}{b} = \frac{\sin \psi}{\sin \chi} \text{ and } \frac{a}{c} = \frac{\sin \lambda}{\sin \kappa}.$$

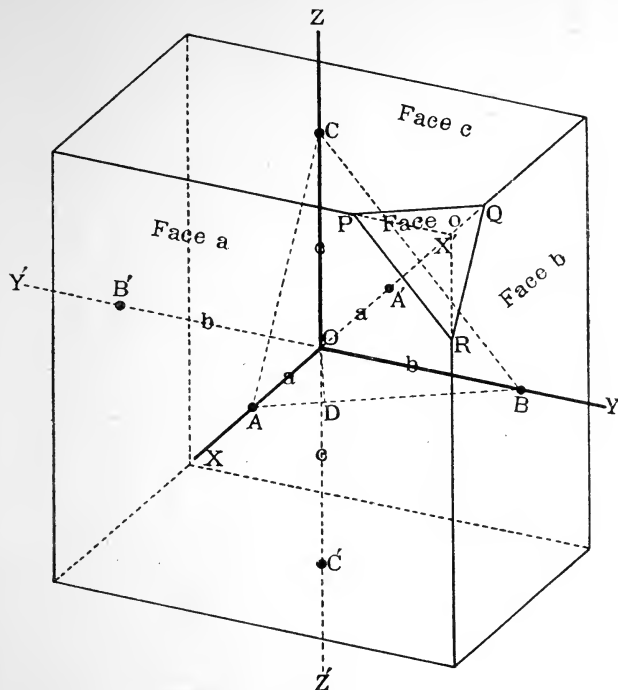


FIG. 42.

When any or all of the axial angles α, β, γ are 90° , the calculation simplifies even further, to the simple tangent of an angle. Thus, when all three are 90° , as in the case of orthorhombic potassium sulphate :

$$\frac{a}{b} = \frac{\sin \theta}{\sin \phi} = \frac{\sin ap}{\sin bp} = \frac{\sin ap}{\cos ap} = \tan ap.$$

$$\frac{c}{b} = \frac{\sin \psi}{\sin \chi} = \frac{\sin cq}{\sin bq} = \frac{\sin cq}{\cos cq} = \tan cq.$$

For the angle θ is equal to $ap = (100):(110)$ when $\gamma = 90^\circ$, and ψ is similarly equal to the arc cq when $\alpha = 90^\circ$. Hence, the natural tangents

of the angles ap and cq at once give us the ratios of the axes a or c to unity (value of axis b), which are the axial ratios required.

It will be obvious that in the case of a triclinic crystal the five most convenient angles to take for measured data are $ab = (100) : (010)$, $bc = (010) : (001)$, $ca = (001) : (100)$, $ap = (100) : (110)$, and $cq = (001) : (011)$.

In the case of an orthorhombic crystal we really have the same five angles as basal data, for the first three are 90° by reason of the symmetry, and the fourth and fifth are the only possible variables, the measured values of which are taken as given basal data.

In the case of a monoclinic crystal, the ratio of two sines also simplifies to a simple tangent in the cases of $\frac{a}{b}$ and $\frac{c}{b}$, which are the only two ratios we really require to calculate, for as b is taken as unity these two ratios give us $a:b:c$ immediately. The sides of the primary triangle ab and bc are still 90° in a monoclinic crystal, the only side which is not 90° being ac . Hence the sum of θ and ϕ and also of ψ and χ is 90° , and the sine of ϕ and of χ can be replaced by $\cos \theta$ and $\cos \psi$, when $\frac{a}{b} = \tan \theta$ and $\frac{c}{b} = \tan \psi$. The angles θ and ψ , however, are not identical with the angular values of the arcs ap and cq , owing to the third axial angle not being 90° , and thus the calculation is not quite so simple as in the case of a rhombic crystal, where all three axial angles are 90° , for θ and ψ have now to be calculated, by the use of Napier's rules, from the two right-angled triangles containing them. An example will be fully worked out in Chapter XVIII.

CHAPTER VIII

CALCULATION OF THE RESULTS OF THE GONIOMETRY OF THE TYPICAL CRYSTAL OF POTASSIUM SULPHATE.

A Complete Morphological Investigation of a Crystal includes (1) the measurement of the angles between the faces along all the principal zones; (2) the calculation of the angles from the essential number—one, two, three or five according to the system of symmetry on which the crystal is constituted—of well-measured angles, which are conveniently termed “basal angles,” and (3) the calculation of the axial angles and axial ratios, that is, of the elements of the crystal.

The general mode of carrying out (3) has just been described in the last section of Chapter VII. The calculation (2) is necessary because it is highly desirable to have this confirmation of the values of angles less frequently measured than the basal angles, or of angles between faces affording less perfect reflections. Also it often happens that certain faces may be missing altogether on a substance like potassium sulphate, but present on the crystals of other salts of the same isomorphous series, such as the sulphates of rubidium, cæsium, and ammonium, and the corresponding selenates; and it is highly desirable to compare the values of all these possible interfacial angles of all the salts of the series with one another. Moreover, besides the interfacial angles, the angles at the corners of the triangles are often needed for the calculation of the crystal elements, as has just been shown at the close of the last chapter.

With this preliminary explanation of the reason for the following calculations, and having in the first section of the last chapter set forth the chief formulæ required, in their simplest guise, we may now pass directly to the calculations for the typical orthorhombic crystal of potassium sulphate, the practical goniometry of which was worked through in Chapter IV., and which forms an excellent and complete example of operation (1).

Calculations of Interfacial Angles of Potassium Sulphate from two Basal Angles.—The symmetry of the orthorhombic system, which will be described in Chapter XV., and shown to consist, when fully developed, in the presence of three rectangular planes of symmetry, requires only two basal angles; given these, all the rest of the angles between actual

or possible faces of the crystal may be calculated, as well as the axial ratios. The three planes of symmetry being at right angles to each other, and their intersections being the axes, all three axial angles α , β , γ are 90° , so that no calculation of axial angles is required for an orthorhombic crystal.

Besides the typical crystal of potassium sulphate the goniometry of which was thoroughly described in Chapter IV., ten others were measured by the author during the crystallographic investigation of that salt. In now proceeding to the calculations, the first thing is to choose which two of the angles measured shall be used as basal angles, and the two on which our choice will fall will be the two which proved uniformly most trustworthy, the faces yielding them having afforded in general the most excellent reflections of the Websky signal-slit, besides which these angles were among those of which the most numerous determinations, afforded by separate faces on all the eleven crystals, were made. The separate values for each angle should be set out in the note-book in a vertical column, the irreproachable "A" values being marked with their A; the arithmetical mean of the whole should then be taken, by adding up the column (if the degrees are the same only the minutes need be summed up) and dividing the result by the number of measurements, and the mean of the "A" values should also be separately found. If care has been taken to exclude altogether from consideration such values as were obtained from untrustworthy images of the signal, the two means should be, and in the case of potassium sulphate were, identical. This in itself is a proof of the careful selection of the crystals measured, and of the discrimination used in accepting the angular values included in the column of which the mean has been taken. Indeed some crystallographers would consider the whole of the accepted values as "A" values, but the author reserves this designation for values derived from truly splendid images, of the kind which would be yielded by optically worked truly plane glass surfaces.

With regard to this taking of means, however, it cannot be too strongly emphasised that this is only to be done *after* the question of the symmetry has been settled, and it is quite certain that the angles the mean of which is being taken are actually such as are intended, in conformity with the proved and accepted symmetry, to be identical in value. In the cases of certain zones on a rhombic crystal such as one of potassium sulphate it is possible for each angle to be repeated four times in the zone, afforded by quite different but symmetrically situated faces, and as the zones themselves may occur in duplicate, symmetrically arranged on each side of a plane of symmetry, each of the angles may thus occur eight times on the same crystal. When ten crystals are measured (eleven were measured in the case of potassium sulphate), the number of repetitions of the same angles may reach as many as eighty. When describing the measurement of the potassium sulphate crystal in Chapter IV., stress was laid on the important rule just referred to as regards the legitimacy of extraction of mean values, and the proper mode of considering the indications of the measurements as to the symmetry

of the crystal was fully discussed. This, however, being all satisfactorily settled, and the whole of the possible eighty values agreed to be of equal value as regards the symmetry, it is quite legitimate to extract their mean.

The two basal angles chosen in the case before us of potassium sulphate are ao and co . The mean values of these angles derived from the crystal measured in Chapter IV., were respectively $43^\circ 52'$ and $56^\circ 12'$. Those derived from the whole eleven crystals were respectively $43^\circ 52'$ and $56^\circ 11'$, in the former case identical and in the latter only $1'$ different. We shall, of course, employ the mean values derived from the whole eleven crystals as the basis of our calculations, which are to serve as a typical set. The angle ao had been measured sixty-eight times, and co no less than eighty times, in all instances from separate and independent faces.

As we shall require the stereographic projection before us continually during the process of making the calculations, as our guiding diagram, Fig. 25 is repeated in Fig. 43.

Basal angles.

$$\begin{aligned}ao &= (100) : (111) = 43^\circ 52', \\co &= (001) : (111) = 56^\circ 11' .\end{aligned}$$

The knowledge of these angles at once implies that of two others, namely :

$$\begin{aligned}oq &= (111) : (011) = 46^\circ 8', \\op &= (111) : (110) = 33^\circ 49' .\end{aligned}$$

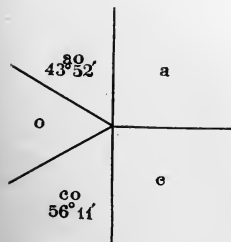


FIG. 44.

For ao and oq together make up 90° , as do also co and op .

To find the interfacial angle $ap = (100) : (110)$.

It will be convenient to commence with this angle, from our knowledge of three parts of the triangle aoc ; for in this triangle the side ac is a right angle thus enabling us to use Napier's rules, and two other sides are the given basal angles ao and co . Moreover, the angle at c in this triangle is actually the angle ap itself. We first draw the usual Napierian diagram, as shown in Fig. 44. Then, by Napier's rules :

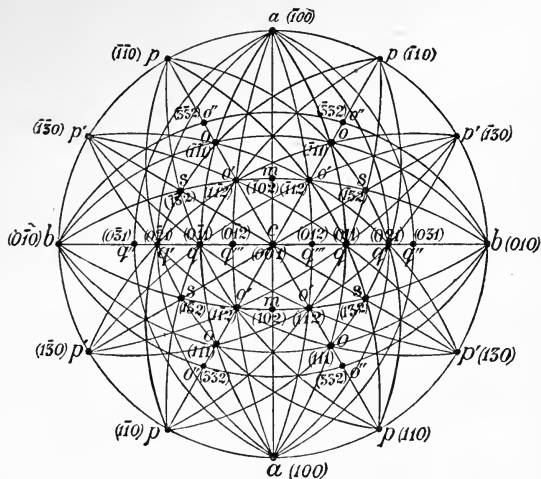


FIG. 43.—Stereographic Projection of Alkali Sulphates and Selenates.

$$\cos 43^\circ 52' = \cos c \sin 56^\circ 11'$$

$$\text{or } \cos c = \frac{\cos 43^\circ 52'}{\sin 56^\circ 11'}$$

Working this out with the aid of a table of logarithms we have :

$$\text{Log. } \cos 43^\circ 52' \quad \bar{1} \cdot 85791$$

$$\text{Log. } \sin 56^\circ 11' \quad \bar{1} \cdot 91951$$

$$\text{Log. } \cos c \quad \bar{1} \cdot 93840$$

$$c = 29^\circ 48' = ap.$$

This value for ap is identical with the measured value, which was the mean of 9 measurements with different a and p faces. The mean of the four values derived from the typical crystal described in Chapter IV. was $29^\circ 49'$.

From our knowledge of ap we can at once get bp , for

$$bp = 90^\circ - 29^\circ 48' = 60^\circ 12'.$$

To find $bp' = (010) : (130)$ and $p'p = (130) : (110)$.

In each quadrant ab of the primitive zone-circle there are four poles, namely, a , p , p' , and b , of which the positions of three are now known, namely a , p , and b . Hence, we can immediately apply the theorem of the anharmonic ratio of four poles in a zone in order to find the fourth, and as one of the angles is 90° it will simplify to the product or quotient of two tangents. The conditions are shown diagrammatically in Fig. 45. Setting down the anharmonic ratio we have :

$$\begin{array}{ccc} & 100 & 010 \\ & \times & \times \\ \sin ap & \cdot \sin bp' & 110 \quad 130 \\ \sin ap' & \cdot \sin bp & 100 \quad 010 \\ & \times & \times \\ & 130 & 110 \end{array}$$

$$\text{or } \frac{\sin ap}{\cos bp'} \cdot \frac{\sin bp'}{\cos ap} = \frac{1}{3} \cdot \bar{1}$$

$$\text{or } \tan ap \cdot \tan bp' = \frac{1}{3}, \text{ from which we get}$$

$$\tan bp' = \frac{1}{3} \cot ap = \frac{1}{3} \cot 29^\circ 48'.$$

Working this out by logarithms we have :

$$\text{Log. } \cot 29^\circ 48' \quad 0 \cdot 24207$$

$$\text{Log. } 3 \quad 0 \cdot 47712$$

$$\text{Log. } \tan bp' \quad \bar{1} \cdot 76495$$

$$bp' = 30^\circ 12'.$$

FIG. 45.

Then $p'p = bp - bp' = 60^\circ 12' - 30^\circ 12' = 30^\circ 0'$.

The mean of 32 independent measurements of bp' was $30^\circ 11'$, and of 30 separate measurements of $p'p$ $30^\circ 0'$, the agreement between the calculated and measured values being again eminently satisfactory. The mean of the four independent values of bp' yielded by the typical crystal was $30^\circ 12'$, and of the four of $p'p$ $29^\circ 59'$.

To find $cq = (001) : (011)$.

In the triangle coq we know the angle at c , for it is the same as bp , namely, $60^\circ 12'$, also the angles $oq = 46^\circ 8'$ and $co = 56^\circ 11'$. Moreover, the angle at q is a right angle, for it is at the intersection of two mutually rectangular zones $[cqb]$ and $[aoq]$. Setting out, therefore, the Napierian diagram, Fig. 46, we find that the most convenient pair of known angles to employ will be oq and c . Then, by Napier's rules we have :

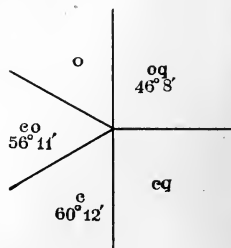


FIG. 46.

Then, by Napier's rules

$$\sin cq = \tan 46^\circ 8' \cot 60^\circ 12'$$

$$\text{Log. } \tan 46^\circ 8' \quad 0.01719$$

$$\text{Log. } \cot 60^\circ 12' \quad \bar{1}.75793$$

$$\text{Log. } \sin cq \quad \bar{1}.77512 \quad cq = 36^\circ 34'.$$

The mean of 16 measured angles cq was $36^\circ 36'$, and of the four values afforded by the typical crystal $36^\circ 37'$.

To find $bq' = (010) : (021)$ and $q'q = (021) : (011)$.

We first find bq' from the anharmonic ratio of the four poles b, q', q , and c , the whole angle bc being a right angle. With the aid of the diagram given in Fig. 47 we see that :

$$\frac{\sin 36^\circ 34'}{\sin cq'} \cdot \frac{\sin bq'}{\sin bq} = \frac{011}{001} \cdot \frac{021}{010}.$$

$$\begin{array}{cc} 001 & 010 \\ \times & \times \\ 011 & 021 \\ \times & \times \\ 021 & 011 \end{array}$$

Now $\sin cq' = \cos bq'$, and $\sin bq = \cos cq$, hence :

$$\frac{\sin 36^\circ 34'}{\cos bq'} \cdot \frac{\sin bq'}{\cos 36^\circ 34'} = \frac{1}{2} \cdot \frac{1}{1}, \text{ or}$$

$$\tan 36^\circ 34' \tan bq' = \frac{1}{2}, \text{ or}$$

$$\tan bq' = \frac{1}{2} \cot 36^\circ 34'.$$

$$\text{Log. } \cot 36^\circ 34' \quad 0.12973$$

$$\text{Log. } 2 \quad 0.30103$$

$$\text{Log. } \tan bq' \quad \bar{1}.82870 \quad bq' = 33^\circ 59'.$$

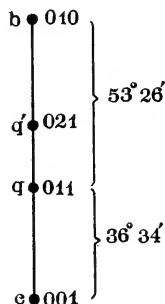


FIG. 47.

This value is identical with the mean value of 30 independent measurements, and the mean of the four bq' angles on the typical crystal was $33^\circ 58'$, only $1'$ removed.

Then $q'q = bq - bq' = 53^\circ 26' - 33^\circ 59' = 19^\circ 27'$.

The mean of 16 measured values was $19^\circ 26'$, and of the 4 values derived from the typical crystal $19^\circ 25'$.

To find $bq'' = (010) : (031)$ and $q''q' = (031) : (021)$.

The pole q'' is obviously of the same character as q' in the zone $[bqc]$, but q'' has the position corresponding to the indices (031), whereas q' has that corresponding to (021). Hence we can find bq'' by considering it as the second pole in the zone of four poles just dealt with, instead of q' , and we can use the same formula, only substituting $\frac{2}{3}$ for $\frac{1}{2}$ as the numerical value of the right-hand side obtained by cross-multiplication of the indices. The final formula then reduces to :—

$$\tan bq'' = \frac{2}{3} \cot 36^\circ 34'.$$

$$\text{Log. } \cot 36^\circ 34' \quad 0.12973$$

$$\text{Log. } 3 \quad 0.47712$$

$$\text{Log. } \tan bq'' \quad \bar{1}.65261 \quad bq'' = 24^\circ 12'.$$

Then $q''q' = bq' - bq'' = 33^\circ 59' - 24^\circ 12' = 9^\circ 47'$.

The above value of bq'' is identical with that of the single measurement of this angle obtained with the typical crystal, and the mean value derived from 8 measurements was only $1'$ removed, namely, $24^\circ 13'$.

The value of $q''q'$ measured once on the typical crystal was $9^\circ 44'$, as was also the mean of 8 independent values derived from 8 different crystals, each of which exhibited only one face of this form.

It is useful and interesting to mention here that another face is found

developed in the zone $[cqb]$ besides $q' = (021)$ and $q'' = (031)$, namely, $q''' = (012)$, between c and q , in the cases of certain other salts of the isomorphous series of alkali sulphates and selenates, to which potassium sulphate belongs, namely, on ammonium sulphate, rubidium sulphate, thallium sulphate, and caesium selenate. It is shown on the stereographic projection in Fig. 43, which is the general figure for the series. It was not, however, observed on the eleven crystals of potassium sulphate investigated. Being a possible face, however, and occurring on certain members of the series, it was desirable to calculate the value of the angles cq''' and $q'''q$ for all the salts of the series so that a comparison of these angles could, as in the cases of all the others, be instituted. This is an excellent and most pertinent example of the value of such calculations of crystal angles from basal angles.

We shall, therefore, include a calculation of the position of the pole q''' .

To find $cq''' = (001) : (012)$ and $q'''q = (012) : (011)$.

The position of the rarer form q''' in the zone $[cqb]$ is found by considering it as the fourth pole in that zone quadrant. The diagram is given in Fig. 48, and the corresponding anharmonic ratio is:—

	010	001
	\times	\times
$\frac{\sin 53^\circ 26'}{\sin bq'''} \cdot \frac{\sin cq'''}{\sin 36^\circ 34'} =$	$\frac{011}{010} \cdot \frac{012}{001}$	
	\times	\times
	012	011
$\frac{\sin 53^\circ 26'}{\cos cq'''} \cdot \frac{\sin cq'''}{\cos 53^\circ 26'} =$	$\frac{1}{2} \cdot \frac{\bar{1}}{\bar{1}}$	
	$\tan 53^\circ 26' \tan cq''' = \frac{1}{2}$	
	$\tan cq''' = \frac{1}{2} \cot 53^\circ 26'$	
Log. cot $53^\circ 26'$	$\bar{1} \cdot 87027$	
Log. 2	0.30103	
Log. tan cq'''	$\bar{1} \cdot 56924$	$cq''' = 20^\circ 21'$

Then $q'''q = cq - cq''' = 36^\circ 34' - 20^\circ 21' = 16^\circ 13'$.

To find $co' = (001) : (112)$ and $o'o = (112) : (111)$.

We can here again use the anharmonic ratio of four poles in a zone, of which the extreme poles c and p are 90° apart, the diagram for which is given in Fig. 49. From this, using $\cos co'$ instead of $\sin po'$, and $\cos 33^\circ 49'$ instead of $\sin 56^\circ 11'$, we at once get:

	110	001
	\times	\times
$\frac{\sin 33^\circ 49'}{\cos co'} \cdot \frac{\sin co'}{\cos 33^\circ 49'} =$	$\frac{111}{110} \cdot \frac{112}{001}$	
	\times	\times
	112	111
$\tan 33^\circ 49' \tan co' = \frac{1}{2}$		
$\tan co' = \frac{1}{2} \cot 33^\circ 49'$		
Log. cot $33^\circ 49'$	0.17401	
Log. 2	0.30103	
Log. tan co'	$\bar{1} \cdot 87298$	$co' = 36^\circ 44'$

Then $o'o = co - co' = 56^\circ 11' - 36^\circ 44' = 19^\circ 27'$.

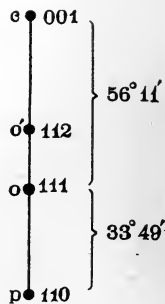


FIG. 49.

The mean of the eight values of the latter angle measured on the typical crystal was identical with this value $19^{\circ} 27'$, and the mean of 37 measurements on the whole eleven crystals was only $1'$ removed, namely, $19^{\circ} 26'$. The mean of 8 angles co' on the typical crystal, and also of 35 measurements of co' on the whole of the crystals, was $36^{\circ} 45'$, only $1'$ removed again from the calculated value.

To find $po'' = (110) : (332)$ and $o''o = (332) : (111)$.

Using again the formula for four poles in a zone, and choosing as the three other than o'' the poles c and p , 90° apart, and o , as shown diagrammatically in Fig. 50, we have :

$$\begin{aligned} \frac{\sin po''}{\sin 33^{\circ} 49'} \cdot \frac{\sin 56^{\circ} 11'}{\sin co''} &= \frac{110}{111} \cdot \frac{001}{111} \\ \frac{\sin po''}{\cos 56^{\circ} 11'} \cdot \frac{\sin 56^{\circ} 11'}{\cos po''} &= \frac{2}{1} \cdot \frac{1}{3} \\ \tan po'' \tan 56^{\circ} 11' &= \frac{2}{3} \\ \tan po'' &= \frac{2}{3} \cot 56^{\circ} 11'. \end{aligned}$$

Log. 2	0.30103
Log. $\cot 56^{\circ} 11'$	$\bar{1}.82599$
	0.12702
Log. 3	0.47712
	$\bar{1}.64990$

$po'' = 24^{\circ} 4'.$

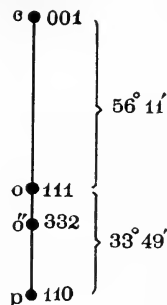


FIG. 50.

Then $o''o = po - po'' = 33^{\circ} 49' - 24^{\circ} 4' = 9^{\circ} 45'.$

These values were only once measured, namely, on the typical crystal described in Chapter IV., the face o'' being very rarely developed, and still more rarely with brilliantly reflecting surfaces. The measured values were $24^{\circ} 0'$ and $9^{\circ} 49'.$

To find $bo = (010) : (111).$

This angle is conveniently found from the right-angled triangle boc , in which the side bc is the right angle, and the side co and the angle at c (which is the same as the angle bp on the primitive circle) are known. From the Napierian diagram, Fig. 51, and Napier's rules we get :

$$\begin{aligned} \cos bo &= \cos 60^{\circ} 12' \sin 56^{\circ} 11' \\ \text{Log. } \cos 60^{\circ} 12' & \quad \bar{1}.69633 \\ \text{Log. } \sin 56^{\circ} 11' & \quad \bar{1}.91951 \\ \text{Log. } \cos bo & \quad \bar{1}.61584 \quad bo = 65^{\circ} 37'. \end{aligned}$$

Then

$$oo = (111) : (1\bar{1}1) = 2[90^{\circ} - (65^{\circ} 37')] = 2(24^{\circ} 23') = 48^{\circ} 46'.$$

The mean values of the measured angles were respectively $65^{\circ} 36'$ and $48^{\circ} 47'$, which show very satisfactory concordance with the calculated values.

To find $bo' = (010) : (112).$

This angle is similarly found from the triangle $bo'e$, which has likewise the same right angle for the side bc and the same angle at c , and co' is also known. The Napierian diagram is given in Fig. 52, and, applying Napier's rules, we obtain :

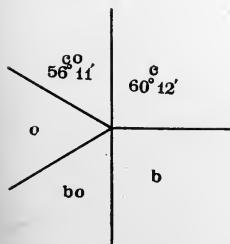


FIG. 51.

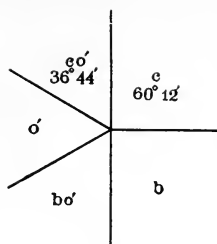


FIG. 52.

$$\cos bo' = \cos 60^\circ 12' \sin 36^\circ 44'$$

$$\text{Log. } \cos 60^\circ 12' \quad \bar{1}.69633$$

$$\text{Log. } \sin 36^\circ 44' \quad \bar{1}.77677$$

$$\text{Log. } \cos bo' \quad \bar{1}.47310 \quad bo' = 72^\circ 42'.$$

Then

$$o'o' = (112) : (1\bar{1}2) = 2[90^\circ - (72^\circ 42')] = 2(17^\circ 18') = 34^\circ 36'.$$

These values are identical with the mean of all the measured angles bo' and $o'o'$.

To find $ao' = (100) : (112)$.

This is achieved from the right-angled triangle $ao'c$, in which the side ac is the right angle, the side co' is known, and the angle at c is also known because it is the same as the angle ap on the primitive circle.

The diagram is given in Fig. 53, and by Napier's rules we obtain from it :

$$\cos ao' = \cos 29^\circ 48' \sin 36^\circ 44'$$

$$\text{Log. } \cos 29^\circ 48' \quad \bar{1}.93840$$

$$\text{Log. } \sin 36^\circ 44' \quad \bar{1}.77677$$

$$\text{Log. } \cos ao' \quad \bar{1}.71517 \quad ao' = 58^\circ 44'.$$

Then

$$o'o' = (112) : (\bar{1}12) = 2[90^\circ - (58^\circ 44')] = 2(31^\circ 16') = 62^\circ 32'.$$

The first of these angles is identical with the mean measured ao' , and the mean value of the measurements of $o'o'$ was $62^\circ 34'$.

We now come to the calculation of the angles in the subsidiary zones the zone-circles of which pass through the poles of the prism faces p and p' .

To find $po = (110) : (1\bar{1}1)$ and $q'p = (021) : (\bar{1}\bar{1}0)$ in the zone $[poq'p]$.

The angle po can be found from the right-angled triangle $p = (110)$, $o = (1\bar{1}1)$, $p' = (1\bar{1}0)$, the diagram being given in Fig. 54. By Napier's rules we have

$$\cos po = \cos 33^\circ 49' \cos 59^\circ 36'$$

$$\text{Log. } \cos 33^\circ 49' \quad \bar{1}.91951$$

$$\text{Log. } \cos 59^\circ 36' \quad \bar{1}.70418$$

$$\text{Log. } \cos po \quad \bar{1}.62369 \quad po = 65^\circ 8'.$$

The angle $q'p$ can similarly be found from the right-angled triangle $b = (010)$, $q' = (021)$, $p = (110)$, the diagram being given in Fig. 55. From this we obtain :

$$\cos q'p = \cos 60^\circ 12' \cos 33^\circ 59'$$

$$\text{Log. } \cos 60^\circ 12' \quad \bar{1}.69633$$

$$\text{Log. } \cos 33^\circ 59' \quad \bar{1}.91866$$

$$\text{Log. } \cos q'p \quad \bar{1}.61499 \quad q'p = 65^\circ 40'.$$

Then

$$oq' = [180 - (65^\circ 8' + 65^\circ 40')] = (180^\circ - 130^\circ 48') = 49^\circ 12'.$$

The mean values of the measured angles po , oq' , and $q'p$ were respectively $65^\circ 9'$, $49^\circ 11'$, and $65^\circ 40'$, the concordance being eminently satisfactory.

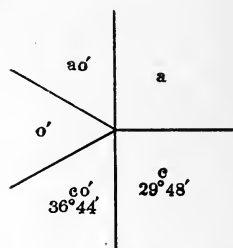


FIG. 53.

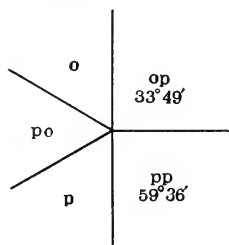


FIG. 54.

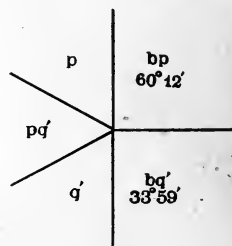


FIG. 55.

To find $po' = (110) : (\bar{1}\bar{1}2)$ and $qp = (0\bar{1}1) : (\bar{1}\bar{1}0)$ in the zone $[po'qp]$.

The angle po' is found from the triangle $p = (110)$, $o' = (\bar{1}\bar{1}2)$, $p = (\bar{1}\bar{1}0)$, in which the angle at $p = (\bar{1}\bar{1}0)$ is a right angle, the side $p = (\bar{1}\bar{1}0) : o' = (\bar{1}\bar{1}2)$ is known to be $90^\circ - co' = 53^\circ 16'$, and the side $pp = 2ap = 59^\circ 36'$.

The diagram is set out in Fig. 56, and by the rules we at once obtain :

$$\begin{aligned} \cos po' &= \cos 53^\circ 16' \cos 59^\circ 36' \\ \text{Log. } \cos 53^\circ 16' & \quad \bar{1}.77677 \\ \text{Log. } \cos 59^\circ 36' & \quad \bar{1}.70418 \\ \hline \text{Log. } \cos po' & \quad \bar{1}.48095 \quad po' = 72^\circ 23'. \end{aligned}$$

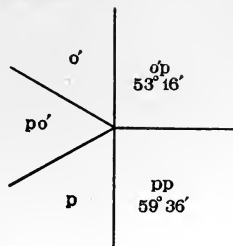


FIG. 56.

The angle pq is obtained from the triangle bqp , in which the angle at b is a right angle, bq is $90^\circ - cq = 53^\circ 26'$ and bp is $60^\circ 12'$.

The diagram will be found in Fig. 57, and from the rules the equation required is :

$$\begin{aligned} \cos pq &= \cos 53^\circ 26' \cos 60^\circ 12' \\ \text{Log. } \cos 53^\circ 26' & \quad \bar{1}.77507 \\ \text{Log. } \cos 60^\circ 12' & \quad \bar{1}.69633 \\ \hline \text{Log. } \cos pq & \quad \bar{1}.47140 \quad pq = 72^\circ 47'. \end{aligned}$$

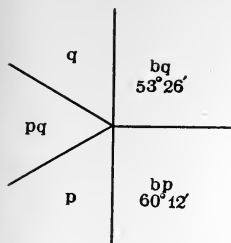


FIG. 57.

Then

$$o'q = [180^\circ - (72^\circ 23' + 72^\circ 47')] = (180^\circ - 145^\circ 10') = 34^\circ 50'.$$

The mean measured values of these three angles were respectively $72^\circ 23'$, $72^\circ 46'$, and $34^\circ 52'$.

To find $p'o = (130) : (111)$, $oo' = (111) : (\bar{1}\bar{1}2)$, $o'q' = (\bar{1}\bar{1}2) : (0\bar{2}1)$, and $q'p' = (0\bar{2}1) : (\bar{1}\bar{3}0)$ in the zone $[p'oo'q'p']$.

The angle $p'o$ is found from the triangle $p = (110)$, $o = (111)$, $p' = (130)$, in which the angle at p is a right angle, and the sides op and pp' are respectively known to be $33^\circ 49'$ and $30^\circ 0'$. The diagram is given in Fig. 58, and from Napier's rules we get :

$$\begin{aligned} \cos p'o &= \cos 33^\circ 49' \cos 30^\circ 0' \\ \text{Log. } \cos 33^\circ 49' & \quad \bar{1}.91951 \\ \text{Log. } \cos 30^\circ 0' & \quad \bar{1}.93753 \\ \hline \text{Log. } \cos p'o & \quad \bar{1}.85704 \quad p'o = 43^\circ 59'. \end{aligned}$$

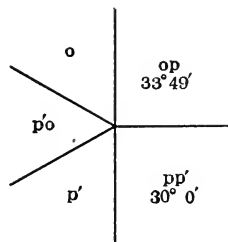
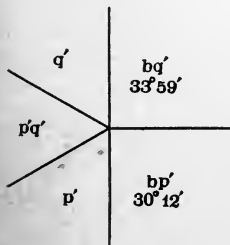


FIG. 58.

The angle $p'q'$ may next be conveniently found from the triangle $b = (010)$, $q' = (0\bar{2}1)$

$p' = (130)$, in which there is a right angle at b , and the sides bq' and bp' are known to be respectively $33^\circ 59'$ and $30^\circ 12'$. The Napierian diagram is given in Fig. 59, and from the rules we obtain :

$$\begin{aligned} \cos p'q' &= \cos 33^\circ 59' \cos 30^\circ 12' \\ \text{Log. } \cos 33^\circ 59' & \quad \bar{1}.91866 \\ \text{Log. } \cos 30^\circ 12' & \quad \bar{1}.93665 \\ \hline \text{Log. } \cos p'q' & \quad \bar{1}.85531 \quad p'q' = 44^\circ 13'. \end{aligned}$$



Thus far we have been able to calculate the whole of the angles with the aid only of the principle of four poles in a zone and Napier's rules

for right-angled triangles. We now meet with a case, in the calculation of the angle $q'o'$, which requires the use of an oblique-angled triangle, $c=(001)$, $q'=(0\bar{2}1)$, $o'=(1\bar{1}2)$, in which we know the two sides $cq'=56^\circ 1'$ and $co'=36^\circ 44'$, and the angle between them at c , for this is the same as bp on the primitive circle, $60^\circ 12'$. The case is that of (b) section (3) of the last chapter, and employing the formulæ there given we have:

$$\begin{aligned}\tan \theta &= \tan 36^\circ 44' \cos 60^\circ 12' \\ \cos q'o' &= \frac{\cos 36^\circ 44' \cos (56^\circ 1' - \theta)}{\cos \theta} \\ \text{Log. } \tan 36^\circ 44' & \quad \bar{1}.87290 & \text{Log. } \cos 36^\circ 44' & \quad \bar{1}.90386 \\ \text{Log. } \cos 60^\circ 12' & \quad \bar{1}.69633 & \text{Log. } \cos 35^\circ 40' & \quad \bar{1}.90978 \\ \text{Log. } \tan \theta & \quad \bar{1}.56923 & & \quad \bar{1}.81364 \\ \theta &= 20^\circ 21'. & \text{Log. } \cos \theta & \quad \bar{1}.97201 \\ & & \text{Log. } \cos q'o' & \quad \bar{1}.84163 \quad q'o' = 46^\circ 1'.\end{aligned}$$

Then the remaining angle of the zone, oo' , is the difference of 180° and the sum of the three angles $p'o'$, $p'q'$, and $q'o'$ just calculated, that is:

$$oo' = (111) : (1\bar{1}2) = 180^\circ - 134^\circ 13' = 45^\circ 47'.$$

These values for $p'o'$, $p'q'$, and oo' are identical with the mean measured angles, and the measured mean for $q'o'$ was $46^\circ 0'$, only $1'$ removed from the calculated mean.

To find $p'o' = (130) : (112)$, $o'q = (112) : (0\bar{1}1)$, and $qp' = (0\bar{1}1) : (\bar{1}30)$ in the final zone $[p'o'qp']$.

The value of $p'o'$ can be readily found from the right-angled triangle $p=(110)$, $o'=(112)$, $p'=(130)$, in which the right angle is at p , and op and pp' are respectively known to be $53^\circ 16'$ and $30^\circ 0'$.

The diagram is given in Fig. 60, and by the rules we

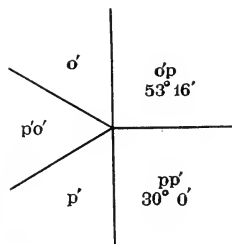


FIG. 60.

get:

$$\begin{aligned}\cos p'o' &= \cos 53^\circ 16' \cos 30^\circ 0' \\ \text{Log. } \cos 53^\circ 16' & \quad \bar{1}.77677 \\ \text{Log. } \cos 30^\circ 0' & \quad \bar{1}.93753 \\ \text{Log. } \cos p'o' & \quad \bar{1}.71430 \quad p'o' = 58^\circ 48'.$$

The angle $p'q$ can also be readily obtained from the triangle $b=(010)$, $q=(011)$, $p'=(130)$, in which there is a right angle at b , and $bq=53^\circ 26'$ and $bp'=30^\circ 12'$.

From the diagram in Fig. 61 and Napier's rules we have:

$$\begin{aligned}\cos p'q &= \cos 53^\circ 26' \cos 30^\circ 12' \\ \text{Log. } \cos 53^\circ 26' & \quad \bar{1}.77507 \\ \text{Log. } \cos 30^\circ 12' & \quad \bar{1}.93665 \\ \text{Log. } \cos p'q & \quad \bar{1}.71172 \quad p'q = 59^\circ 0'.$$

Then, finally,

$$o'q = [180^\circ - (58^\circ 48' + 59^\circ 0')] = 180^\circ - 117^\circ 48' = 62^\circ 12'.$$

The value for $p'o'$ is identical with the mean of all the measured values. The mean measured value of $o'q$ was $62^\circ 14'$, and that of qp' was $58^\circ 58'$, in each case only $2'$ removed from the calculated values.

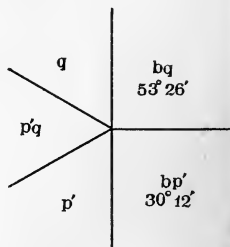


FIG. 61.

The whole of the interfacial angles of all the zones shown on the stereographic projection, Fig. 43, page 103, have now been calculated from

the two basal angles $ao = (100):(111) = 43^\circ 52'$, and $co = (001):(111) = 56^\circ 11'$, as given data. The calculation has been somewhat long, and the logarithms have been actually quoted, as it was desired to work out a typical case completely from beginning to end with the minimum of trouble to the reader. It will have given a good idea how really simple such crystallographic calculations are, when the two principles of four poles in a zone and Napier's rules are grasped. Only in the case of one angle have these two principles proved inadequate, and required to be supplemented by the solution of an oblique-angled triangle. A very large proportion of crystals will prove to be similarly simple, and even in the cases of monoclinic crystals there will only be a very few angles for which oblique-angled triangles will have to be solved, and in every case they can be solved by the simple formulæ quoted in section (3) of the last chapter. A typical case of a monoclinic crystal will be worked through as an example in Chapter XVIII. The cases of triclinic crystals will, of course, require the solution of numerous oblique-angled triangles, but it will be very rarely that the simple formulæ given in section (3) will require to be supplemented by others taken from a text-book of spherical trigonometry, and the principle of four poles in a zone will be of frequent application. A typically difficult triclinic crystal will also be calculated through in Chapter XX. A crystal or a couple of crystals, in fact, will be worked out for each system of symmetry, in the chapter immediately following that describing the forms and classes of the system. In these subsequent cases, however, in order to save valuable space, the logarithms will not be quoted, but only the result of the working out of the equation given.

Calculation of the Ratio of the Axes.—In the case before us, of orthorhombic potassium sulphate, the ratios of the lengths of the axes a and c respectively to that of the axis b (considered as unity) are immediately given by the tangents of the angles of inclination of the two primary faces $p = (110)$ and $q = (011)$ to a and c respectively, as was shown near the close of the last chapter when considering the general mode of calculating axial ratios. Proceeding, however, from first principles, the conditions will be rendered clear by Fig. 62, which shows

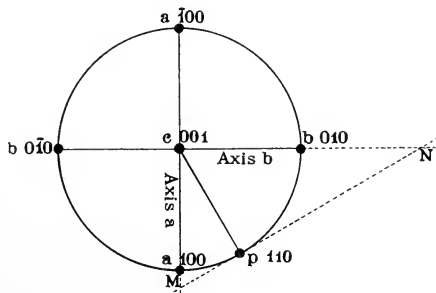


FIG. 62.

the axes a and b , the poles of the a and b faces, and the pole of one of the faces of $p = \{110\}$. If the normal to the latter, cp , be drawn, and then a tangent to the circle at the point p , that is, a perpendicular to the normal, it will be obvious that the plane $p = (110)$ is represented in plan by this tangent, and that it cuts the axes a and b prolonged at the points M and N ; the lengths cM , cN will represent, therefore, the

unit axial lengths, that is to say, the parameters, so $\frac{cM}{cN}$ will be the ratio of the axes $a:b$. But this is the trigonometrical tangent of the angle cNM , and this angle is equal to the angle Mcp . Now this latter angle is the angle measured on the goniometer between the normals to $a = (100)$ and $p = (110)$, in other words it is the angle $ap = (100):(110)$.

Hence:

$$\frac{a}{b} = \tan ap = \tan 29^\circ 48' \\ = 0.5727.$$

Taking out the natural tangent of $29^\circ 48'$ from a book of mathematical tables, we find it to be 0.5727. This, therefore, is the value of the ratio of the axes a and b . In a similar manner the ratio $\frac{c}{b}$ is the tangent of the angle $cq = (001):(011)$. That is:

$$\frac{c}{b} = \tan cq = \tan 36^\circ 34' = 0.7418.$$

The ratios of the three axes are thus:

$$a:b:c = 0.5727:1:0.7418.$$

Tabular Presentation of Results.—Having now completed our morphological investigation of the typical orthorhombic crystal of potassium sulphate, we should conclude by summarising the results in concise tabular form. In doing so, the results will be incorporated for the whole eleven crystals of this substance, which were employed in the author's complete investigation, and of which the crystal which has been worked through here as a typical one was the most perfect, in order that the table shall form an example of the manner in which such results for any substance crystallographically investigated should be presented for publication. The form of the table is that which has been agreed upon by the leaders of the subject in this country, and with the approval of Prof. von Groth, the distinguished founder and editor of the *Zeitschrift für Krystallographie*.

MORPHOLOGY OF CRYSTALS OF POTASSIUM SULPHATE, K_2SO_4 .

Crystal-System: Orthorhombic. **Crystal-Class** 8, rhombic holohedral.

Habit: Short prismatic, or tabular parallel $b = \{010\}$, illustrated respectively by Figs. 1 and 2 (pages 3 and 5).

Ratio of Axes: $a:b:c = 0.5727:1:0.7418$.

Forms observed: $a = \{100\}$, $b = \{010\}$, $c = \{001\}$, $p = \{110\}$, $p' = \{130\}$,
 $q = \{011\}$, $q' = \{021\}$, $q'' = \{031\}$, $o = \{111\}$, $o' = \{112\}$,
 $o'' = \{332\}$.

Table of Angles: In the following table the basal angles are marked with an asterisk, and the angles in the same zone are recorded in their proper sequence and connected by an inclusive zonal bracket.

Morphological Angles of Potassium Sulphate.

Angle Measured.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$\left\{ \begin{array}{l} ap = (100):(110) \\ pp' = (110):(130) \\ p'b = (130):(010) \end{array} \right.$	$\left\{ \begin{array}{l} 29 \\ 30 \\ 32 \end{array} \right.$	$\left\{ \begin{array}{l} 29 \ 45-29 \ 52 \\ 29 \ 55-30 \ 4 \\ 30 \ 8-30 \ 15 \end{array} \right.$	$\left\{ \begin{array}{l} 29 \ 48 \\ 30 \ 0 \\ 30 \ 11 \end{array} \right.$	$\left\{ \begin{array}{l} 29 \ 48 \\ 30 \ 0 \\ 30 \ 12 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ 0 \\ 1 \end{array} \right.$
$\left\{ \begin{array}{l} cq''' = (001):(012) \\ q'''q = (012):(011) \\ cq = (001):(011) \\ qq' = (011):(021) \\ cq' = (001):(021) \\ q'q'' = (021):(031) \\ q''b = (031):(010) \\ q'b = (021):(010) \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ 16 \\ 16 \\ 29 \\ 7 \\ 8 \\ 30 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ 36 \ 31-36 \ 41 \\ 19 \ 20-19 \ 31 \\ 55 \ 55-56 \ 9 \\ 9 \ 40-9 \ 55 \\ 24 \ 7-24 \ 19 \\ 33 \ 52-34 \ 5 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ 36 \ 36 \\ 19 \ 26 \\ 56 \ 2 \\ 9 \ 44 \\ 24 \ 13 \\ 33 \ 59 \end{array} \right.$	$\left\{ \begin{array}{l} 20 \ 21 \\ 16 \ 13 \\ 36 \ 34 \\ 19 \ 27 \\ 56 \ 1 \\ 9 \ 47 \\ 24 \ 12 \\ 33 \ 59 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ 2 \\ 1 \\ 1 \\ 3 \\ 1 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} fao = (100):(111) \\ oq = (111):(011) \end{array} \right.$	$\left\{ \begin{array}{l} 68 \\ 58 \end{array} \right.$	$\left\{ \begin{array}{l} 43 \ 45-44 \ 0 \\ 46 \ 4-46 \ 15 \end{array} \right.$	$\left\{ \begin{array}{l} 43 \ 52 \\ 46 \ 8 \end{array} \right.$	$\left\{ \begin{array}{l} * \\ 46 \ 8 \end{array} \right.$	$\left\{ \begin{array}{l} - \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} fao' = (100):(112) \\ o'o' = (112):(\bar{1}12) \end{array} \right.$	$\left\{ \begin{array}{l} 27 \\ 7 \end{array} \right.$	$\left\{ \begin{array}{l} 58 \ 38-58 \ 47 \\ 62 \ 30-62 \ 38 \end{array} \right.$	$\left\{ \begin{array}{l} 58 \ 44 \\ 62 \ 34 \end{array} \right.$	$\left\{ \begin{array}{l} 58 \ 44 \\ 62 \ 32 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ 2 \end{array} \right.$
$\left\{ \begin{array}{l} fbo = (010):(111) \\ oo = (111):(\bar{1}\bar{1}\bar{1}) \end{array} \right.$	$\left\{ \begin{array}{l} 73 \\ 33 \end{array} \right.$	$\left\{ \begin{array}{l} 65 \ 32-65 \ 41 \\ 48 \ 40-48 \ 52 \end{array} \right.$	$\left\{ \begin{array}{l} 65 \ 36 \\ 48 \ 47 \end{array} \right.$	$\left\{ \begin{array}{l} 65 \ 37 \\ 48 \ 46 \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$
$\left\{ \begin{array}{l} fbo' = (010):(112) \\ o'o' = (112):(112) \end{array} \right.$	$\left\{ \begin{array}{l} 25 \\ 7 \end{array} \right.$	$\left\{ \begin{array}{l} 72 \ 38-72 \ 48 \\ 34 \ 31-34 \ 41 \end{array} \right.$	$\left\{ \begin{array}{l} 72 \ 42 \\ 34 \ 36 \end{array} \right.$	$\left\{ \begin{array}{l} 72 \ 42 \\ 34 \ 36 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} co' = (001):(112) \\ o'o = (112):(111) \\ co = (001):(111) \\ oo'' = (111):(332) \\ o''p = (332):(110) \\ op = (111):(110) \end{array} \right.$	$\left\{ \begin{array}{l} 35 \\ 37 \\ 80 \\ 1 \\ 1 \\ 84 \end{array} \right.$	$\left\{ \begin{array}{l} 36 \ 38-36 \ 51 \\ 19 \ 16-19 \ 33 \\ 56 \ 6-56 \ 17 \\ - \\ - \\ 33 \ 42-33 \ 57 \end{array} \right.$	$\left\{ \begin{array}{l} 36 \ 45 \\ 19 \ 26 \\ 56 \ 11 \\ 9 \ 49 \\ 24 \ 0 \\ 33 \ 49 \end{array} \right.$	$\left\{ \begin{array}{l} 36 \ 44 \\ 19 \ 27 \\ * \\ 9 \ 45 \\ 24 \ 4 \\ 33 \ 49 \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \\ - \\ 4 \\ 4 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} po = (110):(\bar{1}\bar{1}\bar{1}) \\ oq' = (\bar{1}\bar{1}\bar{1}):(\bar{0}21) \\ qp = (\bar{0}21):(\bar{1}\bar{1}0) \end{array} \right.$	$\left\{ \begin{array}{l} 55 \\ 45 \\ 49 \end{array} \right.$	$\left\{ \begin{array}{l} 65 \ 2-65 \ 13 \\ 49 \ 6-49 \ 15 \\ 65 \ 32-65 \ 50 \end{array} \right.$	$\left\{ \begin{array}{l} 65 \ 9 \\ 49 \ 11 \\ 65 \ 40 \end{array} \right.$	$\left\{ \begin{array}{l} 65 \ 8 \\ 49 \ 12 \\ 65 \ 40 \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} po' = (110):(\bar{1}\bar{1}2) \\ o'q = (\bar{1}\bar{1}2):(\bar{0}\bar{1}\bar{1}) \\ qp = (\bar{0}\bar{1}\bar{1}):(\bar{1}\bar{1}0) \end{array} \right.$	$\left\{ \begin{array}{l} 21 \\ 15 \\ 23 \end{array} \right.$	$\left\{ \begin{array}{l} 72 \ 19-72 \ 27 \\ 34 \ 49-34 \ 57 \\ 72 \ 43-72 \ 52 \end{array} \right.$	$\left\{ \begin{array}{l} 72 \ 23 \\ 34 \ 52 \\ 72 \ 46 \end{array} \right.$	$\left\{ \begin{array}{l} 72 \ 23 \\ 34 \ 50 \\ 72 \ 47 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ 2 \\ 1 \end{array} \right.$
$\left\{ \begin{array}{l} p'o = (130):(111) \\ oo' = (111):(\bar{1}\bar{1}2) \\ o'q' = (\bar{1}\bar{1}2):(021) \\ oq' = (111):(021) \\ q'p' = (021):(130) \end{array} \right.$	$\left\{ \begin{array}{l} 46 \\ 23 \\ 20 \\ 39 \\ 42 \end{array} \right.$	$\left\{ \begin{array}{l} 43 \ 54-44 \ 6 \\ 45 \ 44-45 \ 52 \\ 45 \ 53-46 \ 7 \\ 91 \ 38-91 \ 55 \\ 44 \ 5-44 \ 23 \end{array} \right.$	$\left\{ \begin{array}{l} 43 \ 59 \\ 45 \ 47 \\ 46 \ 0 \\ 91 \ 47 \\ 44 \ 13 \end{array} \right.$	$\left\{ \begin{array}{l} 43 \ 59 \\ 45 \ 47 \\ 46 \ 1 \\ 91 \ 48 \\ 44 \ 13 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} p'o' = (130):(112) \\ o'q = (112):(\bar{0}\bar{1}\bar{1}) \\ qp' = (\bar{0}\bar{1}\bar{1}):(\bar{1}30) \end{array} \right.$	$\left\{ \begin{array}{l} 19 \\ 12 \\ 18 \end{array} \right.$	$\left\{ \begin{array}{l} 58 \ 43-58 \ 53 \\ 62 \ 10-62 \ 18 \\ 58 \ 55-59 \ 1 \end{array} \right.$	$\left\{ \begin{array}{l} 58 \ 48 \\ 62 \ 14 \\ 58 \ 58 \end{array} \right.$	$\left\{ \begin{array}{l} 58 \ 48 \\ 62 \ 12 \\ 59 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ 2 \\ 2 \end{array} \right.$
Total number of measurements	1160				

The excellent agreement between the calculated and observed values of the angles, as indicated by the fact that the greatest difference is only 4' and that only in the cases of three angles does it exceed 2', affords the best possible testimony to the accuracy of both the practical measurements and the calculations. It is of the character which may be expected when care has been taken that the crystals employed shall be of the highest attainable state of perfection.

CHAPTER IX

CRYSTALS AS HOMOGENEOUS STRUCTURES.

THE geometrical regularity in accordance with a definite plan of symmetry which has been experimentally shown in previous chapters to be a characteristic of crystals, the constancy of the interfacial angles, the zonal relationships, and particularly the rationality of intercepts and of facial indices, all unite in compelling us to the conclusion that these interesting phenomena, connected with the exterior shape of crystals, owe their origin to the fact that the material units of which the crystalline substance is composed are themselves arranged in an orderly manner, according to some regular structural plan. In other words, we conclude that a crystal is a homogeneous structure, the essential nature of which is that the arrangement about any one structural unit (in general, the chemical molecule) is the same as about every other. It will be obvious, therefore, that a knowledge of the nature of all the possible types of homogeneous structures must be of immense importance in enabling us to understand correctly the real nature of crystals.

It will be shown later, in the chapter (XXXI.) devoted to the volume relationships of crystals, that we arrive, as the result of the purely practical experimental investigation of crystals, at a system of points, the centres of gravity of the chemical molecules, as representing the crystal structure. Upon such an assumption the whole of the phenomena can be readily explained, and the relative separation of these points along the axial directions of the crystal can be actually determined for any definitely related series of substances, such as the members of an isomorphous series of salts. Indeed, it appears likely that we may eventually be able to compare these structural dimensions, the separation of the points representing the structural units, for all kinds of substances, even the most diverse in character, with each other. But this lies with the future.

Before passing, therefore, to the systematic discussion of the thirty-two classes of crystal symmetry, it will be both advisable and interesting to review briefly the great progress which has been made of late years in our knowledge of homogeneous structures in general, especially as this purely geometrical work appears now to be fairly complete.

The important pioneer work of the Abbé Haüy, Professor of the

Humanities at the University of Paris, has already been alluded to in Chapter I., in reference to his belief, now proved to be a fact, that difference in chemical composition is accompanied by difference in crystalline form. The results of his remarkable researches were first communicated to the French Academy and then shortly afterwards, in the year 1784, published in a book entitled *Essai d'une théorie sur la structure des cristaux*, of which the author is in possession of one of the original copies. It is shown therein that all the varieties of crystalline forms can be referred to a few simple types of symmetry, and proofs are offered that all the apparently different forms of the same substance are based on one of these simple fundamental forms. Moreover, Haüy enunciated the laws of symmetry, discovered the law of rational indices, and pointed out the simple relation between crystalline form and cleavage, the latter phenomenon having been discovered four years previously by Bergmann and Gahn. Besides these indubitable facts, which will ever render the name of Haüy famous as the "father of crystallography," he also originated a most interesting theory as to the nature of the fundamental structure of crystals, which was for long discredited, but which is now proved to have a substratum of truth; for he imagined that the ultimate particles, which he called "*molécules intégrantes*," were the primitive parallelepipeda themselves, which in the most general case were of unequal angles and three dimensions of sides, but gradually became, through the various stages of symmetry, the cube of right angles and equal sides. It is quite conceivable that a solid structure built up of such minute elementary parallelepipeda would develop on its exterior surface only such planes as would be formed by the structure terminating in some regular manner, for instance, by each line of parallelepipeda being longer or shorter than the next one by some small rational number, such as one, two, three or four parallelepipeda. Thus in Fig. 63 the plane AB is formed by the terminations of either horizontal or vertical lines of parallelepipeda, each of which is one parallelepipedon longer horizontally (passing from A to B) than the preceding

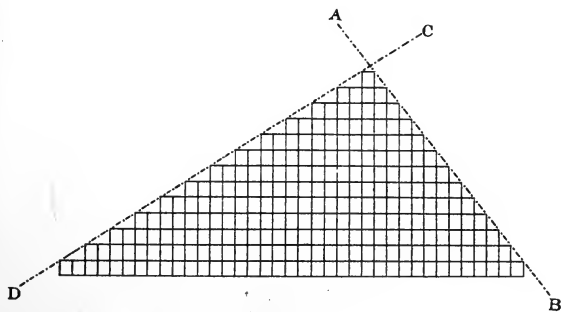


FIG. 63.

one, whereas the plane CD is formed by the terminations of similar lines, but each of which is two parallelepipeda longer than the preceding one. It is easy, for instance, to build up a pyramid, which may represent the half of an octahedron, from a large number of small cubes, and grossly mechanical considerations such as these may be of assistance to the

mind in grasping some germ of the truth regarding the reason for the formation only of such faces as have rational indices, composed of very small whole numbers.

But the advance of physics has gradually made it clear that the chemical molecules cannot be thus arbitrarily considered as solid parallelepipeda, closely packed without interspaces, for it would then be difficult to account for the well-known molecular movements accompanying elastic compression and thermal expansion and contraction. It is, in fact, certain that solids resemble liquids and gases in consisting of molecules separated by interspaces in which their proper movements occur. Instead, therefore, of considering the actual shape of the molecule as being that of the primitive parallelepipedon, the idea of shape is eliminated, and the molecule is considered as being replaced by its "sphere of motion or influence," which may be represented graphically by a point, its centre of gravity or mean position about which the motion occurs, or any other point so long as analogous points are chosen in all the molecules. It is just as easy to consider the crystal as built up of

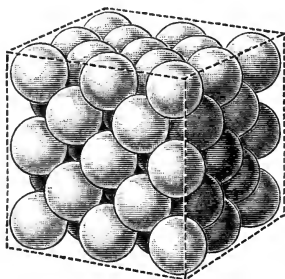


FIG. 64.

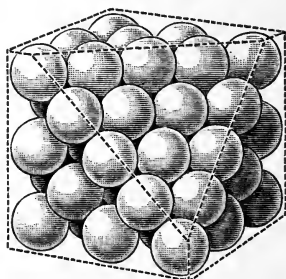


FIG. 65.

spheres as of cubes, as shown in Figs. 64 and 65, and the explanation of rational indices is as simple on the one basis as on the other. The relative distances apart of the representative points are, of course, determined by the intermolecular forces.

We thus come to the same conclusion as that referred to in the second paragraph of this chapter as being derived from a study of the volume relationships, namely, that **a crystal may be considered as an orderly assemblage of points, the centres of gravity or other representative analogous points of the chemical molecules.** This modernised interpretation of Haüy's idea of "*molécules intégrantes*" is in harmony with all the known facts of morphological crystallography, and offers a complete explanation of the observed types of crystal symmetry and of the law of rational indices.

Fig. 66 represents such an orderly assemblage of points, of the most general character, in which the distances 1 to 2, 1 to 3, and 1 to 4, separating them along the three main axial lines, which we may consider as the lengths of the morphological axes a , b , c , are unequal, and the interaxial angles α , β , γ are also unequal. These three axial

directions are distinguished by the different kinds of thin lines in the figure, the a direction being marked by broken lines, the b direction by broken-and-dotted lines, and the c direction by continuous lines. If we consider the point 1 as the origin, $\overline{X\overline{X}}$, $\overline{Y\overline{Y}}$, and $\overline{Z\overline{Z}}$, drawn in thick broken-and-dotted lines, may be regarded as the axes themselves. But any other of the points may be equally well regarded as the origin. This type of assemblage would correspond to a crystal of the **triclinic** (otherwise called **anorthic**) variety, which possesses no symmetry other than that about the centre. When, however, any two of the

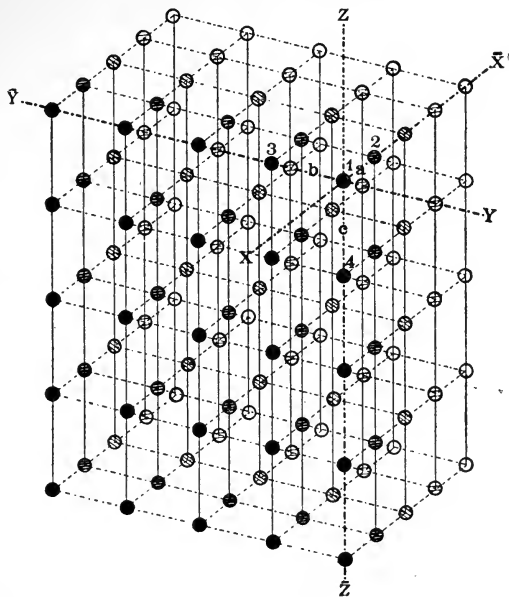


FIG. 66.—Triclinic Space-lattice.

interaxial angles, say α and γ , are right angles, one of the axes, in this case b , being then perpendicular to the plane containing the other two (which remain inclined to each other in that plane), the type becomes that of a **monoclinic** crystal. When all three interaxial angles α , β , γ are right angles, but the distances separating the points along the axial directions (that is, the axial lengths) remain unequal, we have the **orthorhombic** kind of crystal symmetry. When the three angles are all right angles and also two of the axes, usually chosen as a and b , are of equal length, the symmetry ascends to that of the **tetragonal** order. When, further, all three axes are equal, in addition to being all arranged at right angles to each other, the maximum possible symmetry is attained, known as **cubic**. There are still two more possibilities. For all three axes may be of equal length, and all three angles equal to each other but not right angles; this is the case of **trigonal** symmetry. The other case is similar to that of the tetragonal variety, only that the two equal axes are inclined at 60° instead of 90° to each other; this is the case of **hexagonal** symmetry.

The chief characteristic of such a parallel net-like arrangement of points, which is usually termed a "**space-lattice**," is that the environment about any particular point is the same as about any other, although, and this will be shown directly to be a very important point, the strictly parallel arrangement of the units represented by the points, although frequently the case, is not absolutely essential for this condition.

Now a crystal is just such a homogeneous structure,—that is, an assemblage in which the environment about every molecular unit or its representative point is the same as about every other,—formed by the symmetrical arrangement of an indefinitely large number of spheres of atomic influence, the cluster of which constituting each molecule may be considered as represented by a single point of the space-lattice; and it will be shown in the next chapter that there are seven styles of crystal architecture, or “**systems of symmetry**,” corresponding precisely to the seven space-lattices above described.

The proper chemical proportions of the atoms of the different elements present in the molecule are assured in the whole structure by the fact that the latter is of such a character that it may be partitioned into cells identical in composition and configuration with the chemical molecule, the arrangement of the atomic spheres of influence within the cell being such as corresponds with the stereometric arrangement of the atoms in the molecule, and also, as we shall presently see, with a particular homogeneous arrangement of points representing the atoms which corresponds in turn to a specific subdivision or “class” of the system of symmetry.

The problem of the nature of the crystallographic structural units, as to whether we were to consider the chemical molecules or the atoms as such units, had for many years been unnecessarily complicated by the idea held by many physicists that in solid substances we were dealing with edifices composed of units which were aggregates of chemical molecules. Attempts had actually been made up to as late as the year 1895 to determine the number of chemical molecules contained in the so-called physical molecule. The idea being in all probability entirely erroneous, and certainly so for the great majority of substances, no definite success attended these efforts, and in a memoir on “The Nature of the Structural Unit”¹ the author proved in the year 1896, from purely experimental data, that—at any rate for the two isomorphous series of the orthorhombic sulphates and selenates of the alkalis, $R_2\overset{S}{Se}O_4$, and the monoclinic double sulphates and selenates containing the same bases along with dyad metals and six molecules of water of crystallisation, $R_2M(\overset{S}{Se}O_4)_2 \cdot 6H_2O$ —the structural unit of the space-lattice was simply the chemical molecule, thus clearing the ground of all further complication, at least as regards these series of substances.

Evolution of the Geometrical Theory of Crystal Structure.—In the year 1830 Hessel made a study of the types of symmetry possible to a solid bounded by plane faces, and then, by excluding all such types as did not fulfil Haüy’s law of rational indices, he found that only 32 types were left as possible to crystals, assuming the law to represent a natural truth. At this time, however, only a comparatively few of these 32 classes of crystals had been actually observed, and Hessel’s conclusion, published in book form in Leipzig under the title *Krystallonomie und*

¹ *Journ. Chem. Soc.*, 1896, 69, 507.

Krystallographie in 1831, remained unnoticed. The same fact, however, was independently discovered by Gadolin in 1867, and Hessel remained unknown until Sohncke fortunately alighted upon his book so late as 1891. But Gadolin having published his results both in a memoir¹ and in book form at Helsingfors, they were seen and at once taken up. In the previous year Victor von Lang, the well-known mineralogist of Vienna, who for some years was with us in England at the British Museum, had published the first edition of his *Lehrbuch der Krystallographie* (Vienna), in which he had thoroughly laid down the laws of symmetry, and although he did not arrive at all the 32 classes, his method was capable of reaching that result had he carried it to its logical conclusion. In subsequent memoirs and further editions of his book the completion was eventually carried out, so that between Gadolin and von Lang the geometry of the symmetry of the 32 classes of crystals, as concerns their exterior configuration, was by the year 1870 thoroughly well established.

As regards the internal structure, the first authentic conception of a crystal as a network of molecular points, quite independent of any shape the molecules may possess, was advanced by Seeber in the year 1824, in a memoir in Gilbert's *Annalen der Physik*, vol. 76, p. 229. The idea was subsequently further developed by Dana in 1836, Brewster in 1839, Delafosse in 1843, and Forster in 1855, and it is now generally accepted that crystal structure consists in the similar repetition of the ultimate identically similar molecular units throughout the space occupied by the crystal, without any regard whatsoever to the shape or constitution of these units. At the same time it is true, and will be afterwards proved, that the stereometric arrangement of the chemical atoms in the molecule has a profound influence on the choice of the particular one of the 32 crystal classes which a specific chemical compound shall exhibit.

The problem thus simplified by the elimination of the shape of the molecules was to discover the various possible modes of homogeneous repetition of the ultimate parts, that is, to ascertain what types of homogeneous partitioning of space are possible, and then to see if these account for the observed 32 varieties of crystal species.

From this point of view Frankenheim in the year 1842 investigated the possible parallelepipedal networks of points, and came to the conclusion that there are 15 possible different symmetrical arrangements, of the nature of space-lattices, of the ultimate particles in space, the shape of the latter being altogether excluded from consideration. This, however, leaves entirely untouched the so-called hemimorphous and hemihedral classes of crystals, that is, crystal species showing only partially the symmetry of their system or genus, the former (hemimorphous crystals) being differently terminated at the two ends of a principal axis. Now these species lacking the full symmetry of their genera are as much forms of definite symmetry as are the seven holohedral forms which display the full symmetry of the seven crystal systems, the seven genera of crystallography. Hence, Frankenheim's

¹ *Acta Soc. Scient. Fennicae*, 1867, 9, 1.

proposition was only the beginning of a full and comprehensive theory, which should account for the whole of the 32 class species.

In 1848 Auguste Bravais gave rigid proofs for Frankenheim's space-lattices, and showed that two of the 15 are identical, so that there are only 14 distinct space-lattices. These he assigns to seven systems, according to the number and nature of the axes of symmetry which pass through a given point of the space-lattice, and these seven systems are identical with those into which crystals naturally fall. In his "*Mémoire sur les systèmes formés par les points distribués régulièrement sur un plan ou dans l'espace*" and "*Études cristallographiques*,"¹ he not only develops these definite results but goes further by suggesting that the shape of the units represented by the points determines the hemihedral or other forms which only partially represent the symmetry of the system. He further states that the geometrical arrangement of the constituent atoms is the same round the centre of gravity of each molecule. Thus Bravais stipulates a fixed relationship, same-way orientation, and rigidity of the parts of the molecule, and he considered that the process of crystallisation consisted in the rotation of the molecules in such a way as to bring about this uniform orientation about every molecular centre or point of the space-lattice.

This hypothetical fixed position of the atoms in the molecule of the crystalline solid is now an experimentally proved fact as regards certain holohedral crystals, such as our typical crystal of potassium sulphate, which is one of the most interesting results of the author's own investigations.

Bravais also introduced the fruitful conception of a "coincidence movement," a movement by which an assemblage of points is brought again into coincidence with itself, point with point; it is as if there had been two duplicate assemblages in coincident superposition, and one of them had been given a movement of translation or rotation, or both these movements combined, such that when it came to rest all the points of the two assemblages became superposed again, although not so that the same points were in superposition as at first, but so that the appearance was that of a single assemblage only, just as at first. That is, unless one had been told that a movement had occurred one would not have suspected it. This conception has been of inestimable value in the study of symmetry, and is one rich in mathematical results of considerable importance.

The chief work of Bravais is undoubtedly, however, his definite proof of the geometrical existence of the fourteen separate space-lattices, and the explanations which he offers by their means of the experimental law of rational indices, of the homogeneity of the crystal, and of the development in actual fact of seven generic systems of crystal symmetry. For all the fourteen space-lattices have the property of homogeneity in common, their planes of points are all possible crystal faces complying with the law of rational indices, and the symmetry of each space-lattice is that of one or other of the seven crystal systems. Indeed the nature of the space-lattice formed by the arrangement of the molecular centres

¹ *Journal de l'École polytechnique*, Paris, 1850, 19, 127, and 1851, 20, 102 and 197.

of a crystalline substance, or of any analogous points which may be taken within the molecules, one from each, to represent the molecules instead of their centres of gravity, determines both the crystal system and the obedience of the facial planes developed by the crystal to the law of rational indices. These weighty facts alone would entitle Bravais to a large share of the honour due to the discoverers of the true theory of the homogeneous partitioning of space.

The next step forward was the enunciation of the principle of the symmetrical repetition in space by Wiener in 1869, in his *Grundzüge der Weltordnung*. Broadly stated it is, that regularity of arrangement of analogous atoms consists in every atom having the remaining atoms arranged about it in the same manner. In other words, homogeneity consists primarily in the continual repetition throughout space of the same relation between an elementary atom and the entire structure, regarded as unlimited. Wiener described several specific kinds of regularly symmetrical repetitions of identically similar points in space, as examples of his new principle.

But the exhaustive discussion of such assemblages we owe to a pure mathematician, Camille Jordan, who later in 1869 published a "Mémoire sur les groupes de mouvements"¹ in which he defined the possible types of regular repetition in space of so-called identical parts, the term "identical" here meaning more than merely "ordinarily similar," namely, "perfectly alike." He showed that when such identical repetition occurs in a rigid system, a definite group or series of movements may be employed in regard to it, each movement being such that while shifting the system it leaves the appearance the same as before, every point being moved to a position previously occupied by a homologous point. The great advantage of this method is its complete generality, no further limitation being required than that which prescribes the kind of repetition presented by the homologous points of the structure.

It is interesting and somewhat remarkable that Jordan's work was entirely conceived without any reference whatsoever to crystals, a testimony to the value of pure mathematics in paving the way for the explanation of natural phenomena. The method was applied to crystals by Leonhard Sohncke in his *Entwicklung einer Theorie der Krystallstruktur*, Leipzig, 1879, and numerous later memoirs, an especially important one,² embodying his maturer ideas, being published in the year 1892. Leaving out of consideration such parts of Jordan's work as were not applicable to crystals, Sohncke showed that when we drop the limitation of the 14 space-lattices of Bravais, that the units are all arranged parallel and similarly orientated to each other, and admit Wiener's principle and Jordan's more general definition of homogeneity, that the arrangement is merely such that the environment about every point of the assemblage is the same as about every other, we at once admit 65 point-systems, that is, we are able to arrange points in space in 65 typical ways. Each of these 65 "Sohncke regular point-systems,"

¹ *Annali di Matematica*, Milan [2], 1869, 2, 167.

² *Zeitschr. für Kryst.*, 1892, 20, 445.

except the least symmetrical of them, which has no axes, consists of a homogeneous assemblage of points symmetrically and identically arranged about axes of symmetry, which in some cases are screw axes, so that the points surround them in a spiral manner, the resultant of a rotation and a translation. They consist, in general, of two or more interpenetrating identical space-lattices, and, in fact, the atomic points of a Sohncke system may always be grouped together in molecular sets such that the centres of gravity (or any other representative points analogously taken, one from each molecule) of the sets constitute a Bravais space-lattice, which obviously governs the crystal-system to which the crystal shall conform, while the atomic point-system itself determines the particular crystal-class. Moreover, it is clear also that the law of rational indices must be equally applicable to a Sohncke point-system as to a space-lattice, as the former can be resolved into two or more identical space-lattices.

But this earlier work of Sohncke was not complete, for it did not cover the cases of hemimorphous crystals, those which are differently terminated at the two ends of a prism axis, nor all the cases of enantiomorphous crystals, those the two varieties of which are related as a right and a left hand, mirror-images as it were of each other. To explain these, Sohncke at first resorted to the supplementary hypothesis that molecules are endowed with polarity, an assumption which is unsatisfactory, unnecessary, and now known to be erroneous. As far as his 65 point-systems are concerned, however, Sohncke was on sure ground, and after the next step had been indicated by the other three workers to whom reference must now be made, Sohncke himself was the first to recognise their advance, at any rate as regards the work of Schönflies, and applied it to complete his own method.

The final step in the building up of the geometrical theory of crystal structure affords one of the most remarkable instances of independent discovery on record, for no fewer than three different investigators, E. von Fedorow in Russia, A. Schönflies in Germany, and W. Barlow in England, during the years 1890 to 1894 were simultaneously contributing to the complete solution of the problem. They introduced another factor and principle, namely, mirror-image symmetry, repetition as an image seen by reflection from a plane, or "enantiomorphism" (existence in two "enantiomorphous" forms, one the mirror-image of the other) as it is technically termed. A large number of additional types of homogeneous structure are at once admitted by the application of this new principle, amounting altogether to 165, thus rendering the total number of possible homogeneous structures, inclusive of the 65 Sohncke point-systems (themselves inclusive of the 14 Bravais space-lattices), no less than 230. All these fall into one or other of the 32 classes of crystals, and thus these classes are completely explained on the simple and obviously correct assumption that a crystal is a homogeneous structure, the points representing the structural units of which are arranged in accordance with some one or other of the 230 geometrically possible types of such homogeneous structures.

This new property of enantiomorphism or mirror-image symmetry is exhibited by the crystalline forms which are not accounted for among Sohncke's 65 regular point-systems, and both von Fedorow and Schönflies established independently and more or less simultaneously that a definition of the symmetrical repetition of parts which includes enantiomorphous similarity as well as identity leads to types which, as just stated, belong to the whole 32 crystal classes.

Pierre Curie, whose name is so familiar to us in connection with the discovery of radium, had previously made it clear that as regards the exterior symmetry of crystals the whole of the 32 classes could only be fully accounted for by admitting as elements of symmetry not only the well-known planes and axes of symmetry, but also a combined plane and axis of alternating symmetry, that is, a new element of symmetry involving a combined rotation about an axis and reflection across a plane, both operations being completed before the second crystal face is arrived at. Curie further deduced two kinds of repetitions, one which left everything exactly as before, and another in which the new units are the mirror-images of the old units, and if he had only followed up the subject from the internal structural point of view, instead of confining himself to merely the external geometrical aspect, he might have anticipated the work of Schönflies, von Fedorow, and Barlow, and derived directly from his propositions the whole of the 230 types of homogeneous structures, which the labours of these three investigators have resulted in bringing to light.

The first in point of time to establish the existence of the whole 230 types of homogeneous structures was von Fedorow, but unfortunately the three memoirs in which his results are recorded were at first published only in the Russian language, in the *Transactions of the Russian Mineralogical Society* for the years 1885, 1888, and 1890 respectively. Thus it came about that his ideas remained for some years unknown outside Russia. Indeed it was not until he published an abstract¹ in German in 1893, after the publication of the book of Schönflies, *Krystallsysteme und Krystallstruktur*, at Leipzig in 1891, that the work of von Fedorow became generally known. The work of Barlow was published later, his chief memoirs being "Ueber die geometrischen Eigenschaften homogener starrer Structuren und ihre Anwendung auf Krystalle,"² and a later memoir³ published in the year 1897. The methods by which von Fedorow, Schönflies, and Barlow arrived at the same goal are interestingly different, and afford an additional testimony to the accuracy of their conclusions, the same exact number of 230 types being the outcome of each. The memoirs are very voluminous, but their essential points may be stated with brevity.

VON FEDOROW, after developing the known elements of symmetry, the axis, the plane, and the combination of the two, Curie's alternating symmetry, proceeds to show further that two more elements of symmetry are possible, a screw axis and a glide

¹ *Zeitschr. für Kryst.*, 1893, 21, 679.

² *Ibid.*, 1894, 23, 1, and 1895, 25, 86.

³ *Proc. Roy. Dublin Society*, 1897, 8, 527.

plane. Repetition about a screw axis consists of rotation around and translation along the axis, while repetition about a glide plane consists of reflection across and translation parallel to a plane. He next conceives that crystal structure is space partitioned into and filled with parallelohedra, which are equal figures ranged parallel to each other. When plane-faced, they are bounded by parallel pairs of plane faces. Von Fedorow then shows that the number of plane-faced parallelohedra is restricted to five, the cube, rhombic dodecahedron, elongated dodecahedron, cubo-octahedron, and hexagonal prism, the first being a tri-parallelohedron, the last a tetra-parallelohedron, the second and third hexa-parallelohedra, and the cubo-octahedron a hepta-parallelohedron. They are represented in this ascending order at *a, b, c, d, e* in Fig. 67. From these he derives a considerable number of deformed parallelohedra, exhibiting the symmetry of the seven crystal systems, 23 in all, namely, three cubic and three tetragonal, one hexagonal, and four each for the trigonal (rhombohedral), orthorhombic, monoclinic, and triclinic systems. When each of these parallelohedra is represented by a point, the 14 Bravais space-lattices together with 9 compounded point-systems are produced, the latter formed by interpenetrating space-lattices. It is interesting that the cubo-octahedron, von Fedorow's hepta-parallelohedron, that is, a figure composed of seven pairs of parallel faces, is identical with the tetrakaidecahedron of Lord Kelvin, the parallel-faced cell of most

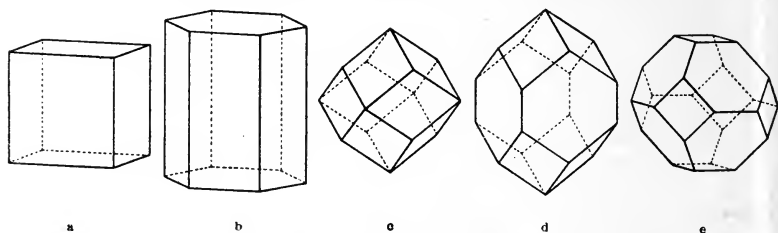


FIG. 67.—Parallelohedra of von Fedorow.

general form into which Lord Kelvin showed that space may be partitioned with a minimum superficial area for a given volume, in his Boyle lecture to the Oxford Junior Scientific Club in 1893, following up and collecting together the results of several of his previous memoirs, particularly one on the "Homogeneous Division of Space,"¹ and an earlier one "On the Division of Space with Minimum Partitional Area."²

Von Fedorow thus appears to have three distinct things in view, namely, the partitioning of space by similar parallelohedra, the close fitting of these bodies so as entirely to fill the space thus partitioned, and the homogeneous assemblages of points produced by considering each parallelohedron as represented by a point, such points being chosen analogously from all the similar parallelohedra. So far, we have seen, this latter aspect of his theory of crystal structure brings us to the Bravais space-lattices and to nine other systems which are combinations of interpenetrating space-lattices.

He now goes further, by assuming that his parallelohedra may be made up of a number of identically similar polyhedra, which he calls "stereohedra," and these may be differently orientated. The analogous points representing these stereohedra constitute a regular point-system, the points of which may obviously be grouped into clusters the mutual arrangement of which is that of a space-lattice, for the parallelohedra corresponding to the clusters form by their representative points a space-lattice, as we have just seen. Employing the method of coincidence movements, the stereohedra and their representative points first arrived at by von Fedorow corresponded to the 65 regular point-systems of Sohncke, and so far there was no real advance, beyond

¹ *Proc. Roy. Soc.*, 1894, 55, 1.

² *Phil. Mag.*, 1887, 24, 503.

the fact that an interesting theory of crystal structure had been put forward, which agreed with Sohncke's truly valid conclusions and gave us at the same time a physical interpretation of them in the division of the crystal into corresponding cells, a reversion more or less to Haüy's conception of the structural units having a definite shape.

But von Fedorow went further in deriving also those stereohedra and their representative points which are the mirror-images of each other, and each corresponding enantiomorphous pair of these he called a "double system." Indeed he distinguishes altogether three kinds of space-partitioning cells and their representative points, namely, (1) symmorphous systems, the elementary figures or cells of which have the same symmetry as the system itself; (2) hemisymmorphous systems, consisting of two analogous simple systems, which together make up a double system, the latter not being symmorphous; and (3) asymmorphous systems, in which adjacent figures are differently orientated. After investigating every kind of homogeneous structure possible on these assumptions, both of cells and of points representing them, he came to the conclusion that there were 230 such systems.

SCHÖNFLIES treated the subject from another standpoint. He began by defining regularity of structure on Wiener's lines by saying that every molecule of an assemblage has the remaining molecules arranged collectively about it in a manner which may be either that of identity or of mirror-image resemblance. As only such of Jordan's groups of movements are concerned in crystal structure as are capable of producing a space-lattice, Schönflies only applies mirror-image operations to such groups. In doing so he employs not only the ordinary planes and axes of symmetry and the screw axes of Sohncke, but also gliding planes of symmetry, which involve reflection over a plane combined with translation parallel to the plane, and the planes of alternating symmetry of Curie. He further proves that in the possible groups of space-operations the only symmetry axes are those characteristic of space-lattices, namely, digonal, trigonal, tetragonal, and hexagonal axes. Carrying out, therefore, all the possible space-operations which conform with these premises, and which lead not only to the 65 Sohncke point-systems but also to point-systems involving mirror-image repetition, and which, moreover, involve also two opposite kinds of screw motions, Schönflies arrived at the same final goal as von Fedorow, namely, that there are 230 groups of such space-operations possible, corresponding to 230 homogeneous structures possible to crystals.

Just as von Fedorow connected his main result,—the discovery of these 230 types of homogeneous structures which fall into and account for the whole 32 classes of crystals,—with a hypothesis of his own regarding crystal structure itself, so also Schönflies regarded his similar main result as a stepping-stone to a molecular theory of crystallised matter. He, too, imagines each representative point as placed within a cell, an elementary cell or "Fundamentalbereich" as he calls it, within which lies the molecule, which does not necessarily fill the cell with its chemical matter; moreover, the form of the cell itself is in general indeterminate, but subject to certain conditions which indicate a polyhedral character. In general, also, there will be two kinds of such polyhedral cells, which are mirror-images of each other. Schönflies is thus obviously attempting to ascertain what conditions of shape and character of the chemical molecule are the basis of the constitution of crystals, and although we may not follow him in his hypothetical work in this direction, particularly in view of certain new experimental facts which will have to be recorded in later chapters as the result of the author's investigations, still the fame of Schönflies is assured as having, in a manner altogether different from that of von Fedorow, discovered and defined the 230 types of homogeneous structures which alone are possible to crystals, whether the units be considered as cells or as points. If we accept the work of both von Fedorow and Schönflies as certainly valid when points are taken to represent the crystal units, whatever they are, we have accepted a solid fact, which

must form the basis of the true and complete theory of crystal structure whenever it is developed and proved.

BARLOW arrived at the same conclusion, that there are 230 kinds of homogeneous structures which belong to the 32 classes of crystal symmetry, from yet a third aspect. He takes his stand on a precise definition of a homogeneous structure, which he gives in the following words, which will be found in a summary of his work¹ given to the Mineralogical Society on November 19, 1895. "A homogeneous structure is one every point within which, if we regard the structure as without boundaries, has corresponding to it an infinitude of other points whose situations in the structure are precisely similar, so that all of the infinite number of geometrical point-systems respectively obtained by taking all similarly situated points are regular infinite point-systems, defined by Sohncke as systems of points such that the arrangement about any one of these points of the rest of the points of the system is the same as it is about any other of them." So far, then, Barlow follows the ideas of both Wiener and Sohncke, but defines the homogeneous structure in more precise terms, and does not encumber it with any further postulate about the corresponding partitioning of space into cells.

The essence of Barlow's work is thus its generality, neither points nor molecules, nor anything in particular, being its foundation, but merely the geometric conception of the arrangement of identically similar parts in a homogeneous rigid structure of any kind whatsoever. Indeed in his paper to the Mineralogical Society (*loc. cit.*) he goes so far in getting rid of the idea of shape of the units as to represent them by plaster models of the human hand, and numerous possible symmetrical arrangements of right and left hands are figured in the plates accompanying his paper. In another memoir he employs a pile of cannon balls as an example of a homogeneous structure, and in yet another he uses a stack of cubes. Perhaps the latter is most instructive of all, as to the nature of his method. He directs attention to a corner of any one of these cubes, and to the fact that it is a point in which eight different cubes of the stack touch each other. Next he indicates some particular point within the cube, which may be selected anywhere, but most instructively near the corner in question. He then shows that there are two other points equidistant from the same corner from each of which the appearance of the whole structure is the same as from that first point; also that there are three other similar points near each of the other seven corners of the cube, from which the outlook is precisely the same. There are thus 24 points within this selected cube from which the stack of cubes would look exactly the same to the eye of a fly, for instance, situated at each of these points in succession, assuming that the whole structure were transparent and non-refractive. The 24 points thus located form a 24-point group of Sohncke.

But Barlow goes a stage further, for he shows that there are three other points near each corner of the cube, from each of which the appearance of the stack is the same as before, except that everything is now inverted, just as if its reflection were being observed in a mirror. These points together make up a second Sohncke 24-point group. He regards identity with its own image as the only property of homogeneity which can raise the symmetry of a homogeneous structure. Starting, therefore, in this manner with the 65 Sohncke point-systems, which his method likewise first arrives at, he inquires which are capable of possessing the additional property of identity with their own images, and how many arrangements there are in which this may be so, and finally arrives at the 230 possible varieties of homogeneous structures.

The above example afforded by a stack of cubes is a very simple case, but it is easy to introduce the sort of complication which many crystals show by imagining each cube to have subsidiary points similarly distributed about it in two ways, firstly so that the appearance of the whole structure from any one of the second 24 points above referred to is the same as from one of the first set of 24 points, and secondly so that

¹ *Mineralogical Magazine*, 1895, 11, 119.

the appearance is not the same. In both cases the structure is obviously a homogeneous one, points in each cube being taken analogously, but the former structure has the full cubic symmetry, being identical with its own image, while the latter has not, but corresponds to a so-called hemihedral class of the cubic system, the pentagonal-icositetrahedral class of which cuprite, the lower oxide of copper, Cu_2O , is a characteristic naturally occurring example.

Barlow gives many reasons for regarding as untenable the arguments of von Fedorow in support of his attempt to determine the shape of the ultimate structural units of crystals, and deprecates all reference to the nature of the crystal faces in connection, at any rate, with the investigation of the nature of homogeneous structures. More recently Barlow himself, however, has advanced, in collaboration with Pope, a theory of space-packing as he assumes it to occur in crystals, that of the closest packing of spheres, the sizes of which are proportional to the chemical valency of the atoms, the spheres of influence of which they are supposed to be. But in his arguments concerning homogeneous structure he most wisely avoids all speculative theories, and confines himself to the strict geometrical problem in hand. He says, "We have to ask ourselves what significance Fedorow attaches to his partitioning of a crystal structure? Is it suggested that when a crystal dissolves, its parts will under any circumstances retain shapes which, if fitted together, fill space, or, to put it more precisely, that the solid angles and edges of the cells are none of them vacuous or structureless in the unbroken structure, but always occupied by matter which, when the structure breaks up, can retain its situation relatively to other matter belonging to the same cell? Such a proposition is so unlikely that to state it is almost to refute it. But if the partitioning into cells of a definite outline capable of filling space is, in the case of crystals, a mere geometrical fiction, what stress can be laid on the relative inclinations of the plane faces of the cells, and what significance is there in the cell walls having plane faces at all? Instead of dwelling on differences between the various kinds of plane-faced cells,—an investigation which may or may not ultimately prove to be of any importance,—we ought to strive after some altogether broader treatment of the subject, which will classify the various kinds of partitioning possible in a perfectly general manner, without absolutely rejecting any. The fact that bodies which form crystals are capable of passing to a liquid state and back again to the crystallised condition without breaking up into their constituents, and the evidence we often have of the survival in the liquid of some portion of the symmetry of arrangement of parts previously in the solid, notably in the cases of those bodies which rotate the plane of polarisation, lead us to conclude that crystallised matter can be broken up into particles or units which are all alike, and each of which has parts or properties that have some definite arrangement relatively to one another. In some cases there is evidence of the survival in the dissolved crystal of two kinds of arrangement, which are enantiomorphous, *e.g.*, racemic compounds can mechanically or otherwise be shown to be composed of two isomers which are respectively right-handed and left-handed, but otherwise alike."

To the evidence adduced as above by Barlow of the survival in the liquid substance of some portion of the crystalline properties, such as the capability of rotation of the plane of polarisation and the exhibition of enantiomorphism, may now be added that offered by the viscous and more mobile oily substances which have been shown by Lehmann to act as so-called "liquid crystals." It has also been abundantly proved, from the author's own researches, that the elementary atoms in the chemical molecule do reveal themselves in the crystal as occupying distinct and definite positions in the molecule, and it is highly probable that the **class** of crystal symmetry which a substance displays is due to this specific orientation of the atoms, while the **system** to which the crystal belongs is the function of the space-lattice formed by the arrangement of the whole molecules or their representative points.

On the general lines now thus indicated Barlow proceeded to investigate all the

possible kinds of homogeneous structures, as already stated. In doing so it may be useful to say that he took into account (1) the different sets of axes of rotation present, (2) the centres of symmetry (a centre of symmetry being equivalent to Curie's rotation about an axis combined with reflection across a plane), (3) the planes of symmetry which have the enantiomorphously related points directly opposite on their two sides, (4) the symmetrically placed planes with accompanying centres, such as those of the two types of structure identical with their own mirror-images, which have no centre or planes of symmetry, and (5) the points and lines of intersection of these various elements of symmetry. He pointed out that each of the regular point-systems of Sohncke represents a class of homogeneous structure which is not identical with its own mirror-image. Further, that the additional property of identity with mirror-image can be displayed by homogeneous structures in three different ways, leading to still other types of symmetry, either (1) across a centre of symmetry so that they are oppositely orientated in every direction, or (2) across a plane or planes of either ordinary or gliding symmetry, or (3) they are opposite each other with reference to one direction and are at the same time orientated at right angles to each other, this method of repetition having an axis of alternating symmetry as used by Curie. Barlow only requires this last mode of repetition to explain the tetartohedral symmetry of the tetragonal system, the other 229 classes being derived by the methods already previously quoted.

Barlow's method of deriving the additional 165 types of homogeneous structures, beyond the 65 Sohncke point-systems, possesses the advantage of distinguishing clearly the enantiomorphous types from those possessing mirror-image symmetry, and of showing the mutual relations of the two enantiomorphous types of which a double system consists; and it also indicates the exact positions of the centres and planes of symmetry. The points which lie at these centres or on axes or planes of symmetry Barlow calls "singular points," and these are very few in number compared to the other points. For instance, when a point occupies the centre of each cube in the stack of cubes referred to on page 126, there are two sets of 24 points, and thus 48 in all, surrounding each such central or "singular" point.

The extraordinary agreement between these three independent investigators, von Fedorow, Schönflies, and Barlow, renders it practically certain that the problem of the homogeneous partitioning of space, as it is possible in crystals, has now thus received a complete solution. With the establishment of the 230 types of homogeneous structures, each compatible with and referable to one of the 32 classes of crystals, the geometrical theory of crystal structure has attained what in all probability will prove to be finality. It must be quite understood, however, that the purely geometrical problem only is referred to. The moment we begin to invest the space-units or cells with definite shapes, such as the parallelehedra of von Fedorow and of Lord Kelvin, the "Fundamentaltbereiche" of Schönflies, or even the spheres of influence of Barlow, we simultaneously introduce controversial questions. Standing quite clear of these, and representing the structural units of the crystals by points or nodes, we are on absolutely sure geometrical ground. Whatever may be the parts of which a crystal consists, they must be arranged according to one or other of the 230 possible arrangements of such points as types of homogeneous structures, of which Sohncke systems and Bravais space-lattices are special cases. Moreover, as Barlow suggests, the material occupying the *Fundamentaltbereiche* of Schönflies, or represented by a generalised point-system, may always be

supposed to be grouped about the nodes of the underlying space-lattice if required, so that what were at first regarded as independent units come to be considered as the parts of a composite unit, which is represented by a nodal point of the space-lattice.

Now Lord Kelvin, in his Boyle lecture at Oxford in 1893, laid special emphasis on the importance of the Bravais space-lattice, and there can be no doubt that his far-seeing and prescient judgment in this matter is entitled to great weight. Indeed the whole of the remarkable investigations which have now been detailed emphasise this point, whether it were the intention of the authors to do so or not. The space-lattice it undoubtedly is which determines both the crystal system and the compliance with the law of rational indices.

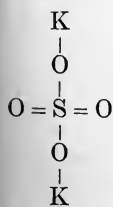
After the completion of the work of von Fedorow, Schönflies, and Barlow, and mainly owing to a study of the results of Schönflies, with which alone he was at first acquainted, Sohncke further developed his theory of crystal structure in two important communications.¹ This more generalised form embodies many of the results of von Fedorow, Schönflies, and Barlow, and if modified in a manner suggested by Barlow, by stipulating that the component point-systems which Sohncke refers to shall have all their coincidence-movements in common, it covers the cases of all but one or two of the crystal classes. In this more generally acceptable form it has been recommended by von Groth, in his Address to the British Association at Cambridge in 1904. As quoted by him it is expressed as follows :

A crystal—considered as indefinitely extended—consists of n interpenetrating regular point systems, each of which is formed from similar atoms ; each of these point systems is built up from n interpenetrating space-lattices, each of the latter being formed from similar atoms occupying parallel positions. All the space-lattices of the combined system are geometrically identical or are characterised by the same elementary parallelepipedon.

We may add to this statement the following corollary :

The combined system conforms to some particular one of the 230 possible homogeneous arrangements in space, and determines which one of the 32 classes of crystal symmetry shall be exhibited in the exterior form of the crystal ; the space-lattice determines the crystal system, and the interfacial angles, and brings about the fulfilment of the law of rational indices.

The actual arrangement in space, for instance, in the case of our typical crystal of potassium sulphate, the chemical formula of which may be graphically expressed as in the margin, is probably that of a structure consisting of three regular point systems, the first being of sulphur atoms, the second of four times as many oxygen atoms, and the third of twice as many potassium atoms as there are sulphur atoms ; the three systems probably interpenetrate in such a manner as to produce equilibrium, being so arranged that the sulphur-atom system lies centrally with respect to the



¹ *Zeitschr. für Kryst.*, 1888, 14, 435, and 1892, 20, 445.

other two, and the potassium-atom system in such wise as would affect chiefly the vertical direction. For this orientation corresponds to the directional changes which occur when the sulphur is replaced by selenium, and when the potassium is replaced by rubidium or caesium. The arrangement of each cluster of seven atoms corresponding to the molecule K_2SO_4 would thus correspond to the stereometric arrangement of the atoms. Each of these seven atoms composing the molecule gives rise to a space-lattice, and all these seven interpenetrating space-lattices are geometrically identical, so that the atom is the unit of the space-lattice, although it may be considered equally well as the representative point of the whole molecule. The combined system of the seven, however, corresponding to the cluster of atoms in the whole molecule, determines that the symmetry is that of the holohedral class, rather than that of any other class, of the orthorhombic system; while the space-lattice formed by each or any one of the seven atoms, analogously chosen from all the clusters of seven throughout the whole assemblage (and which chosen atom indeed may be taken as a point representing the molecule equally as well as does the centre of gravity of the molecule), determines that the system is the orthorhombic one, and fixes the interfacial angles and the crystal elements.

One of the chief results of the author's comparative work on the alkali sulphates and double sulphates and the analogous selenates has, in fact, been to prove beyond all shadow of doubt that the atoms and their enveloping spheres of influence do occupy distinct and characteristically orientated portions of space within the crystal structure. Two opposing forces acting between such atomic centres are alone necessary for the stability of the structure, a repulsive force dependent on the kinetic energy of the atoms, and an attractive force of the combined nature of gravity and chemical affinity, in accordance with the well-known selective character of chemical combination.

This modified Sohncke method has the great merit of simplicity, and of emphasising the undoubted importance of the space-lattice. It is equivalent to the assumption that two or more regular point-systems, consisting of different kinds of atoms, may be interlaced, interpenetrating in such a manner as to produce equilibrium, the condition for which is that the different point-systems shall be space-lattices of identical dimensions. Moreover, it agrees perfectly with the experimental facts indicating definite orientations for the different atoms in the molecules. For as each or any atom in the chemical molecule may be considered as representative of the molecule, just as well as the centre of gravity or any other representative point, so all the atoms composing the molecule may be considered as points, each set of analogous atoms, one from each molecule, forming a regular point-system. As the atoms are definitely orientated the interlaced point-systems, equal in number to the number of atoms in the molecule, will together form one of the 230 possible types of homogeneous structure.

This later development of his theory by Sohncke agrees for the

great majority of cases with the conclusions of von Federow, Schönflies, and Barlow. But Sohncke has considered the division of the molecules, in the few outstanding cases which his theory does not explain, into two enantiomorphous sets very improbable. The overwhelming number of cases where all agree, as just mentioned, are those in which, on Sohncke's theory, the molecules are all congruent, and Sohncke considered congruency of the molecules to be an essential condition of crystal structure. But the few remaining cases are so completely explained by the assumption of the three other investigators that two enantiomorphous sets of molecules are simultaneously present in the crystal—as there are, for instance, when there is “external compensation” between the two opposite kinds of molecules in a racemic compound¹—that there would appear to be more ground for accepting their assumption as correct, especially as it is geometrically possible, than for rejecting it, and no other explanation would appear to be geometrically valid.

A very able mathematical review of the whole subject has been made by H. Hilton in his *Mathematical Crystallography*.² On this particular point, where the opinion of a mathematician is of special importance, experimental evidence not yet being decisive one way or the other, Hilton says:

“The individual molecules have no symmetry in general” (which the author takes to mean that Hilton agrees that the stereometric arrangement of the atoms in the molecule, while definite, is not necessarily symmetrical); “they are all congruent if the space-group has no operation of the second sort, if, however, it has such an operation the molecules may be divided into two sets, such that the molecules of either set are congruent to all the molecules of the same set and enantiomorphous to those of the other. Some authors have objected to this division of the molecules of a crystal into two enantiomorphous sets as improbable. It is, in fact, in most cases possible to avoid having recourse to this division, . . . but the weakness of the” (alternative) “method” (of grouping into congruent molecules) “lies in the fact that it cannot be used to explain the structure of crystals of every class, without supposing that these congruent molecules may interlace with one another, an idea which hardly commends itself to a physicist.” But Hilton concludes his analysis by adding: “The series of points representing the constituent parts of a crystalline solid are the points of a series of similar and similarly orientated lattices, any one of which represents the translations of the group. We may therefore apply without appreciable modification Bravais' explanation of cleavage, the occurrence of crystal faces, and the law of rational indices.”

These views of Hilton will be observed to be in full agreement with those already expressed in previous pages by the author.

Several attempts have been made to allocate specific types of homogeneous structure to particular crystallised substances. Thus the two enantiomorphous forms of quartz, right-handed and left-handed, have been referred by Sohncke to two of his types which are similar, except that one has a right-handed screw and the other a left-handed screw coincidence movement. The author has also shown that the

¹ A compound of similar nature to racemic acid, which is the optically inactive molecular compound of dextro (right-handedly optically active) and lævo (left-handedly rotating) tartaric acids.

² Oxford, 1903, p. 251.

isomorphous group of sulphates and selenates of the alkalis, of which potassium sulphate, K_2SO_4 , is the first member, are referable to a particular one of Barlow's types, which, while really of rhombic symmetry, is of pseudo-hexagonal (slightly deformed hexagonal) type, in accordance with the pronounced pseudo-hexagonal nature of the crystals. It is only in cases where the optical properties of a substance have been exhaustively worked out, however, as in the cases of quartz and of the potassium sulphate group of salts, that such attempts are likely to meet with success. For generally there are several isomorphous types possible among the 230, and only a large amount of experimental data can enable us to decide between them.

Having thus now dealt with the whole of the original investigations into the nature of homogeneous crystal structure, we may leave the subject with a few practical reflections. In agreement with the view of von Groth, who has had unique means of becoming acquainted with the whole of the details of the published memoirs of all the workers concerned, in his capacity of editor of the *Zeitschrift für Krystallographie*, we may accept the Sohnckian theory of regular point-systems, in the latest form given to it by Sohncke and as perfected by von Fedorow, Schönflies, and Barlow, as affording a fully satisfactory explanation of crystal structure, by providing us with 230 distinct types of homogeneous structures, the total possible, and which among them account for all the 32 symmetry classes of actual crystals. Sohncke is undoubtedly correct as far as he is in agreement with von Fedorow, Schönflies, and Barlow, and for the rest, where Sohncke's method is insufficient, the agreement of these three latter observers among themselves is a guarantee that their purely geometrical work also bears the stamp of accuracy. On the border line, we have the choice of either the method of Sohncke or of that of the three other investigators, for those cases of assemblages in which the possibility of the coexistence of two enantiomorphous sets of molecules is involved; and if we accept the suggestions of Barlow for the modification of the method of Sohncke, all are practically brought into line.

If the crystal be that of a chemical element, there is obviously only one kind of atom and one regular point-system; but when the crystal is that of a chemical compound the arrangement of the combined system corresponds with the equilibrium of the forces with which similar and dissimilar atoms act on each other, that is, on the stereometric arrangement of the atoms within the molecule, including in such equilibrium any mechanical or other physical forces which may likewise be operative in the production of the solid crystal. On fusion, solution, or evaporation of the crystal the system breaks up into separate freely moving molecules, when they lose their parallel or enantiomorphous arrangement which they had assumed with respect to each other in the crystal, except in the interesting cases of the "liquid crystals" brought to our notice by Lehmann, in which the molecules appear to retain their regular orientation in the liquid state.

The following may prove to be a useful summarised statement of the

geometrical theory of crystal structure which has now been discussed in the foregoing pages.

The 14 space-lattices of Bravais comprehend all cases of same-way orientated homogeneity. In Sohncke's 65 point-systems the points employed to show the symmetry are all identically related to the set of symmetrical operations—rotations, screw movements, etc.—which characterise the system. A Sohnckian system can always be regarded as consisting of a number of identical space-lattices symmetrically interlaced, but, in most cases, having different orientations. For the existence of rotations involves the presence of groups of two or more points in a system, and if these groups have the same orientation the group-centres form a Bravais space-lattice. Schönflies, von Fedorow, and Barlow widened the definition of homogeneity to include “enantiomorphous similarity”—that of a right-hand to a left-hand glove. They pointed out that likeness to its own mirror-image is not a necessary property of any Sohnckian system, and by discriminating the number of ways in which this property can be superimposed on the systems of coincidence movements they added 165 systems, distinguishable from the enantiomorphous representatives of the 65 Sohncke systems, thus making a total of 230 types each having its own distinct system of coincidence operations.

In concluding this chapter on the geometrical theory of crystal structure, reference must be made to an important practical consequence of the satisfactory establishment of that theory which immediately followed, namely, the abandonment of the previous mode of description of the crystal classes as holohedral (possessing the full symmetry of the system), hemihedral (in which half the full number of faces were supposed to be suppressed), and tetartohedral (in which one-fourth only of the holohedral faces were supposed to persist). It had been the custom to regard those classes of the 32 which did not exhibit the complete symmetry of one of the seven systems as derived by the suppression or latency of some of the faces, and the mode in which the suppression occurred, for it could generally happen in more than one way, determined the particular class. Even before the advent of the epoch-making work of Sohncke, Victor von Lang, as already mentioned, had, in 1866, published his celebrated treatise (*Lehrbuch der Krystallographie*) in which the real elements of symmetry were indicated. In subsequent editions his work was strengthened and completed, embodying certain supplementary details arrived at by the mathematical genius of Gadolin. Yet still, until even so late as the publication of the second edition of the *Physikalische Krystallographie* of P. Groth, Leipzig, 1885, the classification into holohedral, hemihedral, and tetartohedral classes still persisted. But after another decade, the year 1895, succeeding the completion of the work of Sohncke, von Fedorow, Schönflies, and Barlow, saw the publication both of the third edition of Groth's book, and of another most valuable treatise, the *Morphology of Crystals*, by N. Story-Maskelyne, in both of which the 32 classes were treated as representing distinct types of symmetry, and not merely

as fully or partially developed forms. Owing to the fact that the greater portion of the latter book had been a very long time in course of preparation, the transformation was but partial, for while insisting that axes and planes of symmetry are the true elements of symmetry, and clearly defining their real meaning, the terms holohedral, hemihedral, and tetartohedral were still employed. But in the third edition of Groth's book the 32 varieties were classified in their seven systems entirely by the method of and with reference to their development of planes and axes of symmetry, including the combination of the axis and plane as a plane of alternating symmetry as defined by Curie. Moreover, in a memoir in the *Neues Jahrbuch* for 1896, p. 495, a further strengthening was given to the new mode of description by Viola, who derived the elements of symmetry of the 32 classes of crystals by use of an entirely new method, that of quaternions. Thus supported in several ways from both the geometrical and the purely mathematical sides, the new mode of description, based upon the recognition of axes and planes of symmetry as the true elements of crystal symmetry, is now regarded as the only logical one, and the only one in full conformity with the completed geometrical theory of the homogeneous partitioning of space in a crystalline medium. Each of the 32 classes is described as definitely endowed with and characterised by its own specific planes and axes of symmetry, or by the absence of such; and the crystals conforming to subsidiary classes of any system, exhibiting less than the maximum symmetry and full number of faces possible to that system and formerly termed hemihedral or tetartohedral, are regarded as just as fully perfect individuals as those conforming to the class of maximum symmetry, and as exhibiting the complete symmetry which distinguishes their particular class, any idea of suppression of faces being altogether eliminated.

Thus the mode of description of the exterior symmetry of crystals, as well as the theory of their internal structure, are now both established on a logical and scientific basis.

It has been considered essential to deal at some length with this most important modern advance in our ideas of crystal structure, inasmuch as all future work both in chemistry and in the physics of solids will have to take account of it, and all practical crystallographic investigations must, to be of value, be carried out with these facts borne in mind. For it is only by organised experimental effort, directed to the study of definitely chemically related substances, that an adequate number of facts of indisputable authority can be marshalled to enable us to carry our present knowledge further to its logical conclusion, when we shall be enabled to predict the crystalline form from the given chemical constitution. So far as the geometrical theory has been taken in the preceding pages, it is in full accordance with experimental facts as well as with mathematical geometry, the one confirming the other to a degree which ensures that the truth has so far been arrived at.

We are now, therefore, in a position to proceed on sure ground to consider the thirty-two types of actual crystals, and the seven classes of symmetry into which they naturally fall.

CHAPTER X

CRYSTAL SYMMETRY AND ITS 32 TYPES.

It has been shown in Chapter IV. that a crystal is characterised by the ratio $a:b:c$ between the lengths of its axes a , b , c , and also by the angles α , β , γ between those axes; further, it will have been gathered from the preceding chapter that these axes may be regarded not only as the edges of intersection of the three primary facial planes, but also as those of the elementary parallelepipedon of the Bravais space-lattice according to which the crystal structure is built up. It is, however, also characterised by its degree of symmetry, and certain of the above elements are fixed by that symmetry. The remainder have to be determined by goniometrical measurement, as described in Chapter IV., followed by calculation after the manner shown in Chapter VIII. It is necessary, therefore, that we should next study the types of crystal symmetry, and that we should do so in the light of the geometrical knowledge now acquired concerning homogeneous structures in general. First of all it is advisable to define the elements of symmetry.

Elements of Symmetry.—The true elements of symmetry are “planes of symmetry” and “axes of symmetry.” The so-called “centre of symmetry” is not a true element of symmetry, although it is frequently convenient to use the term in connection with those holohedral triclinic crystals the faces of which are always duplicated in parallel pairs. Centrosymmetry, however, is merely the effect of rotation for 180° or for 60° about an axis of symmetry followed by reflection across a plane of symmetry.

Planes of Symmetry.—If we imagine an ideal crystal with all the faces of each form equally developed, a “plane of symmetry” is such a plane as, passing through the centre of the crystal, would bisect it into equal halves. To every point on the crystal there corresponds a precisely analogous point at the same distance on the other side of the plane of symmetry. Also, when the one half is placed with the section plane against a plane mirror, the other half is apparently reproduced by reflection, so that either half may be considered as the mirror-image repetition of the other. A monoclinic crystal has only one such plane of symmetry, but a cubic crystal has no less than nine planes of symmetry, and between these limiting types there are several intermediate ones, which will presently be discussed.

Planes of symmetry are best recognised on the stereographic projection, for we have there the geometrical plan of the angular arrangement of the faces, uncomplicated by the usually promiscuously unequal development of the separate faces of the same form, which prevents the attainment, except by a very rarely, if ever, occurring chance, of the ideal equality of faces.

Every symmetry plane of a crystal is a possible face of the crystal, and the normal to a plane of symmetry is always a possible interfacial edge on the crystal.

Axes of Symmetry.—Besides reflection-repetition over a plane of symmetry, repetition of parts may occur by rotation about an axis.

An axis of symmetry is perpendicular to a possible face of the crystal, and is always a possible interfacial edge on the crystal.

There are four kinds of axes of symmetry, and only four, namely, digonal or binary, trigonal or ternary, tetragonal or quaternary, and hexagonal or senary; and they are such that rotation of the crystal about them for respectively 180° , 120° , 90° , and 60° brings the crystal again into a position of like aspect to that which it presented when in its original position.

Thus, when an octahedron is rotated for 90° about any one of the three diagonals, each face is brought into the position previously occupied by the adjoining one, and this will happen four times, at each interval of 90° , in a complete revolution of 360° . Such an axis of symmetry is a **tetragonal axis** or **axis of fourfold symmetry**.

These tetragonal axes of the octahedron are identical with the three rectangular axes of the cube and of the cubic system, and each of them is an axis in which four planes of symmetry intersect, namely, two of the three axial planes of symmetry and two of the six diagonal planes of symmetry bisecting the angles between the axial planes; such an axis of symmetry is therefore a tetragonal axis in two senses, and is called a **ditetragonal axis**.

Each of the four diagonals of the cube is similarly a **trigonal axis**, or **axis of threefold symmetry**, because during a complete rotation of 360° of the cube about any one of these diagonals it is brought three times into a position which is indistinguishable from its original one. Moreover, as each of these four diagonals forms the line of intersection of the three planes of diagonal symmetry which pass through the same two corners as the diagonal in question, these trigonal axes of the cube are also so in two senses, and are therefore termed **ditrigonal axes**.

Similarly a **hexagonal axis** or **axis of sixfold symmetry** is such that a sixth of a complete revolution about it brings the crystal to exhibit again the same aspect, and it is termed a **dihexagonal axis** when it is also the intersection of the six planes of symmetry of a hexagonal crystal.

Lastly, a **digonal axis** or **axis of twofold symmetry** is such that half a complete revolution about it restores the original appearance of the crystal, and it is a **didigonal axis** when it also forms the line of intersection of two symmetry planes.

Indeed we may now define in general terms di- n -gonal symmetry, by stating that an n -gonal axis which is the line of intersection of n planes of symmetry is termed a di- n -gonal axis of symmetry.

In a similarly general manner we may also define four other terms currently employed concerning elements of symmetry, namely, holoaxial, equatorial, alternating, and polar symmetry, and which will also indicate four rules governing combinations of the elements of symmetry.

Holoaxial symmetry is present when an n -gonal axis is only accompanied by axes of digonal symmetry perpendicular to it, planes of symmetry being absent.

Equatorial symmetry is produced when an n -gonal axis is perpendicular to a plane of symmetry.

Alternating symmetry is arrived at by simultaneous rotation about an n -gonal axis and reflection across an equatorial plane. The equatorial plane in such a case is called a **plane of compound symmetry**.

Polar symmetry is that caused by the presence of an n -gonal axis which is perpendicular neither to a plane of symmetry nor to an axis of even symmetry. The two polar ends of a crystal endowed with only polar symmetry may be quite different from each other.

Tesseral symmetry is a term sometimes applied to the highest type of all possible crystal symmetry, that of the cubic system. It is specifically characterised by the four ditrigonal axes of symmetry.

Having now explained the terms currently used for the designation of the various elements of symmetry and their modes of combination, we can logically proceed to the classification of crystals according to their symmetry. It will be most instructive to show first of all how the other 31 classes of crystals are gradually evolved from the asymmetric class exhibiting absolutely no symmetry, by the introduction first of axes of symmetry alone, and then by the addition of planes of symmetry, until we eventually arrive at the perfect symmetry of the cubic system, in which we have the fullest possible combination of both elements of symmetry.

THE 32 CLASSES OF CRYSTALS.

Evolution of the other 31 Symmetric Classes from an Asymmetric Crystal (Class 1).

The crystals of class 1, known as asymmetric, possess neither axes nor planes of symmetry, nor even exhibit the phenomenon of centrosymmetry due to the successive operation of both. Crystals of this class only satisfy the one condition of following the law of rational indices and the rules relating to zones.

The crystals of the whole of the other 31 classes possess symmetry, the degree of which continually ascends until we reach a holohedral crystal of the cubic system. Symmetry is unaltered by change of temperature, so that it is quite independent of the latter; for equal deformations are suffered in the symmetrical parts when the

temperature alters. The crystals possessing symmetry may be first arranged in the following three groups:

- (1) Those the symmetry of which is brought about by rotation round an axis or axes of symmetry, but which possess no planes of symmetry.
- (2) Those possessing a plane or planes of symmetry.
- (3) Those the symmetry of which is derived by rotation about an axis of symmetry followed by mirror-image reflection from the plane perpendicular to the axis. This kind of symmetry plane is called a plane of compound symmetry, to distinguish it from an ordinary symmetry plane or plane of simple symmetry. It should be clearly understood that both operations occur in their entirety before reaching the second face. As in two of the three classes involved it results in centro-symmetry, crystals of this kind are sometimes said to possess a centre of symmetry. Centro-symmetrical crystals are, however, only special cases of the general principle underlying the symmetry of group (3).

There are 10 classes of group (1), and together with the asymmetric class 1 (making altogether 11 classes) they are distinguished by the property already alluded to of **enantiomorphism**, which is brought about in all 11 classes by the absence of any plane of symmetry. An enantiomorphous crystal is one which by reflection in a plane mirror yields the image of a crystal similarly symmetrical and having equal angles, but laterally inverted; it cannot be got by rotation to resemble the first precisely, but behaves as a left hand does to a right one. The actual complementary crystal corresponding to this image, would, analogously, furnish by reflection an image resembling the first variety. Both varieties of the crystals of one and the same chemical compound are known in the cases of many real substances.

The class number given for each class in the now following list refers to its order in the next list to be given, that in which the classes are arranged in their proper order in the seven crystal systems, and these class numbers are those which will be adhered to throughout this book. The order in which the classes are set out in this present list is merely one designed to render clear their evolution.

GROUP 1.

10 Enantiomorphous Classes without Planes of Symmetry but with one or more Axes of Symmetry.

First come 4 classes with only one axis of symmetry, namely:

Crystals with 1 digonal	axis of symmetry .	Class	4
„ „ 1 trigonal	„ „ . .	„	16
„ „ 1 tetragonal	„ „ . .	„	9
„ „ 1 hexagonal	„ „ . .	„	23

Next we have 6 classes with more than 1 axis of symmetry, namely:

- Crystals with 1 digonal axis and 2 other digonal axes intersecting at 90° in the perpendicular plane. Class 6.
- „ „ 1 trigonal axis and 3 other digonal axes inclined at 60° in the perpendicular plane. Class 18.
- „ „ 1 tetragonal axis and 4 other digonal axes inclined at 45° in the perpendicular plane. Class 11.
- „ „ 1 hexagonal axis and 6 other digonal axes inclined at 30° in the perpendicular plane. Class 24.
- „ „ 3 perpendicular digonal axes and 4 trigonal axes meeting each other at equal angles. Class 28.
- „ „ 3 perpendicular tetragonal axes and 4 trigonal axes meeting each other at equal angles and also 6 digonal axes. Class 29.

It may be stated in general terms that: If there be another axis of symmetry inclined to an n -gonal axis, there must be n such axes.

Enantiomorphism is generally exhibited by substances which are capable of rotating the plane of polarised light, such as the two optically active (right- and left-handed) varieties of quartz and of tartaric acid, the right-handed and the left-handed varieties always corresponding to oppositely enantiomorphous crystals, which are the mirror-images of each other. The optical activity is probably due to enantiomorphism of the chemical molecules themselves, and therefore to the structure of the crystals.

The theory of point-systems reviewed in the preceding chapter provides completely for such cases, the arrangement of the points being screw-spiral-wise, either right or left. Such a structure can originate by the aggregation of molecules which are themselves enantiomorphous, in which case both the crystals and their solutions are optically active, as in the example, tartaric acid, just quoted, and as in the case of strychnine sulphate. But it may also happen that the enantiomorphism only applies to the point system, and not to the chemical molecules, in which case the solution is inactive and the crystals alone rotate the plane of polarisation to the right or left, as in the case of sodium chlorate. There are also regular point-systems which exhibit the symmetry of one of the above 11 enantiomorphous classes, yet do not possess such screw axes, and are consequently optically inactive, both as regards the crystals and their solutions, as in the case of barium nitrate.

Hence, although all optically active substances afford enantiomorphous crystals, the inverse of the rule is not true, for enantiomorphous crystals are not always optically active. Some specific examples will be described and illustrated in later chapters, and the whole of Chapter XLIX. will be devoted to the phenomenon of rotation of the plane of polarisation and the mode of measuring it.

GROUP 2.

18 Classes of Crystals with a Plane or Planes of Symmetry, 17 of which have also an Axis or Axes of Symmetry.

First come 4 classes of equatorial symmetry, that is, in which an n -gonal axis is perpendicular to a plane of symmetry.

Crystals with a digonal axis and a symmetry plane perpendicular to it. Class 5.

„ „ trigonal axis and a symmetry plane perpendicular to it. Class 19.

„ „ tetragonal axis and a symmetry plane perpendicular to it. Class 12.

„ „ hexagonal axis and a symmetry plane perpendicular to it. Class 25.

Next come 4 other classes having an n -gonal symmetry axis combined with as many symmetry planes as the n indicates.

Crystals with a digonal axis and 2 symmetry planes intersecting in it at 90° . Class 7.

„ „ trigonal axis and 3 symmetry planes intersecting in it at 60° . Class 20.

„ „ tetragonal axis and 4 symmetry planes intersecting in it at 45° . Class 13.

„ „ hexagonal axis and 6 symmetry planes intersecting in it at 30° . Class 26.

2 other classes are formed by adding to the two first of these four :

2 digonal axes perpendicular to the first and bisecting the angles of the 2 symmetry planes. Class 14.

3 digonal axes perpendicular to the trigonal axis and bisecting the angles of the 3 symmetry planes. Class 21.

We have next 4 further classes by combination of the first four classes of this group with the next four ; that is, in which in addition to the presence of an n -gonal axis and n symmetry planes intersecting in it, the plane perpendicular to the n -gonal axis is also a plane of symmetry. The four classes are respectively numbers 8, 22, 15, and 27.

Two further classes, numbers 30 and 31, are afforded in the special cases of the 3 perpendicular digonal axes of the ninth class of this group (class 14) being of equal length ; they are then the cube normals, and combinations of these axes with planes of symmetry are possible in two ways, according to one of which (class 30) the 3 cube faces are planes of symmetry, and according to the other of which (class 31) the 6 dodecahedron faces are the symmetry planes. In both cases the normals to the octahedron faces are trigonal symmetry axes.

Finally we have one class of the highest possible crystal symmetry,

in which both kinds of planes, parallel to the 3 cube faces and to the 6 dodecahedron faces respectively, are symmetry planes. In this case the 3 cube normals are tetragonal axes of symmetry, the 4 octahedron normals are trigonal axes, and the 6 dodecahedron normals are digonal symmetry axes. This class is number 32.

The above are the 17 possible classes possessing both axes and planes of symmetry. To make up the 18 classes containing symmetry planes we have also 1 class of crystals characterised by the presence of 1 symmetry plane, without any symmetry axis. This is class 3.

We have now, in a manner which it is hoped has made clear the mode of evolution, dealt with 1 asymmetric class (class 1), 10 classes of group 1, and 18 classes of group 2, making altogether 29 classes.

The remaining 3 classes, forming group 3, of the whole 32 possible classes of crystals are those characterised by a plane of compound symmetry, or in other words, those formed by rotation about an axis followed immediately by reflection across a plane (a plane of compound symmetry). The axis must be one of even gonality, that is, di-, tetra-, or hexagonal, a trigonal axis being impossible under such circumstances.

GROUP 3.

The first case, that of class 2, is formed by rotation about a digonal axis, which may be the normal of any face on the crystal, the plane of compound symmetry being parallel to the face. It is illustrated in Fig. 68, in stereographic projection, the plane of the primitive circle being parallel to the face in question. The centre is then the projection of the pole of the digonal axis of symmetry. If any other face (*hkl*) on the upper hemisphere of the crystal be represented by the dot, its equivalent face in the lower hemisphere, represented by the small ring, is arrived at by rotation of the dot-pole round the centre for 180° and then reflection at the plane of compound symmetry, the plane of the paper. The two faces are parallel and opposite, and may be said to be centro-symmetrical, but such a centre of symmetry is not essential to the presence of a plane of compound symmetry, as will appear when the next class is discussed. Von Fedorow has shown that the "centre of symmetry" is not a true element of symmetry, but merely the accompaniment of the operation of the general principle of reflection at a plane of compound symmetry in this special case of class 2 and a further one, class 17, the last of the three of this group.

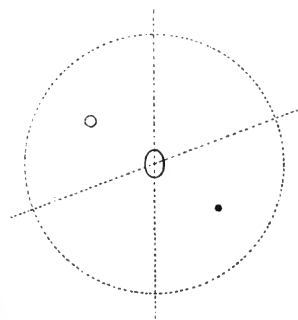


FIG. 68.

The second case of this group, class 10, illustrated in Fig. 69, occurs when the axis of symmetry is a tetragonal one combined with a plane of compound symmetry. Its pole is at the centre, and the plane of

compound symmetry is again the plane of the primitive circle. If the dot in the right-front quadrant of the projection be the pole of a face (hkl) in the upper hemisphere, rotation for 90° followed by reflection at the plane of compound symmetry brings us to the second face in the lower hemisphere, represented by a ring in the left-front quadrant; repetition of the process brings us to a face in the upper hemisphere again, represented by the dot-pole in the back-left quadrant, the opposite quadrant to that started with, and two further repetitions bring us successively to a ring-pole-face in the lower hemisphere and to the original dot-pole-face in the upper hemisphere once more. The tetra-

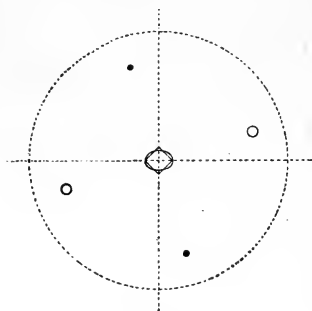


FIG. 69.

gonal axis of compound symmetry is thus also a digonal axis of ordinary symmetry. There are here no parallel faces, and there is, therefore, no centre of symmetry.

The third and last class of this group, class 17, involves the presence of a hexagonal axis and plane of compound symmetry, by the operation of which repetition occurs at every 60° with a change of hemisphere each time, the sixth repetition recovering the original position. Fig. 70 will render the case clear, the centre being the pole of the axis, and the three dots and three rings alternately arranged being the poles of six faces in the successive sextants, situated alternately in the upper and lower hemispheres. The phenomenon of centro-symmetry happens here to be for the second time an accompaniment of the simultaneous operation of an axis of symmetry and a plane of compound symmetry.

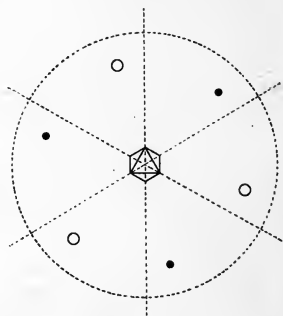


FIG. 70.

The 32 classes of crystal symmetry the evolution of which has now been followed are the only ones possible, if, as we must, we accept the law of rational indices as an experimental fact, irrespective of any theory of crystal structure. In practice, no crystal has ever yet been discovered which could not be referred to one or other of these 32 classes.

Division of the 32 Classes into

7 CRYSTAL SYSTEMS.

We have already seen that a *form* consists of all the faces of equal value with respect to the symmetry exhibited by the crystal, and it

will now be apparent that the number of such faces belonging to one and the same form will be the greater the higher the degree of symmetry. The axes are usually so chosen that all the faces of a form will have the same numbers to represent their indices, the signs only being different for the faces in the different octants. In the classes of higher symmetry all the faces having similar index numbers will belong to the same form, but not so in those of lower symmetry, in which cases only half, a quarter, or an eighth of such faces will belong to the same form, and in the asymmetric class only a single face belongs to a form. When the choice of axes is thus made, so that all the faces of equal value as regards the symmetry have equal index numbers, it is found that this choice happens in a similar way in the cases of several of the 32 classes, and that there are seven such ways. In other words, we find that the 32 classes fall into seven groups or "systems," the characteristic of each of which is a similar arrangement of crystal axes. It should be clear that the term axes in this paragraph refers to the fundamental crystallographic axes, parallel to 3 chosen interfacial edges, and the unit lengths of which are determined by the intercepts cut off along them by a parametral plane parallel to a fourth face inclined to all three axes. They may or may not be axes of symmetry. These seven systems have already been incidentally mentioned in connection with the theory of space-lattices and regular point-systems. They are the triclinic, monoclinic, rhombic, trigonal, tetragonal, hexagonal, and cubic systems. In the following scheme these seven systems are each separately analysed, their characteristics are indicated, the number of classes which each includes is specified, the actual class numbers (which in this list occur in their proper sequence) are given, and the symmetry elements of each of the 32 classes are recorded, together with the type of their symmetry. An example of each class is also quoted.

I. *The Triclinic or Anorthic System.*

Characterised by 3 inclined (otherwise than at 90° , 60° , 45° , or 30°) unequal axes, a , b , c .

$$a : b : c = ? : 1 : ? \quad \alpha = ? \quad \beta = ? \quad \gamma = ?$$

Includes 2 classes.

Class 1. Asymmetric class. No symmetry.

Example—Calcium thiosulphate, $\text{CaS}_2\text{O}_3 \cdot 6\text{H}_2\text{O}$.

Class 2. Pinakoidal or holohedral class. A digonal axis and a plane of compound symmetry perpendicular to it, which are equivalent to a centre of symmetry.

Example—Copper sulphate, $\text{CuSO}_4 \cdot 5\text{H}_2\text{O}$.

II. *The Monoclinic or Monosymmetric System.*

Characterised by 3 unequal axes, two of which, a and c , are inclined, but the third b is perpendicular to those two.

$$a : b : c = ? : 1 : ? \quad \alpha = \gamma = 90^\circ, \beta = ?$$

Includes 3 classes.

Class 3. Domal class. A plane of symmetry.

Example—Potassium tetrathionate, $\text{K}_2\text{S}_4\text{O}_6$.

Class 4. Sphenoidal class. Type, digonal polar. A digonal axis.

Example—Tartaric acid, $C_4H_6O_6$, two oppositely enantiomorphous forms, dextro and lævo.

Class 5. Prismatic or holohedral class. Type, digonal equatorial. A symmetry plane and a digonal axis perpendicular to it.

Example—Potassium magnesium sulphate, $K_2Mg(SO_4)_2 \cdot 6H_2O$.

III. The Rhombic or Orthorhombic System.

Characterised by 3 unequal but rectangular axes, a , b , c .

$$a : b : c = ? : 1 : ? \quad \alpha = \beta = \gamma = 90^\circ.$$

Includes 3 classes.

Class 6. Bisphenoidal class. Type, digonal holoaxial. 3 rectangular digonal axes of symmetry.

Example—Magnesium sulphate, $MgSO_4 \cdot 7H_2O$.

Class 7. Pyramidal class. Type, di-digonal polar. A digonal axis and 2 mutually perpendicular planes of symmetry parallel to it.

Example—Ammonium magnesium phosphate, $NH_4MgPO_4 \cdot 6H_2O$.

Class 8. Bipyramidal or holohedral class. Type, di-digonal equatorial. 3 rectangular digonal axes and 3 planes of symmetry perpendicular to each other.

Example—Potassium sulphate, K_2SO_4 .

IV. The Tetragonal System.

Characterised by 3 rectangular axes, of which two, a and b , are equal and are, therefore, both written as a , but the third, the vertical axis c , is different. Includes all crystals with one tetragonal axis of symmetry.

$$a : a : c = 1 : 1 : ? \quad \alpha = \beta = \gamma = 90^\circ.$$

Includes 7 classes.

Class 9. Pyramidal class. Type, tetragonal polar. One tetragonal axis of symmetry.

Example—Wulfenite, $PbMoO_4$.

Class 10. Bisphenoidal class. Type, tetragonal alternating. One tetragonal axis and a plane of compound symmetry perpendicular to it.

Example—Only instance yet observed is the compound $2CaO \cdot Al_2O_3 \cdot SiO_2$.

Class 11. Trapezohedral class. Type, tetragonal holoaxial. One tetragonal axis and 4 digonal axes in the plane perpendicular to it.

Example—Strychnine sulphate $(C_{21}H_{22}N_2O_2) \cdot H_2SO_4 \cdot 6H_2O$.

Class 12. Bipyramidal class. Type, tetragonal equatorial. One tetragonal axis and a plane of symmetry perpendicular to it.

Example—Scheelite, $CaWO_4$.

Class 13. Ditetragonal-pyramidal class. Type, ditetragonal polar. One tetragonal axis and 4 symmetry planes intersecting in it.

Example—Iodosuccinimide, $C_4H_4O_2NI$.

Class 14. Scalenohedral class. Type, ditetragonal alternating. One tetragonal axis and a plane of compound symmetry perpendicular to it; 2 rectangular digonal axes lying in the latter plane; 2 symmetry planes intersecting in the tetragonal axis and which bisect the angle of the two digonal axes.

Example—Potassium dihydrogen phosphate, KH_2PO_4 .

Class 15. Ditetragonal-bipyramidal or holohedral class. Type, ditetragonal equatorial. One tetragonal axis and 4 symmetry planes intersecting in it; also a symmetry plane and 4 digonal axes all perpendicular to the tetragonal axis.

Example—Zircon, $ZrSiO_4$.

V. *The Trigonal System.*

Characterised by 3 equal and equally inclined axes, a, b, c , all three written, therefore, as a . Includes all crystals with one trigonal axis of symmetry.

$$a : a : a = 1 : 1 : 1 \quad \alpha = \beta = \gamma = ?$$

Includes 7 classes.

Class 16. Pyramidal class. Type, trigonal polar. One trigonal axis of symmetry.

Example—Sodium periodate, $\text{NaIO}_4 \cdot 3\text{H}_2\text{O}$.

Class 17. Rhombohedral class. Type, hexagonal alternating. One trigonal axis of symmetry which is also a hexagonal axis of compound symmetry.

Example—Diopside, CaH_2SiO_4 .

Class 18. Trapezohedral class. Type, trigonal holoaxial. One trigonal axis and 3 digonal axes in the plane perpendicular to it.

Example—Quartz, SiO_2 .

Class 19. Bipyramidal class. Type, trigonal equatorial. One trigonal axis and a plane of symmetry perpendicular to it.

No example yet found.

Class 20. Ditrigonal-pyramidal class. Type, ditrigonal polar. One trigonal axis and 3 symmetry planes intersecting in it.

Example—Tourmaline, $\text{H}_6\text{Na}_2\text{Fe}_4\text{B}_6\text{Al}_3\text{Si}_{12}\text{O}_{63}$.

Class 21. Ditrigonal-scalenohedral class. Type, dihexagonal alternating. One trigonal axis and 3 symmetry planes intersecting in it; also 3 digonal axes in the plane perpendicular to the trigonal axis.

Example—Calcite, CaCO_3 .

Class 22. Ditrigonal-bipyramidal or holohedral class. Type, ditrigonal equatorial. A trigonal axis and 3 symmetry planes intersecting in it; also a symmetry plane and 3 digonal axes all perpendicular to the trigonal axis.

Example—Benitoite, $\text{BaTiSi}_3\text{O}_9$.

VI. *The Hexagonal System.*

Characterised by 3 equal axes a, a, a , lying in the same plane and inclined at 60° to each other, and a fourth axis, the vertical axis c , perpendicular to them and unequal to them in length. This latter is a hexagonal axis of symmetry.

$$a : a : a : c = 1 : 1 : 1 : ?$$

Includes 5 classes.

Class 23. Pyramidal class. Type, hexagonal polar. One hexagonal axis of symmetry.

Example—Strontium antimonyl tartrate, $\text{Sr}(\text{SbO})_2(\text{C}_4\text{H}_4\text{O}_6)_2$.

Class 24. Trapezohedral class. Type, hexagonal holoaxial. One hexagonal axis and 6 digonal axes in the plane perpendicular to it.

Example—Barium antimonyl tartrate with pot. nitrate, $(\text{C}_4\text{H}_4\text{O}_6)_2(\text{SbO})_2\text{Ba} \cdot \text{KNO}_3$.

Class 25. Bipyramidal class. Type, hexagonal equatorial. One hexagonal axis and one plane of symmetry perpendicular to it.

Example—Apatite, $\text{Ca}_5\text{F}(\text{PO}_4)_3$.

Class 26. Dihexagonal-pyramidal class. Type, dihexagonal polar. One hexagonal axis and 6 symmetry planes intersecting in it.

Example—Greenockite, CdS .

Class 27. Dihexagonal-bipyramidal or holohedral class. Type, dihexagonal equatorial. One hexagonal axis and 6 symmetry planes intersecting in it; also a symmetry plane and 6 digonal axes perpendicular to the hexagonal axis.

Example—Beryl, $\text{Be}_3\text{Al}_2(\text{SiO}_3)_6$.

VII. *The Cubic System.*

Characterised by 3 equal rectangular axes, a , b , c , all three written, therefore, as a .

$$a : a : a = 1 : 1 : 1 \qquad \alpha = \beta = \gamma = 90^\circ.$$

Includes the 5 classes with 3 rectangular digonal or tetragonal axes of symmetry and 4 trigonal axes of symmetry.

Class 28. Tetrahedral-pentagonal-dodecahedral class. Type, tesseral polar. 3 equal rectangular digonal axes and 4 equal trigonal axes equally inclined to the others.

Example—Barium nitrate, $\text{Ba}(\text{NO}_3)_2$.

Class 29. Pentagonal-icositetrahedral class. Type, tesseral holoaxial. 3 equal rectangular tetragonal axes; 4 trigonal axes equally inclined to the tetragonal ones; also 6 digonal axes each bisecting the angle between two of the tetragonal axes.

Example—Cuprite, Cu_2O .

Class 30. Dyakis-dodecahedral class. Type, tesseral central. 3 equal rectangular digonal axes and 4 equal trigonal axes equally inclined to them; also 3 planes of symmetry perpendicular to the digonal axes.

Example—Pyrites, FeS_2 .

Class 31. Hexakis-tetrahedral class. Type, ditesseral polar. 3 equal rectangular digonal axes and 4 equal trigonal axes equally inclined to them; also 6 symmetry planes bisecting the angles of the symmetry planes of class 30.

Example—Tetrahedrite, Cu_3SbS_3 .

Class 32. Hexakis-octahedral or holohedral class. Type, ditesseral central. 3 equal rectangular tetragonal symmetry axes, 4 equal trigonal axes equally inclined to them, and 6 digonal axes each bisecting the angle between two of the tetragonal axes; also 3 symmetry planes perpendicular to the tetragonal axes, and 6 symmetry planes bisecting the angles between the first 3 symmetry planes. This is the highest possible combination of elements of symmetry.

Example—Fluorspar, CaF_2 .

CHAPTER XI

THE CUBIC SYSTEM, ALSO CALLED REGULAR OR TESSERAL.

Three equal rectangular crystallographic axes. Symmetry elements characteristic of system: 3 rectangular digonal or tetragonal axes of symmetry and 4 trigonal axes of symmetry.

THIS highest system of crystal symmetry consists of 5 classes, the lowest of which possesses the essential minimum elements of symmetry mentioned above, namely, 4 trigonal axes and 3 digonal axes of symmetry, the latter arranged at right angles to each other and coincident with the crystallographic axes. These elements of symmetry, and the repetitions of the general pole (hkl) to which by their operation they give rise, are shown in the stereographic projection Fig. 71. The poles

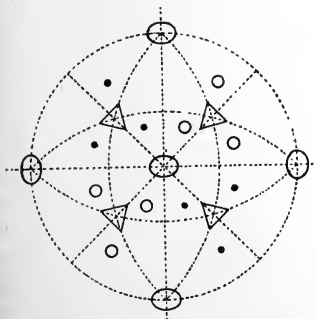


FIG. 71.—Class 28.

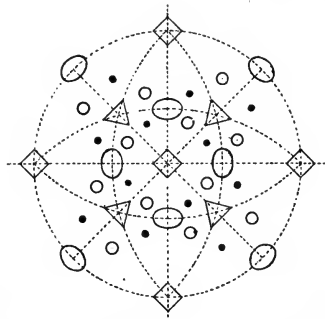


FIG. 72.—Class 29.

or points of emergence of the trigonal axes are indicated by small triangles, and the poles of the digonal axes by little ellipses. The facial poles in the upper hemisphere above the plane of the paper are indicated by dots as usual, and those of the lower hemisphere by miniature rings. If the digonal axes are replaced by tetragonal ones, the poles of emergence of which from the sphere may be indicated by little squares, we have the second of the five classes, the projection of which is given in Fig. 72, the figure being then seen to be symmetrical also about 6 new digonal axes arranged diagonally. In both figures the

crystallographic axes are indicated by the two rectangular diameters parallel to the edges of the page, and by the centre, which represents in the projection the diameter of the sphere perpendicular to the page.

In the tetragonal system to be considered in Chapter XIII., which most closely resembles the cubic system, the characteristic of the system is a single tetragonal axis. Similarly, the trigonal system described in Chapter XXIII. possesses a single trigonal axis, and the hexagonal system considered in Chapter XXI. is endowed with a single hexagonal axis of symmetry. It is the possession of this single vertical axis of trigonal, tetragonal, or hexagonal symmetry that unites the three systems named after these axes in the optically uniaxial class of crystals, conferring as it does special optical properties in the vertical direction. But the cubic system is unique in possessing more than a single axis of higher symmetry than digonal. For the degree of symmetry goes up by leaps and bounds the moment we introduce more than one such axis of higher symmetry, so that even the lowest class of the cubic system becomes automatically possessed of four trigonal axes by the mere act of attempting to add a second one; and the lowest class but one comes into possession of three tetragonal axes when we attempt to add one more to the single one of the tetragonal system.

The lowest class of the cubic system, class 28, is distinguished, therefore, by having more than one trigonal axis of symmetry, a single trigonal axis being the characteristic of the trigonal system. The addition of one more can only be effected in such a manner as to produce by the operation of the pair a solid which is really symmetrical about four such trigonal axes. Four, therefore, is the smallest number of trigonal axes that we can have in operation, the moment we step above a single trigonal axis. These four are the normals to the regular octahedron or tetrahedron faces, their angle of mutual intersection being the same as the crystallographic angle over the edges of the tetrahedron, $109^{\circ} 28'$. The poles of emergence of these four trigonal axes are indicated in Fig. 71 by the little triangles, occupying the positions of the octahedron poles. If we take any general facial pole (hkl) in the upper right front octant, rotation for 120° and 240° round the trigonal axis emerging in this octant will cause the facial pole to be repeated twice more at 120° intervals in the octant, making three facial poles altogether in that octant, as indicated by the three dots. If now we rotate the whole for 120° about an adjacent trigonal axis, say that emerging in the left front octant, the three poles of the upper right front octant become coincident with three analogous ones in the lower left front octant, as indicated by the three little rings. This can be readily verified with a model of an octahedron through the centres of the four pairs of parallel faces of which long needles have been perpendicularly inserted, for on rotating the octahedron for 120° , that is until the original aspect is restored, about the needle corresponding to the trigonal axis in question, the upper right front face will be seen to become the lower left front one.

Similarly, a further rotation for 120° about either this same trigonal axis or the trigonal axis emerging in the upper right back octant reproduces the original three dotted poles in the upper left back octant, as again indicated by three dots, and rotation for 120° in the inverse way, or 240° in the same way, about the fourth trigonal axis emerging in the upper left back octant will repeat the first three dotted poles as ring-poles in the lower right back octant. All these effects can be verified in the same manner by the rotation of the octahedron about the corresponding needles.

The total effect is thus to produce a 12-faced solid, the tetrahedral pentagonal dodecahedron, which has two sets, of three faces each, in each hemisphere, alternately arranged in the two hemispheres as regards octants, as will be clear from the projection (Fig. 71). It will be obvious, on inspection of this figure, that there is further symmetry about three mutually rectangular digonal axes, which are identical with the axes of the cube. For, half a revolution about any one of the three will end with the figure being of precisely the same aspect as it was before the rotation.

Class 29 is produced by adding to these symmetry elements of class 28 six more digonal axes parallel to the normals to the faces of the rhombic dodecahedron (a well-known solid which will be fully described a few pages hence), that is, bisecting the angles between the cube normals. Fig. 72 shows the conditions, and it will be observed that the six new axes of symmetry occupy the diagonal diameters of the sphere. Moreover, it will be observed that the three old digonal axes are now indicated by squares, for the figure is such that rotation for 90° about these cube normals recovers the same aspect again, so that these axes are no longer merely digonal but tetragonal. Thus here we jump, as it were, from the single tetragonal axis of the tetragonal system to a class of symmetry possessing three such axes. The faces of the solid are double the number of those of class 28, being 24, every pole of class 28 having in class 29 a fellow in the other hemisphere, by rotation for 180° about one of the six new digonal axes. It is the pentagonal icositetrahedron.

This is the maximum combination of symmetry axes possible to a crystal, and any further advance in symmetry must be brought about by the addition of planes of symmetry. We then obtain three further classes, 30, 31, and 32.

Class 30, represented in stereographic projection by Fig. 73, is produced by adding to class 28 three symmetry planes parallel to the faces of the cube, indicated by the primitive circle and two diameters being drawn in continuous lines. By repetition about these three planes of symmetry the 12 facial poles of Fig. 71 become doubled, each one having a parallel fellow added

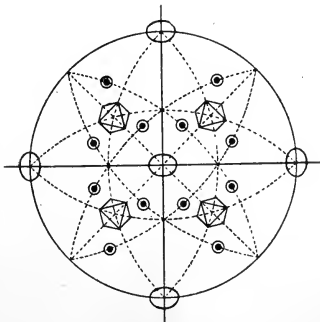


FIG. 73.—Class 30.

at the other side of the centre, so that all the facial-polar positions are occupied by ringed dots. The solid produced is the dyakis dodecahedron. A further effect of the addition of the three planes of symmetry is to render the four trigonal axes also hexagonal axes of compound or reflective symmetry, and the little triangles are surrounded by hexagons to indicate the fact.

Class 31 is formed by adding to the symmetry elements of class 28 six planes of symmetry parallel to the faces of the rhombic dodecahedron, that is, planes at 45° to the cube faces, the angles of intersection of which they consequently bisect. The effect is to add also 12 more faces to the 12 of class 28, but this time by adding three more facial poles to the three already present in each octant, in the case of the four octants occupied by poles in class 28, producing as the solid the hexakis tetrahedron. Moreover, the three cube normals become tetragonal axes of compound symmetry, hence the ellipses indicating their points of emergence from the sphere are interpenetrated by squares in Fig. 74.

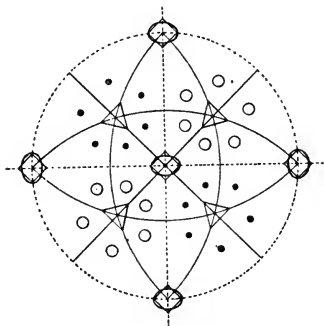


FIG. 74.—Class 31.

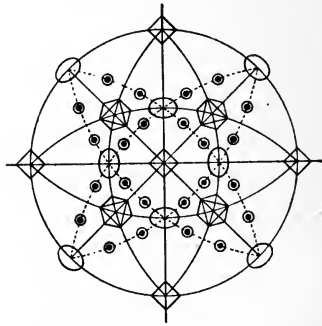


FIG. 75.—Class 32.

Finally, by combining all the elements of symmetry present in all the other four classes, or, which is the same thing, those of classes 29, 30 and 31, we arrive at the class of highest possible crystal symmetry, class 32, indicated in stereographic projection by Fig. 75. In this class we have present the **three tetragonal axes of symmetry**, coincident with the crystallographic axes, the **four trigonal axes of symmetry**, which are also hexagonal axes of compound symmetry, and the **six digonal axes of symmetry**, bisecting the angles of intersection of the tetragonal ones; also **three rectangular planes of symmetry** parallel to the faces of the cube and intersecting in the crystallographic axes and the three tetragonal axes, and **six diagonal planes of symmetry** bisecting the angles of the cube planes, that is, parallel to the faces of the rhombic dodecahedron, intersecting each other in pairs in the crystallographic axes, and intersecting the 3 cubic planes of symmetry in the six digonal axes. The number of faces and of the corresponding facial poles is now again doubled to 48, the solid being the hexakis octahedron, there being 6 poles in each of the 8 octants, those of the upper hemisphere

immediately above those of the lower, so that we have 24 dots each surrounded by a ring.

The general form $\{hkl\}$ of each of these 5 classes, the poles of which are shown on the 5 stereographic projections, Figs. 71 to 75, consists of faces having the same index numbers, when the 3 cube planes are taken as axial planes and the edges of the cube, or, which is the same thing, the cube normals are taken as the axial directions; it is assumed also that the regular octahedron is the parametral form, its faces being perpendicular to the trigonal axes of symmetry, so that the axial lengths are equal.

Having thus traced the evolution of the 5 classes of the cubic system, we are in a favourable position for considering these classes in detail, and studying the nature of the forms possible to each of them.

Class 32.—Class of Highest Crystal Symmetry. Hexakis-Octahedral Class. Holohedral Class of the Cubic System. Type, Ditesseral Central.

This holohedral¹ class of the cubic system of crystals, the elements of symmetry of which have just been enumerated, and which together represent the maximum amount of symmetry developed by crystals, comprises seven simple forms, each being a well-known geometrical solid. They are, respectively, the cube (the axial-plane form), octahedron (the parametral form), the rhombic dodecahedron (with faces parallel to the diagonal planes of symmetry), the hexakis octahedron (the general form $\{hkl\}$), the icositetrahedron, the triakis octahedron, and the tetrakis hexahedron.

(1) The fundamental form, the three pairs of faces of which are chosen as the axial planes intersecting each other at right angles (exactly $90^{\circ}0'$), is the **cube** or regular **hexahedron**, Fig. 76. The normals to its faces, parallel to its edges, are both the three crystallographic rectangular equal axes and the three tetragonal symmetry axes. The form symbol is $\{100\}$, for each pair of faces cuts one of the crystallographic axes only, and is parallel to the other two, so that the intercepts upon those two are ∞ and the indices consequently 0. As regards the six separate sets of facial indices, those of the front face meeting the axis a are identical with those of the form, namely (100). The indices of the parallel face behind are $\bar{1}00$. Those of the two faces meeting the axis b right and left of the centre are respectively (010) and (0 $\bar{1}$ 0), while those of the top and bottom horizontal faces meeting the axis c are respectively (001) and (00 $\bar{1}$). All the faces being of similar value as regards the symmetry, and this being of such a perfect character, it is immaterial which of the three pairs of faces are regarded as the horizontal ones, and which as the front and back ones. Simple cubes are frequently found of rock salt, sodium chloride NaCl, fluor-spar, calcium fluoride CaF₂, and galena, lead sulphide PbS.

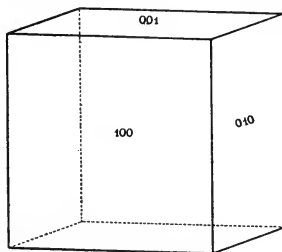


FIG. 76.—The Cube.

¹ It is convenient to retain the term "holohedral" to distinguish the class of crystals exhibiting the full symmetry of a system.

(2) The parametral form is the **octahedron** $\{111\}$, shown in Fig. 77. Each of its eight faces forms an equilateral triangle. All its edges are similar, and the angle across any one of them, that is, the angle between the normals to any two adjacent faces, is $70^\circ 31' 44''$. The four normals are the four trigonal symmetry axes of the cubic system; the twelve equal edges lie (four in each, making a square) in the three axial planes, the three principal planes of symmetry; and each parallel pair of edges is perpendicular to one of the six digonal axes of symmetry of the system, and parallel to one of the six diagonal planes of symmetry. The eight faces have the following symbols: the face in the upper right front octant (111), that in the upper left front octant ($\bar{1}\bar{1}1$), that in the upper left back octant ($\bar{1}\bar{1}\bar{1}$), that in the upper right back octant ($1\bar{1}\bar{1}$), that in the lower right front octant ($11\bar{1}$), that in the lower left front octant ($1\bar{1}\bar{1}$), that in the lower left back octant ($\bar{1}\bar{1}\bar{1}$), and that in the lower right back octant ($1\bar{1}\bar{1}$). The six solid angles are all similar, and are formed by the meeting of four equal edges at angles of 60° .

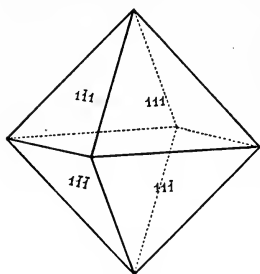


FIG. 77.—The Octahedron.

Spinel, aluminate of magnesia, MgAl_2O_4 , and magnetite, Fe_3O_4 (probably ferrous ferrate $\text{Fe}^{2+}\text{Fe}^{3+}_2\text{O}_4$), are examples of minerals crystallising in octahedra.

(3) The next simplest form is the **rhombic dodecahedron** $\{110\}$, which is represented in Fig. 78. As its name implies, it is composed of twelve congruent faces of rhombic shape, each being a regular rhombus, the plane angles of which are equal to the octahedron angle, $70^\circ 31' 44''$, and its supplement, $109^\circ 28' 16''$. Each face is inclined to two of the crystallographic axes only, and is parallel to the third axis. It meets the two axes in question at 45° , thus cutting off equal lengths from them. Hence the intercepts are always ∞ for one axis and equal definite lengths along the two others, so that the indices of each face are composed of one cipher and two ones. The four faces parallel to the vertical axis c have the indices respectively (110) right front, $(\bar{1}\bar{1}0)$ left front, $(\bar{1}\bar{1}0)$ left back, and (110) right back. The four faces parallel to the back-to-front a axis are respectively (011) right upper, and $(0\bar{1}\bar{1})$ right lower; and $(0\bar{1}1)$ left upper, and $(01\bar{1})$ left lower. The remaining four faces parallel to the lateral b axis are (101) front upper, and $(10\bar{1})$ front lower; and $(\bar{1}01)$ back upper, and $(\bar{1}0\bar{1})$ back lower. The angle between the normals to every adjacent pair of faces is 60° , while the angles between the four faces parallel to any one and the same axis are all 90° . The most interesting fact about the rhombic dodecahedron, however, is that its six pairs of parallel faces are parallel to the six diagonal planes of symmetry of the cubic system, so that it may be said that the dodecahedron is symmetrical to its own faces. Each pair of faces is consequently also normal to one of the six digonal axes of the system. Another interesting property is that, like the cube, an indefinite number of rhombic dodecahedra will pack together in space without leaving any interstices, provided all are of the same size. The solid angles of the dodecahedron are of two kinds, namely six equiangular four-faced quoin angles and eight equiangular three-faced ones, the points of the former being at the ends of the three principal axes, and those of the latter occupying the centres of the octants. Garnet, $\text{R}_3\text{R}_2''(\text{SiO}_4)_3$, is very generally found in rhombic dodecahedra.

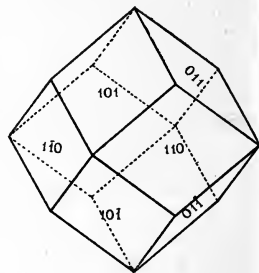


FIG. 78.—The Rhombic Dodecahedron.

The three forms now dealt with, the cube, octahedron, and rhombic dodecahedron, have one property of great importance in common, which distinguishes them from the other four holohedral forms of the cubic system, namely, that the symmetry of the

system only permits of there being one cube, one octahedron, and one rhombic dodecahedron; whereas several distinct representatives of the other four forms are possible and are actually observed in practice.

Before passing on to the consideration of the other four holohedral forms of the cubic system, with their variable character, it will, therefore, be well to refer to the **combinations** of the three permanent forms now studied. Two such combinations of all three forms are shown in Figs. 79 and 80, the cube faces being marked *c*, the octahedron faces *o*, and the rhombic dodecahedron faces *d*. Starting from the cube as the primitive form, it will be observed that the octahedron truncates (replaces symmetrically) the corners of the cube, while the rhombic dodecahedron truncates its edges. A crystal showing all three forms among which the cube predominates, as represented in Fig. 79, appears with both the corners and the edges of the latter truncated; while a combination of all three such as is shown in Fig. 80, in which the

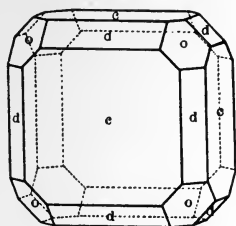


FIG. 79.

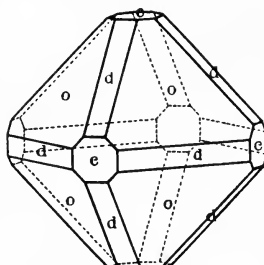


FIG. 80.

Combinations of Cube, Octahedron, and Rhombic Dodecahedron.

octahedron predominates, appears with all the solid octahedral angles truncated by small cube faces, and all the octahedral edges truncated by narrow faces of the dodecahedron. These modifications of edges and solid angles, by a plane making equal angles with the adjacent faces, afford excellent examples of "**truncation**."

Most crystals exhibit more than one form, and combinations of these three simple forms are very common, and indeed some of the other four holohedral forms of the cubic system are often present with them. A practical example, a garnet crystal, showing faces of the rhombic dodecahedron and the icositetrahedron, will be described in the next chapter and its measurement followed.

Before leaving the consideration of these three important simple forms it is advisable that their stereographic projection should be given and thoroughly understood, as a necessary basis of operations for practical work on cubic crystals. It is given in Fig. 81. The poles of the cube faces are marked *c*, those of the octahedron *o*, and those of the rhombic dodecahedron *d*. The primitive circle is, of course, the plane of the crystallographic axes *a* and *b*, the third axis *c* perpendicular to the plane of the paper being represented by the pole in the centre. This

pole is also that of the pair of parallel horizontal cube faces, the poles of the two pairs of vertical faces being at the extremities of the axes a and b on the primitive circle.

The poles of the rhombic dodecahedron are next found as follows. Four of them are situated at the points midway between, that is, 45° from, the poles of the cube on

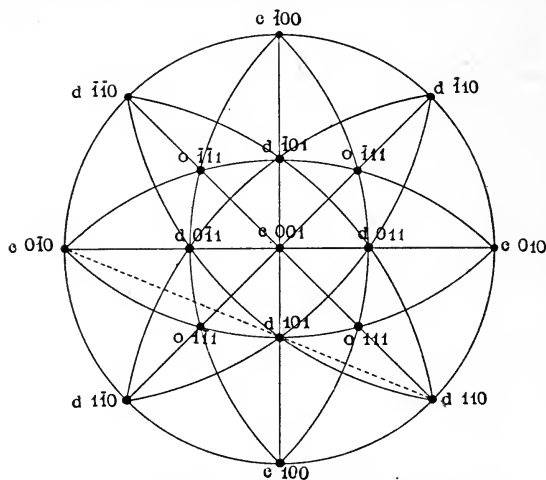


FIG. 81.—Stereographic Projection of Cube, Octahedron, and Rhombic Dodecahedron.

the primitive circle. The remaining eight are situated at the similar 45° positions on the great circles the projections of which are the diameters which represent also the axes a and b and join the poles of the cube faces. To find the projections of the dodecahedron poles on these diameters, one operation only is sufficient, and it will incidentally also afford us the means of locating the poles of the octahedron. The operation required is an application of the rule given on p. 83 in Chapter VI., namely, to find first the pole of the zone, and then

to join it to a point marked off along the primitive circle at the required angle (here 45°) from the end of the diameter representing the zone. To find, for instance, the pole (101) on the zone circle represented by the vertical diameter; the pole of the zone circle may be either extremity of the horizontal diameter on the primitive circle. Let it be the left extremity which is taken, coincident with the facial pole (010) of the cube. The 45° points have already been marked off along the primitive circle in finding the poles of the dodecahedron which lie on that circle; we have, therefore, merely to join the pole (010) to the pole of one of the two dodecahedron faces already found in the right primitive semicircle, say (110). The line thus drawn cuts the vertical diameter, representing the zone circle on which we desire to find one of the dodecahedron poles, at the particular dodecahedron pole required, (101). From the centre, with the distance between the poles (001) and (101) as radius, we have only to mark off the three other similar positions, one ($\bar{1}01$) above the centre along the same zone circle, and the other two (011) and ($0\bar{1}1$) at the symmetrical positions on the zone circle represented by the horizontal diameter. Each of the four positions thus found as the projections of dodecahedron faces represents two faces, one in the upper hemisphere, marked by a dot, and another vertically beneath it in the lower hemisphere, which might, if desired, be marked by a miniature ring surrounding the dot, and having similar indices except that the c index is negative. If we now draw circular arcs through these four poles just found and the poles of both cube and dodecahedron on the primitive circle, so as to connect up all the possible zones, as shown in Fig. 81, we find that the arcs terminating at the cube poles intersect each other in pairs on the diagonal diameters, and these points of intersection of three zones are the poles of the octahedron faces. This can be immediately proved by calculating the angle between any two of these octahedron poles on the projection, say between (111) and ($1\bar{1}1$), when it will be found to be the accepted octahedron angle. The calculation is very simple,

if we consider the half angle $(111):(101)$. For in the triangle made by this arc with the central pole (001) the angle at (001) is 45° , the arc $(001):(101)$ is also 45° , and the angle at (101) is a right angle. Constructing a Napierian diagram, as shown in Fig. 82, we have by Napier's rules:—

$$\begin{aligned}\cos o &= \sin 45^\circ \cos 45^\circ; \sin do = \cot o \tan 45^\circ. \\ &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \quad \text{Log. cot } 60^\circ \quad \bar{1} \cdot 76144 \\ o &= 60^\circ \quad \text{Log. tan } 45^\circ \quad 0 \cdot 00000 \\ &\quad \text{Log. sin } do \quad \bar{1} \cdot 76144 \\ &\quad do = 35^\circ 15' 9''\end{aligned}$$

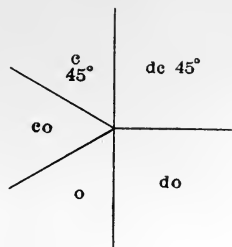


FIG. 82.

It will be shown directly to be instructive to know the value of this angle at o . But if we had not needed it we might have calculated the angle do more directly from the Napierian diagram thus:

$$\sin 45^\circ = \tan do \cot 45^\circ, \text{ or } \tan do = \sin 45^\circ \tan 45^\circ.$$

$$\begin{aligned}\text{Log. sin } 45^\circ &\quad \bar{1} \cdot 84949 \\ \text{Log. tan } 45^\circ &\quad 0 \cdot 00000 \\ \hline \text{Log. tan } do &\quad \bar{1} \cdot 84949 \\ do &= 35^\circ 15' 9''\end{aligned}$$

Hence the whole arc oo , $(111):(1\bar{1}1)$, twice do , is $70^\circ 31' 8''$, which is the angle between the normals to any two adjacent octahedron faces.

The angular value of the arc co , $(001):(111)$, may also be found from the same triangle:

$$\begin{aligned}\cos co &= \cos do \cos dc \\ &= \cos 35^\circ 15' 9'' \cos 45^\circ \\ \text{Log. cos } 35^\circ 15' 9'' &\quad \bar{1} \cdot 91195 \\ \text{Log. cos } 45^\circ &\quad \bar{1} \cdot 84949 \\ \hline \text{Log. cos } co &\quad \bar{1} \cdot 76144 \\ co &= 54^\circ 44' 1''\end{aligned}$$

Thus the arc $c=(001):o=(111)$ is $54^\circ 44' 1''$, the half supplement of the octahedron angle. It is also the complement of the angle do just found, for $do=(110):(111)$ is the same as the latter, $(101):(111)$, as is proved by this calculation. The angle at $o=(111)$ between the two arcs just calculated has also been shown above to be 60° . As each of these arcs represents an edge, the arc co corresponding to that formed by the truncation of the upper right front octahedron face by the top cube face, or the parallel right front horizontal edge of the octahedron, and the arc oo to the front upper edge of the octahedron, this angle of 60° is the angle between two adjacent edges of the octahedron face, that is, the face is an equilateral triangle.

Hence we now know the whole of the angular dimensions of every part of the stereographic projection, both interpoles (interfacial) arcs and interzonal angles.

As we now thoroughly understand, therefore, the stereographic projection of these simple holohedral forms of the cubic system, we are in a more favourable position to appreciate the expression of the symmetry of this highest class 32 of the cubic system, as given in Fig. 83, which is Fig. 75 repeated here for convenience of reference.

The poles of emergence on the surface of the sphere of projection of the various axes of symmetry are indicated in the manner proposed by

Gadolin. The 3 tetragonal axes, identical with the crystallographic axes, are indicated by little squares, the 4 trigonal axes, which also act as hexagonal axes of compound symmetry, are represented by little triangles surrounded by hexagons, and the 6 digonal axes are marked by elliptical rings. The 3 tetragonal poles are identical with those of the cube faces, the 4 trigonal with those of the octahedral faces, and the 6 digonal with those of the rhombic dodecahedron. The small ringed dots also given on the projection are the poles of the general holohedral form $\{hkl\}$ of the system, the dot representing the face on the upper hemisphere and the ring that on the lower one.

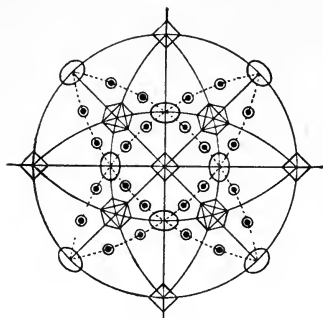


FIG. 83.—Class 32.

The actual hexakis octahedron shown is the commonest one $\{321\}$. This form is the next to be considered, but before passing to it reference must be made to the attempt in Fig. 84 to exhibit the axes and planes of symmetry of the cubic system in perspective about a cube. Owing to their number, it is not easy to do so with great clearness in a single diagram (except as just shown, in stereographic projection), but by marking the three types of symmetry axes with the Gadolin symbols as in the case of the stereographic projection, and by emphasising the three rectangular planes of symmetry coincident with the crystallographic axial planes by means of thicker lines, to distinguish them from the six diagonal planes of symmetry drawn in thin lines, the elements of the perfect symmetry of class 32 of the cubic system may be grasped. A comparison with Fig. 83 will be found very helpful, and the two figures are placed near

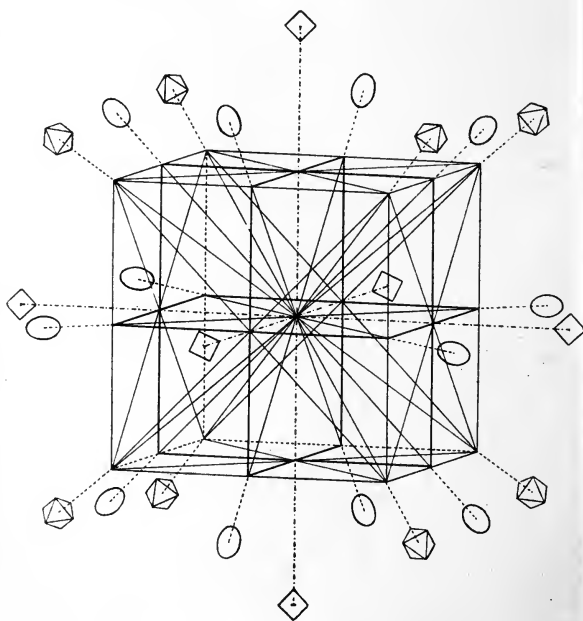


FIG. 84.—Axes and Planes of Cubic Symmetry.

of symmetry coincident with the crystallographic axial planes by means of thicker lines, to distinguish them from the six diagonal planes of symmetry drawn in thin lines, the elements of the perfect symmetry of class 32 of the cubic system may be grasped. A comparison with Fig. 83 will be found very helpful, and the two figures are placed near

to each other for the purpose of facilitating such a comparison. The high order of symmetry which is displayed in the cubic system will be very apparent from this diagram.

(4) We now come to the consideration of the general holohedral form of the system above referred to, $\{hkl\}$, the **hexakis octahedron**. As its name implies, it may be considered as an octahedron each face of which is replaced by 6 others, thus producing a solid of 48 faces. Or it may be more generally regarded as the figure demanded by the full symmetry of the cubic system when any one face (hkl) is assumed to exist. That is, any face (hkl), cutting off unequal intercepts from the three axes, requires to be associated with 47 others in order to fulfil the conditions of full cubic symmetry. The hexakis octahedron $\{321\}$ is shown in Fig. 85.

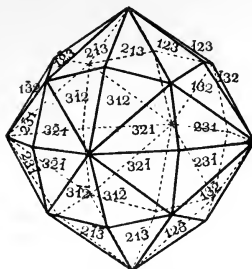


FIG. 85.—The Hexakis Octahedron $\{321\}$.

Each facial pole occupies one of the continuous-line triangles of the stereographic projection shown in Fig. 83. The faces themselves, when all are equally developed, are shaped as scalene triangles, the edges of intersection which form these triangles lying in the various planes of symmetry. There are thus three kinds of edges, the triangles being all similar, and there are consequently three correspondingly different sets of angles, over these edges, between the normals to adjacent faces. These three angles as measured goniometrically are given below for the two commonest representatives of the form, $\{321\}$ and $\{421\}$.

		Angle over Longest Edge.	Angle over Shortest Edge.	Angle over Medium Edge.
For {321}	.	21° 47'	21° 47'	31° 0'
For {421}	.	17 45	35 57	25 13

The edges of medium length lie in the three axial planes of symmetry, and the sections of the crystal made by these planes are octagons of equal edges but not regular, the angles being of two alternating kinds.

In the representation of the form {321} in Fig. 85 the indices of the various faces on the front hemisphere are inserted, in order to indicate the character of the permutations of the three numbers which occur. This particular hexakis octahedron has the peculiarity that, as indicated by the above angular values, the angles over two kinds of edges, the longest and the shortest, are equal.

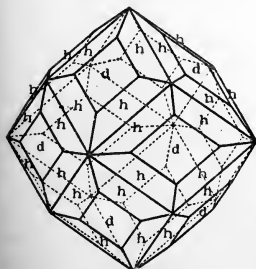


FIG. 86.—Combination of Hexakis Octahedron and Rhombic Dodecahedron.

The solid angles are of three kinds, six eight-faced the points of which are on the crystallographic axes, eight six-faced meeting in the four trigonal axes, and twelve four-faced meeting in the 6 digonal axes of symmetry.

All faces having three unequal indices belong to this form.

The hexakis octahedron is scarcely known in its simple form, for, although many diamonds appear to be hexakis octahedra, their faces are usually so rounded that it is not possible to verify the presence of the form by measurement. But it is often found in combination with other of the holohedral forms of the cubic system, for instance in spinel and garnet. Fig. 86 shows a garnet exhibiting a combination of the hexakis octahedron {321}, with the letter *h*, with the rhombic dodecahedron {110}, and *d*. The edges *hd* are parallel to the longer edges *hh* of

the hexakis octahedron $\{321\}$, the rhombic dodecahedron faces replacing the four-faced solid angles of this particular hexakis octahedron in a symmetrical manner; the poles of $\{321\}$ are consequently situated on the dodecahedral arc zones, those shown in dotted lines in Fig. 83, that is, the faces of $\{321\}$ belong to the dodecahedral zones.

(5) When two of the index numbers in the symbol $\{hkl\}$ become equal the number of permutations of the indices, and therefore the number of faces comprising the form, is reduced to half, namely, 24. There are two new varieties of simple forms thus produced, in which the two equal indices are respectively less or greater than the third. The former case is that of the icositetrahedron, the latter that of the triakis octahedron.

The **icositetrahedron**, the commonest representative of which, $\{211\}$, is shown in Fig. 87, possesses only two kinds of edges, half being of each kind. There are 24 longer edges lying in the axial planes, of which every four make up a solid angle the point of which is on the crystallographic axis, and 24 shorter edges lying in the 6 diagonal planes of symmetry, of which the eight sets of three each meet in a solid angle on the trigonal axes of symmetry. The faces also group themselves into six other sets of four each, two longer and two shorter edges in each set, meeting in the digonal axes of symmetry. The icositetrahedron may be regarded as derived from the hexakis octahedron by the fusion into a single face of the two faces meeting in each of the longest of the three kinds of edges possessed by that 48-faced solid.

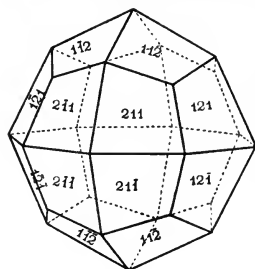


FIG. 87.
The Icositetrahedron $\{211\}$.

Leucite and garnet exhibit characteristically the icositetrahedron $\{211\}$. An icositetrahedral crystal of garnet showing also faces of the rhombic dodecahedron will be fully worked through as a practical goniometrical example in the next chapter.

The angles met with in measurements on the goniometer, between the normals to adjacent faces of the two commonest icositetrahedra, are as under:

	Over the Longer Edges.	Over the Shorter Edges.
For $\{211\}$. . .	48° 11'	33° 33'
For $\{311\}$. . .	35 6	50 29

(6) The **triakis octahedron**, of which the forms $\{221\}$ and $\{331\}$ are those most commonly met with, may be regarded as an octahedron every face of which is replaced by a low three-faced pyramid, or a hexakis octahedron the shortest edges of which have disappeared, the two faces on each side of each such short edge having become one face. Its faces are all isosceles triangles, the 12 longer edges of which all lie in the 3 principal planes of symmetry, four in each, and are actually parallel to (or they may be regarded as identical with) the edges of the octahedron. The form $\{221\}$ is shown in Fig. 88. The angle over these edges, between the normals to each pair of faces forming the edge by their intersection, for the form $\{221\}$, is 38° 57', and for the form $\{331\}$, 26° 32'. The other 24 shorter edges intersect in 8 three-faced solid angles, the points of which lie on the axes of trigonal symmetry. The angle over these shorter edges, between the normals to the adjacent faces forming the edge in each case, is 27° 16' for the form $\{221\}$ and 37° 52' for the form $\{331\}$. The triakis octahedron shows also six eight-faced solid angles, the quoins of which are formed by four longer and four shorter

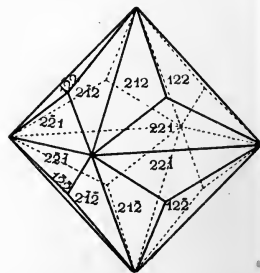


FIG. 88.
The Triakis Octahedron $\{221\}$.

edges, and the points of which are identical with the apices of the octahedron. The diamond sometimes exhibits the form $\{221\}$.

(7) The last holohedral form of the cubic system, the **tetrakis hexahedron**, is produced when one of the indices is zero, that is when each face is parallel to one of the crystallographic axes, and when the other two indices are unlike (if alike the dodecahedron is the solid produced). Its poles all lie on the 3 great circles formed by the intersection of the three axial planes with the sphere, that is, on the primitive circle and the two axial diameters. Its $\{210\}$ representative is shown in Fig. 89. It may be regarded as a cube the faces of which have been replaced by a low four-faced pyramid, or as produced by the disappearance of the medium edges of the hexakis octahedron, the two faces on each side of each such medium edge having coalesced into one. Each face is an isosceles triangle, the 12 longer sides (one in each triangle) being identical with the edges of the cube, and the 24 shorter ones meeting in fours on the crystallographic axes. The goniometrical angle over the longer edges, that is, the angle between the normals to each pair of faces intersecting in a longer edge, in the cases of the two tetrakis hexahedra most frequently met with, has the following values :

For $\{210\}$, $36^\circ 52'$; for $\{310\}$, $53^\circ 8'$.

The angle over the shorter edges is the same in the case of the form $\{210\}$ as it is over the longer edges, namely, $36^\circ 52'$, but in the case of $\{310\}$ it is different, namely, $25^\circ 51'$.

There are also eight six-faced solid angles the apices of which lie on the four trigonal axes, the quoins being composed of three longer and three shorter edges.

The simplest form, $\{210\}$, that represented in Fig. 89, is exhibited by fluor-spar.

Combinations of the three forms last described and the three primary forms are shown in the next three illustrations, Figs. 90, 91, and 92.

Fig. 90 represents a combination of the cube (faces marked *c*), octahedron (*o*), and rhombic dodecahedron (*d*) with the icositetrahedron $\{211\}$, the faces of which are marked

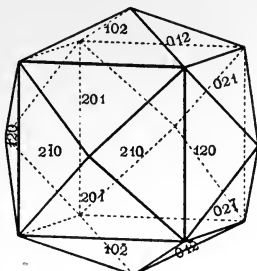


FIG. 89.

The Tetrakis Hexahedron $\{210\}$.

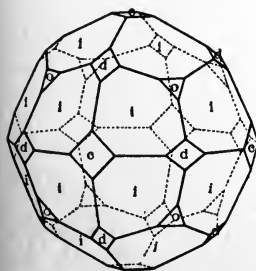


FIG. 90.

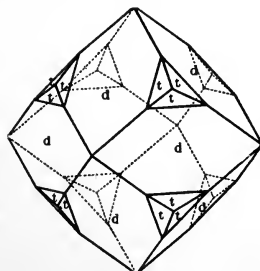


FIG. 91.

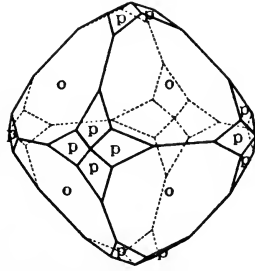


FIG. 92.

Combinations of Holohedral Cubic Forms.

with the letter *i*. The solid angles of the last-named predominating solid are replaced by smaller faces of the three primary forms, and the figure illustrates clearly the fact that the faces of the cube, icositetrahedron, octahedron, and rhombic dodecahedron, always follow each other in the same zone. Also the icositetrahedron $\{211\}$ lies in the same cross-zones with the dodecahedron faces alone, so that the faces of this particular form $\{211\}$ directly replace the dodecahedron edges. Other icositetrahedra do not thus exactly, in parallel fashion, blunt the edges of the dodecahedron, but modify

either the solid angles terminating on the crystallographic axes or those occupying the centres of the octants. Fig. 91 represents a rhombic dodecahedron (d) the three-edged solid angles of which are all modified by small faces, marked t , of the triakis octahedron $\{221\}$. Fig. 92 represents the combination of the octahedron (o) and tetrakis hexahedron $\{210\}$, the solid angles of the preponderating octahedron being all modified by four little faces of the latter form, marked p .

List of Forms in Class 32.

In the following list we have a summary of the forms belonging to class 32, which have now all been fully described.

$\{100\}$ Cube or hexahedron.	$\{hko\}$ Tetrakis hexahedron.
$\{110\}$ Rhombic dodecahedron.	$\{hkk\}$ Icositetrahedron.
$\{111\}$ Octahedron.	$\{hhl\}$ Triakis octahedron.
	$\{hkl\}$ Hexakis octahedron.

Practical Goniometrical Considerations regarding Holohedral Cubic Crystals.—There are three methods of recognising the various forms which are developed on a cubic crystal. We have first the appearance, and it is of immense help in deciphering a complicated crystal showing numerous forms to be able, owing to familiarity with them, to recognise the seven holohedral types now discussed at sight. This would be fairly easy if the various faces were relatively equally developed, as they have been drawn in the illustrations now given. They are represented thus, with a convenient point taken as centre and all the faces drawn in equipoise and equidistant from that centre, in order to exhibit the ideal symmetrical figure in which the planes of symmetry may be at once apparent. They are all drawn to scale, with the axes arranged according to the convention to be explained in Chapter XXV., according to which parallel faces appear as such, in order to avoid the uncertainty which would be introduced as to whether certain faces were intended to be parallel or not, if they were drawn in ordinary perspective. Such ideal crystals are, however, but very rarely seen in actual practice, and in this book it is the author's special purpose to smooth the way to practical work. As the faces, therefore, cannot be expected to be found in exact equipoise, a very mature judgment and familiarity is required to detect the faces of the various forms, and even then, in the end, except perhaps as regards such simple forms as the cube, octahedron and dodecahedron, the second and third methods have to be employed before the crystallographer can be certain as to the forms present.

The second method of recognition of forms is by means of the angles. This is, of course, the only trustworthy test. The angles of all the seven forms, as regards the two commonest representatives in each case where more than one representative is possible, have been given in the present chapter, to simplify the identification. It will be observed that all these angles are fixed by the symmetry, the cubic being the only system in which this is the case.

The third method, that by the position of the pole of any face on the stereographic projection, is bound up with the second method, and the construction of the stereographic projection should be the first act after a few preliminary goniometrical measurements. In the presence

of numerous forms, it becomes difficult to recognise the angles of the various specific forms, owing to intermediate faces of other forms intervening; and several angles require to be added together, in order to obtain the form angle. But when the poles are carefully plotted out on the stereographic projection, it becomes easy to recognise the various forms, both by the actual positions of the poles on the projection and by the angles between them, obtained, if necessary, by adding together the intermediate angles which make them up, as just indicated.

In Fig. 93 is given the stereographic projection of a combination of all the seven holohedral forms of the cubic system. It appears compli-

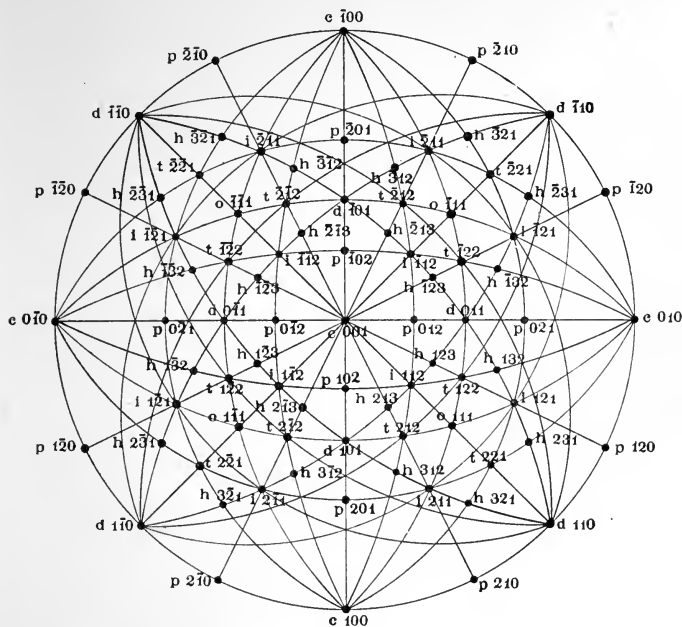


FIG. 93.—Stereographic Projection of the seven Holohedral Cubic Forms.

cated, but as the four quadrants are alike, it is only necessary to study one quadrant. The seven forms are indicated both by their indices and by significant letters, poles marked *c* belonging to cube faces, those marked *o* to the octahedron, *d* stands for the rhombic dodecahedron, *h* for the hexakis octahedron {321}, *i* for the icositetrahedron {211}, *t* for the triakis octahedron {221}, and *p* for the tetrakis hexahedron {210}. The letter *p* is used to designate the last-mentioned form because *t* is already adopted for the triakis octahedron, and because the two hemihedral forms of the tetrakis hexahedron are the pentagonal dodecahedra to be described under class 30. The poles of the faces of these several forms can, therefore, be readily identified.

In the next chapter two actual examples of cubic crystals, a garnet and a crystal of cobaltite, showing several of these forms, will be

goniometrically worked through, their stereographic projections constructed step by step from the measurements, and the determination of the indices of the more complex forms described, together with the mode of carrying out all necessary calculations.

THE SO-CALLED HEMIHEDRAL AND TETARTOEDRAL FORMS OF THE CUBIC SYSTEM.

Class 31.—Hexakis Tetrahedral Class, also called Tetrahedrite Class. Hemihedrism of the Inclined-Face Character. Type, Ditetsseral Polar.

In the crystals of this class the three axial planes are no longer symmetry planes, but the other six diagonal planes of symmetry remain. The effect is to cause the digonal axes also to disappear, and to convert the 3 tetragonal axes into digonal axes of ordinary or simple symmetry, although it will be seen that they still remain tetragonal axes of compound or reflective symmetry. The stereographic projection of the symmetry axes and of the general form (hkl) is given in Fig. 94, which is Fig. 74 repeated here for convenience.

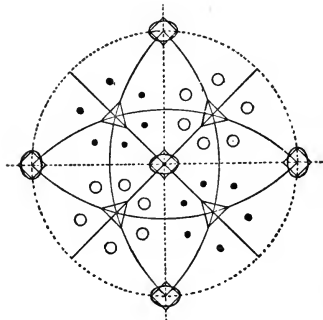


FIG. 94.—Class 31.

The general form in question, called the **hexakis tetrahedron**, is shown in Fig. 95, which represents the form $\{321\}$. It was formerly considered as the hemihedral form of the hexakis octahedron, derived by suppression of the faces in alternate octants. But it is more conformable to the theory of crystals as homogeneous structures to regard it as the solid figure obtained by the repetition of any face (hkl), having 3 different index numbers, about the particular elements of symmetry distinguishing this class. There are really two such solids, a positive and a negative, corresponding to the two possibilities of choice of alternate octants, but the two are seen to be identical when turned about and examined separately, provided the 3 index numbers are the same. No face on this solid has a parallel face. The diamond is occasionally found in hexakis tetrahedra.

The interfacial angles of the hexakis tetrahedron, in the case of $\{321\}$, are the same as in the case of the hexakis octahedron as regards the 3 long and 3 short edges of the six-faced solid angle occupying the middle of the octant and brought over intact from the hexakis octahedron. Over the new hemihedral medium edges the angle is $59^\circ 5'$.

If we next consider what happens to the icositetrahedron, in which we have two equal indices which are less than the third, the common form being $\{211\}$, we find that it becomes the **triakis tetrahedron**, or trigonal dodecahedron, shown in Fig. 96. The two varieties, positive and negative, are found to be identical when examined irrespective of their positions *in situ*. The longer edges are those of the regular tetrahedron, and the angle over these edges, between the normals to the pairs of

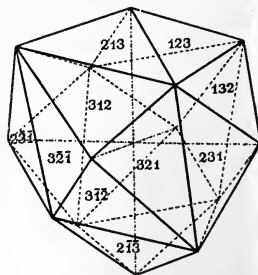


FIG. 95.
The Hexakis Tetrahedron.

faces forming them, in the case of $\{211\}$ is $70^\circ 32'$, the angle of the octahedron. The three faces forming a low pyramid on each tetrahedron face, and meeting in the three shorter edges, have the same inclination over these edges as they have on the original icositetrahedron.

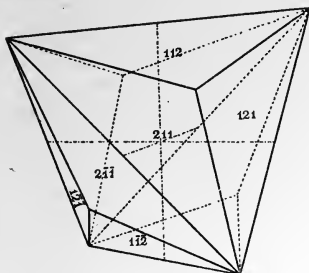


FIG. 96.—The Triakis Tetrahedron.

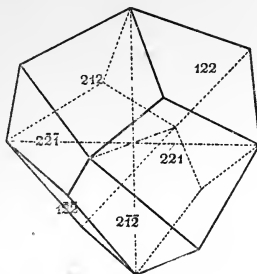


FIG. 97.—The Deltoid Dodecahedron.

The holohedral triakis octahedron, with two equal indices greater than the third, for instance the common form $\{221\}$, becomes the **deltoid dodecahedron**, shown in Fig. 97, under the operation of the symmetry elements of this class. The faces of this solid, as its name implies, are deltoidal four-sided figures, with two pairs of equal sides, the angles of the figure being such that one diagonal joins opposite equal angles and the other unequal ones. The two enantiomorphous forms appear identical

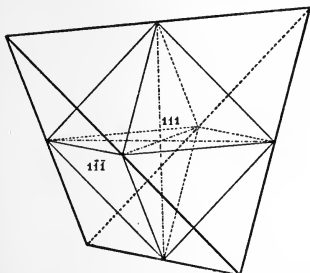


FIG. 98.—Relation of Tetrahedron to Octahedron.

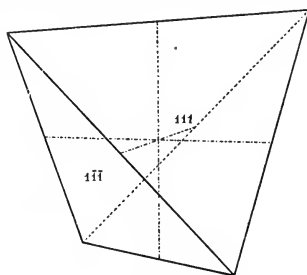


FIG. 99.—The Tetrahedron.

when regarded away from their positions of derivation. In the case of $\{221\}$ the angles over the new hemihedral edges are right angles, the other interfacial angles over the edges of the three-faced pyramids remaining as in the triakis octahedron.

The octahedron under the operation of class 31 symmetry becomes the regular **tetrahedron**, Fig. 99, the manner of origin being illustrated in Fig. 98, in which the tetrahedron is drawn in thick lines and the original octahedron in thin lines. The positive and negative tetrahedra are only distinguishable by their different position when *in situ* as derived. The tetrahedron has six equal edges over which the angle is $109^\circ 28'$, between the normals to the two faces intersecting in the edge. Each face of the tetrahedron is an equilateral triangle.

The tetrahedron is so characteristic a form of grey copper ore, fahl-ore, that this mineral is commonly called tetrahedrite, and the name "tetrahedrite class" is often given to the type of symmetry of class 31. Fig. 100 represents a crystal of tetrahedrite, exhibiting the faces of the positive tetrahedron

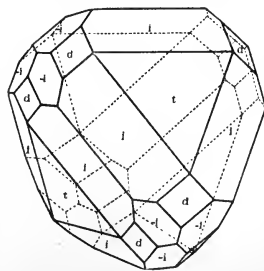


FIG. 100.—Fahl-ore, Tetrahedrite.

$t = \{111\}$, the rhombic dodecahedron $d = \{110\}$, and the positive and negative triakis tetrahedra $i = \{211\}$ and $-i = \{2\bar{1}1\}$. The composition is mainly Cu_3SbS_3 .

To summarise, we have the following forms in this class :

List of Forms in Class 31.

$\{100\}$ Cube or hexahedron.	$\{h k 0\}$ Tetrakis hexahedron.
$\{110\}$ Rhombic dodecahedron.	$\{h k k\}$ Positive triakis tetrahedron ; $\{h \bar{k} k\}$ negative triakis tetrahedron.
$\{111\}$ Positive tetrahedron ; $\{1\bar{1}1\}$ negative tetrahedron.	$\{h h l\}$ Positive deltoid dodecahedron ; $\{h \bar{h} l\}$ negative deltoid dodecahedron.
	$\{h k l\}$ Positive hexakis tetrahedron ; $\{h \bar{k} l\}$ negative hexakis tetrahedron.

The Real Nature of Hemihedrism and Tetartohedrism.—The cube, the rhombic dodecahedron, and the tetrakis hexahedron still retain their full number of faces under the operation of class 31 symmetry. A similar fact will be shown to hold as regards the two former with respect to all the other three classes of the cubic system also.

This fact raises a very interesting question. In discussing the elements of symmetry in Chapter X. it may have been remarked that, while emphasising axes and planes of symmetry as the true elements of symmetry, no mention was made of what has formerly been considered a natural division of crystals into **holohedral**, possessing all the faces of a form, **hemihedral**, in which half the faces of a form are suppressed, and **tetartohedral**, in which a form is represented by only a quarter of the holohedral number of faces. The reason is that our recently acquired full knowledge of the geometry of homogeneous structures has led us to recognise that it is not merely a freak or whim of nature which is occurring, causing the suppression of a half or a quarter of the possible faces of a form, but that it is the definite operation of a distinct and lower combination of elements of symmetry which is occurring ; and that this is owing to the specific stereometric arrangement of the atoms in the molecules of the substance of which the crystal is composed, which only permits them to arrange themselves in accordance with this particular class of symmetry, which latter is high enough to bring the crystal into the system in question, but not into the highest class of that system.

It is evident that this is a much more scientific and trustworthy explanation of the so-called hemihedral and tetartohedral forms, and there is every probability that in it we have now arrived at the true meaning of hemihedrism and tetartohedrism. These two terms may now, therefore, disappear, except as being of historic interest. The term "holohedral," however, may be retained, as it affords a concise expression for the maximum symmetry possible to any system.

An exceptionally interesting prophetic statement was made by Professor Story Maskelyne in his *Morphology of Crystals*, in which, some years before the completion of the work on homogeneous structures, he says with remarkable prescience : "The whole treatment of

crystallographic symmetry on the assumption of planes and axes of symmetry, actual or potential, represents a geometrical abstraction, that needs for its development and due explanation a complete science of position applied to the molecular mass-centres, competent to embrace not merely the relative distribution *inter se*—the *intermolecular* distribution—of the chemical molecules constituting the crystallised substance, but also the *intramolecular* arrangement of the atoms whereof the molecules of the substance are themselves composed. Then the true significance of the ideal planes and axes of symmetry will be understood; and they will assuredly retain a place in the explanation of crystalline symmetry, since they rise into recognition directly from the fundamental principle of rationality of indices and are controlled by its consequences.”

The literal fulfilment of this prediction has now come to pass, and axes and planes of symmetry are no longer mere geometrical abstractions but the proved elements of crystal symmetry, based on the solid groundwork of incontrovertible knowledge concerning the homogeneous partitioning of space, and the possible modes of packing therein, of the chemical molecules in space-lattices and of the atoms of which they are composed in regular point-systems.

The fact that the cube and rhombic dodecahedron are present on a crystal, the other forms of which are those of class 31 (or as we shall also see, of classes 30, 29, and 28 either), does not mean, therefore, that we are to regard these forms on the crystal as holohedral, and therefore as belonging to class 32, and only the others as belonging to class 31. The cube and rhombic dodecahedron (as well as also the tetrakis hexahedron in the case of the class 31 now under discussion) being permitted to have all their faces developed under the strict symmetry conditions exclusively dominating class 31, it is certain that the **whole** crystal, on which the development of these two forms is observed alongside the forms indubitably belonging to class 31, has been developed under the operation of the elements of symmetry solely governing class 31; for it is unthinkable that adjacent precisely similar molecules of one and the same definite chemical substance under like conditions of temperature, pressure and environment, can be deposited according to two different types of homogeneous structure. It is the very essence of the geometrical theory of crystal structure based on homogeneous partitioning, unquestionably supported from the practical experimental standpoint by the results of the author's investigations of the alkali sulphates, selenates, and double salts, that, given these identical conditions, a substance of definite chemical composition is characterised by an equally definite crystalline form. Polymorphism, the only other alternative, only enters into the range of possibility when those conditions are different.

It is, therefore, certain that the cube and rhombic dodecahedron on such crystals as we are discussing are present as forms of class 31 and not as forms of the holohedral class 32. It may also occur, moreover, and indeed very often does occur, that the two kinds of hexakis tetrahedron, triakis tetrahedron, deltoid dodecahedron, or simple

tetrahedron, are both simultaneously developed on a crystal, and to such approximately equal extents that at first sight the holohedral forms from which they are derived appear to be present, combined in all probability with cube or rhombic dodecahedron faces or both, to render the verisimilitude still more complete. But the deception is as a rule only momentary, for close inspection such as accompanies goniometrical investigation usually at once reveals the fact that the faces of the positive and negative complementary forms differ in their power of reflecting light, due to their different physical character, one of the complementary forms being composed of dull faces and the other of brilliant ones. Moreover, the two kinds of faces frequently bear striations or other markings, which are characteristically different in the two cases. Hence, the physical character of the faces affords an excellent criterion of their grouping into positive and negative complementary forms of separate individuality, or of their combined nature as a complete holohedral form. For instance, in the case of the two complementary triakis tetrahedra $i = \{211\}$ and $-i = \{2\bar{1}1\}$ of tetrahedrite, shown on the crystal represented in Fig. 100, the faces of the former are striated perpendicularly to the dodecahedron edge which it truncates, while the latter is striated parallel to the dodecahedral edge.

Further, the faces of the cube and rhombic dodecahedron on crystals of the classes of lower symmetry than class 32 also frequently bear striations of a definitely distinctive character. For instance, the mineral blende, sulphide of zinc, ZnS , belongs to the class 31 under discussion, and although it usually exhibits clearly tetrahedral forms, it is sometimes found in cubes the faces of which, however, exhibit diagonally arranged striations, due to rapidly recurring attempts at producing a tetrahedron, always succeeded by persistent and more extensive growth of the cube. On the other hand, iron pyrites, FeS_2 , belongs to the next class to be considered, class 30, and is frequently also found in cubes; but these are striated parallel to alternate pairs of the edges, that is, to one of the crystallographic axes in each case, in accordance with the symmetry of that class, in which the three axial planes are planes of symmetry, and so that the striations on each face are at right angles to those on the adjacent faces. (See Fig. 105 in description of class 30.) Thus these markings on the faces of the cubes of the two substances indicate that they belong to different classes of the cubic system, and classes which are other than the holohedral class 32.

If natural markings are absent, treatment of the faces by a very minute quantity of a solvent for the crystallised substance will generally produce "etched figures," of different oppositely enantiomorphous shape, on the faces of the two positive and negative forms of the lesser symmetry, and also figures on the faces of the cube and rhombic dodecahedron which are not identical with those produced on a truly holohedral crystal, and which indicate more or less clearly the class of symmetry to which the crystal belongs.

Class 30.—Dyakis Dodecahedral Class, also called Pyrites Class. Parallel-Faced Hemihedrism. Type, Tesseral Central.

The crystals of this class retain the 3 equal digonal axes and 4 equal trigonal axes of class 31, but the 6 diagonal planes of symmetry disappear and instead the crystal is symmetrical about the 3 axial planes, the faces of the cube.

The general form $\{hkl\}$, in accordance with this symmetry, the form which may be regarded as derived from the hexakis octahedron, is the **dyakis dodecahedron**, which gives its name to the class. The $\{321\}$ representative is shown in Fig. 102, and the

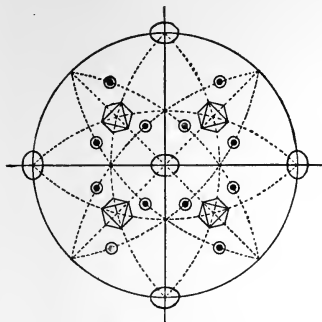


FIG. 101.—Class 30.

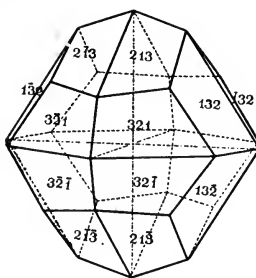


FIG. 102.—The Dyakis Dodecahedron.

stereographic projection, on which the axes of symmetry of the class are also shown, is given in Fig. 101, which is Fig. 73 repeated here for convenience of reference.

The dyakis dodecahedron consists of 12 parallel pairs of trapezoidal faces, and the four edges bounding each face are of three kinds, that is, two are equal and the other two are unequal and different from the equal pair. The longest and the shortest lie in two of the 3 symmetry planes, and the pair of equal intermediate edges meet a third intermediate edge, in which two adjacent faces intersect, to form an equiangular 3-faced solid angle in the middle of each octant, the points lying on the trigonal axes, which latter are also in this class hexagonal axes of compound symmetry. The positive and negative forms only differ so long as they are *in situ*, that is, as regards their positions; a quarter of a revolution of either brings it into the position of the other, when it cannot be distinguished from it. In the case of $\{321\}$ the angle over any longest edge is $30^\circ 0'$, over any shortest edge $64^\circ 37'$, and over the intermediate edges $38^\circ 13'$.

The cube, octahedron, rhombic dodecahedron, icositetrahedron, and triakis octahedron retain their full number of faces under the operation of the symmetry of this class, so that unless they bear superficial markings such as striations these forms cannot be distinguished from class 32 forms, produced under the operation of the full symmetry of the cubic system.

The form $\{hk0\}$, however, which is the tetrakis hexahedron in the case of the fully symmetrical class 32, becomes under the action of the lower symmetry of class 30 the **pentagonal dodecahedron**, shown in Fig. 103, which represents the positive or left one having the indices $\{210\}$ of the two possible

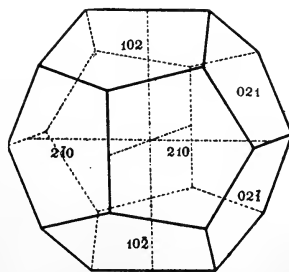


FIG. 103.
The Pentagonal Dodecahedron.

varieties which together correspond to the tetrakis hexahedron $\{210\}$, and which are only different when *in situ* as derived. The faces of this solid are pentagons with four equal sides and a fifth side of different length, which is considerably longer in the case of $\{210\}$, and which is parallel to one of the crystallographic axes. For $\{210\}$ the angles over these longer edges are $53^{\circ} 8'$, and over the shorter edges $66^{\circ} 25'$.

The pentagonal dodecahedron $\{210\}$ is so particularly characteristic of iron pyrites, FeS_2 , that not only is it often called the pyritohedron, but also the whole of class 30 is frequently termed the pyrites class. Fig. 104 represents a typical crystal of pyrites as obtained from the island of Elba, which exhibits the pentagonal dodecahedron $\{210\}$ in combination with the cube and octahedron.

This common variety of the pentagonal dodecahedron of pyrites exhibits positive thermo-electric properties, a current flowing **from** the copper wire with which contact is being made with the crystal and connection thereby established to a delicate galvanometer, **to** the crystal when the latter is gently warmed. Now it is exceedingly interesting that this variety of the pentagonal dodecahedron is striated parallel to the cube edges, as is indicated by the linear shading on Fig. 104. Moreover, when pyrites is found in simple cubes, they are always striated, the striations upon any face being

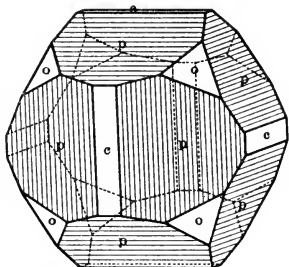


FIG. 104.—Iron Pyrites, Striated Pyritohedron with Cube and Octahedron.

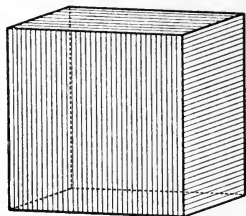


FIG. 105.—Striated Cube of Pyrites.

at right angles to those on the adjacent faces, as shown in Fig. 105. These striations are likewise due to repeated attempts to form the pentagonal dodecahedron $\{210\}$, attempts which only succeed to a relatively slight extent, the cube resuming its growth almost immediately, and thus forming the vastly predominating partner in the oscillatory combination. On the other hand the much rarer variety of pentagonal dodecahedron of pyrites $\{120\}$ which exhibits negative thermo-electric properties, the current passing from the crystal to the copper wire during the experiment just referred to, shows striations in the directions perpendicular to those of the first variety of positive nature, that is, at right angles to the singular edge, the longest in $\{210\}$, of the pentagon and parallel to the line of symmetry of the pentagonal face. The oscillatory alternation of the cube and pentagonal dodecahedron $\{210\}$ is responsible for the common positive striations, while rapid alternation of the pentagonal dodecahedron $\{120\}$ with the dyakis dodecahedron $\{412\}$ or with the icositetrahedron $\{211\}$ produces the rarer negative striations. Indeed the dyakis dodecahedron and the icositetrahedron generally actually predominate in the cases of negatively thermo-electrical pyrites crystals, while the cube and pentagonal dodecahedron predominate on positive crystals.

In this very striking manner are the left (positive) $\{210\}$ and right negative $\{120\}$ varieties of the pentagonal dodecahedron distinguished in the typical case of pyrites, thus affording most important confirmatory evidence of the separate nature of the two forms.

The dyakis dodecahedron and the pentagonal dodecahedron are

the only two new forms introduced by the operation of class 30 symmetry.

In brief, we have the following forms to distinguish in this class :

List of Forms in Class 30.

{100} Cube or hexahedron.	{hkk} Icositetrahedron.
{110} Rhombic dodecahedron.	{hhl} Triakis octahedron.
{111} Octahedron.	{hkl} Left dyakis dodecahedron ; {khl}
{hko} Left pentagonal dodecahedron ;	right dyakis dodecahedron.
{kho} right pentagonal dodecahedron.	

Class 29.—Pentagonal Icositetrahedral Class, also called Cuprite Class. Hemihedrism, Plagihedral or Gyrohedral. Type, Tesseral Holoaxial.

This class is distinguished by possessing all the axes of symmetry of the cubic system but none of the planes of symmetry.

There is only one new solid introduced, however, and this is the **pentagonal icositetrahedron** which gives its name to the class, and which is the class 29 representative of the fully symmetric class 32 hexakis octahedron. The cube, octahedron, rhombic dodecahedron, icositetrahedron, triakis octahedron, and tetrakis hexahedron are all represented by their full number of faces under the action of class 29 symmetry. Hence, they cannot be distinguished from the fully symmetric class 32 forms unless they possess facial markings or actually show the pentagonal icositetrahedron developed. As this is a very rare occurrence, undoubted examples of the class are equally rare. Cuprite has been shown by H. A. Miers to be one of these few cases. The pentagonal icositetrahedron is enantiomorphous, and as such the two possible varieties are distinct forms of the solid, which cannot be brought into identity by rotation. They behave as mirror-images of each other, or as a right-hand glove does to a left-hand one. The two varieties, corresponding together to the {321} hexakis

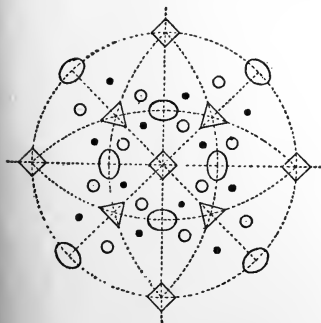


FIG. 106.—Class 29.

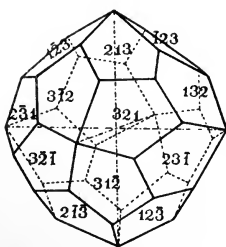


FIG. 107.

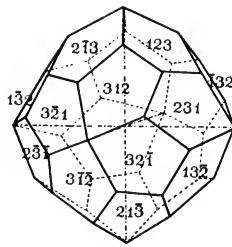


FIG. 108.

Left and Right Pentagonal Icositetrahedra.

octahedron, are shown in Figs. 107 and 108. In the previous illustration, Fig. 106, representing the symmetry elements, the repeated Fig. 72, the stereographic projection of one of them is given, that which contains the face (hkl), in this case (321), and which is usually termed the positive or left variety, together with the projection of the symmetry axes. The projection of the other variety, right or negative, would be obtained by replacing all the dots by rings and *vice versa*, thus changing the hemisphere of every facial pole represented on the projection. Each of the 24 faces

of the pentagonal icositetrahedron is a five-sided figure, no two sides being of equal length.

To summarise, we have the following forms in this class :

List of Forms in Class 29.

$\{100\}$ Cube or hexahedron.	$\{hhl\}$ Triakis octahedron.
$\{110\}$ Rhombic dodecahedron.	$\{hkl\}$ Left pentagonal icositetrahedron ;
$\{111\}$ Octahedron.	$\{lhl\}$ right pentagonal icositetrahedron.
$\{hko\}$ Tetrakis hexahedron.	
$\{hkk\}$ Icositetrahedron.	

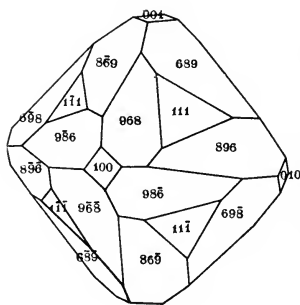


FIG. 109.—Cuprite.

In Fig. 109 a crystal of cuprite from Cornwall measured by Miers is represented, on which the faces of the pentagonal icositetrahedron $\{896\}$ are developed, in combination with faces of the cube and octahedron. By the kindness of Principal Miers the author is enabled to reproduce his drawing. Ammonium chloride, NH_4Cl , has also been shown by Slavik to be occasionally deposited from solution in well-developed pentagonal icositetrahedra of the form $\{943\}$, which exhibit cleavage parallel to the cube faces.

Class 28.—Tetrahedral Pentagonal Dodecahedral Class, also called Ullmannite Class. Tetartohedral Class. Type, Tesseral Polar.

This class of crystals possesses the minimum elements of symmetry which it is possible for a crystal belonging to the cubic system to exhibit, as was clearly explained in the introduction to this chapter. The three equal and mutually rectangular crystallographic axes are no longer tetragonal axes of symmetry, but merely digonal ones; the four trigonal axes of symmetry are retained but become polar, while the six digonal ones disappear altogether. There are no planes of symmetry.

The symmetry elements are clearly shown in the stereographic projection given in Fig. 110, the repeated Fig. 71, which also shows the disposition of the poles of the most general form $\{hkl\}$ of the class, the only new solid which the class introduces. It is termed the **tetrahedral pentagonal dodecahedron**, hence the name of the class. This solid occurs in no less than four modifications, of which, however, only two are truly enantiomorphously distinct, the other two being brought by rotation to resemble these first two exactly, when moved out of their positions of derivation from the general holohedral form $\{hkl\}$, the hexakis

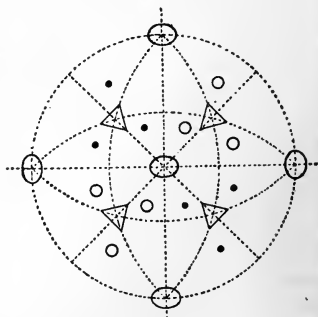


FIG. 110.—Class 28.

octahedron. The two distinct forms are shown in Figs. 111 and 112. The first of these two figures corresponds to the form given in the stereographic projection, and is known as the left positive tetrahedral pentagonal dodecahedron. The second illustration represents the right positive tetrahedral pentagonal dodecahedron. The

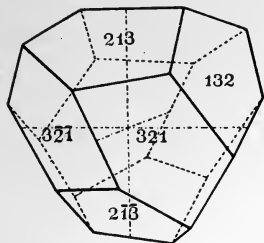


FIG. 111.

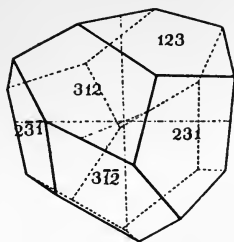


FIG. 112.

Left and Right Positive Tetrahedral Pentagonal Dodecahedra.

two forms are the mirror-images of each other, by reflection across a hypothetical rhombic dodecahedron face, which is equivalent in the stereographic projection to rotation about one of the diagonal diameters bisecting the crystallographic axial diameters. The negative right tetrahedral pentagonal dodecahedron is obtained by reflection of the left positive form shown in the projection across a cube face, or rotation about one of the crystallographic axial diameters. It is the one which is produced if the poles indicated in the projection by dots are changed into rings, and the rings into dots. Similarly, from the right positive a left negative form is obtained by reflection across a cube face. The right and left varieties of the same sign are the enantiomorphous forms, but the two right forms or the two left forms only differ in their position when *in situ*, and may be brought by rotation for 90° about a crystallographic axis into indistinguishable identity. The faces of both solids are unsymmetrical pentagons, in which they differ characteristically from the ordinary pentagonal dodecahedron.

The cube and rhombic dodecahedron are represented by their full number of faces even under the operation of these minimum elements of cubic symmetry.

The octahedron is represented by the tetrahedron, the positive and negative varieties only differing in position while *in situ*.

The icositetrahedron, $\{hkk\}$ where h is greater than k , yields the two triakis tetrahedra, positive and negative. The triakis octahedron $\{hhl\}$ affords the positive and negative deltoid dodecahedra. Lastly, the tetrakis hexahedron $\{hko\}$ affords the left and right pentagonal dodecahedra.

Thus the only novelty in this class is the tetrahedral pentagonal dodecahedron, the four modifications of which have been described, and which may be considered as together making up, when *in situ*, the holohedral class 32 hexakis octahedron. We have, therefore, the following forms to record as belonging to class 28 :

List of Forms in Class 28.

- $\{100\}$ Cube.
 $\{110\}$ Rhombic dodecahedron.
 $\{111\}$ Positive tetrahedron ; $\{1\bar{1}1\}$ negative tetrahedron.
 $\{hk0\}$ Left pentagonal dodecahedron ; $\{kh0\}$ right pentagonal dodecahedron.
 $\{hkk\}$ Positive triakis tetrahedron ; $\{h\bar{k}k\}$ negative triakis tetrahedron.
 $\{hhl\}$ Positive deltoid dodecahedron ; $\{h\bar{h}l\}$ negative deltoid dodecahedron.
 $\{hkl\}$ Left positive tetrahedral pentagonal dodecahedron ; $\{khl\}$ right positive tetrahedral pentagonal dodecahedron ; $\{h\bar{k}l\}$ right negative tetrahedral pentagonal dodecahedron ; $\{k\bar{h}l\}$ left negative tetrahedral pentagonal dodecahedron.

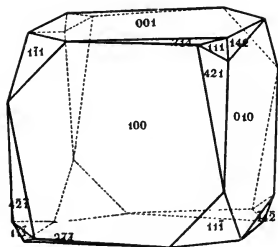


FIG. 113.—Barium Nitrate.

Among minerals, ullmannite, NiSbS , crystallises according to the symmetry of this class. Among chemically prepared substances, barium nitrate, $\text{Ba}(\text{NO}_3)_2$, and sodium chlorate, NaClO_3 , are noteworthy examples. Fig. 113 shows a crystal of barium nitrate measured by Wulff, which

consists of a combination of the cube with a left positive tetrahedral pentagonal dodecahedron $\{421\}$ and the two tetrahedra $\{111\}$ and $\{1\bar{1}1\}$.

CHAPTER XII

TWO PRACTICAL EXAMPLES OF CUBIC CRYSTALS, GARNET AND COBALTITE.

(1) *Garnet. Class 32, Cubic Holohedral.*

THE important garnet group of minerals has the general chemical composition $R''_3R'''_2(\text{SiO}_4)_3$, being silicates in which R'' may be calcium, magnesium, iron, or manganese, and R''' may be aluminium, iron, or chromium, isomorphously replacing each other. Indeed not only may different garnets have different R'' and R''' metals as their dominating elements, but one and the same garnet usually contains two or more metals of each R'' or R''' group more or less replacing each other. The general crystallographic form is the rhombic dodecahedron, but the icositetrahedron is almost as frequently and as largely developed, so that it is also characteristic of the garnet group.

The particular garnet chosen for the purpose of these typical cubic measurements is a small one, 3 to 4 millimetres in diameter, very dark red in colour, and almost an ideally perfect icositetrahedron in shape, the faces of this form being nearly equally developed. Smaller faces, apparently of the rhombic dodecahedron, and subsequently to be shown to be so in fact, may be observed truncating the twelve solid angles which are formed in each case by two longer and two shorter icositetrahedral edges. Its appearance is shown in Fig. 114, and its stereographic projection in Fig. 115.

The former figure was drawn to scale after the completion of the measurements, by the general conventional method which will be fully described in Chapter XXV., the actual construction being represented in Fig. 372. The mode of constructing the stereographic projection, as far as the rhombic dodecahedron is concerned, was fully gone into on page 154 of Chapter XI., and the additional construction necessary for the insertion of the icositetrahedral poles and the zone-circles including them will be described during the course of the measurements.

The shape of the crystal being so obviously that of the icositetrahedron, and the small truncations being so clearly situated at the apparently correct positions for faces of the rhombic dodecahedron,

the following course was suggested and adopted in order to verify that these two forms were really those present.

The plan of operations involved five stages.

(1) To ascertain that the 12 small dodecahedral faces, marked *d*, were arranged in 3 mutually rectangular zones, 4 *d*-faces in each zone, at equal angular intervals of 90° . These 3 zones are indicated in the stereographic projection by the primitive circle and the two cubic axial diameters parallel to the edges of the page.

(2) To verify that the dodecahedral faces also lie in 4 zones in which the angles are all 60° , there being 6 *d*-faces in each. These zones are indicated by the arc-zones similar to that containing (110) and (101), terminating at the *d*-poles on the primitive circle. If the icositetrahedron present be $\{211\}$, as will be shown to be the case, these zones will also contain *i*-faces alternating with

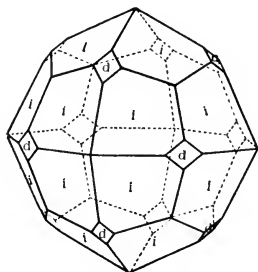


FIG. 114.

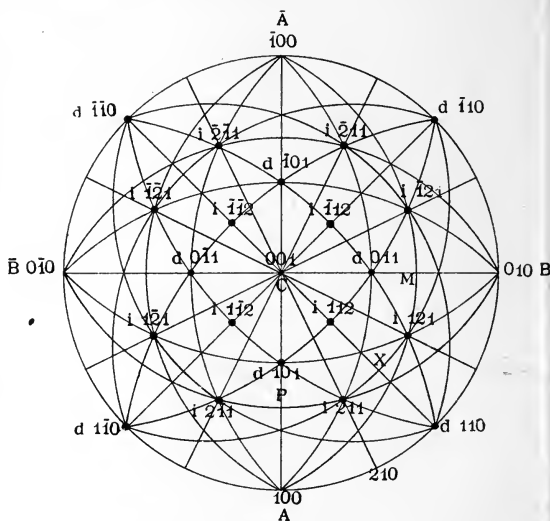


FIG. 115.

the *d*-faces and at exactly intermediate distances, so that there will be 12 angles of 30° in each zone instead of merely 6 of 60° .

(3) To determine whether the angles over the longer edges of the icositetrahedron are $48^\circ 11'$, in which case the form is $\{211\}$, as stated on page 158 of Chapter XI., or whether they are $35^\circ 6'$ corresponding to $\{311\}$. They will be shown to have the former value, and there are 12 such zones containing none but *i*-faces, 4 in each zone, and every *i*-face is common to two such zones, as it is bounded by two longer (and two shorter) edges. Four of these zones are indicated by the arc-zones such as $(211):(2\bar{1}1)$, which terminate at the ends of the principal diameters.

(4) To similarly ascertain whether the angles over the shorter icositetrahedral edges are $33^\circ 33'$ corresponding to $\{211\}$, or $50^\circ 29'$ the angle proper to $\{311\}$. The former value will be shown to be that present, and there are 12 such zones, each containing four *i*-faces and also two *d*-faces. Four of these zones are indicated by the arcs

such as $(1\bar{1}0):(211)$ terminating at the d -poles on the primitive circle.

(5) It was also advisable to obtain confirmatory measurements across the 6 main quoins or solid angles of the icositetrahedron, that is, to measure the angle between either pair of opposite i -faces of each quoin. These angles should be $70^\circ 32'$ (the same as the octahedral angle) if the icositetrahedron present be $\{211\}$; and the zones containing them, two such angles in each, will also each contain a pair of dodecahedral faces at $54^\circ 44'$ from each of the adjacent i -faces of the zone (the half-supplement of the octahedral angle). The two secondary diameters at 45° to the edges of the page indicate 2 of the 6 zones in question, and the other 4 are indicated by the arc-zones such as $(101):(121)$ terminating at the ends of the principal diameters. The measurement of this fifth class of zone will finally confirm that the d -faces and the i -faces are mutually arranged as they should be if the forms are truly those of the rhombic dodecahedron $\{110\}$ and the icositetrahedron $\{211\}$.

Commencing now the measurements on the lines thus planned out, by measuring the angles between the four small apparently dodecahedral faces of one of the zones referred to under (1), we set the crystal up on the wax holder of the goniometer by one of the 6 axial quoins, so as to bring one of the cubic crystallographic axes vertical, just as the crystal stands in Fig. 114, and with the four little d -faces to be measured approximately parallel to the axis of the goniometer and parallel in pairs to the two adjusting movements. Adjusting now these faces accurately with the aid of the tangent screws, and finally centring the crystal, we proceed to the actual measurement. The following readings were obtained for the three such zones on the crystal, treated precisely similarly in this manner, the crystal being reset for the second and again for the third so as to bring the second and third crystallographic axes in turn vertical and parallel to the goniometer axis.

3 PRINCIPAL DODECAHEDRAL ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.	Circle Reading.	Angle.
$d \ 360^\circ \ 0' A$ $d \ 269 \ 59$ $d \ 179 \ 59 A$ $d \ 90 \ 0 A$ $d \ 0 \ 0 A$	$dd \ 90^\circ \ 1'$ $dd \ 90 \ 0$ $dd \ 89 \ 59 A$ $dd \ 90 \ 0 A$	$d \ 271^\circ \ 12'$ $d \ 181 \ 13$ $d \ 91 \ 21$ $d \ 1 \ 14$ $d \ 271 \ 12$	$dd \ 89^\circ \ 59'$ $dd \ 89 \ 52$ $dd \ 90 \ 7$ $dd \ 90 \ 2$	$d \ 340^\circ \ 15' A$ $d \ 250 \ 14 A$ $d \ 160 \ 15 A$ $d \ 70 \ 14$ $d \ 340 \ 15$	$dd \ 90^\circ \ 1' A$ $dd \ 89 \ 59 A$ $dd \ 90 \ 1$ $dd \ 89 \ 59$

It will be clear from these angular values that the faces are indeed either those of the rhombic dodecahedron or of the cube (assuming cubic symmetry); or of a tetragonal or rectangular rhombic prism, the angles being all so close to 90° that there can be no doubt of nature's intention to make them precisely $90^\circ 0'$. That they are really faces of the rhombic dodecahedron and not of the cube or other rectangular form is rendered probable by the fact that the four faces of the three zones were all different ones, there being 12 faces altogether. The values obtained in the first and third of the zones never differ by more than a minute from 90° , and the two middle values of the second zone only show more difference because the faces affording them were of poorer

quality, some slight malformation resulting in blurred images of the Websky signal. In a case like this we must judge the symmetry absolutely by the individual values, those marked "A" by preference, as being fully trustworthy; for to take the mean would be to obtain $90^{\circ} 0'$ anyhow, a result perfectly worthless as regards any indication of the symmetry, although after having decided the latter we may take the mean as representing the final measured value of the angle.

We may next pass on to the measurement of a zone of type (2), consisting of both *d*- and *i*-faces, say that containing the right *d*-face shown in Fig. 114 along the equatorial edges of the crystal (the face (110)), the *i*-face which forms the right upper one of the central four shown in the figure (the face (211)), and the adjoining upper *d*-face along the middle crystal edges of the vertical plane (the face (101)). Setting the crystal so that this zone was approximately adjusted parallel to the goniometer axis, and then completing the final adjustment with the aid of the tangent screws, the following measurements were obtained. Alongside are also given those for a second similar zone, that corresponding to the other diagonal diameter of the projection.

DIAGONAL DODECAHEDRAL ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.
<i>d</i> 360° 0'A	<i>di</i> 30° 0'	<i>d</i> 360° 0'A	<i>di</i> 30° 5'
<i>i</i> 330 0	<i>id</i> 30 0	<i>i</i> 329 55	<i>id</i> 29 55
<i>d</i> 300 0 A	<i>di</i> 30 3	<i>d</i> 300 0 A	<i>di</i> 30 2
<i>i</i> 269 57	<i>id</i> 29 54	<i>i</i> 269 58	<i>id</i> 29 55
<i>d</i> 240 3	<i>di</i> 30 6	<i>d</i> 240 3	<i>di</i> 30 1
<i>i</i> 209 57	<i>id</i> 29 55	<i>i</i> 210 2	<i>id</i> 30 6
<i>d</i> 180 2 A	<i>di</i> 30 0	<i>d</i> 179 56	<i>di</i> 30 0
<i>i</i> 150 2	<i>id</i> 30 6	<i>i</i> 149 56	<i>id</i> 29 56
<i>d</i> 119 56	<i>di</i> 29 56	<i>d</i> 120 0	<i>di</i> 29 56
<i>i</i> 90 0	<i>id</i> 29 58	<i>i</i> 90 2	<i>id</i> 29 58
<i>d</i> 60 2	<i>di</i> 30 4	<i>d</i> 60 4	<i>di</i> 30 4
<i>i</i> 29 58	<i>id</i> 29 58	<i>i</i> 30 0	<i>id</i> 29 58
<i>d</i> 0 0 A		<i>d</i> 0 2 A	
	<i>dd</i> 60° 0'A		<i>dd</i> 60° 0'A
	<i>dd</i> 59 57		<i>dd</i> 59 57
	<i>dd</i> 60 1		<i>dd</i> 60 7
	<i>dd</i> 60 6		<i>dd</i> 59 56
	<i>dd</i> 59 54		<i>dd</i> 59 56
	<i>dd</i> 60 2		<i>dd</i> 60 2

The two other similar zones were also measured, but the angular values obtained were so close to those above given that it is unnecessary to set them forth at length here, the two quoted being ample for the purpose of determining the symmetry and deciding as to whether the angles are truly 60° and 30° . An inspection of the lists at once shows that there can be no doubt on the point, the greatest discrepancy being only $7'$, and in the two cases where absolutely irreproachable "A" images of the signal were concerned, the values were precisely $60^{\circ} 0'$. We can consequently definitely decide that nature intended the angles to be truly $dd = 60^{\circ} 0'$ and $di = 30^{\circ} 0'$, that the conditions of (2) for the forms {110} and {211} are absolutely fulfilled, and that the *d*-faces of the three rectangular zones first measured are truly those of the rhombic dodecahedron. Having thus decided the question of the symmetry, and not before, we may be permitted to record the mean of the values as the final measured value in the case of each of the two angles *dd* and *di*.

It may be here remarked that the faces of the icositetrahedron were

slightly striated, parallel to the axis of the zone now in question, that is, parallel to the edges of intersection with the two adjacent dodecahedron faces belonging to the same zone. The images were therefore occasionally slightly affected by this circumstance, hence the small deviations of a few minutes from exactly $30^{\circ} 0'$ in the angles. The effect of these striations will be shown to disappear entirely when we come to measure the zones perpendicular to them, across the quoins, under (5), so that absolutely no doubt is introduced by the fact of the slight striation.

We next tackle the question of the value of the icositetrahedral angles across the longer edges, referred to under (3).

Setting the crystal almost upright on the wax as in Fig. 114, but just tilted over slightly so as to bring one of the longer edges, say that which passes upwards from the central quoin in the figure, parallel to the goniometer axis, and then adjusting the two faces on each side of and forming the edge, with the aid of the tangent screws, we find that on rotating the crystal on the goniometer the zone comprises only these two and the parallel pair of icositetrahedral faces. It is unnecessary to measure all 12 such zones, four of them, judiciously selected, say two involving one quoin and two involving another of the 6 quoins in a complementary manner, will be fully adequate to afford us definite information as to the symmetry. Four such were measured, and the values in the cases of a pair of them are recorded below.

LONGER-ICOSITETRAHEDRAL-EDGE ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.
$i \begin{cases} 360^{\circ} & 0' \\ 311 & 51 \\ 180 & 4 \\ 131 & 51 \\ i & 0 & 0 \end{cases}$	$ii \begin{cases} 48^{\circ} & 9' \\ 131 & 47 \\ 48 & 13 \\ ii & 131 & 51 \end{cases}$	$i \begin{cases} 360^{\circ} & 0' \\ 311 & 53 \\ 180 & 0 \\ 131 & 45 \\ i & 0 & 0 \end{cases}$	$ii \begin{cases} 48^{\circ} & 7' \\ 131 & 53 \\ 48 & 15 \\ ii & 131 & 45 \end{cases}$

None of the images were of "A" quality, owing to the slight striation, which was sufficient in these zones to blur the images somewhat; but the latter were quite certain to within two or three minutes, so that the values of ii may be taken as trustworthy within those approximate limits. It will be seen at once that the value of the angle over the longer edges is very close indeed to $48^{\circ} 11'$, the theoretical angle for $\{211\}$, and as a matter of fact the mean of all the values obtained is exactly $48^{\circ} 11'$. For the values in the other pair of similar zones measured were very close to those recorded above and the mean afforded was identical. The measurements of these longer-edge zones, therefore, fully establish the identity of the icositetrahedron as $\{211\}$.

Passing on now to the measurement of the shorter-edge zones, as indicated under (4) to be advisable, a very slight further tilting of the crystal will enable one of them to be set conveniently on the wax, and with the aid of the tangent screws the adjustment to the axis of the goniometer of the shorter edge selected is only a matter of a moment or two. The measurement of four of these zones will also suffice, and the actual readings for two of them are recorded below. Instead of only containing the adjusted pair of icositetrahedral faces and the parallel pair, however, each of these

zones is found to contain also a parallel pair of the little dodecahedral faces, each symmetrically placed in the centre of the obtuse angle between the icositetrahedral faces.

SHORTER-ICOSITETRAHEDRAL-EDGE ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} d \ 360^\circ \ 0' \\ i \ 286 \ 51 \\ i \ 253 \ 16 \\ d \ 180 \ 0 \\ i \ 106 \ 42 \\ i \ 73 \ 9 \\ d \ 0 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} di \ 73^\circ \ 9' \\ ii \ 33 \ 35 \\ id \ 73 \ 16 \\ di \ 73 \ 18 \\ ii \ 33 \ 33 \\ id \ 73 \ 9 \end{array} \right.$	$\left\{ \begin{array}{l} i \ 360^\circ \ 0' \\ i \ 326 \ 28 \\ d \ 253 \ 15 \\ i \ 179 \ 59 \\ i \ 146 \ 28 \\ d \ 73 \ 13 \\ i \ 0 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} ii \ 33^\circ \ 32' \\ id \ 73 \ 13 \\ di \ 73 \ 16 \\ ii \ 33 \ 31 \\ id \ 73 \ 15 \\ di \ 73 \ 13 \end{array} \right.$

It will be clear that the angle ii across the shorter edges of the icositetrahedron corresponds to the theoretical angle $33^\circ 33'$ for $\{211\}$. The four individual values of ii are all within $2'$ of this angle, and their mean is precisely $33^\circ 33'$. The values obtained from two other of these twelve zones, judiciously selected again so as to afford the maximum confirmation of the symmetry, yielded almost identical values, leaving no room for doubt that the icositetrahedron present is $\{211\}$.

We come now, finally, to the measurement of the angles across the quoins, as suggested in (5). Let us commence, say, by measuring the zone containing the upper right (211) icositetrahedral face, of the four central faces shown in Fig. 114, and the lower left face ($2\bar{1}\bar{1}$). The complete zone will include also the two parallel faces at the back of the crystal as it is represented in the figure, and also the two little dodecahedral faces on the upper right and lower left margin of the figure, that is, those two at the ends of the 45° -diagonal which passes from the right upper to the left lower intermediate corners on the outside periphery of the figure, namely, (011) and ($0\bar{1}\bar{1}$). The setting of the crystal on the wax for this zone is a very easy matter, the crystal being merely tilted so as to bring the diagonal just referred to horizontal, and one of the icositetrahedral faces can be adjusted with one tangent screw, and one of the dodecahedral faces with the other, to which it should preferably have been set preliminarily as nearly parallel as possible.

When the adjustment was completed for each of the four of these six zones which in turn were measured, it was found, as already suggested would be the case, that all effect of striation had disappeared, the striations being now horizontal and parallel to the zone plane and thus perpendicular to the signal-images, which latter were absolutely sharp. The readings obtained were consequently of great precision, affording the most absolute confirmation of the supposition that the forms present were the rhombic dodecahedron and the icositetrahedron $\{211\}$. Those for two of the zones are given below.

ICOSITETRAHEDRAL CROSS-QUOIN ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} d \ 360^\circ \ 3'A \\ i \ 305 \ 20 \ A \\ i \ 234 \ 48 \ A \\ d \ 180 \ 3 \ A \\ i \ 125 \ 19 \ A \\ i \ 54 \ 48 \ A \\ d \ 0 \ 4 \ A \end{array} \right.$	$\left\{ \begin{array}{l} di \ 54^\circ \ 43'A \\ ii \ 70 \ 32 \ A \\ id \ 54 \ 45 \ A \\ di \ 54 \ 44 \ A \\ ii \ 70 \ 31 \ A \\ id \ 54 \ 44 \ A \end{array} \right.$	$\left\{ \begin{array}{l} d \ 360^\circ \ 0'A \\ i \ 305 \ 17 \ A \\ i \ 234 \ 45 \ A \\ d \ 180 \ 0 \ A \\ i \ 125 \ 18 \ A \\ i \ 54 \ 45 \ A \\ d \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} di \ 54^\circ \ 43'A \\ ii \ 70 \ 32 \ A \\ id \ 54 \ 45 \ A \\ di \ 54 \ 42 \ A \\ ii \ 70 \ 33 \ A \\ id \ 54 \ 45 \ A \end{array} \right.$

The separate values of the angle ii across the quoin (between opposite i -faces) are thus seen to be within a single minute of the theoretical angle for the icositetrahedron $\{211\}$, namely $70^\circ 32'$, and the angles di and id indicating the position of the two dodecahedron faces with respect to the faces of $\{211\}$ are also all within $2'$ of the theoretical equal angles of $54^\circ 44'$, assuming the little faces to be truly those of the rhombic dodecahedron. The values for the other two zones of this type measured gave also the same within $2'$.

We have now very thoroughly explored this crystal of garnet, and it only remains to present the main results in tabular form, provided we accept as well known the theoretical values of the angles of the icositetrahedron $\{211\}$, as recorded in the last chapter. But there is no occasion to assume them at all, for they are quite easily calculated from first principles, and as it will be an excellent exercise so to calculate them, we shall proceed to do so, before presenting the table of results in which the measured angles are comparatively recorded alongside the theoretical ones. In the cubic system there are no elements to calculate, for the axial angles α , β , and γ are all 90° , and the lengths of the axes are all equal, and may be regarded as unity. Moreover, no basal angle or angles are necessary for the calculation of the angles of cubic crystals, however complicated, for the perfect symmetry of the system provides that the angles of the simplest of the primary forms, the rhombic dodecahedron, are 90° and 60° , and the angles between the faces of this form and those of the axial-plane form, the cube, are 45° . These data are adequate to enable us to calculate any angle between any faces of any cubic crystal whatsoever. It has already been shown in the last chapter (page 155) how we may calculate the octahedron angle, $70^\circ 32'$, from these premises fixed by the nature of the symmetry. We will now proceed, therefore, on similar lines to calculate the angles between adjacent faces of the icositetrahedron $\{211\}$ and between those of this form and adjacent ones of the rhombic dodecahedron. The stereographic projection given in Fig. 115 shows all the necessary triangles. It was constructed in the manner already described in the last chapter for Fig. 81.

To find the angles $di=(101):(112)$ and $ii=(112):(\bar{1}\bar{1}2)$.

We can find both from the same right-angled triangle $C=(001):d=(101):i=(112)$, for Ci is the half of ii . The right angle is at i , and if we construct the Napierian diagram for the triangle, as shown in Fig. 116, we may derive from it the following equations, by Napier's rules:

$$\begin{aligned} \sin di &= \sin 45^\circ \sin 45^\circ, & \cos 45^\circ &= \cot 45^\circ \tan Ci, \\ &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \text{or } \tan Ci &= \cos 45^\circ \tan 45^\circ; \\ &= \frac{1}{2} & &= \frac{1}{\sqrt{2}} \cdot 1 \\ di &= 30^\circ. & & \end{aligned}$$

$$Ci = 35^\circ 16'.$$

$$\text{Angle over quoin, } ii = 2Ci = 70^\circ 32'.$$

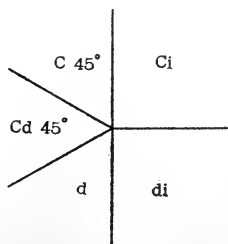


FIG. 116.

Thus the angle $ii=(112):(\bar{1}\bar{1}2)$ over the top quoin in Fig. 114, or the similar angle over any other of the 6 quoins, such, for instance, as the central quoin $(211):(2\bar{1}\bar{1})$, is

$70^\circ 32'$, which is the same as the octahedron angle, a fortuitous occurrence owing to the high degree of symmetry.

The angle $di=(110):(112)$ of the same zone is $90^\circ - 35^\circ 16' = 54^\circ 44'$.

The angle $di=(101):(112)$ has been shown above to work out to 30° . Hence, the complementary angle of the zone, $dd=(101):(\bar{1}\bar{1}0)$, is 60° , for $di=(\bar{1}\bar{1}0):(112)$ is 90° . Thus we independently prove one of the dodecahedron angles previously assumed. In fact, all the angles of cubic crystals follow from the simple assumption that the poles of the rhombic dodecahedron are at 45° from the cube poles, that is, that they bisect the cube angles of 90° .

To find the angle $H=(211):(2\bar{1}\bar{1})$, the angle over the longer edges.

We can most conveniently find the half of the angle, namely, iP , where P is the point of intersection of the arc with the diameter AC, from the right-angled triangle APi , in which the right angle is at P, the angle at A is 45° , and the side Ai is $35^\circ 16'$, being half the angle ii over the quoin at A. The conditions are set forth in the Napierian diagram Fig. 117, from which we deduce by the rules:

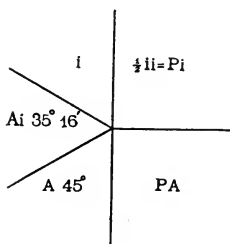


FIG. 117.

$$\sin \frac{1}{2} ii = \sin 35^\circ 16' \sin 45^\circ;$$

$$\frac{1}{2} ii = 24^\circ 54\frac{1}{2}';$$

$$\text{and } H = 48^\circ 11'.$$

Thus the icositrahedral angle (for $\{211\}$) over the longer edges works out to be $48^\circ 11'$, the theoretical angle quoted in the last chapter.

The angle over the other long edge of the initial face (211) , between that face and $(2\bar{1}\bar{1})$, may be equally easily calculated. For the half of this angle is represented by the portion of the diameter $Ci=(001):(211)$ which lies between (211) and the primitive circle, which it intersects at the possible pole of (210) , and we can find the arc $(211):(210)$ from the triangle $i=(211):(210):d=(110)$, in which we know that the angle at (210) is a right angle, the angle at d is the complement of $35^\circ 16'$, and di is 30° . If we work out the Napierian equation we shall obtain as the result that $\text{Log. sin } (211:210) = \bar{1}.61091$, which corresponds to $24^\circ 54\frac{1}{2}'$, and thus affords us the same value $48^\circ 11'$ for the double angle ii , the angle over the second long edge, as we have just found for the angle over the first long edge.

To find the angle $H=(211):(121)$ over the shorter edges of the icositrahedron $\{211\}$.

We can find the half again most conveniently, iX , from the triangle formed by these two points (i and X) and $d=(110)$, in which we know that the angle at X is a right angle, the arc $id=30^\circ$, and the angle at d is $35^\circ 16'$ (same as $Ci=(001):(\bar{1}\bar{1}2)$). The Napierian diagram is given in Fig. 118, and from it we get:

$$\sin iX = \sin 30^\circ \sin 35^\circ 16'$$

$$iX = 16^\circ 46.7' = \frac{1}{2} ii.$$

$$\text{Angle over short edge } H = 33^\circ 33'.$$

We can now finally confirm that the angles over all the 6 quoins are the same, taking the case of the angle between (121) and $(\bar{1}2\bar{1})$ as a second example. We can get $\frac{1}{2} ii$ from the triangle BiM , where M is the intersection of $(121):(\bar{1}21)$ with the diameter CB. In this triangle the angle at M is a right angle, that at B is 45° , and iM is half the angle over the long edge, namely $24^\circ 54\frac{1}{2}'$.

If we construct a Napierian diagram we can deduce thence the equation:

$$\sin iB = \frac{\sin 24^\circ 54\frac{1}{2}'}{\sin 45^\circ}$$

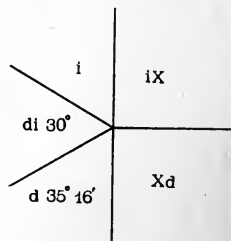


FIG. 118.

like that of Fig. 81, already fully explained in the last chapter, except that eight additional arc-zones are constructed, one on each side of each of the four principal diameters, so as to terminate at the ends of each diameter at its intersection with the primitive circle. Those arc-zones which terminate at cube poles contain the angles over the longer icositetrahedral edges, and those which end at d -poles on the primitive circle contain the angles over the shorter edges of the icositetrahedron.

Their construction can be accomplished directly by the compasses, by trial, so as to pass through the required poles, all of which are already fixed by the intersection of the arcs and diameters drawn in Fig. 81. Or they can be more systematically drawn by the usual geometrical method of constructing an arc to pass through three given points, the three determinative facial poles of the particular zone, finding the centre of the arc by the intersection of the two bisecting perpendiculars to the two lines connecting respectively the first and second and the second and third polar points.

Four subsidiary diameters may also be drawn, to pass through the poles of the icositetrahedron not already situated on the diagonal diameters at 45° to the edges of the page. This then completes the stereographic projection.

It now only remains to present the results for garnet in concise tabular form.

Table of Results for Garnet, $R''_3R'''_2(SiO_4)_3$.

Crystal-System : Cubic. **Class :** 32, cubic holohedral.

Habit : Icositetrahedral.

Forms observed : Rhombic dodecahedron $d = \{110\}$, and icositetrahedron $i = \{211\}$, the latter form largely predominating.

Interfacial angles : as tabulated in the following list.

MORPHOLOGICAL ANGLES OF GARNET.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$dd = (110) : (1\bar{1}0)$. .	12	$89^\circ 52' - 90^\circ 7'$	$90^\circ 0'$	$90^\circ 0'$	$0'$
$\left\{ \begin{array}{l} dd = (110) : (101) \\ \text{or } (101) : (011) \end{array} \right.$. .	24	$59 \ 54 - 60 \ 6$	$60 \ 0$	$60 \ 0$	0
$\left\{ \begin{array}{l} di = (110) : (211) \\ \text{or } (101) : (112) \end{array} \right.$. .	48	$29 \ 54 - 30 \ 6$	$30 \ 0$	$30 \ 0$	0
$\left\{ \begin{array}{l} di = (110) : (112) \\ \text{or } (101) : (121) \end{array} \right.$. .	16	$54 \ 42 - 54 \ 45$	$54 \ 44$	$54 \ 44$	0
$\left\{ \begin{array}{l} ii = (112) : (\bar{1}\bar{1}2) \\ \text{or } (121) : (\bar{1}2\bar{1}) \end{array} \right.$. .	8	$70 \ 31 - 70 \ 33$	$70 \ 32$	$70 \ 32$	0
$\left\{ \begin{array}{l} ii = (211) : (2\bar{1}\bar{1}) \\ \text{or } (121) : (\bar{1}21) \end{array} \right.$. .	8	$48 \ 7 - 48 \ 15$	$48 \ 11$	$48 \ 11$	0
$\left\{ \begin{array}{l} di = (\bar{1}\bar{1}0) : (211) \\ \text{or } (110) : (2\bar{1}\bar{1}) \end{array} \right.$. .	16	$73 \ 9 - 73 \ 18$	$73 \ 14$	$73 \ 13$	1
$\left\{ \begin{array}{l} ii = (211) : (121) \\ \text{or } (2\bar{1}\bar{1}) : (\bar{1}21) \end{array} \right.$. .	8	$33 \ 31 - 33 \ 35$	$33 \ 33$	$33 \ 33$	0

This concludes our investigation of garnet. It has been a very complete one as regards the morphology of the crystal; for being the first practical example of a cubic crystal, it was considered advisable to work it out in detail, in order to be of maximum use to those who desire to carry out a similar series of measurements on a garnet of like character, which can usually be obtained from any mineral dealer of standing.

(2) *Cobaltite. Arsenosulphide of Cobalt, CoAsS . Class 30, Pyrites Class of Cubic System.*

Cobaltite is a mineral closely resembling pyrites in crystalline character, the pyritohedron, the pentagonal dodecahedron $\{210\}$, being usually a prominent form as in the case of pyrites. The cube and octahedron are frequently associated with the pyritohedron.

The crystal of cobaltite selected as our second example of a cubic

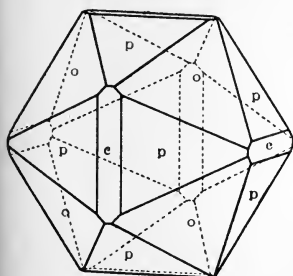


FIG. 119.

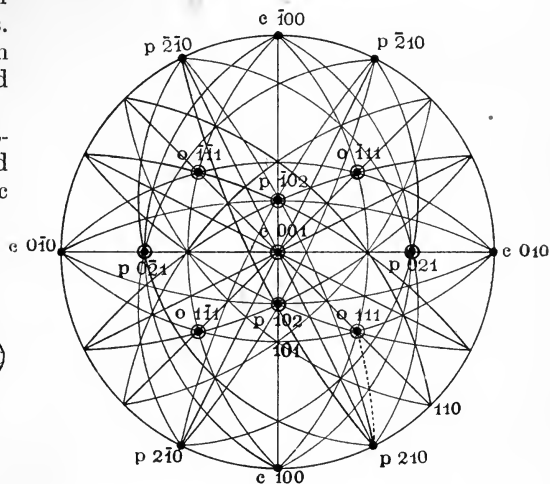


FIG. 120.

crystal was a beautiful little specimen, of a type which is readily obtained from the best mineral dealers (hence its choice as an example), roughly spherical in shape like the garnet, and of about $3\frac{1}{2}$ millimetres diameter. It possessed a brilliant grayish-white metallic lustre, affording excellent reflections of the goniometer signal, and was altogether eminently suitable for goniometrical investigation. It is illustrated in Fig. 119, which was drawn to scale in the usual conventional manner after the completion of the measurements, the construction being described in Chapter XXV., and illustrated in Fig. 376. Its stereographic projection is given in Fig. 120, the method of constructing which will presently be described.

At first sight the crystal appears to be of a highly complicated character, consisting of a large number of triangular facets, not suggestive of any well-known cubic form. Indeed the crystal has been chosen as an example largely because of its apparent complication, and for the

purpose of showing how simple it really becomes after the first few measurements have been made, clearly indicating the nature of the forms present. Careful preliminary examination with a pocket lens revealed the fact that the crystal possessed six sets of faces, each consisting of three faces, and of markedly similar nature, arranged about the positions of possibly cubic axes, one set at each end of each axis. Each set consisted of a central face somewhat narrow, tailing off on each long side into a triangular face, having its apex outwards, inclined to the strip-face at a relatively small angle, like a pair of pointed wings; and the edge between the strip-face and each of the two triangular faces was usually blunted by striae, having the effect of making the two triangular faces appear as if they were joined by a curved face. The two triangular faces at the ends of the curve, however, were themselves beautifully plane and gave excellent reflections of the signal, and the central strip-face between them (separated from them by the curved parts consisting of numerous striae) also gave "A" reflections. The circumstance of the occurrence of the striations, however, served well to identify these 6 symmetrically (cubically) placed sets of 3 faces, and it appeared not improbable that the strip-faces might actually prove to be faces of the cube, in which case the triangular faces into which they tailed at each side would be those of a pentagonal dodecahedron. The striations themselves lent additional weight to this probability, because one of the most characteristic features of the pyritohedron {210} is the striation parallel to the singular long edges (those which would be replaced by cube faces if present), as shown in Fig. 104, due to repeated alternation of attempts to produce faces of the cube and pentagonal dodecahedron, the striation being often so deep and stepped as to result in apparent curvature over the edge instead of there being a sharp edge.

The remaining triangular faces on the crystal were then seen to be eight in number, one in each octant, symmetrically placed; this would agree perfectly with the supposition that they were the faces of the octahedron. It was the fact that they were about the same size as the pyritohedron faces, and that they joined the latter at the corners, which gave the crystal the appearance of being composed of the faces of a single unknown form having triangular faces, the cube faces being mostly narrow strips only, and hence not apparent at first sight, not sensibly modifying the six edges between the pairs of triangular faces where they occurred.

Having thus from superficial observation with a pocket lens arrived at the preliminary idea that the crystal may prove to be composed of faces of three forms belonging to the cubic system, the cube, octahedron, and pentagonal dodecahedron {210}, we proceed to the measurements.

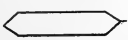







It was obvious that the six sets of possibly cube-pyritohedron faces already alluded to as being so marked, were arranged in pairs along three zones, and the proper course to take was clearly to measure these three zones first, when we should arrive at a confirmation or otherwise of our supposition as to the nature of the faces, from the magnitude

of the angles found. It has already been shown in the last chapter (p. 168) that the angle over the singular long edge of the pentagonal dodecahedron {210}, which is the edge that may be modified by a cube face, is 53° 8'. If the cube face be present between the two pentagonal dodecahedral faces it will bisect this angle, that is, the angles between the strip-like cube face and the two adjacent triangular faces of the pentagonal dodecahedron will each be 26° 34'.

We commence the measurements, therefore, by setting up the crystal on the wax by one of the cube edges, so that a zone containing two other sets of supposed pyritohedron-cube faces is adjusted for measurement, precisely, in fact, as the crystal appears in Fig. 119. It will be most convenient also to arrange the two vertical strips of cube faces (which are parallel to each other at the front and back of the crystal respectively) parallel to one of the adjusting movements, when it will be found that the other adjusting movement is parallel to two other similar strip-like cube faces which belong to this zone, horizontally arranged, however, as regards their long edges and belonging to two other sets of the faces in question, lying in a zone perpendicular to the one about to be measured, the cube faces being common to both zones.

After the completion of the measurements for the zone thus adjusted, the crystal was reset in a position 90° from the first setting, and after the adjustment and measurement of this second similar cube zone the third of these zones was set, by another rearrangement of the crystal on the wax holder at a position 90° to both the first and second settings, and then also adjusted and measured. The readings obtained are set forth below. In order to distinguish clearly the various faces in a case like this it is an excellent plan to make little drawings of their shapes, showing their relative sizes, opposite the readings and label-letters, as is shown in the case of the first of the three zones recorded.

3 CUBE ZONES.

Shape of Face.	Circle Reading.	Angle.
	<i>c</i> 360° 0'A	<i>cp</i> 63° 20'A <i>cc</i> 89° 57'A
	<i>p</i> 296 40 A	
	<i>c</i> 270 3 A	<i>pc</i> 26 37 A
	<i>p</i> 243 31	<i>cp</i> 26 32
	<i>c</i> 180 0 A	<i>pc</i> 63 31 <i>cc</i> 90 3 A
	<i>p</i> 116 34	<i>cp</i> 63 26
	<i>c</i> 90 0 A	<i>pc</i> 26 34 <i>cc</i> 90 0 A
	<i>p</i> 63 25 A	<i>cp</i> 26 35 A <i>cc</i> 90 0 A
	<i>c</i> 0 0 A	<i>pc</i> 63 25 A

[TABLE.

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} c \ 360^\circ \ 0' \\ p \ 333 \ 23 \ A \\ c \ 269 \ 53 \ A \\ p \ 206 \ 30 \\ c \ 179 \ 55 \ A \\ p \ 153 \ 22 \\ c \ 89 \ 55 \ A \\ p \ 26 \ 34 \\ c \ 0 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} cp \ 26^\circ \ 37' \\ pc \ 63 \ 30 \ A \\ cp \ 63 \ 23 \\ pc \ 26 \ 35 \\ cp \ 26 \ 33 \\ pc \ 63 \ 27 \\ cp \ 63 \ 21 \\ pc \ 26 \ 34 \end{array} \right.$	$\left\{ \begin{array}{l} c \ 360^\circ \ 0' \ A \\ p \ 296 \ 35 \\ c \ 270 \ 0 \ A \\ p \ 243 \ 29 \ A \\ c \ 179 \ 58 \ A \\ p \ 116 \ 36 \ A \\ c \ 90 \ 0 \ A \\ p \ 63 \ 27 \ A \\ c \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} cp \ 63^\circ \ 25' \\ pc \ 26 \ 35 \\ cp \ 26 \ 31 \ A \\ pc \ 63 \ 31 \ A \\ cp \ 63 \ 22 \ A \\ pc \ 26 \ 36 \ A \\ cp \ 26 \ 33 \ A \\ pc \ 63 \ 27 \ A \end{array} \right.$
	$\left\{ \begin{array}{l} cc \ 90^\circ \ 7' \\ cc \ 89 \ 58 \ A \\ cc \ 90 \ 0 \ A \\ cc \ 89 \ 55 \end{array} \right.$		$\left\{ \begin{array}{l} cc \ 90^\circ \ 0' \ A \\ cc \ 90 \ 2 \ A \\ cc \ 89 \ 58 \ A \\ cc \ 90 \ 0 \ A \end{array} \right.$

There can be no doubt from these measurements that the *c*-faces are those of the cube or other rectangular solid, the angles in all three mutually perpendicular zones being evidently intended to be 90° . To take the mean of all these values in the neighbourhood of 90° would not only be useless as regards any indication of symmetry, but positively misleading, for to divide the sum of four angles together making up 360° by four, is to obtain as the result in any case $90^\circ \ 0'$. We must look, therefore, to the individual values for the indication of the symmetry. In 5 cases the angles actually observed, of "A" quality, were exactly $90^\circ \ 0'$, and in the other 5 "A" cases the greatest divergence was only $3'$. Hence, we have excellent ground for concluding that nature intended these for cube or other rectangularly inclined faces.

As regards the position of the *p*-faces between the supposed cube faces, the means of the twelve values of the two angles *cp* are respectively $26^\circ \ 34'$ and $63^\circ \ 26'$. It is obvious that the individual values of the acuter angle are all within $3'$ of the mean value $26^\circ \ 34'$, and that we can, therefore, accept this mean angle as representing the truth. This, however, is the exact angle which has been shown to occur between the faces of the cube and of the pentagonal dodecahedron {210}. The whole angle *cc* having been accepted as $90^\circ \ 0'$, the more obtuse angle *cp* must be the difference, the complement, namely, $63^\circ \ 26'$; the 12 individual values of the angle show a maximum divergence (in one case only) of $6'$ from the mean. But it cannot be too strongly emphasised that no deduction concerning the symmetry can be legitimately drawn from these mean values. On the contrary, we must look for any indication of real validity to the individual values, particularly to the relative values of the angles adjacent to any specific cube face, that is, the angles given by the relative positions of the signal-images yielded by two *p*-faces, one on each side of the *c*-face in question, with respect to the image yielded by the *c*-face, which is thus common to the two angles.

The fact that we have accepted the *c*-faces as being mutually rectangularly inclined by no means proves cubic symmetry. They might equally well be the faces of a prism belonging to the tetragonal or rhombic system. In order to prove cubic symmetry we have to show that the faces inclined to the *c*-faces are equally so inclined in all three

rectangular zones. If the inclination were only the same along two the symmetry would probably be tetragonal, and if different along all three zones it would very likely be rhombic.

Regarding now the individual values in pairs, one on each side of a *c*-face in this manner, we do come to the conclusion that every such pair of angles is so nearly equal that we may fairly assume it to have been nature's intention to have made them all exactly so. If we take first the only case of a pair of adjacent *cp* angles having "A" values, the case in the third zone where a pair of such adjacent values are $26^{\circ} 36'$ and $26^{\circ} 33'$, the difference is only $3'$, and the difference from the mean of the whole 12 values is only $1'$ in the one case and $2'$ in the other. If we take all the 5 pairs in turn other than this sixth pair, we observe that the differences of the two angles from each other are respectively, $5'$, $1'$, $3'$, $2'$, and $4'$. These variations are sufficiently small to be legitimately ascribed to slight disturbances during the formation of the crystal, and the angular values may, therefore, be taken as equal with respect to the symmetry. Having thus decided, **and not before**, we may set down the mean value in the table of angles as representing the ideal angle intended by nature.

Further, respecting the symmetry. We have found that the three mutually rectangular zones (represented by the primitive circle and the two diameters parallel to the edges of the page) exhibit similar angles, each showing four angles of $26^{\circ} 34'$ and four of $63^{\circ} 26'$, together making up four right angles. The symmetry, therefore, is of cubic character. But it is not of full cubic character; for, as will be plain from the stereographic projection, while there is symmetry about the three axial planes (parallel to the faces of the cube), there is not also symmetry to the 45° -diagonal planes of symmetry (parallel to the faces of the rhombic dodecahedron). The poles of the faces of the form *p* are only symmetrical to the two diameters parallel to the page edges, and not also symmetrical to the diameters inclined at 45° to the edges of the page. Hence the symmetry is that of class 30, the pyrites class, and not that of class 32, the holohedral class. The form *p* is consequently that of the pentagonal dodecahedron {210} and not that of the tetrakis hexahedron having the same symbol and double the number of faces; the latter would have shown a further pair of *p*-poles at $26^{\circ} 34'$ on each side of the central *c*-pole on the horizontal diameter of the projection, and also a further pair at $63^{\circ} 26'$ on each side of the centre on the vertical diameter, and also four more faces in the zone represented by the primitive circle. - That is, we should have had the angle $26^{\circ} 34'$ repeated 8 times in each of the three zones measured, instead of only 4 times, if the *p*-form had been represented by the entire number of faces possible to full cubic symmetry, in other words, had the form been the tetrakis hexahedron.

Hence, our conclusion from the study of these three first measured zones is that the symmetry is that of class 30 of the cubic system, the pyrites class, and that the forms exhibited in those three zones are the cube and the pentagonal dodecahedron {210}, represented in the

stereographic projection and in the drawing of the crystal by the letters *c* and *p* respectively.

We pass next to the measurement of the zones containing the cube faces and those which we surmise to be octahedron faces, marked *o* in the figures.

Setting the crystal on the wax holder of the goniometer, and arranging the holder in its socket so that a cube face (one of the broader strips for choice) and an adjacent octahedron face can be adjusted parallel to the goniometer axis (setting by preference the much larger and more brilliantly reflecting octahedron face parallel to one of the adjusting movements), and then adjusting and centring these two faces, we find on rotation round the zone that the latter consists of two parallel *c*-faces and four *o*-faces, the latter in two parallel pairs. The measurements obtained from such a zone are set forth below, together with those for two other similar zones which were also measured. There are indeed six such zones altogether, all of which were measured, but three will be ample for us to record here, in order to afford adequate evidence of the symmetry. Two of the zones are indicated by the diagonal 45°-diameters in the projection, and the four others by the arc-zones constructed about the rectangular axial diameters, one pair about each, as fully described in the last chapter in connection with the construction of the stereographic projection (Fig. 81) of the three simple cubic forms, the cube, octahedron, and rhombic dodecahedron.

6 CUBE-OCTAHEDRON ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.	Circle Reading.	Angle.
<i>c</i> 360° 0'A	<i>co</i> 54° 44'A	<i>c</i> 360° 0'A	<i>co</i> 54° 41'A	<i>c</i> 360° 0'A	<i>co</i> 54° 44'A
<i>o</i> 305 16 A	<i>oo</i> 70 32 A	<i>o</i> 305 19 A	<i>oo</i> 70 33 A	<i>o</i> 305 16 A	<i>oo</i> 70 32 A
<i>o</i> 234 44 A	<i>oc</i> 54 46 A	<i>o</i> 234 46 A	<i>oc</i> 54 44 A	<i>o</i> 234 44 A	<i>oc</i> 54 43 A
<i>c</i> 179 58 A	<i>co</i> 54 42 A	<i>c</i> 180 2 A	<i>co</i> 54 44	<i>c</i> 180 1 A	<i>co</i> 54 41 A
<i>o</i> 125 16 A	<i>oo</i> 70 31 A	<i>o</i> 125 18	<i>oo</i> 70 31	<i>o</i> 125 20 A	<i>oo</i> 70 31 A
<i>o</i> 54 45 A	<i>oc</i> 54 44 A	<i>o</i> 54 47 A	<i>oc</i> 54 47 A	<i>o</i> 54 49 A	<i>oc</i> 54 48 A
<i>c</i> 0 1 A		<i>c</i> 0 0 A		<i>c</i> 0 1 A	

These excellent readings at once assure us that we are indeed dealing with the combination of the cube and the octahedron. The values of the angle *oo* are all within one minute of the exact angle of the octahedron, 70° 32', and the cube-octahedron angle *co* is only once as much as 4' removed from the theoretical value 54° 44'. The readings for the other three zones were equally good and as fully confirmatory of this conclusion. The images of the goniometer signal afforded by the octahedron faces were particularly excellent, very brilliant, single, and clearly defined.

In the last chapter the pentagonal dodecahedron {210} was stated to exhibit an angle over the shorter edges of 66° 25'. This we should next proceed to verify on our crystal.

Suppose we start by re-setting the crystal so that we can measure the angle between the face (210), which is that *p*-face to the right of the front *c*-face (vertical strip) in Fig. 119, and the *p*-face (102), which lies immediately above the front *c*-face in the figure. Adjusting these two faces parallel to the goniometer axis, and then exploring

the zone by rotation of the axis, we find that the zone comprises also two other parallel faces, at the back of the crystal in the figure. Three such zones were measured, and the readings for them are recorded below; namely, besides the zone just described, a second one containing the same p -face (210) and the p -face (10 $\bar{2}$), which is the bottom face in the figure, and a third one containing again the initial p -face (210) and the p -face (021) which is the upper one on the right side of the figure. Two other similar zones on a different part of the crystal were also measured, but the three named, and recorded below, are ample for our purpose.

SHORTER PENTAGONAL-DODECAHEDRAL-EDGE ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} p \ 360^\circ \ 0' \\ p \ 246 \ 29 \\ p \ 180 \ 0 \\ p \ 66 \ 24 \text{ A} \\ p \ 0 \ 1 \text{ A} \end{array} \right.$	$pp \ 66^\circ \ 29'$ $pp \ 66 \ 23 \text{ A}$	$\left\{ \begin{array}{l} p \ 360^\circ \ 0' \text{ A} \\ p \ 246 \ 26 \text{ A} \\ p \ 180 \ 0 \text{ A} \\ p \ 66 \ 26 \\ p \ 0 \ 0 \end{array} \right.$	$pp \ 66^\circ \ 26' \text{ A}$ $pp \ 66 \ 26$	$\left\{ \begin{array}{l} p \ 360^\circ \ 0' \text{ A} \\ p \ 293 \ 41 \\ p \ 180 \ 1 \\ p \ 113 \ 33 \\ p \ 0 \ 0 \end{array} \right.$	$pp \ 66^\circ \ 19'$ $pp \ 66 \ 28$

The mean of these 6 values of the acute angle pp is $66^\circ 25'$, exactly the theoretical value of the angle across the shorter edges (edges other than the singular long edge which in our crystal is replaced by a strip cube face) of the pentagonal dodecahedron {210}. The individual values are adequately near to the mean and to each other for us to accept the mean as the angle intended by nature; no question of symmetry is involved as regards the two values from one and the same zone, as they are afforded by faces obviously intended to be parallel, and the mean value for each of the three zones is practically identical (the means being respectively $66^\circ 26'$, $66^\circ 26'$, and $66^\circ 24'$). The first two zones are the pair of arc-zones about the diameter $p = (210) : c = (001) : p = (2\bar{1}0)$ and very close to it; the third is one of the wider arc-zones terminating at the ends of the same diameter in the stereographic projection, and it will be shown presently that the shorter or the longer arcs pp of both these types of zone are equal, in accordance with the symmetry of class 30, and with the fact that the angles across all four shorter edges of the pentagonal dodecahedron are equal.

There is only one further type of zone that obviously invites measurement, namely, that containing the angle between adjacent p - and o -faces. Six such zones were actually measured on the crystal before us, and the readings for three of them are given below.

They contained respectively the angles po between the p -face (102) above the central cube strip in Fig. 119 and the right-hand primary o -face (111) adjoining, between the same p -face (102) and the adjacent left o -face (1 $\bar{1}$ 1), and between the right central initial p -face (210) and the right primary o -face (111) adjoining. All these zones contain this angle op twice, yielded by two pairs of parallel o - and parallel p -faces. Its value will be shown during the course of the calculations to be $39^\circ 14'$. The measurements for the three zones specifically mentioned were as follows:

OCTAHEDRAL-PENTAGONAL-DODECAHEDRAL ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} p \ 360^\circ \ 0' A \\ o \ 320 \ 47 \ A \\ p \ 179 \ 59 \\ o \ 140 \ 48 \ A \\ p \ 0 \ 0 \ A \end{array} \right.$	$po \ 39^\circ \ 13' A$ $po \ 39 \ 11$	$\left\{ \begin{array}{l} o \ 360^\circ \ 0' A \\ p \ 320 \ 47 \ A \\ o \ 180 \ 3 \\ p \ 140 \ 48 \\ o \ 0 \ 0 \ A \end{array} \right.$	$op \ 39^\circ \ 13' A$ $op \ 39 \ 15$	$\left\{ \begin{array}{l} p \ 360^\circ \ 0' A \\ o \ 320 \ 42 \\ p \ 179 \ 59 \\ o \ 140 \ 44 \\ p \ 0 \ 0 \ A \end{array} \right.$	$po \ 39^\circ \ 18'$ $po \ 39 \ 15$

The mean of these six values of the acute angle op , the angle between adjacent o - and p -faces, is $39^\circ 14'$, which is identical with the theoretical angle assuming the o -faces to be those of the octahedron and the p -faces to be those of the pentagonal dodecahedron {210}. The individual values are also adequately close to indicate clearly intended identity with the mean value.

We have now very fully measured this crystal of cobaltite, and shown that the measurements agree perfectly with the assumption that the forms present are those of the cube, octahedron, and pentagonal dodecahedron {210}. The theoretical values of the angles have been assumed to be known, but, as in the case of garnet, we shall now no longer take this for granted, but actually calculate those angles which have not already been calculated during our investigation of garnet, or previously in the discussion of the three primary forms of the cubic system in the last chapter. The octahedron angle $oo = 70^\circ 32'$ has already been calculated (p. 155), together with its half supplement, the angle between adjacent cube and octahedron faces, $co = 54^\circ 44'$. The construction of the stereographic projection has also been fully described (p. 154) and given in Fig. 81, as far as the cube and octahedron faces are concerned. The angle $54^\circ 44'$ just referred to is that between the central cube face (001) of the projection and each of the four octahedron faces surrounding it, or between the parallel c -face underneath, (00 $\bar{1}$), and the four octahedron faces in the lower hemisphere. All these eight o -poles lie on the diagonal 45° -diameters. It is also the angle of the arc co of each of the pair of arc-zones terminating at the ends of the axial diameters parallel to the edges of the page.

The additional construction shown in Fig. 120 for cobaltite is only of a very simple character, and can be described as we proceed with the calculations.

To calculate the position of the facial poles of the pentagonal dodecahedron {210}, that is, to find $cp = (100) : (210)$.

This can best be done by utilising the principle of the anharmonic ratio of four poles in a zone, fully explained in Chapter VI. The four poles in question are $c = (100)$, $p = (210)$, $d = (110)$ which is not present but the position of which is known to be 45° from either cube face, and $c' = (010)$ which we can temporarily call c' to distinguish it from (100). The conditions are shown in Fig. 121, and applying the principle of the anharmonic ratio, we have the equation :

$$\begin{aligned} & \begin{array}{cc} 010 & 100 \\ \times & \times \\ \sin 45^\circ & \sin cp \\ \sin c'p & \sin 45^\circ \end{array} = \begin{array}{cc} 110 & 210 \\ 010 & 100 \\ \times & \times \\ 210 & 110 \end{array} \\ & \frac{\sin cp}{\cos cp} = \frac{1}{2} \cdot \frac{1}{1} \\ & \tan cp = 0.5 \\ & cp = 26^\circ 34'. \end{aligned}$$

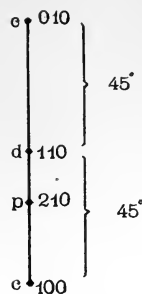


FIG. 121.

It is thus seen that the anharmonic equation gives immediately the value of the tangent of the required angle cp , which is exactly 0.5. Now 0.5000 is the value of the tangent of $26^\circ 34'$, as obtained from a table of natural tangents. This is the angle which we have already assumed as the theoretical one for $\{210\}$, so that the accuracy of that assumption is now definitely proved. Double the angle, $53^\circ 8'$, is the angle over the singular long edges of the pentagonal dodecahedron $\{210\}$, as stated in the last chapter, these six edges being replaced by the cube faces in our crystal.

We can now place the poles of the pentagonal dodecahedron in position in the stereographic projection, beginning with those on the primitive circle, namely, (210) at $26^\circ 34'$ to the right of (100) , and $(2\bar{1}0)$ at the same angle to the left. We draw next a couple of diameters to pass through the central pole (001) and terminate at (210) and $(2\bar{1}0)$. The other terminations on the upper part of the primitive circle are the poles $(\bar{2}10)$ and $(\bar{2}\bar{1}0)$.

The other p -poles will lie on the axial diameters parallel to the edges of the page. To find one of those on the horizontal diameter, say (021) , we mark off the angle $26^\circ 34'$ from (010) along the primitive circle, either up or down (indeed it will be most useful to mark both off, and to draw diameters through the centre to them, as we shall subsequently require these diameters), and join the point so marked off to the pole of the zone, that is, to (100) if the marking had been upwards or to $(\bar{1}00)$ if it had been downwards. Where this line cuts the horizontal diameter is the pole (021) , and also the lower pole $(02\bar{1})$ represented by the ring. The poles $(0\bar{2}1)$ and $0\bar{2}\bar{1}$ may also be inserted at the same distance to the left of the centre as the pair just found are to the right, along the horizontal diameter.

The poles (102) and $(10\bar{2})$, and $(\bar{1}02)$ and $(\bar{1}0\bar{2})$, lie on the vertical axial diameter, and can be found by the same process, and in fact by using the same marked-off point at $26^\circ 34'$ from (010) , for we have only to join this to $(0\bar{1}0)$, the proper pole of the zone on which the desired pole lies, in order to get, at the intersection of the joining line with the vertical diameter, the position of the poles $(\bar{1}02)$ and $(10\bar{2})$, or (102) and $(10\bar{2})$, according as the marking off had been upwards or downwards.

We have now found the positions of all the poles of the crystal faces, and it only remains, as far as the stereographic projection is concerned, to connect them by further arcs representing the other zones measured, as actually shown in Fig. 120, either by the trial method with the compasses, or by the more systematic geometrical method of first finding the proper centre from which to construct the required arc to pass through three given poles belonging to the zone.

The indices of the pentagonal dodecahedron are proved to be $\{210\}$ by the mode of calculating the angle cp ; for we assumed those indices in the calculation, and the result of the latter, $26^\circ 34'$, coincides absolutely with the measured angle.

There are only two further angles to calculate, owing to the perfection of cubic symmetry, namely, the angle over the shorter edges of the pentagonal dodecahedron, which we have stated in the last chapter to be $66^\circ 25'$ and confirmed by measurement, and of which we may take as example either $(210) : (021)$ or $(210) : (102)$, or both;

and the acute angle between adjacent octahedron and pentagonal dodecahedron faces, which we have found by measurement to be $39^{\circ} 14'$. We shall, therefore, proceed now to calculate these two final angles from the calculated data already acquired.

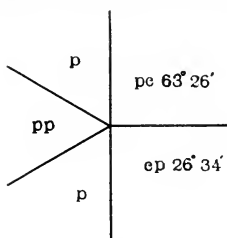


FIG. 122.

To find $pp = (210) : (021)$ or $(210) : (102)$.

We utilise the right-angled triangle $p = (210) : p = (021) : c = (010)$, or the similar one having the same elements $p = (210) : p = (102) : c = (100)$, the Napierian diagram for either of which is given in Fig. 122, from which we derive the equation:

$$\cos pp = \cos 63^{\circ} 26' \cos 26^{\circ} 34' \\ pp = 66^{\circ} 25'.$$

The angle pp across the shorter edges of $\{210\}$ is thus proved to be $66^{\circ} 25'$, as stated in the last chapter, and as found by measurement, a further proof that the form p is indeed $\{210\}$.

To find $op = (111) : (102)$ or $(111) : (210)$.

These two angles should be identical in value if the symmetry be truly cubic. To find the first we utilise the oblique angled triangle $c = (100) : p = (102) : o = (111)$, in which we know that $cp = 63^{\circ} 26'$, $co = 54^{\circ} 44'$, and the included angle at $c = 45^{\circ}$. Employing formula (b) of Chapter VII, we have:

$$\tan \theta = \tan 54^{\circ} 44' \cos 45^{\circ}; \text{ and } \cos op = \frac{\cos 54^{\circ} 44' \cos (63^{\circ} 26' - \theta)}{\cos \theta} \\ \theta = 45^{\circ}. \quad op = 39^{\circ} 14'.$$

To find the second, namely, $o = (111) : p = (210)$, we can use the right-angled triangle $opd = (111) : (210) : (110)$ (the letter d not being given on the projection as the face (110) is not developed), in which we know that od is $35^{\circ} 16'$ (half the octahedron angle between (111) and $(11\bar{1})$), that dp is $45^{\circ} - 26^{\circ} 34' = 18^{\circ} 26'$, and that the angle between them at d is a right angle. The Napierian diagram is given in Fig. 123, and from it we derive:

$$\cos op = \cos 18^{\circ} 26' \cos 35^{\circ} 16' \\ op = 39^{\circ} 14'.$$

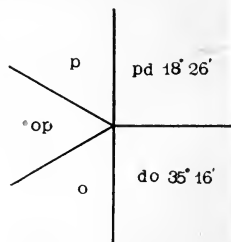


FIG. 123.

We thus see that both the angles under consideration between the primary o -face (111) and different adjacent p -faces are $39^{\circ} 14'$, the exact value of the angle as measured.

This is in complete accordance with the symmetry of class 30 of the cubic system, and the proof now achieved brings to an end our morphological investigation of the crystal.

The results are printed in concise form in the following statement:

Table of Results for Cobaltite, CoAsS.

Crystal - System: Cubic. **Class:** 30, pyrites class, parallel-faced hemihedral or tesseral central class.

Habit: Approximately spherical, due to more or less equal development of triangular faces of pyritohedron and octahedron, and blunting of pyritohedron edges by cube-face strips.

Forms observed: Cube $c = \{100\}$, octahedron $o = \{111\}$, and pentagonal dodecahedron $p = \{210\}$ (pyritohedron). Two latter predominate, octahedron somewhat the more prominently. Cube faces narrow, some-

times mere strips, and the adjacent faces of {210} are striated parallel to the strips.

Interfacial angles : as given in the following table.

MORPHOLOGICAL ANGLES OF COBALTITE.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$\left\{ \begin{array}{l} cc=(100):(010) \\ cp=(100):(210) \\ pc=(210):(010) \end{array} \right. . . .$	12	89° 55'-90° 7'	90° 0'	90° 0'	0'
	12	26 31-26 37	26 34	26 34	0
	12	63 20-63 31	63 26	63 26	0
$\left\{ \begin{array}{l} co=(001):(111) \\ \text{or } (100):(111) \end{array} \right. . . .$	24	54 41-54 48	54 44	54 44	0
$\left\{ \begin{array}{l} oo=(111):(\bar{1}\bar{1}\bar{1}) \\ \text{or } (111):(\bar{1}\bar{1}1) \end{array} \right. . . .$	12	70 31-70 33	70 32	70 32	0
$\begin{array}{l} pp=(210):(021) \\ \text{or } (210):(102) \end{array} . . .$	10	66 19-66 29	66 25	66 25	0
$\begin{array}{l} op=(111):(102) \\ \text{or } (111):(210) \end{array} . . .$	12	39 11-39 18	39 14	39 14	0

Owing to the excellence of the faces of this crystal, and the consequent clear definition of the signal images, the agreement between the observed and calculated angles is exceptionally good.

CHAPTER XIII

TETRAGONAL SYSTEM.

Three rectangular crystallographic axes, two being equal and lying in the horizontal plane, the third being vertical. Characteristic of system, one tetragonal symmetry axis.

THE tetragonal system includes seven classes, the lowest of which in order of symmetry possesses only the minimum essential of the system, namely, the tetragonal axis of symmetry. This latter is identical in direction with the vertical (the third and singular) crystallographic axis, the two equal crystallographic axes being arranged equatorially.

The seven classes are derived in a manner which arranges them in three groups of two, two, and three classes respectively.

The first two classes possess only the tetragonal axis of symmetry, but in the one case it is of simple, and in the other of compound symmetry. In the latter case we have class 10, the stereographic projection of which as regards elements of symmetry and the general form $\{hkl\}$ has already been shown in Fig. 69, which is here repeated in Fig. 125;

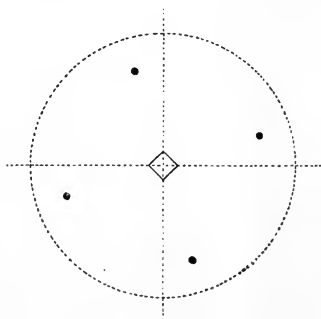


FIG. 124.—Class 9.

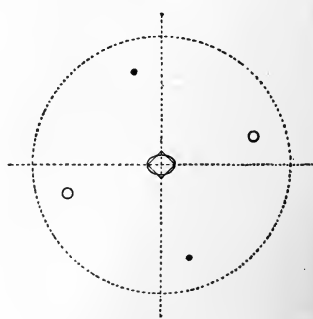


FIG. 125.—Class 10.

and in the former case we have class 9, represented in stereographic projection in Fig. 124. The effect of the symmetry being only compound in class 10 is to render the tetragonal axis only equivalent to a digonal axis of simple symmetry, the reflection across the equatorial

plane which accompanies the operation of the tetragonal axis, rotation for 90° , causing two of the poles of the general form to be thrown into the lower hemisphere, thus producing a bisphenoidal solid represented by two poles in each hemisphere, arranged alternately to each other. In class 9, where the symmetry is simple tetragonal, the four poles are situated in the same hemisphere, and the tetragonal axis is thus of polar character, and the faces developed on the crystal at the two ends of the vertical axis may consequently belong to quite different forms.

If we add to the single element of symmetry exhibited in classes 9 and 10 a digonal axis in the equatorial plane, the four additional poles introduced by its operation are found to be also symmetrical to a second digonal axis at right angles to the first, the two being coincident with the horizontal crystallographic axes; two new classes, 11 and 14, are thus produced, the stereographic projections of which are given in Figs. 126 and 127. In class 14, it is found that the disposition

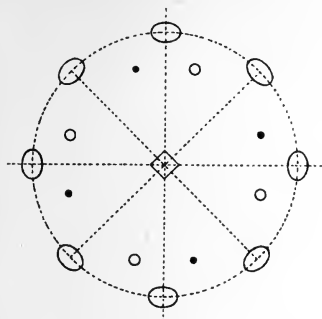


FIG. 126.—Class 11.

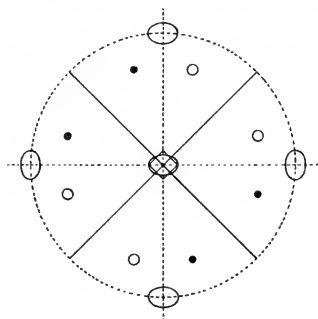


FIG. 127.—Class 14.

of faces brought about by the operation of these further elements of symmetry is such that there is also symmetry about two vertical planes intersecting in the tetragonal axis, and bisecting the angles between the two vertical crystallographic axial planes. In class 11 no symmetry planes are developed by the addition of the two rectangular digonal axes, but there is symmetry also produced about another pair of digonal axes, bisecting the angles between those already added.

Three further classes are then derived from class 9, which possesses the minimum of truly tetragonal symmetry, namely the tetragonal axis of simple symmetry itself, by the addition of a plane of equatorial symmetry (class 12), a vertical plane of symmetry (class 13), or both (class 15). In the case of class 12 the addition of the equatorial plane of symmetry perpendicular to the tetragonal axis simply results in removing the polar character of that axis, the two ends of the crystal, upper and lower, being now symmetrical. Its stereographic projection is given in Fig. 128. In class 13 (Fig. 129) the addition of a plane of symmetry parallel to the vertical axis causes the repetition of each of the four poles of the general form $\{hkl\}$ across that symmetry plane,

and the effect is to make the crystal and its projection symmetrical also about three other planes of symmetry also parallel to the vertical

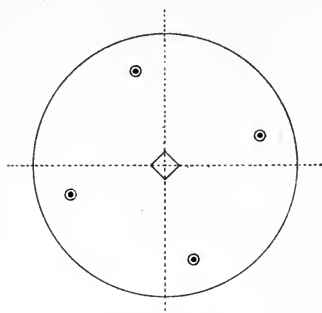


FIG. 128.—Class 12.

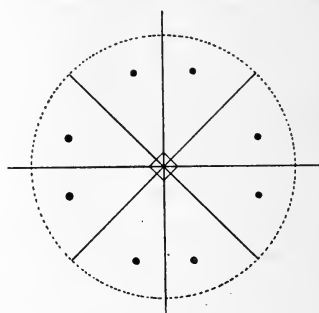


FIG. 129.—Class 13.

tetragonal axis and intersecting each other and the first plane in it, the four planes being arranged at 45° from each other.

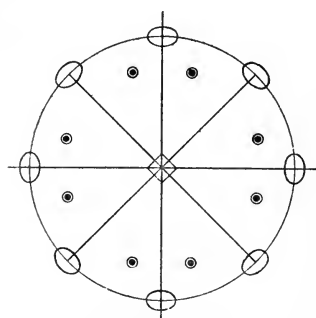


FIG. 130.—Class 15.

Finally, by combining classes 12 and 13, or in other words, adding to class 9 both the equatorial plane and the four vertical symmetry planes, we find that the effect is also to cause the presence of four digonal axes where the four vertical planes intersect the equatorial plane, and thus we have combined in this class, 15, represented in Fig. 130, the whole of the elements of symmetry possible in the tetragonal system. Class 15 is, therefore, the holohedral class, or class of highest tetragonal symmetry.

It will be observed that the tetragonal axis is polar only in classes 9 and 13, so that these classes are the hemimorphic ones.

*Class 15.—Ditetragonal-Bipyramidal Class. Holohedral Class.
Type, Ditetragonal Equatorial.*

The crystals of this highest class of tetragonal symmetry are distinguished, as above indicated, by possessing the one essential tetragonal axis, and 4 symmetry planes (of which two are crystallographic axial planes) intersecting in it, thus rendering it a ditetragonal axis, and which are arranged at 45° to each other; there is also a fifth equatorial plane of symmetry perpendicular to the others and to the tetragonal axis, and there are consequently also 4 digonal (di-digonal) axes lying in this equatorial plane, at its intersection with each of the 4 other symmetry planes. These elements of symmetry are clearly shown by the stereographic projection given in Fig. 130, the plane of projection being

the equatorial plane, which forms the third crystallographic axial plane; the tetragonal axis is represented by the pole in the centre. The two vertical crystallographic axial planes are represented by the back-to-front and right-and-left diameters. There are seven forms in this holohedral class, which will be considered in turn.

(1) **The basal pinakoid, $\{001\}$.** The form $\{100\}$ does not, as in the cubic system, include the six faces parallel to the crystallographic axial planes, the different nature of the vertical axis now causing these faces to be divided into two forms, consisting of the four vertical faces in one case and the two horizontal ones in the other, which latter is now under consideration. Hence, this form is an open one consisting only of the two parallel faces (001) and $(0\bar{0}\bar{1})$, and is, therefore, termed the basal pinakoid. Its pole occupies the centre of the projection, and may be considered as the special form produced when the general pole (hkl) , situated inside the primary triangle in the projection formed by two adjacent digonal axial radii and the 45° -segment of the primitive circle which they cut off, migrates to that corner of the triangle which occupies the centre of the projection, where the digonal axes intersect.

(2) **Tetragonal prism of the first order, $\{110\}$,** also called **protoprism**. This form is also an open one like the basal pinakoid, but consists of four faces, (110) , $(\bar{1}\bar{1}0)$, $(\bar{1}10)$, and $(1\bar{1}0)$, which are parallel to the two vertical planes of symmetry bisecting the angles between the two vertical crystallographic axial planes, and the edges of intersection of which are parallel to the vertical axis. The poles of the four faces are consequently situated on the primitive circle at the ends of the diameters arranged at 45° to the axial diameters. The faces thus make equal intercepts on the horizontal crystallographic axes a and b , represented in the projection by the axial diameters just referred to parallel to the edges of the page, a running back and front and b laterally right and left. It will be obvious that, given one such pole at the intersection of one of the 45° -diameters with the primitive circle, the symmetry demands that there shall be three others, by repetition over the crystallographic axial planes of symmetry. This form may be considered as the special case which occurs when the general pole (hkl) migrates to that corner of the primary triangle of the projection which is formed by the intersection of an interaxial diameter with the primitive circle. With the basal pinakoid end faces, this form makes a closed rectangular prism, of square horizontal section, and only differing from the cube by the elongation or shortening of the vertical dimension.

It is represented as thus closed in Fig. 131. The axes employed in this and the succeeding tetragonal figures are those of an actual tetragonal substance, anatase, for which the axial ratio of the two equal horizontal axes to the vertical axis is $a:c = 1:1.7771$.

(3) **Tetragonal prism of the second order, $\{100\}$.** When the faces of the tetragonal prism are parallel to the vertical crystallographic axial planes, as in the case of the vertical cube faces, we have this second variety of tetragonal prism of square section produced, identical in shape with that of the first order and differing only in position. The poles of the form are situated at the points of intersection of the primitive circle and the crystallographic axial diameters, so that the form may be considered as the special case produced when the general pole (hkl) migrates to this corner of the primary 45° -triangle. The front face is (100) , the back face parallel to it $(\bar{1}00)$, the right-hand side face is (010) , and the left-hand face parallel to it is $(0\bar{1}0)$. The form as closed by the basal pinakoid is shown in Fig. 132. In an actual crystal showing

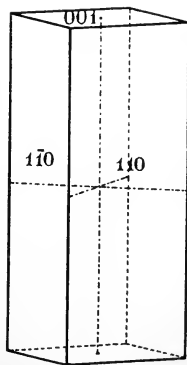


Fig. 131.—The First Order Tetragonal Prism.

both forms of tetragonal prism, or different crystals showing them separately, there will usually be surface markings, striations, or other physical differences to distinguish the two from one another.

(4) **Ditetragonal prism, $\{h k 0\}$.** Any face the pole of which is situated on the primitive circle (that is, which is parallel to the vertical crystallographic axis), in any position other than at the intersection with one of the 4 principal diameters shown in the projection, Fig. 130, must, by reason of the symmetry of this class 15, be repeated over each of the 4 symmetry planes represented by those diameters, so as to form an eight-sided prism, which has always a section showing two alternating kinds of angles, a less and a greater. For the case where the pole would be half-way between any two principal diameters is an impossible one, corresponding to irrational values of the first

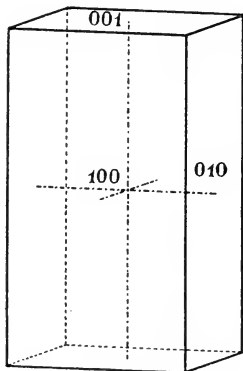


FIG. 132.—The Second Order Tetragonal Prism.

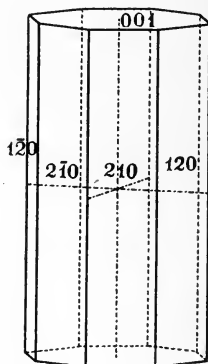


FIG. 133.—The Ditetragonal Prism.

two indices, their ratio being $= \tan 22\frac{1}{2}^\circ = 2.4142$. A truly regular octagonal prism is a frequent natural occurrence on tetragonal crystals, but it is always a combination of the first and second order prisms $\{110\}$ and $\{100\}$ showing the physical differences already alluded to. The ditetragonal prism $\{210\}$ is shown in Fig. 133, combined with the basal

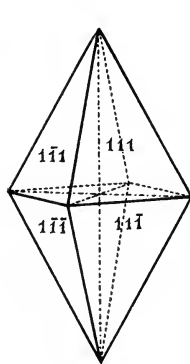


FIG. 134.—First Order Tetragonal Bipyramid.

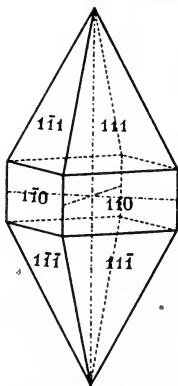


FIG. 135.—Combination of First Order Prism and Pyramid.

plane. It may be considered as the special case produced when the general pole (hkl) migrates on to the primitive circle, but not at an intersection with an axis, that is, on to the 45° -segmental side of the primary triangle.

(5) **Tetragonal bipyramid of the first order, $\{h k l\}$.** When the pole leaves the primitive circle and lies on the great circle represented by a principal diameter bisecting the angle between the horizontal axial diameters, that is, when it lies on the arc between the pole of the basal plane (001) and that of the first order prism $\{110\}$, the form produced is a four-sided pyramid, each face making equal intercepts on the two horizontal axes. When this form is chosen as the parametral form the symbol is $\{111\}$, otherwise the symbol is of the general type $\{h k l\}$, the first two indices being invariably equal.

Each pole on the projection now represents two faces, one in the upper hemisphere and the other in the lower hemisphere, the pyramid being thus a double

one or "bipyramid" pointing upwards and downwards, the two pyramids being mutually joined in, and springing from, a square base occupying the central plane of the solid. It is illustrated in Fig. 134, and adjoining it in Fig. 135 is a drawing illustrating its combination with, and the truncation of its edges by, a prism likewise of the first order. Indeed the prism is the limiting case when the intercept on the c axis becomes increased to infinity, and the faces consequently become parallel to c ; the indices then become $\{hk0\}$, or to give them their simplest form $\{110\}$. Similarly, if the intercept on the vertical axis becomes likewise rationally reduced, we have as the limiting case of reduction to 0, the production of the basal pinakoid $\{001\}$, parallel to the two horizontal axes. In the first case the pyramid becomes longer and sharper, with increasing angle over the basal edges, the l index becoming less and less in comparison with h ; while in the latter case the pyramid becomes shorter and flatter, with diminishing angle over the basal edges, and a greater and greater l index as compared with h .

(6) **Tetragonal bipyramid of the second order, $\{h0l\}$.** When the pole of a face lies

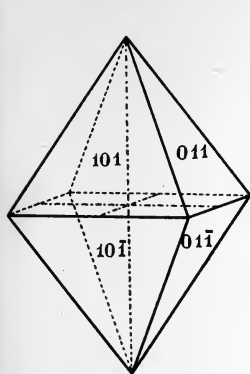


FIG. 136.—The Second Order Tetragonal Bipyramid.

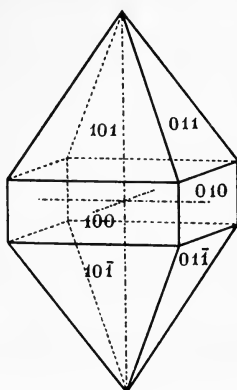


FIG. 137.—Combination of Second Order Prism and Pyramid.

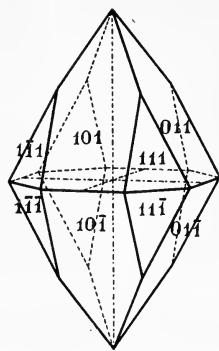


FIG. 138.—Combination of First and Second Order Bipyramids.

on the 90° -arc between (001) and (100) , represented by a crystallographic axial plane semi-diameter of the projection, a four-faced pyramid is again produced, by the repetition of the pole on the other three similar arcs as demanded by the symmetry of the class, and this being duplicated on the lower hemisphere gives us the tetragonal bipyramid of the second order, which is also a double square-based pyramid like that of the first order. Any face, however, will always be parallel to one of the horizontal crystallographic axes, and will consequently have a 0 in its symbol. If the intercepts are the parametral ones on the other two axes, the symbol is $\{101\}$. This form is shown in Fig. 136. The second order prism $\{100\}$ forms the limiting case of increase of the vertical intercept, and the basal pinakoid $\{001\}$ again forms the limiting case of shortening. In the four succeeding illustrations are exhibited several combinations of simple tetragonal forms. In Fig. 137 a tetragonal bipyramid of the second order with the prism of the same order truncating the basal edges is shown. In Fig. 138 a combination of the two primary tetragonal bipyramids of the two different orders $\{111\}$ and $\{101\}$ is represented. Fig. 139 portrays a

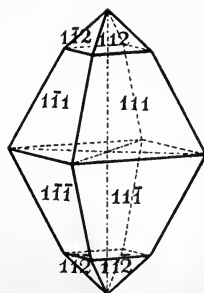


FIG. 139.—Combination of Two First Order Bipyramids.

combination of two bipyramids of the same order but having different values of the l index, $\{111\}$ and $\{112\}$. And in Fig. 140 is represented the primary bipyramid of the first order combined with the primary prism of the second order and the basal pinakoid.

(7) The last holohedral form of the tetragonal system is the general one $\{hkl\}$, the poles of which occupy the interior of the elementary spherical triangles, in positions which are symmetrical to the 4 planes of symmetry represented by the four diameters at 45° , as shown on the stereographic projection in Fig. 141, which is a repetition of Fig. 130, and represents the poles of one of the possible forms $\{312\}$ of anatase, as well as the elements of symmetry. It is the 16-faced **ditetragonal bipyramid**, the appearance of which is represented in Fig. 142, the particular one shown being $\{212\}$. As its name implies it is a doubly terminated pyramid the basal section of which is eight-sided but not

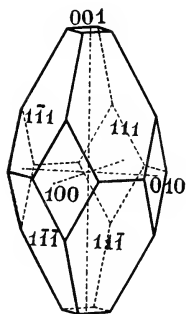


FIG. 140.—Combination of First Order Bipyramid, Second Order Prism, and Basal Pinakoid.

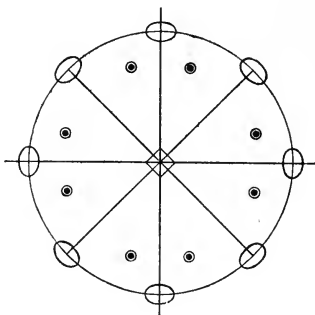


FIG. 141.—Symmetry Elements and General Form of Class 15.

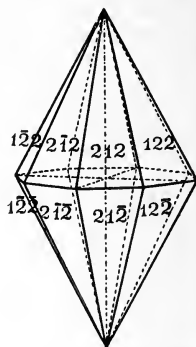


FIG. 142.
The Ditetragonal Bipyramid.

a regular octagon, although the sides are of equal length; it resembles the ditetragonal prism in section, and indeed that form may be regarded as the limiting case of a ditetragonal bipyramid of infinite vertical intercept. The section of both prism and pyramid shows alternately larger and smaller angles. The angles over the basal edges of the ditetragonal bipyramid are, however, all equal, just as are those of the simple pyramids of the first and second orders. As already explained in connection with the prism, a ditetragonal pyramid the base of which is a regular octagon is impossible on account of irrationality. The actual angles over the pyramidal edges resemble the section in exhibiting alternately larger and smaller values. This general form of the tetragonal system, possessing the fullest symmetry appertaining to that system, passes into a simple pyramid of the first order when $h=k$, the two faces from each octant then coalescing into the same plane making equal intercepts on the two horizontal crystallographic axes. It passes into the pyramid of the second order when $k=0$ in the form symbol $\{hkl\}$, a face from each of every two adjacent octants coalescing to form a plane parallel to one of the horizontal crystallographic axes. Finally, if either of these limiting cases is also accompanied by the extension of the intercept on the vertical axis to infinity, the prism of the corresponding order is produced, or if the vertical intercept becomes 0 the ditetragonal bipyramid passes into the basal pinakoid. In all these cases of the evolution of one form from another, it must be remembered that the passage occurs in definite steps, corresponding to specific stages or cases of rationality of the indices h , k , l , and not by very gradual changes of almost insensible magnitude. This completes the detailed study of the holohedral class 15 of tetragonal crystals. The following is a summary of the seven types of forms:

List of Forms in Class 15.

- $\{001\}$ Basal pinakoid. 2 faces.
 $\{110\}$ Tetragonal prism of first order. 4 faces.
 $\{100\}$ Tetragonal prism of second order. 4 faces.
 $\{h k 0\}$ Ditetragonal prism. 8 faces.
 $\{h h l\}$ Tetragonal bipyramid of the first order, including the primary parametral form $\{111\}$. 8 faces.
 $\{h 0 l\}$ Tetragonal bipyramid of the second order, including the primary one $\{101\}$. 8 faces.
 $\{h k l\}$ Ditetragonal bipyramid. 16 faces.

In actual practical work the discrimination between first and second order prisms and pyramids is not possible, unless both forms are developed on the crystal. For rotation of either for 45° converts it

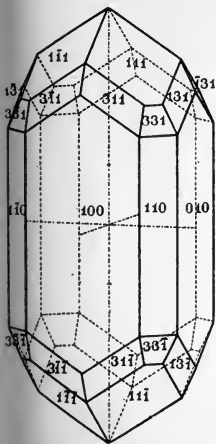


FIG. 143.—Zircon.

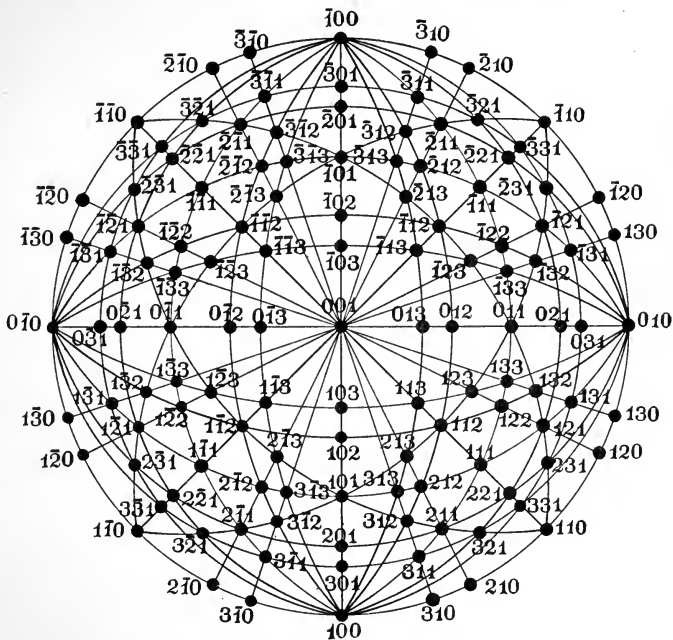


FIG. 144.—Stereographic Projection of Tetragonal Forms.

into the other, and even when both are present it is only possible to decide arbitrarily which is to be considered as of the first order and which of the second. No ambiguity can, however, arise as to which is the vertical axis, on account of the unmistakable particular character of this axis, which is often termed the principal axis, and which will subsequently be shown to confer on the crystal optical properties of a unique character, which render any mistake in recognising it out of the question.

Excellent examples of substances crystallising in the ditetragonal-bipyramidal class are zircon, silicate of zirconium, ZrSiO_4 , cassiterite or

tin-stone, dioxide of tin, SnO_2 , and the two tetragonal forms of titanium dioxide, TiO_2 , both of which, rutile and anatase, belong to this class yet exhibit different forms. A fairly complicated crystal of anatase will be thoroughly worked out in the next chapter. Titanium dioxide is trimorphous, a third rhombic variety, brookite, being a well-known and optically very interesting mineral, the optical characters of which will be described in Chapter XLVIII. A characteristic crystal of zircon is shown in Fig. 143, on which are developed the forms $\{100\}$, $\{110\}$, $\{111\}$, $\{331\}$, and $\{311\}$. The axial ratio $a:c = 1:0.6404$.

In Fig. 144 is given a stereographic projection showing a considerable number of the commoner forms of tetragonal crystals, constructed for the axial ratio of anatase, $a:c = 1:1.7771$. The author has found it of the greatest practical help in the investigation of tetragonal crystals.

Class 14.—Scalenohehral Class. Sphenoidal-Hemihedral Class. Type, Ditetragonal Alternating.

This class corresponds to the hexakis-tetrahedral class 31 of the cubic system. The elements of symmetry are shown in the stereographic projection Fig. 145, which is Fig. 127 repeated here for convenience of reference. The three axial planes are no longer

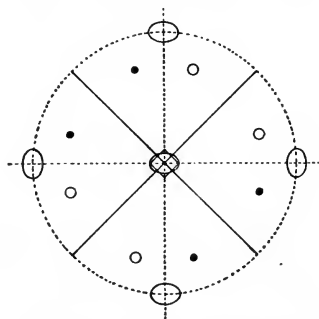


FIG. 145.—Symmetry Elements and General Form of Class 14.

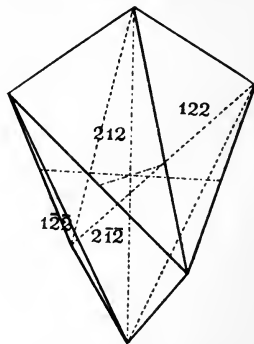


FIG. 146.
The Tetragonal Scalenohedron.

planes of symmetry, the two vertical interaxial planes of symmetry alone persisting; the two horizontal digonal axes which formed the intersections of these latter planes with the equatorial plane also disappear, leaving only the two other digonal axes (the horizontal crystallographic axes) persisting. The main axis coincident with the vertical crystallographic axis remains tetragonal, however, and although the equatorial plane is no longer a plane of simple symmetry, it is one of compound symmetry, so that the tetragonal axis is one of alternating symmetry, or a tetragonal mirror-axis as it is variously called. This will become clear from a consideration of the general form of the class, which will be immediately proceeded to.

The **scalenohedron**, $\{hkl\}$, is an eight-faced solid the poles of which are shown in the projection in Fig. 145. Fig. 146 represents the scalenohedron $\{212\}$. As will be evident from that projection there are two modifications possible, the second being obtained by changing the dots on the projection into rings and *vice versa*. The two forms are not, however, different solids; they differ only in position while *in situ*, and rotation of either for 90° about the vertical axis produces the other. The two forms are the positive $\{hkl\}$, shown in Fig. 146, and the negative, $\{\bar{h}\bar{k}\bar{l}\}$. That the principal axis is a tetragonal one of alternating symmetry is shown by the fact that each pole on the projection if rotated 90° about that axis and then immediately reflected over the equatorial plane will coincide with the pole of another face on the same scalenohedron. And yet in spite of this reflection, there is no centre of symmetry, and this fact has been shown by von Fedorow to displace the "centre of symmetry" from the position in which it has long been regarded as a true element of symmetry, for it only accompanies the operation of a plane of compound symmetry in two special cases, which have been referred to in Chapter X.

The solid angles at the poles of the principal axis are each formed by the meeting of four edges in two pairs, one pair long and acutely inclined and the other pair short and obtusely inclined; the other four edges are of intermediate length. These three kinds of edges correspond to three different interfacial angles, of which that over any pair of long edges is the same as the angle over the pyramidal edges of the holohedral ditetragonal bipyramid.

There is only one further new solid introduced into the tetragonal system by the operation of class 14 symmetry, namely the double-wedge-shaped **bisphenoid**, $\{hhl\}$, which may be regarded as derived from the bipyramid of the first order, just as the scalenohedron may be considered to be derived from the ditetragonal bipyramid. The primary form $\{111\}$ of this solid is illustrated in Fig. 147, and is formed when the pair of facial poles adjacent and symmetrical to one of the 45° (interaxial) planes of symmetry coalesce into one lying in the plane itself, which is expressed in the actual form of the solid by the further flattening into a plane of the two faces on either side of the long edges of the scalenohedron, which in the latter solid meet in that edge to form a very flat ridge.

Each face of the bisphenoid is an isosceles triangle, and the whole solid resembles a tetrahedron, but drawn out, or compressed, according to the value of the axial ratio $a:c$, along the direction of the principal (tetragonal) axis. There are two forms of the bisphenoid as of the scalenohedron, the positive $\{hhl\}$, shown *in situ* in Fig. 147, and the negative $\{\bar{h}\bar{h}\bar{l}\}$, but the two are superposable when rotated 90° out of their positions of derivation from the holohedral pyramid of the first order, $\{hhl\}$. Hence there is only one such solid.

The other forms consistent with the symmetry of this class are all identical with the holohedral forms already described in class 15, the two differences from that class being as we have seen that 2 holohedral forms, the first order bipyramid and the ditetragonal bipyramid, are replaced by the bisphenoid and scalenohedron respectively. The basal pinakoid is the same by virtue of the operation of either of the digonal axes. All three prisms are the same because the general poles of both hemispheres have a prism as their limiting form when they migrate to the primitive circle, a first or second order prism if they go to the ends of the interaxial or axial diameters, and a ditetragonal prism if they go elsewhere on the circle. The tetragonal pyramid of the second order is the same as the holohedral one because when each pair of the general poles migrate to the crystallographic axial diameters

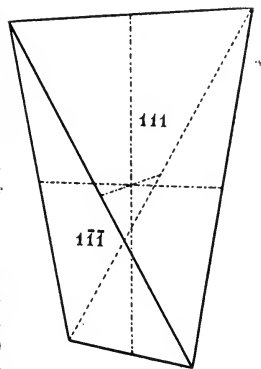


FIG. 147.
The Tetragonal Bisphenoid.

they do so by increasing their separation, one pole going to each semi-diameter; whereas the tetragonal pyramid of the first order becomes the bisphenoid by the coalescence of each pair of poles of the same hemisphere on one half of one and the same interaxial diameter, those of the upper hemisphere on one diameter and those of the lower on the other. We have, therefore, the following seven forms to tabulate as belonging to this class:

List of Forms in Class 14.

- $\{001\}$ Basal pinakoid. 2 faces.
- $\{110\}$ Tetragonal prism of the first order. 4 faces.
- $\{100\}$ Tetragonal prism of the second order. 4 faces.
- $\{hk0\}$ Ditetragonal prism. 8 faces.
- $\{hhl\}$ Positive tetragonal bisphenoid of the first order, including the primary form $\{111\}$; $\{h\bar{h}l\}$ negative tetragonal bisphenoid of the first order, including the primary form $\{1\bar{1}1\}$. Each 4 faces.
- $\{h0l\}$ Tetragonal bipyramid of the second order, including the primary form $\{101\}$. 8 faces.
- $\{hkl\}$ Positive tetragonal scalenohedron; $\{h\bar{k}l\}$ negative tetragonal scalenohedron. Each 8 faces.

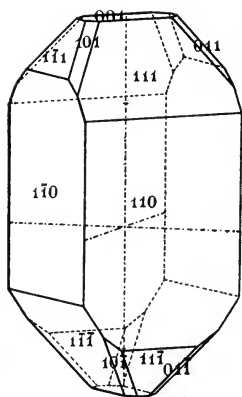


FIG. 148.—Copper Pyrites.

An excellent natural example of a substance crystallising in this class is the common ore of copper, chalcopyrite, copper pyrites, CuFeS_2 , a typical crystal of which, showing faces of the two bisphenoids $\{111\}$ and $\{1\bar{1}1\}$, the former much more developed than the latter, in combination with the prism of the first order $\{110\}$, the bipyramid of the second order $\{101\}$, and the basal pinakoid $\{001\}$, is given in Fig. 148. The ratio of the axis is: $a : c = 1 : 0.9856$.

Among artificial chemical preparations potassium hydrogen phosphate, KH_2PO_4 , affords an example, but as a rule the forms present are those which are identical with the holohedral forms of the tetragonal system, and it is only by the character of the etched figures on these faces that the hemihedrism or prevalence of class 14

symmetry is indicated.

Class 13.—Ditetragonal-Pyramidal Class. Hemimorphic-Holohedral Class. Type, Ditetragonal Polar.

In this class the tetragonal axis and the four symmetry planes intersecting in it operate as in the holohedral class 15, but all other elements of symmetry, including the equatorial plane, disappear entirely. The elements are shown in the projection, Fig. 149.

The general form $\{hkl\}$ is also shown by its poles on the projection, and is the upper open half of a ditetragonal bipyramid. If rings were given instead of dots it would equally well represent the lower half. The two halves, however, are never necessarily both present on a crystal of this class, owing to the absence of the

equatorial plane of symmetry, the tetragonal axis being thus polar. The figure may be closed by one face of the basal plane, the two parallel basal pinakoid faces of the holohedral class belonging in this class to two separate forms, $\{001\}$ and $\{00\bar{1}\}$. The general form is thus the **ditetragonal pyramid**, of which there are these two, upper and lower, modifications, the upper or upwards pointing form being the positive $\{hkl\}$ and the lower or downwards pointing form being the negative $\{h\bar{k}l\}$. No changes are produced in the exterior appearance of the tetragonal prisms by the operation only of this lower degree of symmetry of class 13. For the prisms are equally produced by the migration to the primitive circle of poles in either the upper or the lower hemisphere.

But the bipyramids of the two orders are affected similarly to the ditetragonal bipyramid, of which indeed they are only special cases, that of the first order being the special case when $k=h$, and that of the second order being the case when $k=0$; the graphic interpretation being the migration of the general pole $\{hkl\}$, situated within the 45° -triangle of the diameters representing the great zone circles of the holohedral vertical symmetry planes, to one or other of these zone circles themselves. In both cases the operation of class 13 symmetry causes the upper (upwards pointing) and lower (downwards pointing) pyramids to become two separate forms, the positive $\{hhl\}$ or $\{h0l\}$, and the negative $\{h\bar{h}l\}$ or $\{h0\bar{l}\}$.

In résumé, the following forms are included in this class :

List of Forms in Class 13.

- $\{001\}$ Upper positive basal plane ; $\{00\bar{1}\}$ lower negative basal plane. Each 1 face.
 $\{110\}$ Tetragonal prism of the first order. 4 faces.
 $\{100\}$ Tetragonal prism of the second order. 4 faces.
 $\{hk0\}$ Ditetragonal prism. 8 faces.
 $\{hhl\}$ Upper positive tetragonal pyramid of the first order ;
 $\{h\bar{h}l\}$ lower negative tetragonal prism of the first order. Each 4 faces.
 $\{h0l\}$ Upper positive tetragonal pyramid of the second order ;
 $\{h0\bar{l}\}$ lower negative tetragonal pyramid of the second order. Each 4 faces.
 $\{hkl\}$ Upper positive ditetragonal pyramid ; $\{h\bar{k}l\}$ lower negative ditetragonal pyramid. Each 8 faces.

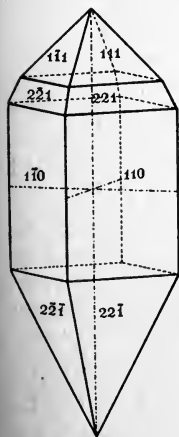


FIG. 150.
Iodosuccinimide.

The absence of centro-symmetry in this class is well shown by crystals of iodosuccinimide, $C_4H_4O_2NI$, first described by von Groth and subsequently by Traube, and an illustration of one of which, prepared by Traube and figured by him on p. 579 of his memoir (*Zeitschr. für Kryst.* 23, 577), is given in Fig. 150. The upper and lower terminations of the first order prism $m = \{110\}$ are seen to be formed by two different pyramids, the primary positive $\{111\}$, and $\{22\bar{1}\}$. The presence also of small faces of the complementary pyramidal form $\{221\}$ might be thought to indicate holohedrism, but the complementary primary

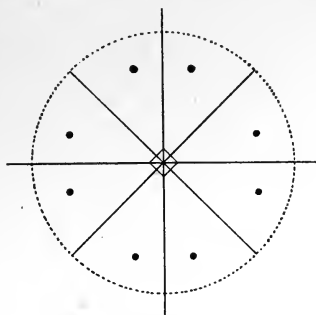


FIG. 149.—Symmetry Elements and General Form of Class 13.

negative pyramid $\{11\bar{1}\}$ is not similarly present. Moreover, the etched figures obtained on the faces of the prism $m = \{110\}$ also indicate clearly the hemimorphic character of the crystals and the absence of the equatorial plane of symmetry. The two ends also exhibit opposite pyroelectrical characters, the upper blunter end being positive and the lower sharper end negative, which affords a further strong confirmation of the supposition that the substance belongs to this class. The axial ratio of the crystals, according to von Groth, is: $a:c = 1:0.8733$.

*Class 12.—Bipyramidal Class. Pyramidal Hemihedral Class.
Type, Tetragonal Equatorial.*

This class of tetragonal crystals is distinguished by possessing only the tetragonal axis and the equatorial symmetry plane perpendicular to it. Both the primary and secondary planes of symmetry parallel to the axial vertical planes and at 45° to them disappear, and with them the

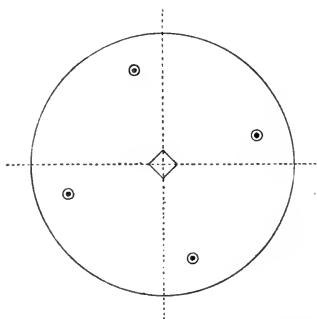


FIG. 151.—Symmetry Elements and General Form of Class 12.

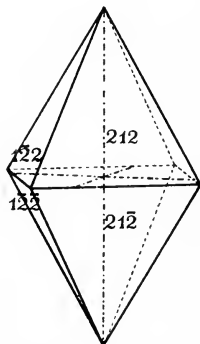


FIG. 152.—Third Order Tetragonal Bipyramid.

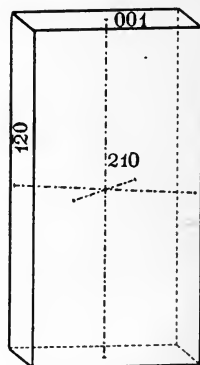


FIG. 153.—Third Order Tetragonal Prism.

four digonal horizontal axes. The stereographic projection in Fig. 151 shows both the symmetry elements and the general form $\{hkl\}$.

This latter is a tetragonal bipyramid composed of four pairs of parallel faces, there being centro-symmetry present; it is indistinguishable as regards outward shape from the other two kinds, the first and second orders, of tetragonal pyramids. Its pole, however, falls neither on a crystallographic axial diameter nor on one at 45° thereto, but, as the projection indicates, somewhere in the triangle formed by those two types of diameter. It is, therefore, a **tetragonal bipyramid of the third order**. It is shown in Fig. 152, the particular form being $\{212\}$. There are, however, two such forms having the same numbers in their indices, and the second variety can be obtained from the first modification $\{hkl\}$ shown in the projection by rotating the solid about either of the crystallographic axes a or b for 180° . Its symbol would have the h and k values of the first modification interchanged, and can be written therefore as $\{khl\}$. To these two forms belong two corresponding **tetragonal prisms of the third order**, the first of which is represented in Fig. 153, the form shown being $\{210\}$ corresponding to the pyramid drawn in Fig. 152. Their form symbols are $\{hko\}$ and $\{k\bar{h}o\}$, and they may be considered as the limiting forms of the corresponding bipyramids when the

poles migrate on to the primitive circle, and the intercept on the c axis becomes infinite, that is, the faces become parallel to that axis.

Other limiting cases are when $k=0$ and when $h=k$, that is when the poles migrate to the two kinds of diameters of the projection, parallel to the edges of the page or at 45° thereto, when we obtain the ordinary second and first order bipyramids respectively, or if also $l=0$ the corresponding prisms. The basal pinakoid $\{001\}$ is also again the limiting form when any of the three bipyramids are flattened until their faces merge in a plane parallel to the a and b crystallographic axes.

We have, therefore, the following

List of Forms in Class 12.

- $\{001\}$ Basal pinakoid. 2 faces.
- $\{110\}$ Tetragonal prism of the first order. 4 faces.
- $\{100\}$ Tetragonal prism of the second order. 4 faces.
- $\{hk0\}$ Tetragonal prism (right-hand form) of the third order; $\{k\bar{h}0\}$ left-hand form of the same. The two solids indistinguishable. Each 4 faces.
- $\{hhl\}$ Tetragonal bipyramid of the first order. 8 faces.
- $\{h0l\}$ Tetragonal bipyramid of the second order. 8 faces.
- $\{hkl\}$ Tetragonal bipyramid (right-hand form) of the third order; $\{k\bar{h}l\}$ left-hand form of the same. The two solids indistinguishable. Each 8 faces.

This class 12 is so characteristic of the symmetry of crystals of scheelite, tungstate of calcium, CaWO_4 , that it is sometimes called the "scheelite class." In Fig. 154 is shown a crystal of scheelite which exhibits a combination of the predominating primary bipyramid of the second order $\{101\}$, with that of the first order $\{111\}$, and two bipyramids of the third order, $\{313\}$ and $\{131\}$. The axial ratio of scheelite is $a:c = 1:1.5359$.

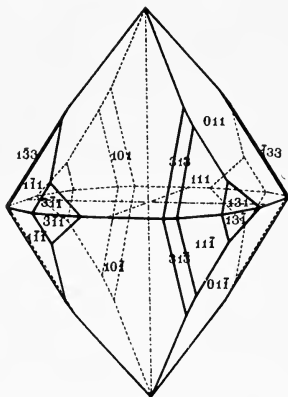


FIG. 154.—Scheelite.

*Class 11. Trapezohedral Class. Trapezohedral-Hemihedral Class.
Type, Tetragonal Holoaxial.*

This class possesses no symmetry planes at all, but all the symmetry axes of the system, namely, the tetragonal principal axis and the four digonal axes in the plane of the horizontal crystallographic axes, which are identical with two of them, the other two being at 45° to these two and thus bisecting their angles.

These symmetry elements and the general form $\{hkl\}$ are shown in the stereographic projection, Fig. 155. This form is the only new solid introduced by the symmetry of this class, and is called the **tetragonal trapezohedron**, an eight-faced solid the four upper and four lower faces of which meet like those of a pyramid in polar edges intersecting in the principal axis at a more or less sharply pointed solid angle, the upper or lower termination of the bipyramid. It is at once distinguished from the tetragonal bipyramid, however, by the zig-zag arrangement of the eight alternately longer and

shorter edges around the middle of the solid, a longer and a shorter going with every two polar edges to form the boundary of each face, which has thus the outline of a trapezoid. The shorter edges are bisected by the horizontal axes. There are two distinct enantiomorphous forms of this solid, shown in Figs. 156 and 157, the former

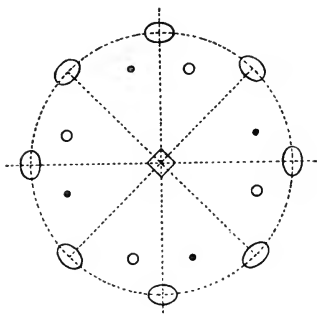


FIG. 155.—Symmetry Elements and General Form of Class 11.

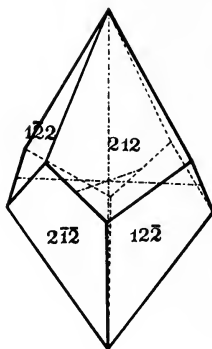


FIG. 156.

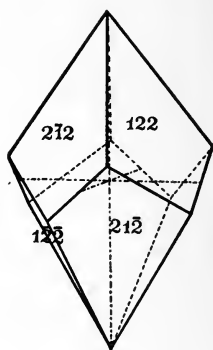


FIG. 157.

Left Positive and Right Negative Tetragonal Trapezohedra.

representing the left or positive form $\{hkl\}$ and the latter the right negative modification $\{khl\}$, the particular trapezohedra being $\{212\}$ and $\{122\}$; the poles of the former are analogous to those shown in the projection Fig. 155 for $\{312\}$, and the poles of the other are obtained by exchanging the dots for rings and *vice versa*.

The two forms are truly enantiomorphous, that is, they cannot be brought to resemble each other by rotation.

The other forms possible to this combination of symmetry elements are identical with the holohedral forms in the cases of the pinakoid, prisms of the two orders, ditetragonal prism, and bipyramids of the two orders, as described in class 15. This fact can be readily verified by considering the effect of the corresponding migrations of the general poles, to the centre, to the primitive circle, or to the digonal-axial diameters, of the projection.

We have, therefore, the following forms to include in this class :

List of Forms in Class 11.

- $\{001\}$ Basal pinakoid. 2 faces.
- $\{110\}$ Tetragonal prism of the first order. 4 faces.
- $\{100\}$ Tetragonal prism of the second order. 4 faces.
- $\{hk0\}$ Ditetragonal prism. 8 faces.
- $\{hhl\}$ Tetragonal bipyramid of the first order. 8 faces.
- $\{h0l\}$ Tetragonal bipyramid of the second order. 8 faces.
- $\{hkl\}$ Left positive tetragonal trapezohedron; $\{khl\}$ right negative tetragonal trapezohedron. Two distinct enantiomorphous solids. Each 8 faces.

Sulphate of strychnine, $C_{21}H_{22}N_2O_2 \cdot H_2SO_4 \cdot 6H_2O$, is the best-known example among the few substances crystallising according to this class of symmetry. It usually exhibits only the first order primary bipyramid $\{111\}$ in combination with the basal pinakoid $\{001\}$, the axial ratio being $a:c = 1:3.312$; but both the crystals and their aqueous solution are optically active, and cleavage plates parallel to the basal plane afford

etched figures with dilute hydrochloric acid which indicate clearly the trapezohedral nature of the symmetry.

*Class 10.—Bisphenoidal Class. Sphenoidal-Tetartohedral Class.
Type, Tetragonal Alternating.*

This class is distinguished by the operation of the equatorial plane as a plane of compound symmetry only, in simultaneous co-operation with the tetragonal axis. That is, as will be clearer from Fig. 158 showing the stereographic projection of the two elements of symmetry and the general form $\{hkl\}$, repetition of the general face (hkl) occurs by a rotation of 90° about the tetragonal axis followed at once (without a face being produced at this stage) by reflection across the equatorial plane. The symmetry is such as would be produced if the planes of symmetry of the scalenohedral class 14 disappeared, together with the digonal axes.

The general form $\{hkl\}$ is a solid which was also produced under the operation of scalenohedral class 14 symmetry, namely, the bisphenoid, already illustrated in Fig. 147. The bisphenoid here produced, however, is not of the first order, as in the case of class 14, but of the third order. This **tetragonal bisphenoid of the third order** has four modifications, indistinguishable after removal out of their positions of derivation. The first variety is the left positive form $\{hkl\}$, the poles of which are shown in the projection. The other three varieties all have the same numbers in their indices, and are the right positive $\{khl\}$, the right negative $\{h\bar{k}l\}$, and the left negative $\{k\bar{h}l\}$.

Besides these four third order bisphenoids, two **bisphenoids of the first order** belong to this class; they are produced when the pole (hkl) migrates on to the 45° (diagonal) diameter and $h=k$, and are termed the positive, $\{hhl\}$, and the negative, $\{h\bar{h}l\}$. They may be regarded as derived from the first order pyramid. Similarly, when the general pole migrates on to a crystallographic axial diameter we have a pair of **bisphenoids of the second order** produced, the right $\{h0l\}$, and the left $\{0kl\}$, which may be considered as derived from the second order pyramid. When, however, the pole (hkl) moves on to the primitive circle we have a tetragonal prism $\{hk0\}$ of the third order, of which there are two varieties, a right $\{hko\}$ and a left $\{k\bar{h}o\}$. The first order and second order prisms, and also the basal pinakoid, remain as in the holohedral class 15. The prisms are the limiting cases of bisphenoids of increasing steepness and diminishing third index in relation to the other two indices. The basal pinakoid, on the other hand, is the limit of all bisphenoids of increasing flatness and increasing third index.

There are, therefore, the following forms in this class:

List of Forms in Class 10.

- $\{001\}$ Basal pinakoid. 2 faces.
- $\{110\}$ Tetragonal prism of the first order. 4 faces.
- $\{100\}$ Tetragonal prism of the second order. 4 faces.
- $\{hk0\}$ Right tetragonal prism of the third order; $\{k\bar{h}o\}$ left tetragonal prism of the third order. Each 4 faces.

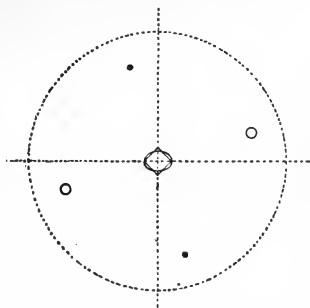


FIG. 158.—Symmetry Elements and General Form of Class 10.

- $\{hhl\}$ Positive tetragonal bisphenoid of the first order; $\{h\bar{h}l\}$ negative tetragonal bisphenoid of the first order. Each 4 faces.
- $\{h0l\}$ Right tetragonal bisphenoid of the second order; $\{0kl\}$ left tetragonal bisphenoid of the second order. Each 4 faces.
- $\{hkl\}$ Left positive tetragonal bisphenoid of the third order; $\{khl\}$ right positive form of same; $\{h\bar{k}l\}$ right negative form; $\{k\bar{h}l\}$ left negative form. Each 4 faces.

The only substance which has yet been observed to crystallise in accordance with this class of symmetry is a silicate of lime and alumina, $2\text{CaO} \cdot \text{Al}_2\text{O}_3 \cdot \text{SiO}_2$, which was isolated by Weyberg¹ in the year 1906. Both varieties of bisphenoids were observed, although rarely on the same crystal, together with the forms $\{001\}$ and $\{110\}$. The etch-figures on the faces of the two latter forms (basal pinakoid and prism), afforded by hydrochloric and nitric acids, exhibited clear evidence of bisphenoidal symmetry.

Class 9.—Pyramidal Class. Hemimorphic-Hemihedral Class.
Type, Pyramidal Polar.

This final class of lowest possible tetragonal symmetry is distinguished by the possession of only one element of symmetry, the indispensable tetragonal axis persistent throughout the system, and which is in this class polar.

As will be evident from the stereographic projection Fig. 159, which exhibits both the symmetry element and the poles of the general form $\{hkl\}$, repetition of the face

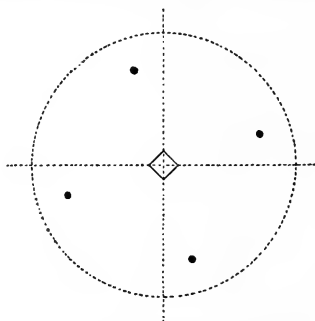


FIG. 159.—Symmetry Element and General Form of Class 9.

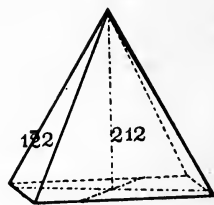


FIG. 160.
The Tetragonal Pyramid.

$\{hkl\}$ about the tetragonal axis results in the production of an open simple equi-four-faced pyramid, the **tetragonal pyramid**, shown in Fig. 160. It may be closed by a single basal plane, for the basal pinakoid in this class falls into two separate forms, $\{001\}$ and $\{00\bar{1}\}$. It is obvious also that to the four-faced upper pyramid $\{hkl\}$ there corresponds a duplicate lower pyramid $\{h\bar{k}l\}$. Moreover, just as in the bipyramidal class 12, in which these two forms together make up one form, there are two other complementary forms (together also making up one form in the bipyramidal class) having the same numbers in their indices, and which have therefore the same interfacial angle, namely, the form $\{khl\}$ of four faces and the corresponding lower form $\{k\bar{h}l\}$. Hence, there are in all four forms of the general type $\{hkl\}$ in this class. They are all pyramids of the third order, indistinguishable except by their positions *in situ*.

¹ *Anzeiger der Akad. der Wiss., Krakau, 1906, 611.*

This is not, however, a case of true tetartohedrism, but merely of hemihedrism having a hemimorphic character. The truly tetartohedral forms belong to class 10 of the tetragonal system.

When $h=k$ and the poles fall consequently on the 45° -diameters, we have an open pyramid of the first order produced, which may be closed by the basal plane. When l is negative, we have the corresponding lower pyramid.

When either h or k is 0, on the other hand, the poles migrate to the crystallographic axial diameters, and we then have the upper and lower pyramids of the second order produced, which also may be closed by the basal plane.

If $l=0$, and the poles migrate consequently on to the primitive circle, the prisms of the first, second, or third orders are formed, of the third of which there are two varieties, $\{hk0\}$ and $\{k\bar{h}0\}$. All these separate varieties of each of the various pyramids and prisms are indistinguishable from each other, except when in their positions of derivation from the holohedral form.

Finally, if the poles approach the centre, the pyramids become flatter and flatter, until when the poles coincide in the centre the upper pyramids yield the basal plane $\{001\}$ and the lower pyramids the second face, now a separate form, of the basal pinakoid $\{00\bar{1}\}$.

There are thus to be distinguished the following forms in the class :

List of Forms in Class 9.

- $\{001\}$ Upper positive basal plane ; $\{00\bar{1}\}$ lower negative basal plane.
- $\{110\}$ Tetragonal prism of the first order. 4 faces.
- $\{100\}$ Tetragonal prism of the second order. 4 faces.
- $\{hk0\}$ Tetragonal right-hand prism of the third order ; $\{k\bar{h}0\}$ tetragonal left-hand prism of the third order. Two solids indistinguishable. Each 4 faces.
- $\{hhl\}$ Upper tetragonal pyramid of the first order ; $\{hh\bar{l}\}$ lower tetragonal pyramid of the first order. Two solids indistinguishable. Each 4 faces.
- $\{h0l\}$ Upper tetragonal pyramid of the second order ; $\{h0\bar{l}\}$ lower tetragonal pyramid of the second order. Two solids indistinguishable. Each 4 faces.
- $\{hkl\}$ Upper right tetragonal pyramid of the third order ; $\{khl\}$ upper left form ; $\{hk\bar{l}\}$ lower right form ; $\{khl\}$ lower left form. Four solids indistinguishable. Each 4 faces.

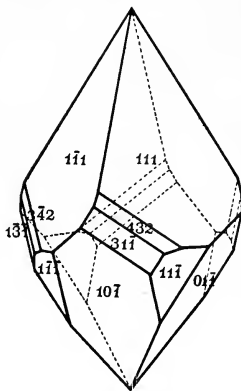


FIG. 161.—Wulfenite.

Wulfenite, the natural molybdate of lead, PbMoO_4 , forms an excellent example of a substance crystallising according to this class of symmetry. Fig. 161 shows a crystal of wulfenite exhibiting a combination of the five pyramids $\{111\}$, $\{10\bar{1}\}$, $\{11\bar{1}\}$, $\{31\bar{1}\}$, and $\{432\}$. The axial ratio is $a : c = 1 : 1.5777$.

CHAPTER XIV

PRACTICAL EXAMPLE OF A TETRAGONAL CRYSTAL.

Class 15.—Tetragonal Holohedral. Anatase, TiO_2 .

THE crystal of anatase which forms our example is a beautifully developed specimen about two millimetres in average diameter, and very rich in faces, no less than forty-eight being developed. Its appearance is shown in Fig. 162, drawn to scale with the aid of the usual preliminary careful freehand drawing, after the completion of the calculations, by the ordinary

conventional method described in Chapter XXV., where the construction will be given in detail. Its stereographic projection is given in Fig. 163, and the process of constructing it will be described stage by stage.

The crystal was brownish-yellow in colour, more or less transparent (translucent), and the faces were exceedingly brilliant, exhibiting the well-known adamantine lustre of anatase.

The exterior shape of the crystal, on the average more or less spherical, was chiefly determined by three forms, which will be shown in the sequel to be the tetragonal prism of the second order $a = \{100\}$,

the tetragonal bipyramid of the first order $z = \{113\}$, and the bipyramid also of the first order $r = \{335\}$. The only other form observable at first sight on the crystal is one of smaller faces, symmetrically replacing the eight solid angles formed by the intersection of the forms a and r ; it is the tetragonal bipyramid of the second order $e = \{101\}$. Here and there, however, the edges of intersection of various adjacent faces of these forms showed reflections as if from other narrow faces, and as a matter of fact the faces of three other forms were present, which will be

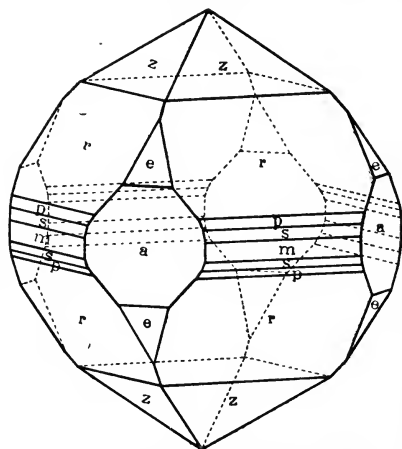


FIG. 162.—The Crystal of Anatase used as Example.

shown to be two further bipyramids of the first order, $p = \{111\}$ the parametral form, and $s = \{221\}$; and the tetragonal prism of the first order $m = \{110\}$. The parametral form is not prominent on this crystal, although it often is so on crystals of anatase from particular localities; notwithstanding this fact, the mode of description of the crystal will be that currently accepted, according to which p is the parametral form $\{111\}$.

On making a careful study of the crystal with the eye and pocket lens, turning it round and round on the little wax cone of the crystal holder, two prominent features were observed indicative of a promising mode of commencing the measurements. Firstly, there was clearly a zone of four tolerably large and well-developed faces arranged at what

appeared to be a right angle to each other. Secondly, there were two mutually similar zones of large faces, crossing each other rectangularly, which appeared to intersect in two sharply pointed four-faced pyramidal terminations, one above the zone plane of the rectangular zone and one below, the points being apparently the ends of the vertical axis

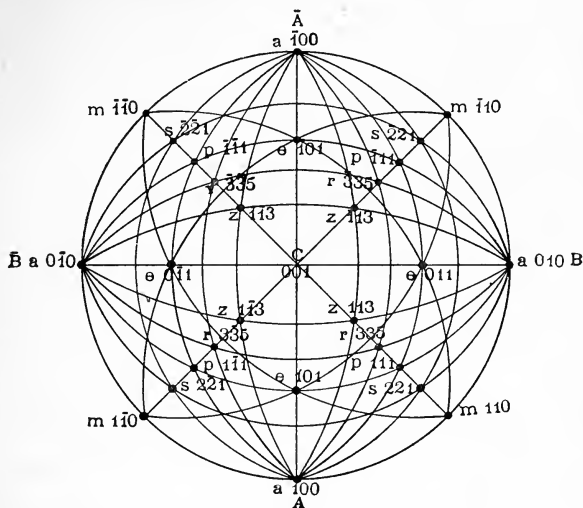


FIG. 163.—Stereographic Projection of the Anatase Crystal.

the four faces just referred to meeting in right angles. The zone planes of these two pyramidal zones appeared to bisect the angles between adjacent rectangular faces, as if they were composed of first-order pyramid faces with poles on diameters of a tetragonal projection arranged at 45° to the axial diameters, assuming the latter to be the normals of the four rectangularly inclined faces. There were four pairs of these large pyramidal faces in each of the two zones, each pair consisting of closely adjacent faces belonging to two pyramids of different altitudes and occupying one octant, so that there were eight of these pairs altogether, apparently symmetrically arranged in the eight octants.

Our obvious first duty was to measure the apparently rectangular zone. For this purpose the crystal was set up on the wax cone by one of the pointed pyramidal ends, the line joining the two such ends being vertical, as in Fig. 162. The four apparently rectangular faces were first adjusted and centred, and then definitely found to belong to the same zone, and four other smaller faces were found to alternate with them in

the zone at apparently equal distances between each pair of them. Proceeding then to measure the zone, the following readings were obtained :

TETRAGONAL PRISM ZONE.

Circle Reading.	Angle.
$\left\{ \begin{array}{l} a \ 359^\circ \ 56' \text{ A} \\ a \ 269 \ 56 \text{ A} \\ m \ 224 \ 56 \text{ A} \\ a \ 179 \ 56 \\ m \ 134 \ 55 \text{ A} \\ a \ 89 \ 55 \text{ A} \\ a \ 359 \ 56 \text{ A} \end{array} \right.$	$\begin{array}{lll} aa \ 90^\circ \ 0' \text{ A} \\ am \ 45 \ 0 \text{ A} \\ ma \ 45 \ 0 & aa \ 90^\circ \ 0' \\ am \ 45 \ 1 & \\ ma \ 45 \ 0 \text{ A} & aa \ 90 \ 1 \\ aa \ 89 \ 59 \text{ A} \end{array}$

It will be clear from these measurements that the four prominent faces of the zone, those marked with the letter *a*, were truly at right angles to each other, that is, composed of two pairs of parallel faces the planes of which were exactly at 90° to each other. Moreover, the intermediate faces marked *m* were situated exactly 45° from them, so that we are justified in assuming that in all probability the zone is either one of a cubic crystal, or is the prism zone of a tetragonal crystal. The whole of the measured values of the angles are exactly either $90^\circ \ 0'$ or $45^\circ \ 0'$ or within $1'$ of these round values, hence there can be no question as to nature's evident intention to erect these faces exactly at $90^\circ \ 0'$ and $45^\circ \ 0'$ to each other.

We may next with greatest advantage measure the two zones each of which is apparently formed by a pair of parallel *a*-faces and four of the eight small faces marked *e*, which have been referred to on the first page of this chapter as replacing the solid angles formed by the intersection of the large faces *a* and *r*. From the situation of these little *e*-planes it appeared probable that they were faces of either the cubic rhombic dodecahedron (in which case the angle *ae* would be 45°) or its tetragonal equivalent, a tetragonal pyramid of the second order.

In order to measure these two zones, the crystal was set in each case on the wax cone by one of the *a*-faces belonging to the complementary zone, that is with this *a*-face horizontal. On adjusting the pair of *a*-faces (perpendicular to this) of the zone to be measured truly vertical and parallel to one of the two adjusting movements, and adjusting one of the small *e*-faces also by the tangent screw of this movement (afterwards completing the adjustment of the *a*-face with the other tangent screw), all four small *e*-faces in question were found, as expected, to belong to the zone, the three other than the one adjusted being also automatically adjusted. The two zones gave the following measurements :

PAIR OF SECOND-ORDER PYRAMID ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} a \ 360^\circ \ 0' \text{ A} \\ e \ 330 \ 35 \\ e \ 209 \ 19 \\ a \ 180 \ 3 \\ e \ 150 \ 35 \\ e \ 29 \ 23 \text{ A} \\ a \ 0 \ 0 \text{ A} \end{array} \right.$	$\begin{array}{l} ae \ 29^\circ \ 25' \\ ee \ 121 \ 16 \\ ea \ 29 \ 16 \\ ae \ 29 \ 28 \\ ee \ 121 \ 12 \\ ea \ 29 \ 23 \text{ A} \end{array}$	$\left\{ \begin{array}{l} a \ 360^\circ \ 0' \text{ A} \\ e \ 330 \ 39 \text{ A} \\ e \ 209 \ 23 \text{ A} \\ a \ 180 \ 5 \text{ A} \\ e \ 150 \ 45 \text{ A} \\ e \ 29 \ 29 \\ a \ 0 \ 4 \text{ A} \end{array} \right.$	$\begin{array}{l} ae \ 29^\circ \ 21' \text{ A} \\ ee \ 121 \ 16 \text{ A} \\ ea \ 29 \ 18 \text{ A} \\ ae \ 29 \ 20 \text{ A} \\ ee \ 121 \ 16 \\ ea \ 29 \ 25 \end{array}$

An examination of these values at once renders it clear that the two zones are similar, that there are only two different angles, symmetrically situated in the zone, and that the angle ae is not 45° but somewhere near $29\frac{1}{2}^\circ$, so that the form e is not the rhombic dodecahedron but in all probability a tetragonal bipyramid of the second order. The proof that the angles ae and ea are all intended to be identical in magnitude is afforded by the "A" values, which are sufficiently close to indicate that the angle ea on one side of any face a is of equal value with the angle ae on the other side of the face a ; and, moreover, that this pair of angles have the same magnitude and same value as regards the symmetry as the pair on the two sides of the parallel a -face. Further, the two pairs of such angles on the two zones are undoubtedly intended to be both of identical magnitude and of equal symmetric value. Having decided this, entirely from the consideration of the individual angular values of the two zones, and **not before**, we may adopt the best method open to us of ascertaining the true magnitude of the angle, by taking the arithmetical mean of all the trustworthy values, which in this case include all the values. The mean value thus obtained for ae and ea is $29^\circ 22'$, and for the larger angle ee $121^\circ 15'$.

Very emphatic caution must again be given as to the taking of mean values of such series of angles. It must never be done without having first, as we have done here, absolutely decided that the angles the mean of which is being taken are of equal symmetric value, as in the case just dealt with, from a consideration of the most trustworthy individual "A" measurements; and this symmetry must be confirmed by the results of the measurement of other zones on the crystal, as will be shown to be true in this case of anatase. Otherwise a grave mistake may be made, and a higher type of symmetry assumed than is actually developed. For there are many cases known in which the actual angles are a few minutes only removed from those which would correspond to a higher system of symmetry. For instance, the prism zone of rhombic potassium sulphate has been shown in Chapter IV. to be only $12'$ removed from 60° , the angle of a hexagonal prism. This pitfall is most alluring, and fatal if fallen into, and too strong a warning cannot be given with regard to it. When, however, the extraction of the mean value has been decided to be truly legitimate, it affords the best mode of eliminating the fortuitous slight variations of a very few minutes due to disturbance of the conditions during growth.

We proceed next to the measurement of the pair of apparently first-order pyramid zones, the two very prominent zones of large faces crossing each other apparently rectangularly and at 45° to the vertical axial-plane zones already measured. On adjusting each of these two zones in turn, it was found that besides the large faces lettered z and r , and the smaller faces of the first-order prism m , there were also automatically adjusted small faces of two other forms, lettered p and s . They were generally only narrow strips, but one p -face was quite well developed and gave an excellent reflection. This form lettered p will

be shown in the sequel to be the important parametral form $\{111\}$. The actual results of the measurements are given below.

PAIR OF FIRST-ORDER PYRAMID ZONES.

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} m \ 360^\circ \ 0'A \\ p \ 338 \ 22 \ A \\ r \ 326 \ 29 \ A \\ z \ 309 \ 55 \ A \\ z \ 230 \ 3 \ A \\ r \ 213 \ 30 \ A \\ m \ 180 \ 0 \\ r \ 146 \ 28 \\ z \ 129 \ 57 \ A \\ z \ 50 \ 3 \ A \\ r \ 33 \ 31 \ A \\ p \ 21 \ 42 \\ s \ 11 \ 15 \ A \\ m \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} mp \ 21^\circ \ 38'A \\ pr \ 11 \ 53 \ A \quad mr \ 33^\circ \ 31'A \\ rz \ 16 \ 34 \ A \quad mz \ 50 \ 5 \ A \\ zz \ 79 \ 52 \ A \\ zr \ 16 \ 33 \ A \quad zm \ 50 \ 3 \\ rm \ 33 \ 30 \\ mr \ 33 \ 32 \quad mz \ 50 \ 3 \\ rz \ 16 \ 31 \\ zz \ 79 \ 54 \ A \\ zr \ 16 \ 32 \ A \quad zm \ 50 \ 3 \ A \\ rp \ 11 \ 49 \quad rm \ 33 \ 31 \ A \\ ps \ 10 \ 27 \quad pm \ 21 \ 42 \\ sm \ 11 \ 15 \ A \end{array} \right.$	$\left\{ \begin{array}{l} z \ 360^\circ \ 0'A \\ z \ 280 \ 7 \ A \\ r \ 263 \ 38 \ A \\ p \ 251 \ 40 \\ s \ 241 \ 14 \ A \\ m \ 230 \ 4 \ A \\ s \ 218 \ 46 \ A \\ p \ 208 \ 26 \\ r \ 196 \ 30 \ A \\ z \ 180 \ 0 \ A \\ z \ 100 \ 7 \ A \\ r \ 83 \ 36 \ A \\ p \ 71 \ 48 \\ p \ 28 \ 21 \ A \\ r \ 16 \ 31 \ A \\ z \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} zz \ 79^\circ \ 53'A \\ zr \ 16 \ 29 \ A \quad zm \ 50^\circ \ 3'A \\ rp \ 11 \ 58 \quad rm \ 33 \ 34 \ A \\ ps \ 10 \ 26 \quad pm \ 21 \ 36 \\ sm \ 11 \ 10 \ A \\ ms \ 11 \ 18 \ A \\ sp \ 10 \ 20 \quad mp \ 21 \ 38 \\ pr \ 11 \ 56 \quad mr \ 33 \ 34 \ A \\ rz \ 16 \ 30 \ A \quad mz \ 50 \ 4 \ A \\ zz \ 79 \ 53 \ A \\ zr \ 16 \ 31 \ A \\ rp \ 11 \ 48 \\ pp \ 43 \ 27 \\ pr \ 11 \ 50 \ A \\ rz \ 16 \ 31 \ A \end{array} \right.$

Similar angles in these two zones so rich in faces are collected together in the following lists, a slight gap being left in each table to indicate where the values from the second zone begin; and the mean value is given below the line.

zz.	zr.	rp.	ps.	sm.	pm.	rm.	zm.
79° 52'A	16° 34'A	11° 53'A	10° 27'	11° 15'A	21° 38'A	33° 31'A	50° 5'A
79 54 A	16 33 A	11 49			21 42	33 30	50 3
	16 31		10 26	11 10 A		33 32	50 3
79 53 A	16 32 A	11 58	10 20	11 18 A	21 36	33 31 A	50 3 A
79 53 A		11 56			21 38		
	16 29 A	11 48	10° 24'	11° 14'		33 34 A	50 3 A
79° 53'	16 30 A	11 50 A			21° 39'	33 34 A	50 4 A
	16 31 A						
	16 31 A	11° 52'				33° 32'	50° 4'
	16° 31'						

It will be clear that, firstly, the two zones are exactly similar and equiangular, and secondly, that in each zone the pyramidal poles s , p , r , and z are repeated in the same order and at the same angles four times in the zone, symmetrically placed as regards the prism poles m . Take, for instance, the four values of the angle zz , two in each zone; there can be no question but that these four angles were intended by nature to be equal, for the two values in one zone were absolutely equal, and in the other zone only two minutes different, 1' on each side of the mean value of the four. Hence we are fully justified in accepting the mean value as the true angle. The measurements obviously indicate that z , r , p , and s are a series of four

tetragonal bipyramids in ascending order of steepness, m being the limiting case of the prism itself, with faces perfectly vertical.

We are now in a position to construct the stereographic projection, Fig. 163 on page 213.

Commencing with the primitive circle, we mark off along it the positions of the four a -poles at the rectangular positions and the four m -poles at the intermediate 45° -positions, and then join opposite poles by diameters. Dots are to be placed at the ends of all the diameters on the primitive circle, in order to mark the positions of the poles a and m . On the diameters parallel to the page edges, the crystallographic axial diameters, we are next to find the positions of the four poles of the second-order bipyramid e . The angle ae has been shown to be $29^\circ 22'$. Suppose we find the position of the e -pole on the front radius ac , which not only represents the axis a but also the quadrant-arc (of a great circle) $ca=(001):(100)$. The two poles of this zone-circle are clearly $a=(010)$ and $a=(0\bar{1}0)$. In accordance with the instructions fully given in Chapter VI., we mark off along the primitive circle from $a=(100)$ an angle $29^\circ 22'$ with the protractor, on either side of $a=(100)$. We then join this point so marked off to the opposite pole of the zone-circle, that is, to that one of its two poles just mentioned which lies on the other side to that on which the point was just now marked off; if we marked off to the right, we join that point to $a=(0\bar{1}0)$. This joining line will then pass through the position of the required pole e on the radius $c=(001):a=(100)$. Having thus found one pole e , we mark off on the other three rectangular radii similar points e at the same distance from the centre, when we shall thus have found the positions of all four e -poles on the projection. We next proceed to find the positions of the first-order bipyramid poles z , r , p , and s . It will be sufficient to find their positions on one diagonal radius, and then to transfer the same distances on to the other 3 radii, from the centre c (actually marked with the axial letter C in Fig. 163, no c -face being developed), in order to get the others. The angles ms , mp , mr , and mz have been shown to be respectively $11^\circ 14'$, $21^\circ 39'$, $33^\circ 32'$, and $50^\circ 4'$; we mark off, therefore, from m along the primitive circle on one side, these successive angular arcs, and then join the points thus marked off to that pole of the zone circle which lies on the other side. Suppose, for instance, we are going to find the positions on the radius $c=(001):m=(110)$; the poles of the zone circle of which this radius is the semi-projection are at the extremity of the perpendicular diameter, on the primitive circle, namely, $m=(1\bar{1}0)$ and $m=(1\bar{1}0)$, and if we mark off the angles ms , mp , mr , and mz to the right from $m=(110)$ we must join the marked points on the primitive circle to $m=(1\bar{1}0)$. The points where these junction lines intersect the radius $c=(001):m=(110)$ are the required positions of the facial poles s , p , r , and z . From the centre c we then mark off along the other three diagonal radii the distances cz , cr , cp , and cs , in order to find the positions of the other facial poles of the same bipyramidal forms z , r , p , and s .

We may next connect each adjacent pair of z -poles, r -poles, p -poles, and s -poles with the a -poles lying in the same zone with them, by a circular arc to represent the zone as shown in Fig. 163. The method of finding the centre for each is, stated generally, that of the simple problem in plane geometry "to construct a circular arc to pass through three given points," the three points being any three of the four poles $azza$, or of $arra$, or of $appa$, or of $assa$; bisecting perpendiculars are drawn to the two imaginary straight lines joining the first and second points, and the second and third points, and where the two perpendiculars intersect is the required centre. This centre, however, must lie on that axial diameter perpendicular to the one which joins the ends aa of the zone-arc in question, so that it is only necessary to draw one of the bisecting perpendiculars, and where it cuts the axial diameter is the centre required. We shall then find that there are 16 such secondary zones altogether, composed of four sets of four equal zones.

This indicates the next step to be taken in the measurements, namely, that each of these four different types of zones should be measured, and at least two of the four similar ones of each type, in order to confirm still more fully the tetragonal nature of the symmetry; and the pair of each measured should not always be the analogous pair, but should be so selected as to get confirmations of the tetragonal arrangement all round the circle, so that the evidence shall be as strong as if all the 16 zones had been measured instead of only eight of them.

Commencing at the apex of the pyramid, we measure first a couple of the zones [azza]. The actual readings and angles obtained are as under:

TWO OF THE FOUR ZONES [azza].

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} z \ 360^\circ \ 0' A \\ z \ 306 \ 0 \ A \\ a \ 242 \ 59 \ A \\ z \ 180 \ 1 \ A \\ z \ 126 \ 1 \ A \\ a \ 63 \ 4 \ A \\ z \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} zz \ 54^\circ \ 0' A \\ za \ 63 \ 1 \ A \\ az \ 62 \ 58 \ A \\ zz \ 54 \ 0 \ A \\ za \ 62 \ 57 \ A \\ az \ 63 \ 4 \ A \end{array} \right.$	$\left\{ \begin{array}{l} z \ 360^\circ \ 0' A \\ a \ 297 \ 3 \ A \\ z \ 233 \ 59 \ A \\ z \ 180 \ 0 \ A \\ a \ 116 \ 58 \ A \\ z \ 54 \ 1 \ A \\ z \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} za \ 62^\circ \ 57' A \\ az \ 63 \ 4 \ A \\ zz \ 53 \ 59 \ A \\ za \ 63 \ 2 \ A \\ az \ 62 \ 57 \ A \\ zz \ 54 \ 1 \ A \end{array} \right.$

It is obvious that there are only two different angles, of about 54° and 63° respectively, and a study of the two lists of similar angles shows clearly that this is again a case where we are justified in taking the two mean values, which are $54^\circ 0'$ and $63^\circ 0'$. In the case of zz , which was measurable with the greater accuracy owing to the higher excellence of the images from the z -faces, the identity is within $2'$ of being absolute, and in the case of the angle az , the slight variations of $3'$ or $4'$ each side of the mean is only owing to the lesser excellence of the images from the a -faces when adjusted for these oblique zones. The two particular zones measured were the intersecting ones [$a = (100):z = (113):z = (\bar{1}13)$] and [$a = (010):z = (113):z = (\bar{1}\bar{1}3)$], so that we have here further confirmation of the tetragonal nature of the symmetry.

Passing next to the zones [arra], the following measurements were obtained with a pair of them, the particular pair being [$a = (100):r = (335):r = (\bar{3}35)$] and [$a = (010):r = (\bar{3}35):r = (335)$], so that this pair of zones will afford complementary evidence to the last pair.

TWO OF THE FOUR ZONES [arra].

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} r \ 360^\circ \ 0' A \\ r \ 287 \ 44 \ A \\ a \ 233 \ 52 \ A \\ r \ 179 \ 56 \ A \\ r \ 107 \ 44 \\ a \ 53 \ 50 \\ r \ 0 \ 1 \ A \end{array} \right.$	$\left\{ \begin{array}{l} rr \ 72^\circ \ 16' A \\ ra \ 53 \ 52 \ A \\ ar \ 53 \ 56 \ A \\ rr \ 72 \ 12 \\ ra \ 53 \ 54 \\ ar \ 53 \ 49 \end{array} \right.$	$\left\{ \begin{array}{l} r \ 360^\circ \ 0' A \\ r \ 287 \ 42 \ A \\ a \ 233 \ 50 \ A \\ r \ 179 \ 58 \ A \\ r \ 107 \ 44 \\ a \ 53 \ 48 \\ r \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} rr \ 72^\circ \ 18' A \\ ra \ 53 \ 52 \ A \\ ar \ 53 \ 52 \ A \\ rr \ 72 \ 14 \\ ra \ 53 \ 56 \\ ar \ 53 \ 48 \end{array} \right.$

As in the case of the previous zone [azza], it is clear that there are only two different angles, symmetrically distributed about the a -poles, and the mean value extracted from the two lists of their individual values are $rr = 72^\circ 15'$ and $ar = 53^\circ 52'$.

Tetragonal symmetry is again borne out by the individual values, especially the "A" values; thus these values for rr from the two zones only differ by $2'$, and three out of four "A" values for ar from both zones are absolutely identical.

We pass next to the important parametral zone [appa], which is of the same character, except that most of the p -readings are derived from very small faces, there having been only one good-sized p -face developed on the crystal, and also that each of these four similar zones also includes two e -faces, each situated halfway between two p -faces. The measurements derived from a pair of these zones are as under, the actual pair chosen having been those containing the best p -faces.

TWO OF THE FOUR PARAMETRAL ZONES [appa].

Circle Reading.	Angle.	Circle Reading.	Angle.
e 360° 0'A	ep 41° 5'A	a 360° 0'A	ap 48° 54'A
p 318 55 A	pa 48 59	p 311 6 A	pe 41 0
a 269 56	ap 48 47	e 270 6	ep 41 8 pp 82° 8'A
p 221 9	pe 41 7	p 228 58 A	pa 48 55 A
e 180 2	ep 41 8 pp 82° 15'	a 180 3 A	ap 48 57
p 138 54	pa 48 58	p 131 6	pe 41 7
a 89 56	ap 48 50	e 89 59	ep 41 1 pp 82 8
p 41 6 A	pe 41 6 A pp 82 11 A	p 48 58	pa 48 58
e 0 0 A		a 0 0 A	

The "A" values of ap and of pe only differ in each case by $1'$, so that we may accept it as a fact that all the ap values were intended to be the same and all the pe angles to be identical. Owing to the less excellent character of some of the p -faces a few of the angles intended to be identical gave rather wider readings than in the cases of those derived from the better faces, but the divergences are not great, and we can legitimately take the mean of symmetrically analogous and equal angles, which will be found to yield the same value as the "A" values alone, namely $ap = 48^\circ 55'$, $pe = 41^\circ 5'$, and $pp = 82^\circ 10'$.

We next pass on to the last of these four secondary arc-zones, namely [assa]. The readings and angles for a pair of them as far as developed are as under:

TWO OF THE FOUR ZONES [assa].

Circle Reading.	Angle.	Circle Reading.	Angle.
a 360° 0'	as 46° 7'	a 360° 0'A	as 46° 1'A
s 313 53	ss 87 47	s 313 59 A	ss 87 51 A
s 226 6	sa 46 4	s 226 8 A	sa 46 8 A
a 180 2 A	as 46 9	a 180 0 A	
s 133 53	ss 87 45		
s 46 8	sa 46 6		
a 0 2			

The readings in the first zone were not quite so satisfactory as usual, on account of the small size of the *s*-faces and the somewhat poorer quality of the images reflected by them. The second zone had only two *s*-faces developed, but they gave "A" images of the signal. We have now accumulated such a mass of evidence, however, that the symmetry is tetragonal, that we may here also with confidence take the mean values as a close approximation to the truth, leaving the calculations from the irreproachable basal angles to check the values arrived at. Even as they stand, the divergence never exceeds 7'. The mean value for *as* = 46° 6', and for *ss* = 87° 48'.

There is still one further set of four zones that it is desirable to measure, namely, those connecting each parallel pair of *m*-faces with four of the *e*-faces, as indicated by the four circular arcs in the projection, drawn from the ends of the diagonals *mm* to pass through the *e*-poles by the usual method of describing a circular arc to pass through three given points, as already described in the cases of the four arc-zones already completed. Two of these four equal zones [*meem*] were measured, namely, [*m* = (110), *e* = (101), *e* = (011), *m* = (110)] and [*m* = (110), *e* = (101), *e* = (011), *m* = (110)]. The record of the observations is as under :

TWO OF THE FOUR ZONES [*meem*].

Circle Reading.	Angle.	Circle Reading.	Angle.
$\left\{ \begin{array}{l} e \ 360^\circ \ 0'A \\ e \ 283 \ 53 \ A \\ m \ 231 \ 54 \ A \\ e \ 180 \ 0 \\ e \ 103 \ 54 \\ e \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} ee \ 76^\circ \ 7'A \\ em \ 51 \ 59 \ A \\ me \ 51 \ 54 \\ ee \ 76 \ 6 \end{array} \right.$	$\left\{ \begin{array}{l} m \ 360^\circ \ 0' \\ e \ 308 \ 3 \ A \\ e \ 231 \ 55 \\ m \ 180 \ 0 \ A \\ e \ 128 \ 2 \ A \\ e \ 51 \ 55 \ A \\ m \ 0 \ 1 \ A \end{array} \right.$	$\left\{ \begin{array}{l} me \ 51^\circ \ 57' \\ ee \ 76 \ 8 \\ em \ 51 \ 55 \\ me \ 51 \ 58 \ A \\ ee \ 76 \ 7 \ A \\ em \ 51 \ 54 \ A \end{array} \right.$

It is quite clear that all four values of *ee* belong to angles of equal value as regards the symmetry, the maximum difference among them being only 2', and the same naturally follows for *me*, although the difference amounts to 5', which, however, is still small enough to indicate nature's slightly disturbed attempt at identity. These measurements finally complete the proof that the symmetry of the crystal is tetragonal, and having now fully convinced ourselves, from the whole of the measurements, of this fact, we can with confidence accept the mean of the *me*-values, namely, 51° 56', and the mean of the *ee*-values, 76° 7', as representing the true magnitudes of those angles.

This completes our practical study of the crystal of anatase, and we may now address ourselves to the calculation of the elements, and of the angles other than the basal angle, for purposes of confirmation.

Calculation of Elements and Angles of Anatase.

Basal angle, *ae* = (100) : (101) = 29° 22'. This angle is very convenient to choose for the one basal angle required in the tetragonal

system, as it enables us immediately to calculate the axial ratio. It was measured eight times in a fully trustworthy manner, with only 5' between the extreme "A" values. Also the "A" value of the supplementary angle in the zone, $ee = (101) : \bar{1}01$, was identical with the value $121^\circ 16'$ which it should possess if $ae = 29^\circ 22'$. For $ee = 180^\circ - 2ae = 180^\circ - 58^\circ 44' = 121^\circ 16'$.

Hence we can take $ae = 29^\circ 22'$ with great confidence as the basal angle.

To find the axial ratio, $a : c$.

We can calculate this fundamental constant directly from first principles. Fig. 164 represents diagrammatically the relations of the face $e = (101)$ to the horizontal axis a and the vertical axis c . The face is inclined so that the angle between the normal to $a = (100)$, which is the axis a , and the normal to the face $e = (101)$ itself, is $29^\circ 22'$, that is the angle aOe is $29^\circ 22'$. But this is also the magnitude of the angle acO , by the principle of similar triangles, and the axial ratio required is that of the two sides Oa and Oc of the right-angled triangle acO to one another, which ratio is obviously the tangent of the angle at c , $29^\circ 22'$, the basal angle.

Now the ratio of the axes in the tetragonal system is always expressed in the form that $a = 1$, and it does not matter whether we consider the front-to-back axis a or the lateral right-and-left axis also lettered a , both being equal. Consequently we shall set out our ratio from Fig. 164 in the following form, which enables us to calculate the value of c compared with $a = 1$.

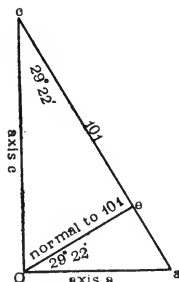


FIG. 164.

$$c : a = Oc : Oa ; \text{ or } \frac{c}{a} = \frac{Oc}{Oa} = \cot 29^\circ 22'.$$

As $a = 1$ we have $c = \cot 29^\circ 22' = 1.7771$.

Or, writing the axial ratio in the form in which it is usually presented :

$$a : c = 1 : 1.7771.$$

This is in absolute agreement with the value of the ratio given by Dana.

The same result is equally well arrived at by the application of the general method of calculating axial ratios, given in Chapter VII., to the case of tetragonal crystals as stated on page 98 of that chapter. It is there shown, and will also be obvious from the above, that as $a = b$ the tetragonal ratio c/a is the same as that for c/b and that :

$$\frac{c}{b} = \frac{\sin \psi}{\sin \chi}.$$

Now ψ is the angle $ce = \frac{1}{2}ee$ (Fig. 163), and χ is the complementary angle ae , the two making up the 90° axial angle ac , so that we have :

$$\frac{c}{a} = \frac{c}{b} = \frac{\sin ce}{\sin ae} = \frac{\sin 60^\circ 38'}{\sin 29^\circ 22'} = \frac{\cos 29^\circ 22'}{\sin 29^\circ 22'} = \cot 29^\circ 22' = 1.7771.$$

Calculation of Interfacial Angles.

To find the position of $p=(111)$, the parametral face, that is, to find $cp=(001):(111)$. (For stereographic projection see Fig. 163.)

We make use of the triangle $cep=(001):(101):(111)$, in which the angle at e is a right angle, the angle at c is 45° , and the side-arc ce is $60^\circ 38'$, the complement of the basal angle. The Napierian diagram for this triangle, constructed as described in Chapter VII., is given in Fig. 165. From this diagram, by Napier's rules, we obtain:

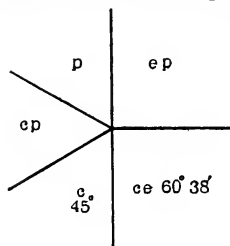


FIG. 165.

$$\begin{aligned} \cos 45^\circ &= \tan 60^\circ 38' \cot cp, \\ \text{or: } \cot cp &= \cos 45^\circ \cot 60^\circ 38'. \quad cp = 68^\circ 18'. \\ \text{Then: } pm &= (111):(110) = 90^\circ - 68^\circ 18' = 21^\circ 42'. \end{aligned}$$

From the same triangle we can find $ep=(101):111$, the diagram at once affording:

$$\begin{aligned} \sin ep &= \sin 45^\circ \sin 68^\circ 18'. \quad ep = 41^\circ 43'. \\ \text{Then: } pp &= (111):(\bar{1}\bar{1}1) = 2 ep = 82^\circ 9', \\ \text{and: } ap &= 90^\circ - 41^\circ 4' = 48^\circ 56'. \end{aligned}$$

The measured angular values of pm , ep , pp , and ap were respectively $21^\circ 39'$, $41^\circ 5'$, $82^\circ 10'$, and $48^\circ 55'$, in all but the first case only $1'$ removed from the calculated value, and in the case of pm only $3'$ removed. The angle cp was not measured on account of the absence of the faces of the basal pinakoid.

To find the position of $z=(113)$, that is, to find $cz=(001):(113)$.

It was surmised that the form z was $\{113\}$, and confirmed by a preliminary use of the anharmonic ratio of four poles in a zone, the 90° -zone [$c=(001)$, $z=(113)$, $p=(111)$, $m=(110)$], taking the measured angles as correct and setting down the indices of z as (hhl) , for h worked out to be 1 and l to be 3. We are now to find the value of the angle cz by the converse process, taking the indices of z to be (113) . The conditions in the zone-quadrant are shown in Fig. 166, with the aid of which we derive the following anharmonic ratio:

$$\begin{aligned} \frac{\sin cz}{\sin 68^\circ 18'} \cdot \frac{\sin 21^\circ 42'}{\sin mz} &= \frac{001}{001} \cdot \frac{110}{110} \\ &= \frac{113}{113} \cdot \frac{111}{111} \\ &= \frac{111}{111} \cdot \frac{113}{113} \\ \frac{\sin cz}{\sin 68^\circ 18'} \cdot \frac{\cos 68^\circ 18'}{\cos cz} &= \frac{1}{1} \cdot \frac{1}{3} \\ \tan cz &= 1/3 \tan 68^\circ 18'. \quad cz = 39^\circ 57'. \end{aligned}$$

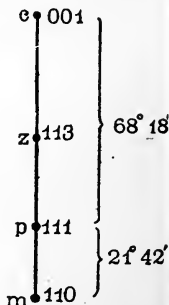


FIG. 166.

This angle could not be measured because of the absence of c -faces. Its double, $zz=(113):(\bar{1}\bar{1}3)$, was measured, however, and found to be $79^\circ 53'$, the calculated value being $79^\circ 54'$. Also its complement, $zm=(113):(110)$, was measured and found to be $50^\circ 4'$, only $1'$ again removed, for:

$$zm = 90^\circ - 39^\circ 57' = 50^\circ 3'.$$

To find the position of $r=(335)$, that is, to find $cr=(001):(335)$.

The indices of the form r were not so easy to surmise as those of z , for this particular form is not so common on anatase, and we are quite unassisted by any possible cross-multiplication of the zones intersecting in any r -pole, there being only one such zone, $[crpm]$, with more than one other known pole situated on it, two being of course required in order to be able to find the indices of the zone for use in cross-multiplication. We have, therefore, to rely entirely on the anharmonic ratio of four

poles, including the as yet unknown r , in the zone $[crpm]$. We can either assume some likely simple indices for r and see if the calculated angles then agree with the measured, or we can set the indices of r down in the ratio as (hhl) and assume the measured angles to be correct, and thus find h and l . The former course was first tried, as if successful at once it was the easier, assuming the indices of r to be (112) , for we knew they should be intermediate between those of z and of p ; if correct, we should have an expression for the anharmonic ratio of the following final form, analogous exactly to that for cz just calculated :

$$\tan cr = 1/2 \tan 68^\circ 18'.$$

This, however, yielded the value for cr of $51^\circ 29'$, whereas half the measured angle rr (c being not developed) was $56^\circ 28'$. Obviously, therefore, the indices of r were not (112) . The indices (223) were then tried, corresponding to the anharmonic ratio :

$$\tan cr = 2/3 \tan 68^\circ 18'.$$

This yielded $59^\circ 10'$ for cr , so here again the correct indices had not been arrived at.

The latter of the two courses above referred to was then proceeded with, as it was desirable to find the indices now at once by an infallible method. Setting, therefore, r down as (hhl) , we have the conditions shown in Fig. 167, from which we derive the anharmonic ratio :

$$\begin{array}{rcccl} & 001 & 110 & & \\ & \times & \times & & \\ \sin 39^\circ 57' & \sin 33^\circ 32' & = & \frac{113}{001} \cdot \frac{hhl}{110} & \\ \cos 33^\circ 32' & \cos 39^\circ 57' & = & \frac{001}{hhl} \cdot \frac{110}{113} & \\ & \times & \times & & \\ & hhl & 113 & & \end{array}$$

$$\tan 39^\circ 57' \tan 33^\circ 32' = \frac{1}{h} \cdot \frac{l}{3} = \frac{l}{3h}; \text{ or } \frac{l}{h} = 3 \tan 39^\circ 57' \tan 33^\circ 32'.$$

$$\frac{l}{h} = 1.6656 = \frac{5}{3} \text{ exactly.}$$

Thus $h=3$ and $l=5$, and the indices are therefore (335) .

Employing now the true indices (335) , we can calculate with confidence the angle cr from the equation analogous to that for cz , namely,

$$\tan cr = 3/5 \tan 68^\circ 18'. \quad cr = 56^\circ 27'.$$

Then also :

$$rm = 90^\circ - cr = 90^\circ - 56^\circ 27' = 33^\circ 33'.$$

This is only $1'$ removed from the measured value of rm , $33^\circ 32'$. The complementary angle cr itself was not measurable on account of the absence of the c -faces, but rr its double was measured, and the half of this $\frac{rr}{2} = 56^\circ 28'$.

Also it follows that : $pr = cp - cr = 68^\circ 18' - 56^\circ 27' = 11^\circ 51'$;
and that : $zr = cr - cz = 56^\circ 27' - 39^\circ 57' = 16^\circ 30'$.

The mean measured values of pr and zr were $11^\circ 52'$ and $16^\circ 31'$, in each case only $1'$ removed from the calculated value.

In a precisely similar manner a preliminary use of the anharmonic ratio of the four poles $[cp sm]$, assuming the correctness of the measured angles involving ms and sp and setting down the indices of s as (hhl) , indicated that the latter were (221) , for h came out to be 2 and l to be 1. We now use these indices to find the position of s by means of the same anharmonic ratio in which the angles involving the other three poles are such as have been already calculated and are thus known with certainty.

To find $ms = (110) : (221)$ and $sp = (221) : (111)$.

The conditions in the quadrant are indicated in Fig. 168. The anharmonic ratio is as follows :

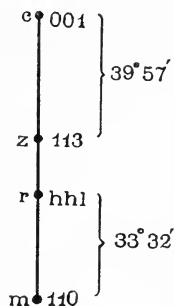


FIG. 167.

$$\frac{\sin ms}{\cos cp} \cdot \frac{\sin cp}{\cos ms} \cdot \frac{221}{110} \cdot \frac{111}{001},$$

$$\tan ms \tan cp = \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2}.$$

Now cp has been shown to be $68^\circ 18'$; therefore:

$$\tan ms = 1/2 \cot 68^\circ 18'. \quad ms = 11^\circ 15'.$$

Then $sp = mp - ms = 21^\circ 42' - 11^\circ 15' = 10^\circ 27'.$

The mean measured values of ms and sp were $11^\circ 14'$ and $10^\circ 24'$, differing only $1'$ and $3'$ respectively from the values just calculated.

Having now calculated all the positions of poles in the two diagonal diametral zones, we may next proceed to calculate the positions of the poles of the forms z , r , and s in the arc-zones connecting them with the a -poles. The fourth of these arc-zones, including also e -poles, has already been calculated, namely, that involving the parametral form $p = \{111\}$. We have proved from the measurements that the symmetry is truly tetragonal, so that we have only to calculate one angle for each form, namely, az , ar , and as , for when each of these is known the other angle zz , rr , or ss is given by the symmetry. Hence we have only now three angles to calculate az , ar , and as ; and we use in the three cases the three triangles formed by $a = (100)$, $m = (110)$, and the pole z , r , or s in question, in which the common side am is 45° and the angle at m is always 90° .

To find $az = (100) : (113)$.

Employing the triangle $a = (100) : m = (110) : z = (113)$, the other necessary known part of the triangle is the side mz , which we have found to be $50^\circ 3'$. The Napierian diagram is given in Fig. 169, and we deduce therefrom by Napier's rules that:

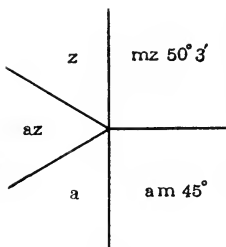


FIG. 169.

$$\cos az = \cos 50^\circ 3' \cos 45^\circ. \quad az = 63^\circ 0'.$$

Then:

$$zz = (113) : (\bar{1}13) = 2(90^\circ - 63^\circ) = 2 \times 27^\circ 0' = 54^\circ 0'.$$

These values for az and zz are identical with the mean measured values.

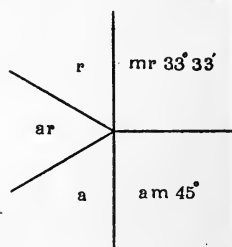


FIG. 170.

To find $ar = (100) : (335)$.

Similarly, the other necessary element of the triangle $a = (100) : m = (110) : r = (335)$ is the side mr , which has already been shown to be $33^\circ 33'$. From the Napierian diagram in Fig. 170 we deduce:

$$\cos ar = \cos 33^\circ 33' \cos 45^\circ. \quad ar = 53^\circ 53\frac{1}{2}'.$$

Then: $rr = (335) : (\bar{3}35) = 2(90^\circ - 53^\circ 53\frac{1}{2}') = 2 \times 36^\circ 6\frac{1}{2}' = 72^\circ 13'.$

The mean measured values of ar and rr were respectively $53^\circ 52'$ and $72^\circ 15'$.

To find $as = (100) : (221)$.

Also similarly, in the triangle $a = (100) : m = (110) : s = (221)$ the other known element besides $am = 45^\circ$ and the right angle at m is ms , which we have found to be $11^\circ 15'$. From these data and the Napierian diagram in Fig. 171 we derive:

$$\cos as = \cos 11^\circ 15' \cos 45^\circ. \quad as = 46^\circ 5'.$$

Then: $ss = (221) : (\bar{2}21) = 2(90^\circ - 46^\circ 5') = 2 \times 43^\circ 55' = 87^\circ 50'.$

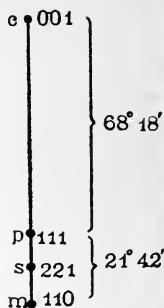


FIG. 168.

The mean measured values of these angles as and ss were respectively $46^{\circ} 6'$ and $87^{\circ} 48'$.

It now only remains to calculate one further set of four equal zones, each including a parallel pair of m -faces and four e -faces, and which are indicated on the projection by the four circular arcs joining the ends of the diagonal diameters mem (the m -poles) with two e -poles, one on each crystallographic axial diameter. There are only two different angles, the same two in all four zones, a fact proved by the measurements and which is in accordance with the tetragonal symmetry, namely, me and ee .

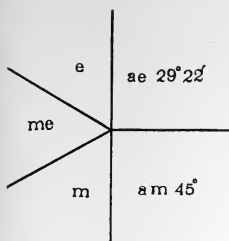


FIG. 172.

To find $me = (110) : (101)$.

We employ the triangle $m = (110) : e = (101) : a = (100)$, which has a right angle at a ; the side am is 45° and the side ae is the basal angle $29^{\circ} 22'$. Constructing the Napierian diagram, as in Fig. 172, we at once derive from it by Napier's rules :

$$\cos me = \cos 29^{\circ} 22' \cos 45^{\circ}. \quad me = 51^{\circ} 57'.$$

$$\text{Then :} \quad ee = (101) : (0\bar{1}1) = 2(90^{\circ} - 51^{\circ} 57') = 2 \times 38^{\circ} 3' = 76^{\circ} 6'.$$

The mean measured values of me and ee respectively were $51^{\circ} 56'$ and $76^{\circ} 7'$, only $1'$ removed from the calculated value in each case.

This completes the calculations for this crystal of anatase; they have throughout been found to confirm the measurements in a most satisfactory manner. It will have been observed that the whole of the calculations have been carried through entirely by the use of Napier's rules for right-angled triangles, which, owing to the high degree of symmetry, have always been available, and the anharmonic ratio of four poles in a right-angled zone. They have illustrated well how simple such calculations really are, and yet they have been carried out to the full extent required by the very complete series of measurements made. The experience gained in working through such an example as is afforded by this crystal of anatase, so richly endowed with faces, will give the student of practical crystallography considerable confidence, and enable him to proceed to the calculations of crystals of a lower order of symmetry without feeling that any insuperable difficulties whatsoever lie before him. Such difficulties as may be found will equally rapidly disappear under the application of the few straightforward rules given in Chapter VII.

We now finally proceed to draw up the table of results for anatase.

TABLE OF RESULTS FOR CRYSTAL OF ANATASE, TiO_2 .

Crystal-system : Tetragonal. **Class :** 15, tetragonal holohedral.

Habit : Bipyramidal, very rich in faces, so developed that the general contour is nearly spherical.

Ratio of axes : $a : c = 1 : 1.7771$.

Forms observed : $a = \{100\}$, $m = \{110\}$, $e = \{101\}$, $p = \{111\}$, $z = \{113\}$, $r = \{335\}$, $s = \{221\}$.

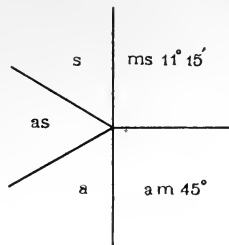


FIG. 171.

No face of $c = \{001\}$ developed, upper and lower apices of bipyramid $z = \{113\}$ pointed.

Interfacial angles: These are tabulated in the following list, the basal angle employed in the calculations being marked with an asterisk.

MORPHOLOGICAL ANGLES OF ANATASE.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$\left\{ \begin{array}{l} aa = (100) : (010) \\ am = (100) : (110) \\ \text{or } (010) : (110) \end{array} \right.$	4 4	$89^\circ 59' - 90^\circ 1'$ 45 0-45 1	90° 0' 45 0	90° 0' 45 0	0' 0
$\left\{ \begin{array}{l} ae = (100) : (101) \\ \text{or } (010) : (011) \\ ce = (001) : (101) \\ \text{or } (001) : (011) \\ ee = (101) : (\bar{1}01) \end{array} \right.$	8 4	29 16-29 28 ... 121 12-121 16	29 22 ... 121 15	* 60 38 121 16 1
$\left\{ \begin{array}{l} cz = (001) : (113) \\ zz = (113) : (\bar{1}\bar{1}3) \\ zr = (113) : (335) \\ rp = (335) : (111) \\ cp = (001) : (111) \\ ps = (111) : (221) \\ sm = (221) : (110) \\ pm = (111) : (110) \\ cr = (001) : (335) \\ rm = (335) : (110) \\ zm = (113) : (110) \end{array} \right.$... 4 8 6 ... 3 3 4 ... 6 6	... 79 52-79 54 16 29-16 34 11 48-11 58 ... 10 20-10 27 11 10-11 18 21 36-21 42 ... 33 30-33 34 50 3-50 5	... 79 53 16 31 11 52 ... 10 24 11 14 21 39 ... 33 32 50 4	39 57 79 54 16 30 11 51 68 18 10 27 11 15 21 42 56 27 33 33	... 1 1 1 ... 3 1 3 ... 1 1
$\left\{ \begin{array}{l} ap = (100) : (111) \\ \text{or } (010) : (111) \\ pe = (111) : (101) \\ \text{or } (111) : (011) \\ pp = (111) : (\bar{1}\bar{1}\bar{1}) \\ \text{or } (111) : (\bar{1}\bar{1}\bar{1}) \end{array} \right.$	8 8 4	48 47-48 59 41 0-41 8 82 8-82 15	48 55 41 5 82 10	48 56 41 4 82 9	1 1 1
$\left\{ \begin{array}{l} az = (100) : (113) \\ \text{or } (010) : (113) \\ zz = (113) : (\bar{1}\bar{1}3) \\ \text{or } (113) : (\bar{1}\bar{1}3) \end{array} \right.$	8 4	62 57-63 4 53 59-54 1	63 0 54 0	63 0 54 0	0 0
$\left\{ \begin{array}{l} ar = (100) : (335) \\ \text{or } (010) : (335) \\ rr = (335) : (\bar{3}\bar{3}5) \\ \text{or } (335) : (\bar{3}\bar{3}5) \end{array} \right.$	8 4	53 48-53 56 72 12-72 18	53 52 72 15	53 53 72 13	1 2
$\left\{ \begin{array}{l} as = (100) : (221) \\ \text{or } (010) : (221) \\ ss = (221) : (\bar{2}\bar{2}1) \\ \text{or } (221) : (\bar{2}\bar{2}1) \end{array} \right.$	6 4	46 1-46 9 87 47-87 51	46 6 87 48	46 5 87 50	1 2
$\left\{ \begin{array}{l} me = (\bar{1}\bar{1}0) : (101) \\ ee = (101) : (011) \end{array} \right.$	6 4	51 54-51 59 76 6-76 8	51 56 76 7	51 57 76 6	1 1

The excellent agreement between the observed and calculated values of the various interfacial angles set forth in the table, indicated by the fact that the maximum difference given in the last column is only 3 minutes, is most satisfactory, and is such as can only be expected from small crystals of a high order of perfection, possessed of brilliant and truly plane faces.

CHAPTER XV

RHOMBIC OR ORTHORHOMBIC SYSTEM.

Three rectangular but unequal crystallographic axes. Characterised by the vertical axis being an axis of only digonal symmetry, in which two planes of symmetry may intersect and to which two other digonal symmetry axes may be perpendicular.

It will be evident from the above statement of the conditions for orthorhombic symmetry that three classes are possible in this system,

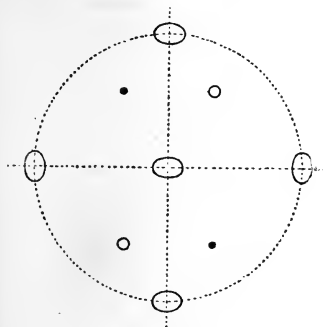


FIG. 173.—Class 6.

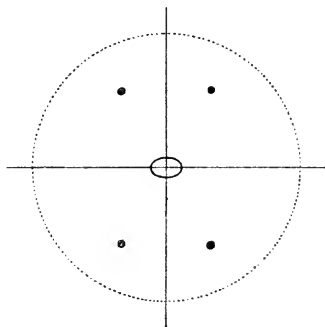


FIG. 174.—Class 7.

namely, one (class 7) in which the vertical crystallographic axis alone is a digonal axis, in which two planes of symmetry intersect; another (class 6) in which the other two crystallographic axes are also digonal axes, but in which no planes of symmetry are present; and a third (class 8) in which besides all three crystallographic axes being digonal axes they are also the lines of intersection of three rectangular symmetry planes. These three classes are comparatively indicated in their class order by the stereographic projections in Figs. 173, 174, and 175.

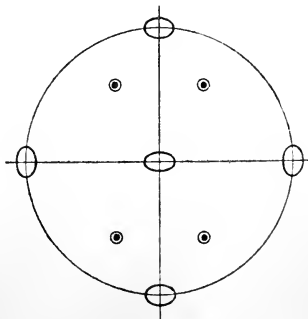


FIG. 175.—Class 8.

*Class 8.—Rhombic-Bipyramidal Class. Rhombic-Holohedral Class.
Type, Di-Digonal Equatorial.*

In this, the highest class of orthorhombic symmetry, the three crystallographic axes are all digonal axes of symmetry, and they are the lines of intersection of three rectangular symmetry planes, identical with the three axial planes. The distinction from the cubic and tetragonal systems is that no two axes are of equal length, and no axis is one of pre-eminent symmetry, such as the principal axis of the tetragonal or hexagonal systems. Hence there is nothing to determine which axis shall be considered the vertical one, except in certain cases of analogy which will be referred to in a later chapter, but after a choice has been made of the vertical axis c , the longer of the two others is always taken as the lateral right-and-left axis b . This axis b is termed the macro-diagonal, and the shorter (front-and-back) axis a is termed the brachy-diagonal; the c -axis is simply referred to as the vertical axis.

The elements of symmetry and the general form $\{hkl\}$ are shown in the stereographic projection, Fig. 175.

The general form, which is produced when the representative facial pole $\{hkl\}$ falls within one of the eight equal octants into which the axial planes divide up the

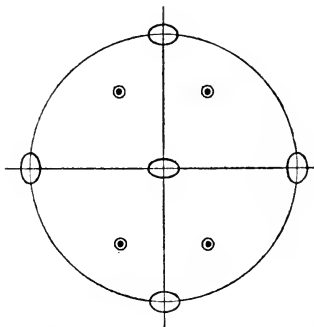


FIG. 175.—Symmetry Elements and General Form of Class 8.

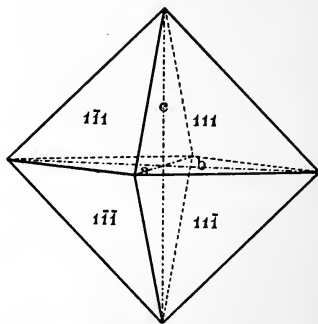


FIG. 176.—The Primary Rhombic Bipyramid.

sphere, consists, owing to the symmetrical repetition of the pole in all the other octants in accordance with the symmetry conditions, of an eight-faced bipyramid, the **rhombic bipyramid**, shown in Fig. 176, composed of two equal-faced and four-faced pyramids arranged base to base. The latter is a regular rhombus, the two pairs of opposite angles being of different magnitudes; in the case of the bipyramid chosen as the parametral form, that shown in Fig. 176, the more obtuse pair of angles are bisected by the brachy-diagonal and the acuter pair by the macro-diagonal. The particular primary rhombic bipyramid shown, $\{111\}$, is that of topaz, the ratio of the axes for which is $a:b:c=0.5287:1:0.9539$. All the succeeding forms of this class shown in Figs. 177-181 are also those of topaz.

There are six special cases, each producing a specific solid characteristic of the system. The first case is when the typical pole $\{hkl\}$ lies on the primitive circle, intermediate between the ends of the axes. Repetition over the symmetry planes

necessitates three other poles on the primitive circle, corresponding to the production of an open four-sided prism, parallel to the vertical axis, the **rhombic prism**; the parametral or primary form is $\{110\}$, the poles of which lie at the ends of the diameters passing through the poles of the parametral bipyramid, so that the section of the prism by the horizontal plane is like that of the latter. Of the general form $\{hk0\}$, the poles of which may lie on either side of these diameters, the macro-diagonal side or the brachy-diagonal side, there will be two corresponding types, which are sometimes distinguished as macro-prisms and brachy-prisms, just as are also termed macro- or brachy-pyramids such rhombic bipyramids other than the parametral one as have a macro- or brachy-character. There is no essential difference, however, in the type of form, so that there is little utility in the employment of these distinguishing terms, which, moreover, are much more properly used in another sense to be explained in the next paragraph. The primary rhombic prism $\{110\}$ is shown in Fig. 177.

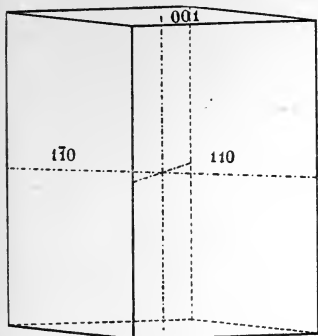


FIG. 177.—The Primary Rhombic Prism.

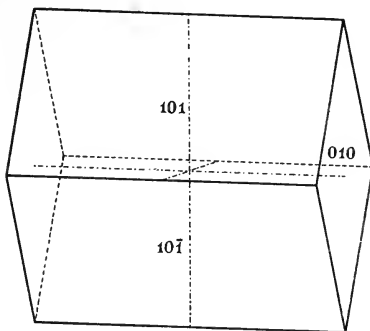


FIG. 178.—The Primary Macro-domal Prism.

The second and third special cases occur when the pole $\{hkl\}$ migrates on to the diametral arcs ac or bc . In the former case a form of four faces parallel to the macro-diagonal is produced, having the form symbol $\{h0l\}$ and generally in the past called the **macro-dome**, but which may more correctly be termed the macro-prism, as it is another form of prismatic character. The term macro-prism has, however, already been appropriated by long usage (initiated by Naumann) for the four-faced prism of a macro-character parallel to the vertical axis as explained in the last paragraph; it is perhaps safest, therefore, and most free from ambiguity to speak of the form $\{h0l\}$ as the **macro-domal prism**, which term renders it quite plain to which type of form we refer. Its primary representative $\{101\}$ is shown in Fig. 178.

The third case, in which the pole lies on the arc bc , is that of a similar form of four prism-like faces parallel to the brachy-diagonal, which similarly is best called the **brachy-domal prism**, its form symbol being $\{0kl\}$. Its primary representative $\{011\}$ is shown in Fig. 179. These two forms correspond together to the tetragonal pyramid of the second order, and may be regarded as forming, when their intercepts on the c axis are alike, the rectangular

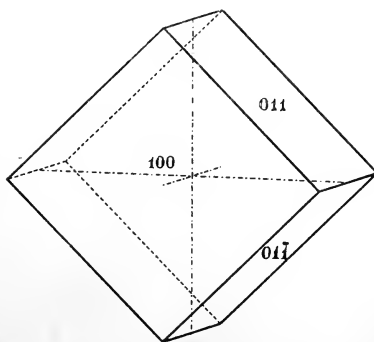


FIG. 179.—The Primary Brachy-domal Prism.

rhombic bipyramid shown in Fig. 180. The two parts, parallel to the two lateral axes respectively, are, however, on account of the merely digonal character of rhombic symmetry, quite distinct forms.

The three remaining special cases are those when the pole $\{hkl\}$ migrates to the ends of the axial diameters on the primitive circle or to the centre of the latter, comparable to the situations of the cube faces, or to those of the tetragonal prism of the second order and the basal plane. In the case of the rhombic system, however, the three pairs of parallel faces, which are thus parallel to the three axial planes, are three separate forms, namely, the **macro-pinakoid** $\{100\}$ the poles of which are situated at the

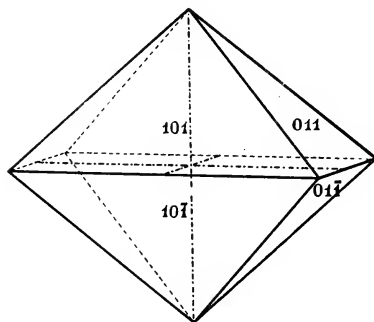


FIG. 180.—The Primary Rectangular Rhombic Bipyramid (two Forms).

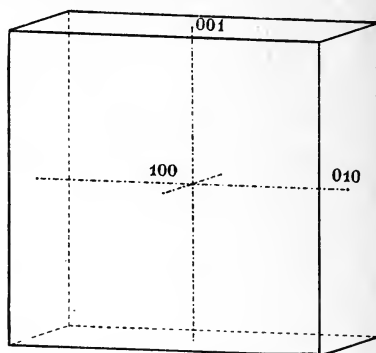


FIG. 181.—The Rectangular Rhombic Prism (three Forms).

ends of the axis a , the **brachy-pinakoid** $\{010\}$ with poles at the terminals of the axis b , and the **basal pinakoid** $\{001\}$ the poles of which occupy the centre of the projection.

The closed solid produced by the combination of these three primary forms is shown in Fig. 181. It is sometimes called a rectangular rhombic prism, each face being a rectangle, and the three edges meeting in each of the eight solid angles are parallel to the three crystallographic axes. The solid actually shown in Fig. 181 looks like a cube, but it is really drawn accurately to the scale of the axial system of topaz, $a : b : c = 0.5287 : 1 : 0.9539$; the b and c axes appear to be equal, but they are really not so, being in the proportion of 1 to 0.9539.

The forms possible to holohedral rhombic symmetry are thus seven in number, made up of the general bipyramid $\{hkl\}$ and six special cases. They are comprised in the following list, in which not only are the above designations of the various forms given, but also those lately proposed for the rhombic, monoclinic and triclinic systems by von Fedorow and adopted by von Groth, according to which the pinakoids are called "first," "second," or "third" when they are parallel to the three respective axial planes, and cut (perpendicularly in this system) the axes a , b and c respectively; while forms parallel to an axis are said to be of the first, second, or third "order" according as the axis is a , b , or c .

List of Forms in Class 8.

- $\{001\}$ Basal pinakoid; or third pinakoid. 2 faces.
- $\{100\}$ Macro-pinakoid; or first pinakoid. 2 faces.
- $\{010\}$ Brachy-pinakoid; or second pinakoid. 2 faces.
- $\{hkl\}$ Rhombic prisms, including the primary rhombic prism $\{110\}$; or rhombic prisms of the third order. Each 4 faces.
- $\{h0l\}$ Macro-domal prisms, including the primary macro-domal prism $\{101\}$; or rhombic prisms of the second order. Each 4 faces.

- $\{0kl\}$ Brachy-domal prisms, including the primary brachy-domal prism $\{011\}$; or rhombic prisms of the first order. Each 4 faces.
- $\{hkl\}$ Rhombic bipyramids, including the primary parametral rhombic bipyramid $\{111\}$. Each 8 faces.

Many excellent examples are available of substances crystallising in this holohedral class of the rhombic system, notably among natural minerals barytes, sulphate of barium BaSO_4 , and topaz, fluoriferous silicate of aluminium $(\text{AlF})_2\text{SiO}_4$, a crystal of which will be fully described in the next chapter. Among chemical salts we have the sulphates of the alkali metals, of which one, sulphate of potassium, K_2SO_4 , has already been fully worked out as a typical practical example of goniometry in Chapters IV. and VIII., and which is illustrated in Figs. 1, 2, and 3. The fairly complicated stereographic projection of potassium sulphate already given in Fig. 25 (page 57) affords an excellent example of the projection of a holohedral rhombic crystal showing most of the types of forms above enumerated.

Class 7.—Rhombic-Pyramidal Class. Rhombic-Hemimorphic Class.
Type, Digonal Polar.

This class retains only one of the three digonal axes, which is conventionally placed vertically for descriptive purposes, and the two mutually rectangular planes of symmetry which intersect in it. The equatorial plane of symmetry thus disappears, together with the two digonal axes lying in it. The symmetry elements are shown in Fig. 182, together with the poles of the most general form $\{hkl\}$.

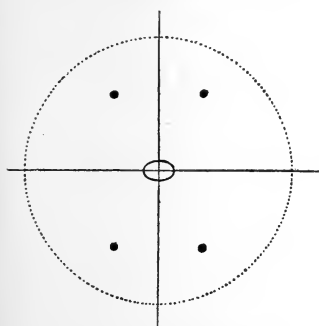


FIG. 182.—Symmetry Elements and General Form of Class 7.

The latter is now only a single open **rhombic pyramid**, either an upper one with poles on the upper hemisphere as shown in the figure, or a lower one which would be represented if the dots were changed into rings. The prisms and pinakoids are the same as in the holohedral class 8, but the two parallel faces of the basal pinakoid now belong to separate forms, the upper and lower basal plane or “pedion,” and Figs. 183 and 184 show two instances of an upper pyramid in combination with a basal plane. Fig. 183

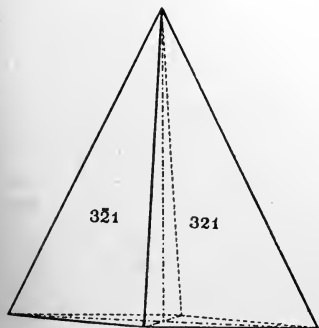


FIG. 183.
 Combinations of Upper Rhombic Pyramids with the Basal Plane.

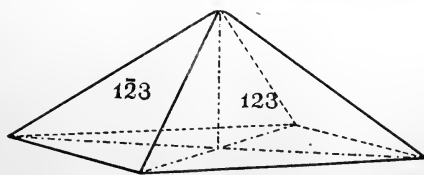


FIG. 184.

represents the form $\{321\}$, and the shorter more obtuse one shown in Fig. 184 is $\{123\}$, both being constructed for the axial ratio of topaz, $a : b : c = 0.5287 : 1 : 0.9539$. The macro-domal and brachy-domal prisms are also divided into an upper and a lower macro-dome or hemi-prism and an upper and a lower brachy-dome or hemi-prism, and these forms are thus truly "domes" in this class, each consisting of two faces meeting in an edge.

The vertical axis c is obviously in this class a polar axis, for quite a different set of forms, pyramidal or domal, may be developed round one end of it than around the other end, and the two ends actually exhibit opposite pyro-electrical properties in the cases of the substances known to crystallise in the class.

We have, therefore, in résumé, the following forms in this class :

List of Forms in Class 7.

- $\{001\}$ Upper basal plane ; or upper third pedion. 1 face.
- $\{00\bar{1}\}$ Lower basal plane ; or lower third pedion. 1 face.
- $\{100\}$ Macro-pinakoid ; or first pinakoid. 2 faces.
- $\{010\}$ Brachy-pinakoid ; or second pinakoid. 2 faces.
- $\{hko\}$ Rhombic prisms, including the primary prism $\{110\}$; or rhombic prisms of the third order. Each 4 faces.
- $\{h0l\}$ Upper macro-domes or hemi-prisms, including the primary $\{101\}$; or upper domes of the second order. 2 faces.
- $\{h0\bar{l}\}$ Lower macro-domes or hemi-prisms, including the primary $\{10\bar{1}\}$; or lower domes of the second order. 2 faces.
- $\{0kl\}$ Upper brachy-domes or hemi-prisms, including the primary $\{011\}$; or upper domes of the first order. 2 faces.
- $\{0k\bar{l}\}$ Lower brachy-domes or hemi-prisms, including the primary $\{01\bar{1}\}$ or lower domes of the first order. 2 faces.
- $\{hkl\}$ Upper rhombic pyramids, including the primary $\{111\}$. 4 faces.
- $\{hk\bar{l}\}$ Lower rhombic pyramids, including the primary $\{11\bar{1}\}$. 4 faces.

Hemimorphite, silicate of zinc, $\text{Zn}_2\text{SiO}_3(\text{OH})_2$, is an excellent example of a substance crystallising in this class.

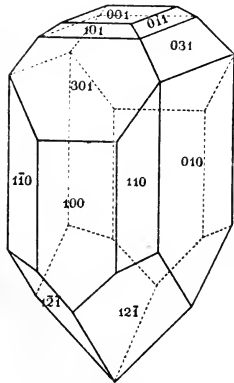


FIG. 185.
Crystal of Hemimorphite.

Fig. 185 represents a crystal of this mineral, which consists of a combination of the macro-pinakoid $\{100\}$, the brachy-pinakoid $\{010\}$, and the primary prism $\{110\}$, together with the following terminal forms : At the upper termination, the "analogous" pyro-electrical pole (becoming positively charged on warming and negatively on cooling, the charge being analogous to the sign of the change of temperature), there are the macro-domes or hemi-prisms $\{101\}$ and $\{301\}$, the brachy-domes or hemi-prisms $\{011\}$ and $\{031\}$, and the upper basal plane $\{001\}$; while at the lower end, the "antilogous" pyro-electrical pole (becoming negatively charged on warming and positively on cooling), only the single pyramid $\{12\bar{1}\}$ is developed. The ratio of the axes, in accordance with

which the figure is accurately drawn, is :— $a : b : c = 0.7835 : 1 : 0.4778$, derived from the measurements of Schrauf. Etched figures on the brachy-pinakoid exhibit the polarity of the two ends very clearly.

Class 6.—Rhombic-Bisphenoidal Class. Rhombic-Hemihedral Class.

Type, Digonal Holoaxial.

This class possesses the full number of digonal axes of symmetry of the system, namely, three, identical with the three unequal rectangular crystallographic axes; there are no planes of symmetry present at all, however. Fig. 186 shows the symmetry elements and the general form $\{hkl\}$.

This latter is a four-faced figure corresponding to the tetragonal sphenoid, and to the regular tetrahedron, but owing to the inequality of the axes its faces are scalene triangles instead of isosceles or equilateral. It is termed the **rhombic bisphenoid**. In the stereographic projection that particular bisphenoid is shown to which the face (hkl) belongs, situated in the top right front octant. Only alternate octants exhibit poles, so that there is a second possible bisphenoid the poles of which would occupy those octants which are unoccupied in the projection of the form shown, and the projection of which would be produced if the two dots were replaced by rings and the two rings by dots. This second form of rhombic bisphenoid is a distinct solid, the mirror-image of the first one, that is, the enantiomorph of the latter. Figs. 187 and 188 show the two

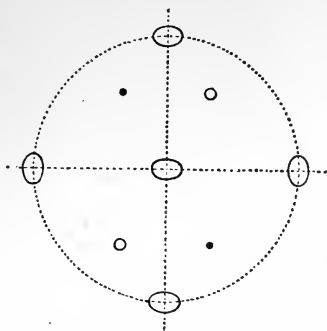


FIG. 186.—Symmetry Elements and General Form of Class 6.

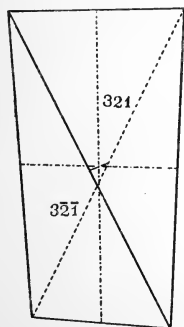


FIG. 187.

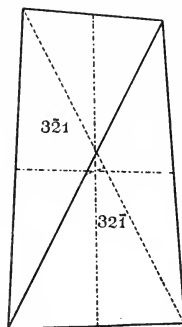


FIG. 188.

Right and Left Rhombic Bisphenoids.

bisphenoids side by side, the first form $\{hkl\}$, termed the right rhombic bisphenoid, in Fig. 187, and the second form $\{h\bar{k}l\}$, called the left rhombic bisphenoid, in Fig. 188. The two solids cannot be brought into coincidence or identity by any amount of rotation. The actual forms shown are $\{321\}$ and $\{3\bar{2}1\}$, constructed for the axial ratio of topaz, the two together thus corresponding to the rhombic bipyramid $\{321\}$, of which the upper half is shown in Fig. 183.

The six possible special cases lead to the same forms as in the holohedral class 8, namely, the two vertical pinakoids, basal pinakoid, rhombic prisms, macro-domal and brachy-domal prisms. The following is, therefore, a summary of the possible forms in the class:

CHAPTER XVI

PRACTICAL EXAMPLE OF A RHOMBIC CRYSTAL.

Class 8.—Rhombic Holohedral. Topaz, $(AlF)_2SiO_4$.

ONE example of this system has already been fully worked out in Chapters IV. and VIII., namely, potassium sulphate, which also belongs to the holohedral class 8. It has been thought advisable, however, to include another example of this very important class of crystals, taken this time from among naturally occurring mineral substances and involving different problems, and topaz has been chosen because of the comparative ease with which excellent small crystals, suitable for goniometrical investigation, may be obtained.

The crystal of topaz employed as this example was somewhat larger than is usually chosen for goniometrical measurement, but the faces were so perfect and the signal-images afforded by them so truly excellent as a rule, that the crystal was found admirably suitable for our purpose. Its general appearance will be gathered from Fig. 190, and its stereographic projection is given in Fig. 191, the method of constructing which will be elaborated during the course of the chapter.

The crystal was a short prism, the prism zone exhibiting eight well-developed faces; it was terminated below only by a cleavage face parallel to the basal pinakoid, but above by an excellent natural face of the same form, $c = \{001\}$, and between this and the prism faces numerous facets of other forms were developed.

A preliminary study of the crystal had made it clear that the faces present were all those of forms known to be characteristic of topaz, a mineral which has been very thoroughly studied. Different observers have all agreed that the prism faces marked p shall be chosen as

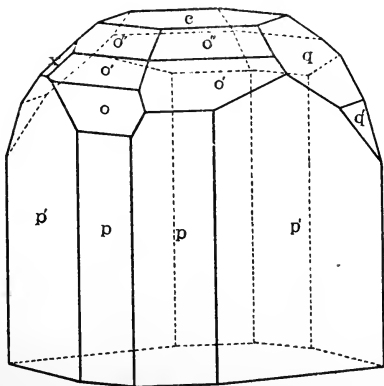


FIG. 190.—The Measured Crystal of Topaz.*

those of the primary prism $\{110\}$, and the other four prism faces, marked p' , from their position with respect to the p -faces, become then those of the prismatic form $\{120\}$, as will be fully proved later on. The macro- and brachy-pinakoids are not usually present and were not developed on the crystal in question.

Between the basal pinakoid $c = \{001\}$ and the prism $p = \{110\}$ the faces of three different pyramids were developed, namely, those marked o ,

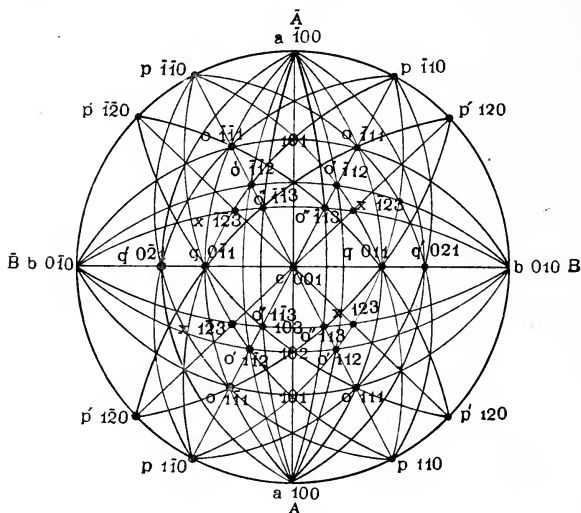


FIG. 191.—Stereographic Projection of Topaz.

o' , and o'' . Of these the first is taken by von Kokscharow, and accepted by von Groth, as the parametral form $o = \{111\}$, but Miers takes the second, o' , as $\{111\}$. There is considerable ground for so doing, for o' is much more frequently developed than o , and on the crystal in question only one o -face, although a very large and well-developed one, was present. We shall adhere, however, to

the original mode of description by von Kokscharow, but as it will form an excellent exercise for the student to work out the whole of the facial symbols on the second plan, a comparison of the two modes of description will be made at the conclusion of the chapter, after the presentation of the results of the measurements and calculations. There is no cleavage other than the principal one parallel to the basal plane to guide us in the choice of the plane which shall determine the length of the vertical axis, so that there is equal ground for choosing either the o -faces or the o' -faces as the parametral ones. Assuming for the purposes of the description, however, that o is $\{111\}$, the form o' will be shown from its position to be $\{112\}$, and the form o'' to be $\{113\}$.

There were also developed the faces of two different forms between the basal pinakoid and the possible brachy-pinakoid $b = \{010\}$, namely the primary and secondary brachy-domal prisms $q = \{011\}$ and $q' = \{021\}$. In addition, there was one face present on the crystal of a pyramidal form in the zone $[c = (001), p' = (120)]$, namely, $x = \{123\}$. All the letters representing these forms are those adopted by von Groth from von Kokscharow.

It will obviously be an advantage to commence with the measurement of the prism zone. The crystal was set up on the wax cone of the goniometer by its broad

basal cleavage plane, and the zone of prism faces adjusted in the usual manner, by arranging one of them parallel to one of the adjusting movements, and then adjusting the signal-images from that face and an adjoining one by means of the two adjusting screws. The whole eight faces belonging to the zone were then found to be automatically adjusted, and gave the following measurements :

PRISM ZONE.

Circle Readings.	Angles.
$\left. \begin{array}{l} p \ 360^\circ \ 0' A \\ p \ 304 \ 17 \ A \\ p' \ 285 \ 30 \\ p' \ 198 \ 40 \\ p \ 179 \ 56 \ A \\ p \ 124 \ 11 \ A \\ p' \ 105 \ 26 \ A \\ p' \ 18 \ 40 \ A \\ p \ 359 \ 58 \ A \end{array} \right\}$	$\begin{array}{l} pp \ 55^\circ \ 43' A \\ pp' \ 18 \ 47 \\ p'p' \ 86 \ 50 \\ p'p \ 18 \ 44 \\ pp \ 55 \ 45 \ A \\ pp' \ 18 \ 45 \ A \\ p'p' \ 86 \ 46 \ A \\ p'p \ 18 \ 42 \ A \end{array}$

The absence of the macro- and brachy-pinakoids renders the proof of the symmetry, with respect to the two axial planes of symmetry intersecting in the vertical axis, not so immediately clear as would be the case if they were present. For it is not yet apparent that the angles pp and $p'p'$ are bisected by the normals to the possible macro-pinakoid $a = \{100\}$ and brachy-pinakoid $b = \{010\}$, and for all the immediate evidence the prism zone alone offers, the positions of the whole of the poles might be rotated together anywhere about the vertical axis so long as the relationships indicated by the measurements were retained. As topaz is so well known, however, we may at once set forth the poles at the measured angles along the primitive circle, beginning with the p -poles at the distance from the possible a -poles of half the angle pp , and with the p' poles distant from the possible b -poles by half the angle $p'p'$. For it will be abundantly shown by the measurements of further zones that the symmetry thus given to the poles of the prism zone is correct.

The fact that the pair of values for pp are practically identical, and that the two $p'p'$ values are similarly within the limits of error allowed for equal angles, does not tell us anything more than that parallel faces are concerned in the angles. But the fact that the four values of pp' are also evidently intended to be equal, the "A" values being within the prescribed limits, is much more significant, for it indicates that the angles in the four quadrants of the primitive circle are equal, an indication of probable rhombic or at least monoclinic symmetry.

We may now tabulate the values of like angles and take the mean in each of the three cases as representing the most probable true value of the angle. The means were $pp = 55^\circ \ 44'$, $pp' = 18^\circ \ 44'$, and $p'p' = 86^\circ \ 48'$.

We may proceed next to measure the three zones perpendicular to the prism zone, all of which contain the basal pinakoid as the best-developed face. Two of these are obviously of similar character, namely the zones $[co''o'p]$, while the third is apparently different, namely, the zone $[cqq']$. It will therefore be most convenient to measure the latter zone first.

The crystal was fixed on the wax cone of the goniometer by the two front p -faces, the prism axis being arranged horizontally and the c -faces vertically and parallel to one of the adjusting screws. On adjusting the signal-image from one of these c -faces and that from an adjacent q -face by means of the two screws, the zone of faces in automatic adjustment was found to consist of a pair of q -faces, a pair of q' -faces, and the c -face and its parallel cleavage face. The measurements afforded were as under :

BRACHY-DOMAL PRISM ZONE.

Circle Reading.	Angle.
$\left\{ \begin{array}{l} q' \ 360^\circ \ 0'A \\ q \ 341 \ 21 \ A \\ c \ 297 \ 42 \ A \\ q \ 254 \ 3 \ A \\ q' \ 235 \ 22 \ A \end{array} \right.$	$\left\{ \begin{array}{l} q'q \ 18^\circ \ 39'A \\ qc \ 43 \ 39 \ A \\ cq \ 43 \ 39 \ A \\ qq' \ 18 \ 41 \ A \end{array} \right.$

The signal-images were all truly excellent, and the angular values afforded leave no doubt whatever that the zone is symmetrical to the axial plane parallel to the possible brachy-pinakoid $b = \{010\}$. The angle cq is $43^\circ 39'$, two identical values having been obtained for the two similar angles on the two sides of the basal pinakoid; and the angle qq may be safely taken to be $18^\circ 40'$, the mean being undoubtedly legitimate to take, as the individual values are only $2'$ apart, and thus within the prescribed limits for identical angles.

We may consequently draw in the zone $[cqq']$ as the horizontal diameter of the primitive circle of the stereographic projection, bisecting the angle $p'p'$, the terminations being the possible brachy-pinakoid poles $b = (010)$ and $b = (0\bar{1}0)$; and we may put in the poles q and q' at their proper positions—by the usual process of drawing a perpendicular diameter, which will pass through $a = (100)$, $c = (001)$, and $a = (\bar{1}00)$, and which in this case should anyhow be shown in the drawing as it is the projection of the axial plane parallel to the brachy-pinakoid—marking off the angles cq and cq' from the top end $a = (\bar{1}00)$ of this latter diameter, along the primitive circle, and joining the points thus marked off to the lower end $a = (100)$ of the diameter, which is the pole of the zone $[cqq']$. The points where the junction lines intersect the diameter $b = (010)$, $c = (001)$, $b = (0\bar{1}0)$, are the required poles q and q' . It will presently be proved that the position of the zone is symmetrical to the prism faces p and p' , that is, that the diameter $[cqq'b]$ does bisect the angle $p'p'$.

We next measure the pair of apparently similar zones $[co''o'p]$.

The crystal may be reset on the wax, maintaining the c -faces vertical and parallel to an adjusting screw, but with the edges cq and qq' now horizontal instead of vertical. By use of the other adjusting screw, first on one side of the centre and then on the other, each of the two zones in question may in turn be adjusted so that the edges co'' and $o''o'$ are vertical, and the two zones thus successively measured. The actual measurements obtained were as follows :

2 SIMILAR DIAMETRAL PYRAMIDAL ZONES.

Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} p \ 360^\circ \ 0'A \\ o \ 333 \ 52 \ A \\ o' \ 315 \ 35 \ A \\ o'' \ 304 \ 15 \ A \\ c \ 270 \ 2 \ A \\ o' \ 235 \ 46 \ A \\ o' \ 224 \ 26 \ A \\ p \ 180 \ 2 \ A \end{array} \right.$	$\left\{ \begin{array}{l} po \ 26^\circ \ 8'A \\ oo' \ 18 \ 17 \ A \\ po' \ 44 \ 25 \ A \\ o'o'' \ 11 \ 20 \ A \\ o''c \ 34 \ 13 \ A \\ co'' \ 34 \ 16 \ A \\ o'o' \ 11 \ 20 \ A \\ o'p \ 44 \ 24 \ A \end{array} \right.$	$\left\{ \begin{array}{l} p \ 360^\circ \ 0'A \\ o' \ 315 \ 35 \ A \\ o'' \ 304 \ 14 \ A \\ c \ 269 \ 59 \ A \\ o'' \ 235 \ 46 \ A \\ o' \ 224 \ 27 \ A \\ p \ 180 \ 2 \ A \end{array} \right.$	$\left\{ \begin{array}{l} po' \ 44^\circ \ 25'A \\ o'o' \ 11 \ 21 \ A \\ o''c \ 34 \ 15 \ A \\ co'' \ 34 \ 13 \ A \\ o'o' \ 11 \ 19 \ A \\ o'p \ 44 \ 25 \ A \end{array} \right.$

Collecting now the measured values of the angles of the two zones which appear to be similar, it becomes clear that the four values of each of the angles co'' , $o'o'$, $o'p$, and co' are within the accepted limits for identity, $o'p$ only showing 1' difference and co' , for which the greatest difference occurs, only 4'. Hence we can legitimately conclude that the angles between the basal pinakoid and the four o'' faces, or between the basal pinakoid and the four o' faces, are equal, and this involves not only symmetry with respect to the two vertical axial planes parallel to $a = \{100\}$ and $b = \{010\}$ but also symmetry with respect to the horizontal axial plane $c = \{001\}$, in short, full rhombic symmetry. The mean of the four values of each of the four angles just referred to may therefore be taken. They are: $co'' = 34^\circ 14'$, $o'o' = 11^\circ 20'$, $o'p = 44^\circ 25'$, and $co' = 45^\circ 34'$. If the crystal were a doubly terminated one, instead of being only singly terminated, a precisely similar arrangement of faces would occur in the lower hemisphere as in the upper one measured, because to every face on the upper hemisphere there would correspond a parallel one in the lower hemisphere, and all poles but those of the primitive circle might thus have rings placed round the dots to represent this fact. Only one face of the primary pyramid o was present, but it was quite a large and brilliant one, affording an excellent image of the signal.

The poles o , o' , and o'' may now be inserted in their proper places in the stereographic projection, on the two diameters pcp , by the same usual method as in the cases of the q and q' poles, the perpendicular diameters required in the construction not being left permanently in, however, after their use as containing the poles of the pcp zones, as they are not zonal diameters. It is only necessary to find the position of one o , one o' , and one o'' pole, for all the others are similar, and can be marked off with the compasses at the same distances from the centre c .

There is one further diametral zone to be measured, namely, that containing the only developed face of the form x , the indices of which will presently be proved to be $\{123\}$. This face was $(1\bar{2}3)$, and it lay in the same zone with, and between, the basal pinakoid $c = (001)$ and the prism-face $p' = (1\bar{2}0)$.

DIAMETRAL ZONE CONTAINING $x = \{123\}$.

Circle Readings.	Angles.
$\begin{cases} c & 360^\circ & 0'A \\ x & 318 & 45 \\ p' & 270 & 0 A \end{cases}$	$\begin{cases} cx & 41^\circ 15' \\ xp' & 48 & 45 \end{cases}$

The position of the pole of this x -face on the diameter cxp' of the projection should next be found, as for the o and q poles, and the dot inserted to mark it.

In further test of the symmetry we ought next to measure all the arc-zones which may be drawn from the poles a and b , at the ends of the axial diameters, to pass respectively through a pair of the poles of each pyramid, that is, through two o -poles, two o' -poles, and two o'' -poles. There will be three such arc-zones on each side of each axial diameter, thus making twelve altogether. As, however, only one pole of the primary pyramid $o = \{111\}$ is developed, and no $a = \{100\}$ or $b = \{010\}$ faces are present, the four zones involving the primary pyramid cannot be measured, except as regards one angle between the single o face ($1\bar{1}1$) and the adjacent q face (011), and the other angles in this zone can only be determined by calculation, an excellent example of the use of the calculations. Hence, the zones of this type to be measured reduce to eight, or nine if we include the single measurement of oq , and the actual measurements obtained with them are now given below. In one of them the single face of the pyramid $x = \{123\}$ occurs.

Circle Readings.	Angles.	Circle Readings.	Angles.
2 ZONES [$ao''o'a$]		2 ZONES [$ao'o'a$]	
$\begin{cases} o'' & 59^\circ 42'A \\ o'' & 0 & 0 A \end{cases}$	$o''o'' 59^\circ 42'A$	$\begin{cases} o' & 78^\circ 20'A \\ o' & 0 & 0 A \end{cases}$	$o'o' 78^\circ 20'A$
$\begin{cases} o'' & 59^\circ 37' \\ o'' & 0 & 0 A \end{cases}$	$o''o'' 59^\circ 37'$ Mean $59^\circ 39\frac{1}{2}'$	$\begin{cases} o' & 78^\circ 18' \\ o' & 0 & 0 A \end{cases}$	$o'o' 78^\circ 18'$ Mean $78^\circ 19'$
2 ZONES [$bo''o''b$]		2 ZONES [$bo'o'b$]	
$\begin{cases} x & 43^\circ 50' \\ o'' & 30 & 29 A \\ o'' & 0 & 0 A \end{cases}$	$xo'' 13^\circ 21'$ $o''o'' 30^\circ 29 A$	$\begin{cases} o' & 39^\circ 0'A \\ o' & 0 & 0 A \end{cases}$	$o'o' 39^\circ 0'A$
$\begin{cases} o'' & 30^\circ 29'A \\ o'' & 0 & 0 A \end{cases}$	$o''o'' 30^\circ 29'A$ Mean $30^\circ 29'$	$\begin{cases} o' & 39^\circ 3'A \\ o' & 0 & 0 A \end{cases}$	$o'o' 39^\circ 3'A$ Mean $39^\circ 1\frac{1}{2}'$

It will be obvious that the two individual values for each pair of angles belonging to the two similar arc-zones on the two sides of an axial diameter are within the limits for intended equality, and that the measurement of these arc-zones fully confirms the supposition that the axial diameters are the projections of planes of symmetry.

The measurement of the angle between the only developed o -face ($1\bar{1}1$) and the adjacent q -face (011) gave the following result:

Circle Readings.	Angle.
$\left\{ \begin{array}{l} o \ 52^{\circ} \ 30' \\ q \ 0 \ 0 \ A \end{array} \right.$	$oq \ 52^{\circ} \ 30'$

These circular arc-zones should all be drawn in the projection, by the usual process of constructing a circular arc to pass through three points:

We may conclude the measurements with two pairs of arc-zones drawn from the ends of each of the cp diameters and one pair drawn from the ends of the cp' diameters, namely, the zones $[pqo'p]$, $[p'q'op]$, and $[p'oo''qp']$. All four of the repetitions of the first of these zones were measured, and gave the following angular values :

4 ZONES $[pqo'p]$

2 terminating each at $p=(110)$ and $p=(\bar{1}\bar{1}0)$. 2 terminating each at $p=(1\bar{1}0)$ and $p=(\bar{1}10)$.

Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} p \ 360^{\circ} \ 0'A \\ q \ 288 \ 50 \ A \\ o' \ 246 \ 20 \ A \\ p \ 180 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} pq \ 71^{\circ} \ 10'A \\ qo' \ 42 \ 30 \ A \\ o'p \ 66 \ 20 \ A \end{array} \right.$	$\left\{ \begin{array}{l} p \ 360^{\circ} \ 0'A \\ q \ 288 \ 49 \ A \\ o' \ 246 \ 18 \ A \\ p \ 180 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} pq \ 71^{\circ} \ 11'A \\ qo' \ 42 \ 31 \ A \\ o'p \ 66 \ 18 \ A \end{array} \right.$
$\left\{ \begin{array}{l} p \ 180^{\circ} \ 0'A \\ o' \ 113 \ 46 \ A \\ x \ 100 \ 12 \\ q \ 71 \ 12 \ A \\ p \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} po' \ 66^{\circ} \ 14'A \\ o'x \ 13 \ 34 \\ xq \ 29 \ 0 \\ o'q \ 42 \ 34 \ A \\ qp \ 71 \ 12 \ A \end{array} \right.$	$\left\{ \begin{array}{l} p \ 180^{\circ} \ 0'A \\ o' \ 113 \ 41 \ A \\ q \ 71 \ 12 \ A \\ p \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} po' \ 66^{\circ} \ 19'A \\ o'q \ 42 \ 29 \ A \\ qp \ 71 \ 12 \ A \end{array} \right.$

It will be evident that the four individual values for each of the angles pq , qo' , and $o'p$ are intended to be equal, that is, that these four different angles on the crystal in each case are of equal value as regards the symmetry. The differences in the case of the first, pq , are only 2', so that the mean $71^{\circ} \ 11'$ can be accepted without hesitation as the true value. In the case of the other two angles, qo' and $o'p$, the differences reach 5' and 6', but the evidence is now so strong that rhombic symmetry is present that here too we are quite justified in regarding the mean values, $qo' = 42^{\circ} \ 31'$ and $o'p = 66^{\circ} \ 18'$, as the expression of the truth, especially as two pairs of the values only differ by 1' and 2'. It will be observed that the face $x = (1\bar{2}3)$ belongs to one of these four similar zones, and the values of the angles $qx = 29^{\circ} \ 0'$ and $xo' = 13^{\circ} \ 34'$ are quite trustworthy, although they depend on a single measurement in each case.

The circular arcs representing these zones should be constructed on the projection in the usual manner already above referred to.

Of the four possible repetitions of each of the other two zones $[p'q'op]$ and $[p'oo''qp']$ only one in each case was measured, namely, that one which included the only developed o -face. The readings are as follows :

Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} p \ 360^\circ \ 0' A \\ q' \ 294 \ 27 \ A \\ o \ 239 \ 40 \ A \\ p \ 180 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} pq' \ 65^\circ \ 33' A \\ q'o \ 54 \ 47 \ A \\ op \ 59 \ 40 \ A \end{array} \right.$	$\left\{ \begin{array}{l} p' \ 180^\circ \ 0' A \\ o \ 148 \ 14 \ A \\ o'' \ 98 \ 42 \ A \\ q \ 59 \ 54 \ A \\ p' \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} p'o \ 31^\circ \ 46' A \\ oo'' \ 49 \ 32 \ A \\ o''q \ 38 \ 48 \ A \\ qp' \ 59 \ 54 \ A \end{array} \right.$

The circular arcs representing these two zones on the stereographic projection, terminating respectively at $p = (110)$ and $p = (\bar{1}10)$ in one case and at $p' = (1\bar{2}0)$ and $p' = (\bar{1}20)$ in the other, and also the two similar zones terminating at the other pairs of p and p' poles, should be constructed, the centres being found in the usual manner for any circular arc on which at least three points are given. This will then complete the stereographic projection, and a fully adequate number of measurements have also now been made to secure a full description of the crystal, and for the determination of its symmetry and elements.

The measured angles should then be tabulated according to zones, in their proper order, in the manner shown in the table of angles given at the end of this chapter, leaving a blank column for the subsequent insertion of the calculated values. The table should also indicate the number of measurements made of each angle, and the limiting values for each angle.

The measurements have fully confirmed our preliminary impression that the symmetry according to which the crystal is built up is rhombic. Having thus satisfied ourselves as to the symmetry, we may pass to the calculations, which will not only afford us confirmation of the indices imputed preliminarily to the faces of forms other than the axial planes and the primary parametral pyramid $o = \{111\}$, but will also give us the crystal elements, as well as enabling us to check the values of those angles the measurement of which was afforded by faces of less than the general excellence of facial development manifested by this truly magnificent crystal.

Calculation of Angles and Elements.

Basal Angles.—For the two essential basal angles it will be convenient to choose:

$$\begin{aligned} pp &= (110) : (1\bar{1}0) = 55^\circ \ 44' \\ cq &= (001) : (011) = 43^\circ \ 39'. \end{aligned}$$

Both angles were measured with particular trustworthiness, the two values of the former being only $2'$ different, or only $1'$ each from their mean, while the two values of the latter were identical with each other.

Axial Ratios.—The values of the ratios $a:b$ and $c:b$ are directly afforded by calculation from the two basal angles, for:

$$\begin{aligned} \frac{a}{b} &= \tan ap = \tan \frac{1}{2}pp = \tan 27^\circ \ 52'; \quad \frac{c}{b} = \tan cq = \tan 43^\circ \ 39'. \\ &= 0.5287. \qquad \qquad \qquad = 0.9539. \end{aligned}$$

Hence: $a:b:c=0.5287:1:0.9539$.

These values for the axial ratios are practically identical with those given by von Kokscharow, which are $0.5285:1:0.9539$.

The proof that the ratio $a:b=\tan ap$, and that the ratio $c:b=\tan cq$, has already been shown to follow from the simplification of the general mode of calculating axial ratios (in the case of triclinic crystals) which is effected when the symmetry is rhombic and the axial angles become 90° . It is fully set forth in Chapter VII. page 99.

We may derive the proof directly from first principles, however, which is always more instructive when possible. For if we draw a diagram showing the conditions, as in Fig. 192, producing Oa to A and Ob to B to meet the tangent ApB drawn to the primitive circle of the stereographic projection at p , it will at once be seen that the angle ΔOp ,—the angle between the normals to $a=(100)$ and $p=(110)$, and which is half the measured angle $pp=(110):(1\bar{1}0)$, namely $27^\circ 52'$,—is equal to the angle at B in AOB , and that the tangent of this angle at B is $\frac{OA}{OB}$. Now OA is the intercept on the axis a cut off by the primary prism face $p=(110)$, and OB is similarly the intercept on the axis b cut off by the same plane.

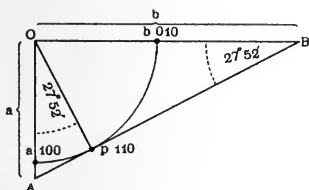


FIG. 192.

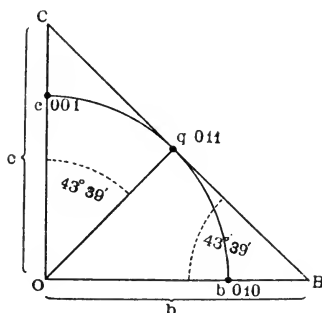


FIG. 193.

Hence: $\frac{OA}{OB}=\frac{a}{b}$, and therefore: $\frac{a}{b}=\tan 27^\circ 52'$.

Similarly, the primary brachy-prism face $q=(011)$ cuts off the intercepts c and b from the vertical axis c and the horizontal right-and-left axis b , as shown in Fig. 193, and the angle at B gives the ratio of these intercepts, for its tangent $\frac{OC}{OB}=\frac{c}{b}$. Now the angle at B is equal to the angle COq , the angle between the normals to the faces $c=(001)$ and $q=(011)$, which has been found by measurement to be $43^\circ 39'$. Hence:

$$\frac{c}{b}=\tan 43^\circ 39'.$$

Calculation of Angles.

Commencing with the angles in the prism zone other than the basal angle ap :

To find $bp'=(010):(120)$ and $p'p=(120):(110)$.

The pole p' is a fourth in the 90° -zone $[app'b]$ in which we know the positions of the other three, so that we can conveniently employ the anharmonic ratio of four poles in a zone. The conditions are defined diagrammatically in Fig. 194, and we can immediately deduce therefrom:

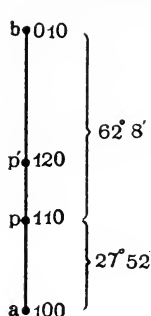


FIG. 194.

$$\frac{\sin ap}{\sin ap'} \cdot \frac{\sin bp'}{\sin bp} = \frac{100}{100} \cdot \frac{010}{010}$$

$$\frac{\sin 27^\circ 52'}{\cos bp'} \cdot \frac{\sin bp'}{\cos 27^\circ 52'} = \frac{1}{2} \cdot \frac{1}{1}$$

$$\tan bp' = \frac{1}{2} \cot 27^\circ 52'. \quad bp' = 43^\circ 24'.$$

$$\text{Then: } p'p = bp - bp' = 62^\circ 8' - 43^\circ 24' = 18^\circ 44'.$$

This value for pp' is identical with the mean of the four measurements obtained of this angle.

We may next take the angles in the zone $[cqq'b]$ other than the basal angle cq , namely:

To find $bq' = (010) : (021)$ and $q'q = (021) : (011)$.

We employ the anharmonic ratio of four poles in a 90° -zone again, as in the last case, and the conditions are similarly set forth in Fig. 195, from which we have:

$$\frac{\sin cq}{\sin cq'} \cdot \frac{\sin bq'}{\sin bq} = \frac{011}{001} \cdot \frac{021}{010}$$

$$\frac{\sin 43^\circ 39'}{\cos bq'} \cdot \frac{\sin bq'}{\cos 43^\circ 39'} = \frac{1}{2} \cdot \frac{1}{1}$$

$$\tan bq' = \frac{1}{2} \cot 43^\circ 39'. \quad bq' = 27^\circ 40'.$$

$$\text{Then: } q'q = bq - bq' = 46^\circ 21' - 27^\circ 40' = 18^\circ 41'.$$

The two measured values of this angle were $18^\circ 39'$ and $18^\circ 41'$, which show a satisfactory agreement.

We may next proceed to find the angles in the zone $[co''op]$, beginning with that of the primary pyramid o .

To find $co = (001) : (111)$.

The right-angled triangle $c = (001)$, $q = (011)$, $o = (111)$ is convenient for this purpose.

The Napierian diagram is given in Fig. 196, and from it we derive:

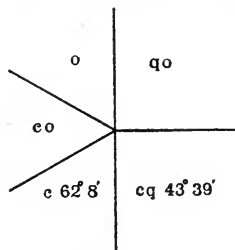


FIG. 196.

$$\cos 62^\circ 8' = \cot co \tan 43^\circ 39'$$

$$\text{or } \tan co = \frac{\tan 43^\circ 39'}{\cos 62^\circ 8'}. \quad co = 63^\circ 54'.$$

$$\text{Then: } op = 90^\circ - co = 90^\circ - 63^\circ 54' = 26^\circ 6'.$$

The only measured value of co was $63^\circ 50'$ and of op $26^\circ 8'$.

Having thus found the position of the primary pyramid we can find that of each of the others, o' and o'' , by considering a pole of each in turn as the fourth pole in the 90° -zone $[cop]$, and employing the anharmonic ratio.

To find $co' = (001) : (112)$.

The conditions are set out in the diagram of Fig. 197, which enables us to form the ratio in the usual manner as follows:

$$\begin{array}{ccc} & 001 & 110 \\ & \times & \times \\ \frac{\sin co'}{\sin co} \cdot \frac{\sin po}{\sin po'} &= \frac{112}{001} \cdot \frac{111}{110}, \\ & \times & \times \\ & 111 & 112 \\ \frac{\sin co'}{\sin 63^\circ 54'} \cdot \frac{\cos 63^\circ 54'}{\cos co'} &= \frac{1}{1} \cdot \frac{1}{2}, \\ \tan co' &= \frac{1}{2} \tan 63^\circ 54'. & co' = 45^\circ 35'. \end{array}$$

Then: $o'p = 90^\circ - co' = 90^\circ - 45^\circ 35' = 44^\circ 25'$.

Also: $o'o = co - co' = 63^\circ 54' - 45^\circ 35' = 18^\circ 19'$.

The mean measured value of co' was $45^\circ 34'$, and of $o'p$ $44^\circ 25'$.

The only measured value of $o'o$ was $18^\circ 17'$.

To find $co'' = (001):(113)$.

The position of o'' can be found in a precisely similar manner, the pole $o'' = (113)$ taking the place of $o' = (112)$ in the anharmonic ratio just worked out. The only difference in the final form of the formula is that the 2 in $\frac{1}{2}$ is replaced by 3, so that it reads as follows:

$$\tan co'' = \frac{1}{3} \tan 63^\circ 54'. \quad co'' = 34^\circ 14'.$$

Then: $o''o' = co' - co'' = 45^\circ 35' - 34^\circ 14' = 11^\circ 21'$.

The mean measured value for co'' was identical with the above value, the most divergent individual values being only $2'$ removed. The mean measured $o''o'$ was $11^\circ 20'$, the outside individual values being $1'$ on each side of this.

We next proceed to calculate the angles in the arc-zones terminating at the poles a and b .

To find $ao = (100):(111)$ and $oq = (111):(011)$.

We can get qo from the right-angled triangle qgo already used for finding co , and for which the Napierian diagram is given in Fig. 196. The latter at once affords us the following equation in accordance with Napier's rules:

$$\sin qo = \sin 62^\circ 8' \sin 63^\circ 54'(co). \quad qo = 52^\circ 33'.$$

Then: $ao = 90^\circ - oq = 90^\circ - 52^\circ 33' = 37^\circ 27'$.

The only measurement obtained of oq was $52^\circ 30'$, which is adequately near for purposes of confirmation; ao could not be measured on account of the absence of a -faces; its double, oao , could not be measured either, as only one o -face was developed. This is, consequently, a case where the calculations are imperatively necessary.

To find $ao' = (100):(112)$ and $ao'' = (100):(113)$.

These angles corresponding to the other two kinds of arc-zones terminating at the a -poles can be similarly found from the two right-angled triangles $a = (100)$, $o' = (112)$,

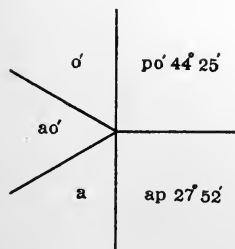


FIG. 198.

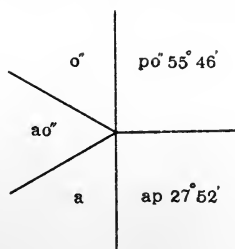


FIG. 199.

$p = (110)$, and $a = (100)$, $o'' = (113)$, $p = (110)$. The Napierian diagrams are given in Figs. 198 and 199, from which we derive the equations:

$$\cos ao' = \cos 44^\circ 25' \cos 27^\circ 52'; \quad \cos ao'' = \cos 55^\circ 46' \cos 27^\circ 52'.$$

$$ao' = 50^\circ 51'(50\frac{1}{2}'), \quad ao'' = 60^\circ 11'(10\frac{1}{2}').$$

Then :

$$o'o' = (112) : (\bar{1}12) = 2(39^\circ 9\frac{1}{2}') = 78^\circ 19',$$

and :

$$o'o'' = (113) : (\bar{1}13) = 2(29^\circ 49\frac{1}{2}') = 59^\circ 39'.$$

The mean measured values for $o'o'$ and $o'o''$ respectively were $78^\circ 19'$ and $59^\circ 39\frac{1}{2}'$ respectively, a most satisfactory agreement; ao' and ao'' were not themselves measured on account of the absence of a -faces.

We next tackle the three analogous zones terminating at the b -poles.

To find $bo = (010) : (111)$, $bo' = (010) : (112)$, and $bo'' = (010) : (113)$.

These three angles may be found in a similar manner from the three right-angled triangles $b = (010)$, $p = (110)$, $o = (111)$, bpo' and bpo'' . Taking first the calculation of bo , from the Napierian diagram in Fig. 200, we have :

$$\cos bo = \cos op \cos pb = \cos 26^\circ 6' \cos 62^\circ 8'. \quad bo = 65^\circ 11'.$$

The calculation of bo' and bo'' from the other two triangles bpo' and bpo'' will be similar, but instead of $\cos op$ we shall have $\cos o'p$ and $\cos o''p$ respectively.

$$\cos bo' = \cos o'p \cos bp = \cos 44^\circ 25' \cos 62^\circ 8'. \quad bo' = 70^\circ 30'.$$

$$\cos bo'' = \cos o''p \cos bp = \cos 55^\circ 46' \cos 62^\circ 8'. \quad bo'' = 74^\circ 45'.$$

FIG. 200.

The angles $oo = (111) : (\bar{1}\bar{1}1)$, $o'o' = (112) : (\bar{1}\bar{1}2)$, and $o'o'' = (113) : (\bar{1}\bar{1}3)$ are then each in turn given by the difference between 180° and twice the angles just given :—

$$oo = 180^\circ - 2bo = 180^\circ - 130^\circ 22' = 49^\circ 38';$$

$$o'o' = 180^\circ - 2bo' = 180^\circ - 141^\circ 0' = 39^\circ 0';$$

$$o'o'' = 180^\circ - 2bo'' = 180^\circ - 149^\circ 30' = 30^\circ 30'.$$

In the last of these three zones, between the faces b and o'' , we meet with a face of the form $x = \{123\}$, so that to complete the zone we must now calculate the position of x .

To find $bx = (010) : (123)$ and $xo'' = (123) : (113)$.

This we can do from the anharmonic ratio of the four poles in the 90° -zone [$b = (010)$, $x = \{123\}$, $o'' = (113)$, and $M = (103)$], the latter pole M being that of a possible but undeveloped face at the intersection of this zone and the zone $[ac]$. The conditions are shown in Fig. 201, from which we can immediately set down :

$$\frac{\sin bx}{\sin bo''} \cdot \frac{\sin Mo''}{\sin Mx} = \frac{010}{113} \cdot \frac{103}{123},$$

$$\frac{\sin bx}{\cos bx} \cdot \frac{\cos bo''}{\sin bo''} = \frac{1}{1} \cdot \frac{1}{2},$$

$$\tan bx = \frac{1}{2} \tan bo'', \text{ that is : } \tan bx = \frac{1}{2} \tan 74^\circ 45'. \quad bx = 61^\circ 25'.$$

$$\text{Then : } xo'' = bo'' - bx = 74^\circ 45' - 61^\circ 25' = 13^\circ 20'.$$

The mean measured value of $o'o'$ was $39^\circ 1'$, and of $o'o''$ $30^\circ 29'$, the maximum difference of the individual values from the calculated being $2'$. The single measured value of xo'' was $13^\circ 21'$, only $1'$ removed from the calculated value.

We are now in a position to find $cx = (001) : (123)$ and $px = (120) : (123)$, the angles in the diametral zone $[caxp]$. For in the triangle $b = (010)$, $c = (001)$, $x = (123)$, we know that bc is a right angle, that the angle at c is the angle $bp' = 43^\circ 24'$, and that the angle bx (just found) is $61^\circ 25'$.

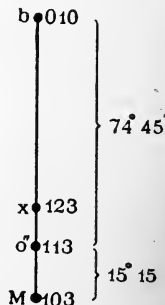
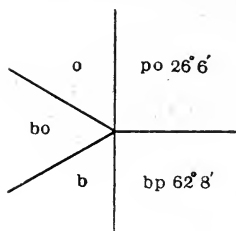


FIG. 201.

Constructing the Napierian diagram, Fig. 202, we derive from it :

$$\cos 61^{\circ} 25' = \sin cx \cos 43^{\circ} 24' \text{ or } \sin cx = \frac{\cos 61^{\circ} 25'}{\cos 43^{\circ} 24'}$$

$$cx = 41^{\circ} 12'.$$

$$p'x = 48^{\circ} 48'.$$

Then, by difference from 90° ,

The single measurements of these two angles, afforded by the one developed x -face, were $41^{\circ} 15'$ and $48^{\circ} 45'$.

We have still three secondary arc-zones to calculate, in order to obtain a complete survey of the angles on the crystal, namely, two zones with circular arcs ending at the p -poles, and one such zone with arcs ending at the p' -poles. The angles in all three zones were measured.

Angles in the Zone [pqxo'p].

To find $pq = (110) : (011)$.

This is a simple calculation from the right-angled triangle $b = (010)$, $q = (011)$,

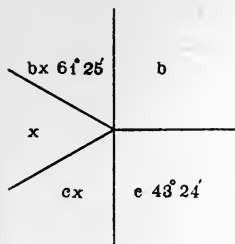


FIG. 202.

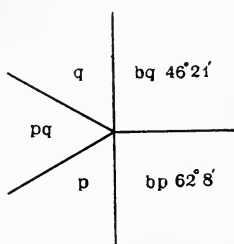


FIG. 203.

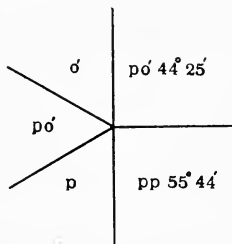


FIG. 204.

$p = (110)$, in which we know $bq = 46^{\circ} 21'$ and $bp = 62^{\circ} 8'$, both directly from the two basal angles. From the diagram in Fig. 203 we get by Napier's rules :

$$\cos pq = \cos 46^{\circ} 21' \cos 62^{\circ} 8'. \quad pq = 71^{\circ} 11'.$$

This is identical with the mean measured value of pq , the individual values only showing a variation of $1'$ on each side of it.

To find $po' = (1\bar{1}0) : (112)$.

We can find this angle from the right-angled triangle $p = (1\bar{1}0)$, $o' = (112)$, $p = (110)$, in which we know $pp = 55^{\circ} 44'$ (basal angle), and $po' = 44^{\circ} 25'$. From the Napierian diagram in Fig. 204 we get :

$$\cos po' = \cos 44^{\circ} 25' \cos 55^{\circ} 44'. \quad po' = 66^{\circ} 17'.$$

The mean of the four measured values was $66^{\circ} 18'$.

To find $qx = (011) : (123)$.

We are obliged here to have recourse to an oblique-angled triangle, $c = (001)$, $q = (011)$, $x = (123)$, in which we know the two sides $cq = 43^{\circ} 39'$ and $cx = 41^{\circ} 12'$, and the included angle at c , which is the angle $bp' = 43^{\circ} 24'$. Employing the formulæ for the calculation of the third side given on page 96 of Chapter VII., we have :

$$\tan \theta = \tan 41^{\circ} 12' \cos 43^{\circ} 24'; \quad \cos qx = \frac{\cos 41^{\circ} 12' \cos (43^{\circ} 39' - \theta)}{\cos \theta}.$$

$$\theta = 32^{\circ} 28'.$$

$$qx = 28^{\circ} 59'.$$

Then : $xo' = (123) : (112) = 180^{\circ} - (po' + qx + qp) = 180^{\circ} - 166^{\circ} 27' = 13^{\circ} 33'.$

The single measured values of qx and xo' were respectively $29^{\circ} 0'$ and $13^{\circ} 34'$.

Angles in the Zone [poq'p].

To find $po = (1\bar{1}0) : (111)$, $oq' = (111) : (021)$, and $q'p = (021) : (\bar{1}10)$.

The first and third of these angles may be readily found from the two right-angled triangles $p = (1\bar{1}0)$, $o = (111)$, $p = (110)$, and $b = (010)$, $q' = (021)$, $p = (\bar{1}10)$,

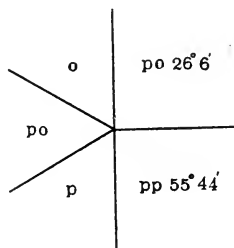


FIG. 205.

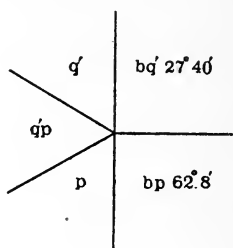


FIG. 206.

for which the Napierian diagrams are given in Figs. 205 and 206, and from which we derive :

$$\cos po = \cos 26^\circ 6' \cos 55^\circ 44';$$

$$po = 59^\circ 38'.$$

$$\cos q'p = \cos 27^\circ 40' \cos 62^\circ 8';$$

$$q'p = 65^\circ 33'.$$

Then : $oq' = 180^\circ - (59^\circ 38' + 65^\circ 33') = 180^\circ - 125^\circ 11' = 54^\circ 49'.$

The three measured angles were respectively $59^\circ 40'$, $65^\circ 33'$, and $54^\circ 47'$.

Angles in the Zone [p'q'o'p'].

To find $p'q = (120) : (011)$.

For this we employ the right-angled triangle $b = (010)$, $q = (011)$, $p' = (120)$, in which it is known that the sides bq and bp' are respectively $46^\circ 21'$ and $43^\circ 24'$. From the Napierian diagram in Fig. 207 we get :

$$\cos p'q = \cos 46^\circ 21' \cos 43^\circ 24'. \quad p'q = 59^\circ 54'.$$

To find $p'o = (120) : (111)$.

This is a side of the right-angled triangle $p' = (120)$, $o = (111)$, $p = (110)$, in which

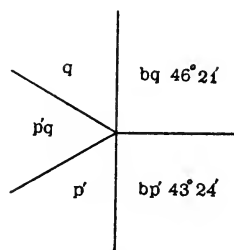


FIG. 207.

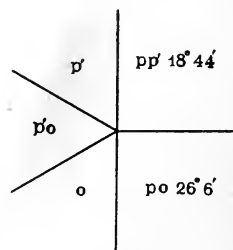


FIG. 208.

po and pp' are known to be $26^\circ 6'$ and $18^\circ 44'$, and from the Napierian diagram in Fig. 208 we derive :

$$\cos p'o = \cos 18^\circ 44' \cos 26^\circ 6'. \quad p'o = 31^\circ 44'.$$

To find $oo'' = (111) : (\bar{1}\bar{1}3)$.

We can find this angle from the oblique triangle $o = (111)$, $c = (001)$, $o'' = (\bar{1}\bar{1}3)$, in which we know the two sides $co'' = 34^\circ 14'$ and $co = 63^\circ 54'$, as well as the included angle at c , for this latter is $p = (110) : p = (\bar{1}\bar{1}0) = 55^\circ 44'$, one of the basal angles. Employing the formulæ (b) of p. 96, Chapter VII., we have :

$$\tan \theta = \tan 34^\circ 14' \cos 55^\circ 44' ; \cos oo'' = \frac{\cos 34^\circ 14' \cos (63^\circ 54' - \theta)}{\cos \theta}$$

$$\theta = 20^\circ 58'. \quad oo'' = 49^\circ 35'.$$

$$\begin{aligned} \text{Then :} \quad o''q &= (\bar{1}\bar{1}3) : (0\bar{1}1) = 180^\circ - (p'q + p'o + oo'') = 180^\circ - 141^\circ 13'. \\ &= 38^\circ 47'. \end{aligned}$$

The measured values of these four angles were $59^\circ 54'$ (identical with the calculated), $31^\circ 46'$ ($2'$ different), $49^\circ 32'$ ($3'$ different), and $38^\circ 48'$ ($1'$ different from the calculated), an agreement as satisfactory as can be expected in the case of angles only once measured, there being but one $o = \{111\}$ face on the crystal.

This completes the whole of the calculations for the crystal of topaz. With only two exceptions, they have been achieved entirely with the aid of the anharmonic ratio of four poles in a simple 90° -zone, and of Napier's rules for right-angled triangles. In the two exceptional cases only has it been necessary to have recourse to oblique triangles, and then in both cases the calculation has been of the simplest kind, that for which the formulæ were given at (b) on p. 96 of Chapter VII.

It now only remains to tabulate the results for topaz in concise form, according to the general scheme set forth at the conclusion of Chapter VIII.

Table of Results for Crystal of Topaz $(AlF)_2SiO_4$.

Crystal-system : Orthorhombic. Class : 8, rhombic holohedral.

Habit : Short prismatic.

Ratio of axes : $a : b : c = 0.5287 : 1 : 0.9539$.

Forms observed : $c = \{001\}$, $p = \{110\}$, $p' = \{120\}$, $q = \{011\}$, $q' = \{021\}$, $o = \{111\}$, $o' = \{112\}$, $o'' = \{113\}$, $x = \{123\}$.

Table of interfacial angles : In the following table the two basal angles are marked with an asterisk.

MORPHOLOGICAL ANGLES OF TOPAZ.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$ap = (100) : (110)$	$27^\circ 52'$...
$pp = (110) : (\bar{1}\bar{1}0)$	2	$55^\circ 43' - 55^\circ 45'$	$55^\circ 44'$	*	...
$pp' = (110) : (120)$	4	18 42-18 47	18 44	18 44	0'
$p'b = (120) : (010)$	43 24	...
$p'p' = (120) : (\bar{1}\bar{2}0)$	2	86 46-86 50	86 48	86 48	0
$cq = (001) : (011)$	2	43 39-43 39	43 39	*	...
$qq' = (011) : (021)$	2	18 39-18 41	18 40	18 41	1
$q'b = (021) : (010)$	27 40	...
$cq' = (001) : (021)$	2	62 18-62 20	62 19	62 20	1

MORPHOLOGICAL ANGLES OF TOPAZ (*continued*).

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$\left\{ \begin{array}{l} co''=(001):(113) \\ o'o'=(113):(112) \\ o'o=(112):(111) \\ co=(001):(111) \\ co'=(001):(112) \\ op=(111):(110) \\ o'p=(112):(110) \end{array} \right.$	$\left\{ \begin{array}{l} 4 \\ 4 \\ 1 \\ 1 \\ 4 \\ 1 \\ 4 \end{array} \right.$	$\left\{ \begin{array}{l} 34 \quad 13-34 \quad 16 \\ 11 \quad 19-11 \quad 21 \\ \dots \\ \dots \\ 45 \quad 32-45 \quad 36 \\ \dots \\ 44 \quad 24-44 \quad 25 \end{array} \right.$	$\left\{ \begin{array}{l} 34 \quad 14 \\ 11 \quad 20 \\ 18 \quad 17 \\ 63 \quad 50 \\ 45 \quad 34 \\ 26 \quad 8 \\ 44 \quad 25 \end{array} \right.$	$\left\{ \begin{array}{l} 34 \quad 14 \\ 11 \quad 21 \\ 18 \quad 19 \\ 63 \quad 54 \\ 45 \quad 35 \\ 26 \quad 6 \\ 44 \quad 25 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ 1 \\ 2 \\ 4 \\ 1 \\ 2 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} cx=(001):(123) \\ xp'=(123):(120) \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} 41 \quad 15 \\ 48 \quad 45 \end{array} \right.$	$\left\{ \begin{array}{l} 41 \quad 12 \\ 48 \quad 48 \end{array} \right.$	$\left\{ \begin{array}{l} 3 \\ 3 \end{array} \right.$
$\left\{ \begin{array}{l} ao=(100):(111) \\ oq=(111):(011) \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 1 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 52 \quad 30 \end{array} \right.$	$\left\{ \begin{array}{l} 37 \quad 27 \\ 52 \quad 33 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 3 \end{array} \right.$
$\left\{ \begin{array}{l} ao'=(100):(112) \\ o'o'=(112):(\bar{1}12) \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 2 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 78 \quad 18-78 \quad 20 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 78 \quad 19 \end{array} \right.$	$\left\{ \begin{array}{l} 50 \quad 51 \\ 78 \quad 19 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} ao''=(100):(113) \\ o''o''=(113):(\bar{1}13) \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 2 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 59 \quad 37-59 \quad 42 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 59 \quad 39 \end{array} \right.$	$\left\{ \begin{array}{l} 60 \quad 11 \\ 59 \quad 39 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} bo=(010):(111) \\ oo=(111):(\bar{1}\bar{1}1) \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} 65 \quad 11 \\ 49 \quad 38 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \end{array} \right.$
$\left\{ \begin{array}{l} bo'=(010):(112) \\ o'o'=(112):(\bar{1}\bar{1}2) \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 2 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 39 \quad 0-39 \quad 3 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 39 \quad 1 \end{array} \right.$	$\left\{ \begin{array}{l} 70 \quad 30 \\ 39 \quad 0 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 1 \end{array} \right.$
$\left\{ \begin{array}{l} bx=(010):(123) \\ xo''=(123):(113) \\ bo''=(010):(113) \\ o''o''=(113):(\bar{1}\bar{1}3) \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 1 \\ \dots \\ 2 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \\ 30 \quad 29-30 \quad 29 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 13 \quad 21 \\ \dots \\ 30 \quad 29 \end{array} \right.$	$\left\{ \begin{array}{l} 61 \quad 25 \\ 13 \quad 20 \\ 74 \quad 45 \\ 30 \quad 30 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ 1 \\ \dots \\ 1 \end{array} \right.$
$\left\{ \begin{array}{l} po'=(1\bar{1}0):(112) \\ o'x=(112):(123) \\ xq=(123):(011) \\ o'q=(112):(011) \\ qp=(011):(\bar{1}10) \end{array} \right.$	$\left\{ \begin{array}{l} 4 \\ 1 \\ 1 \\ 4 \\ 4 \end{array} \right.$	$\left\{ \begin{array}{l} 66 \quad 14-66 \quad 20 \\ \dots \\ \dots \\ 42 \quad 29-42 \quad 34 \\ 71 \quad 10-71 \quad 12 \end{array} \right.$	$\left\{ \begin{array}{l} 66 \quad 18 \\ 13 \quad 34 \\ 29 \quad 0 \\ 42 \quad 31 \\ 71 \quad 11 \end{array} \right.$	$\left\{ \begin{array}{l} 66 \quad 17 \\ 13 \quad 33 \\ 28 \quad 59 \\ 42 \quad 32 \\ 71 \quad 11 \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} po=(1\bar{1}0):(111) \\ oq'=(111):(021) \\ q'p=(021):(\bar{1}10) \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \\ 1 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} 59 \quad 40 \\ 54 \quad 47 \\ 65 \quad 33 \end{array} \right.$	$\left\{ \begin{array}{l} 59 \quad 38 \\ 54 \quad 49 \\ 65 \quad 33 \end{array} \right.$	$\left\{ \begin{array}{l} 2 \\ 2 \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} p'o=(120):(111) \\ oo''=(111):(\bar{1}\bar{1}3) \\ o''q=(113):(0\bar{1}1) \\ qp'=(0\bar{1}1):(\bar{1}20) \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \end{array} \right.$	$\left\{ \begin{array}{l} \dots \\ \dots \\ \dots \\ \dots \end{array} \right.$	$\left\{ \begin{array}{l} 31 \quad 46 \\ 49 \quad 32 \\ 38 \quad 48 \\ 59 \quad 54 \end{array} \right.$	$\left\{ \begin{array}{l} 31 \quad 44 \\ 49 \quad 35 \\ 38 \quad 47 \\ 59 \quad 54 \end{array} \right.$	$\left\{ \begin{array}{l} 2 \\ 3 \\ 1 \\ 0 \end{array} \right.$

The general agreement between the observed and calculated angles is obviously of a satisfactory character.

It has already been remarked that there is an alternative choice of parametral plane possible, to the one due to von Kokscharow made in the preceding description. The parametral form $o=\{111\}$ has been shown to have been only represented by a single face on the crystal under consideration, and it appears in general to be less developed on crystals of topaz than the form $o'=\{112\}$. Owing, therefore, to the

apparent greater importance of the form o' , and to the fact that the cleavage does not help us to decide between the merits of the two forms as possible parametral planes, or at any rate does not make the choice of $o = \{111\}$ imperative, topaz cleaving parallel to the basal pinakoid $c = \{001\}$, Miers, in his *Mineralogy* (page 442) takes o' as the parametral form $\{111\}$. This causes the form o'' to become $\{223\}$, and o to become $\{221\}$. That this is so may readily be proved by setting out the anharmonic ratio of either the four poles $c = (001)$, $o'' = (hhl)$, $o' = (111)$, and $p = (110)$, or $c = (001)$, $o' = (111)$, $o = (hhl)$, and $p = (110)$, in the 90° -arc cp , giving the angles their measured values while o'' and o are represented by (hhl) . It will be found that in the two cases the ratio works out to give values for h and l as follows :

$$\tan 44^\circ 25' \cdot \tan 34^\circ 14' = h/l, \text{ and } \tan 26^\circ 6' \cdot \tan 45^\circ 35' = l/h.$$

The numerical value of the first is 0.6667, or $2/3$, and of the second 0.500 or $1/2$. Hence (hhl) in the first case (that of the face o'') is (223) , and in the second case (that of the face o) $(hhl) = (221)$. The two forms in the zone $[cb]$, namely, q and q' , become respectively $\{021\}$ and $\{041\}$. The former is given at once by cross-multiplication of the zones $[a = (100) : o = (221)]$ and $[c = (001) : b = (010)]$, as q is at the intersection of these two zones ; and the latter can then be found by considering the pole q' as the fourth in the 90° -zone $[c = (001)$, $q = (021)$, $b = (010)]$ and setting out the anharmonic ratio, giving the angles their measured values and calling the indices $(0kl)$. The ratio then works out to :

$$\tan 27^\circ 40' \tan 43^\circ 39' = 2l/k.$$

The numerical value of the product of the two tangents is 0.50, so that :

$$\frac{2l}{k} = \frac{1}{2}, \text{ and } k = 4l.$$

The indices of q' are, therefore $\{041\}$, for the face is parallel to the axis a , so that the value of h is 0.

The effect of thus taking o' as the parametral plane (111) instead of o , on the ratio of the axis is to diminish the length of the c axis by one half relatively to that of the b axis, thus making it 0.477, the value of a/b remaining unaltered. For the length of axis which is given by $\tan cq$, the expression for c/b , is now $2c$ instead of c . The actual values of the axial ratios given by Miers are :

$$a : b : c = 0.528 : 1 : 0.477.$$

This instance of the possibility of two alternative courses, as regards the choice of the parametral plane defining the axial lengths and determining the indices of faces other than the axial planes, is a very typical one. The question of the proper "setting up" of a crystal, that is, as to which axis shall be vertical, which lateral, and which shall run back-and-front, and of the correct choice of the parametral plane, is a very important one, but as yet somewhat obscure. That it has a profound bearing on the correct diagnosis of the internal structure will be obvious, and a subsequent chapter (XXXIII.) will be devoted to discussing it. In the case of topaz in all probability the "setting up" here adopted is the proper one both on von Kokscharow's assumption and on that of Miers, but as to which choice of parametral plane is correct there is as yet no decisive evidence to show, that of cleavage, often of great importance, being here inoperative.

CHAPTER XVII

MONOCLINIC SYSTEM.

Also called Monosymmetric and Oblique System.

Three unequal crystallographic axes, of which one is perpendicular to the other two.

Characterised by possessing either a digonal axis and a plane of symmetry perpendicular to it, or only one of these elements.

THERE are obviously three classes fulfilling these conditions, namely: one, class 3, possessing only a plane of symmetry; a second, class 4, having only a digonal axis; and a third, class 5, possessing both a digonal axis and a plane of symmetry at right angles to it. These three classes are respectively illustrated by the stereographic projections given in Figs. 209, 210, and 211. The plane of symmetry of

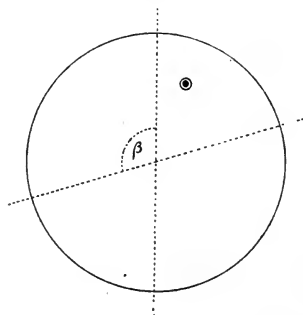


FIG. 209.—Symmetry Elements and General Form of Class 3.

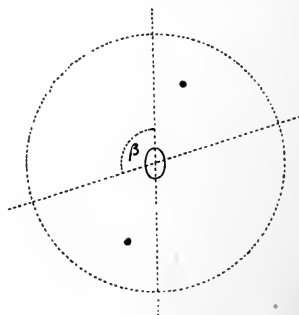


FIG. 210.—Symmetry Elements and General Form of Class 4.

classes 3 and 5 is taken as the plane of projection, that of the paper, and is represented by the primitive circle being drawn in continuous line, as distinguished from the dotted circle of class 4, Fig. 210. The digonal axis is perpendicular to this plane, and its projection is consequently the centre of the circle. The two dotted straight lines represent the vertical and inclined crystallographic axes, lying in the

plane of possible symmetry and inclined to each other at the axial angle β . The third crystallographic axis perpendicular to these two is identical with the digonal axis of symmetry.

*Class 5.—Monoclinic-Prismatic Class. Monoclinic-Holohedral Class.
Type, Digonal Equatorial.*

This holohedral class of the monoclinic system possesses, as its name holohedral implies, both the elements of symmetry referred to above. They are graphically expressed in the stereographic projection given in Fig. 211, which also shows not only the poles of the primary representative $\{11\bar{1}\}$ of the most general form $\{hkl\}$, but also those of the primary pinakoidal forms $\{100\}$, $\{010\}$, and $\{001\}$, together with the section-outline of the actual faces of the forms $\{100\}$ and $\{001\}$ made by the symmetry plane $\{010\}$. It will be observed that a different mode of projection is adopted from that which has been employed in connection with the other systems, the great circle bearing the poles of the faces (100) and (001) parallel to the two crystallographic axial planes bc and ab , and which in fact have been themselves chosen for those axial planes, being the primitive circle instead of that bearing (100) and (010). The reason is one of very great convenience, for the axis b , perpendicular to the other two axes, which is the one symmetry axis and is chosen as b axis because it is so, is thereby brought perpendicular to the plane of the paper; and the latter plane, represented by the primitive circle, is the one symmetry plane of the system, in which lie the other two axes a and c , which may have any mutual inclination, other than 90° , to each other in that plane. The symmetry is thus at once shown by such a mode of projection, and pairs of faces symmetrical to the symmetry plane may be represented by a dot and concentric ring as usual.

The digonal axis, identical, as above stated, with the crystallographic axis b , is represented by the pole in the centre, which also indicates the position of the faces (010) and (0 $\bar{1}$ 0), the symmetry plane being, in fact, the plane $\{010\}$. A horizontal diameter is then drawn, and the pole of the face (100) is placed at the left end of it, its parallel face ($\bar{1}$ 00) being represented by a pole at the right end. The pole of the face (001) is then inserted at the proper position on the primitive circle, so that the angle between the normal to (100), represented by the horizontal diameter, and the normal to the face (001), which may be represented by another diameter, is the actual angle measured on the

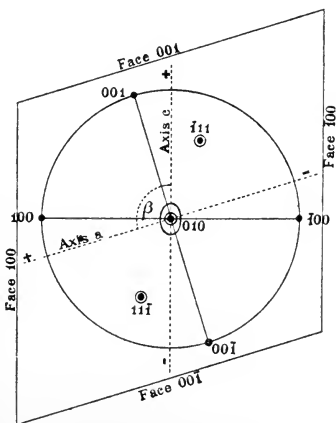


FIG. 211.—Symmetry Elements, Axial Planes, and General Form of Class 5.

goniometer between the faces (100) and (001). This acute angle (100):(001), represented by the acute angle between the diameters, is the supplement of the crystallographic axial angle β , between the two inclined axes, a and c , for the axis a is parallel to the edge (010):(001), and the axis c is parallel to the edge (010):(100), these edges having been chosen deliberately as such axes. The edge (100):(001) is similarly the chosen direction of the symmetry axis b , and is perpendicular to the a and c axes (the two edges just referred to) and to the symmetry plane containing them. This arrangement brings the c axis vertical, as is desirable for reasons of analogy with the more symmetrical systems. The monoclinic differs from these higher systems by the inclination of the a axis, which is tilted downwards towards the left in the projection, as will be clear from the dotted lines indicating these two axes a and c lying in the symmetry plane.

It must be clearly pointed out, however, that this arrangement is only for the purposes of the stereographic projection. For descriptive purposes and pictorial representation the a axis is, as usual, the back-to-front axis, and in this system tilts downwards towards the observer when the crystal is held in position, the axis c being vertical and the symmetry axis b running horizontally right-and-left. The crystal is thus rotated 90° round the vertical axis for the purpose of projection, in order to bring the symmetry plane into coincidence with the plane of the paper.

It must also be clearly understood that the poles (100) and (001) do not lie on the a and c axes, as is usual in the rectangular systems, but that the vertical axis is parallel to the tangent of the circle at the pole (100), that is, parallel to the face represented by the pole, and the inclined a axis is parallel to the tangent at the pole (001), or to the face represented by that pole.

As regards the axial angle β , the only one of the three axial angles which is not 90° , it is equally well expressed either by the actually measured goniometrical angle (100):(001) or by the supplement of that angle already referred to; for the former is equal to the acute angle between the inclined axes, and the latter to the obtuse angle. The obtuse angle is the actual angle between the two primary edges above quoted, and is also the angle in the positive part of the upper hemisphere, and so it has in the past been generally given as the angle β ; as this angle is usually greater than 90° , it has, however, been recently coming into practice to give instead the acute angle as the value of β , as measured directly, between the normals to the faces (100) and (001), on the goniometer.

The symmetry axis b , being at right angles to the vertical axis, is termed the "ortho-diagonal" for descriptive purposes, and the axis a , because inclined to the vertical axis c , is called the "clino-diagonal."

The pair of faces (100) and ($\bar{1}00$), the poles of which are situated at the end of the horizontal diameter in the projection, are a form unto themselves in this system, and being parallel to the ortho-diagonal as well as to the vertical axis the form is termed the **ortho-pinakoid**, its symbol being {100}.

Similarly, the pair of faces (010) and (0 $\bar{1}$ 0), the poles of which lie over each other at the centre (as a dot with concentric ring if it be desired to represent both), and which are parallel to the clino-diagonal and the vertical axis as well as being the facial expression of the symmetry plane itself, also together make up a separate form {010} distinguished by the name **clino-pinakoid**.

Likewise the basal plane (001) and its parallel face (00 $\bar{1}$), both of which are parallel to the inclined axis (clino-diagonal) and the symmetry axis (ortho-diagonal), together make up a third primary form of pinakoidal character, {001}, which may be termed the **basal-pinakoid**.

These three fundamental forms in combination produce the closed prism-like solid shown in Fig. 212, formerly known as "the oblique rectangular prism," which is the monoclinic equivalent of the cube, or of the tetragonal prism of the second order or the rectangular rhombic prism, but with the horizontal pair of faces tilted downwards towards the observer. The two pairs of faces of the ortho- and basal pinakoids are rectangular in shape, but the pair of clino-pinakoids are parallelograms the opposite pairs of equal angles of which are equal respectively to the crystallographic axial angle β and to its supplement.

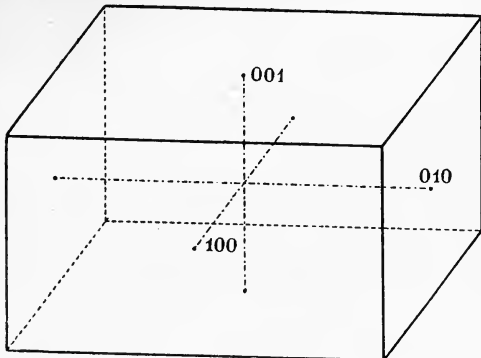


FIG. 212.—Combination of the three Primary Monoclinic Pinakoids forming Oblique Rectangular Prism.

These three forms will have been chosen, at the commencement of the investigation of the crystal, as the axial planes, and their edges of intersection as the directions of the three crystallographic axes.

The general form $\{hkl\}$, of which the primary parametral form $\{111\}$ (which will have been chosen to determine the lengths of the three crystallographic axes) is a special case, consists of four faces, as will be clear from the stereographic projection given in Fig. 211, namely, (hkl) , $(\bar{h}\bar{k}\bar{l})$, $(\bar{h}kl)$, and $(h\bar{k}\bar{l})$; they together make up an open prism-like form of rhombic section, of which one of the two diagonals lies in the symmetry plane, but in combination with the complementary form $\{h\bar{k}l\}$, the actual one shown in Fig. 211, a monoclinic bipyramid is produced. Each of the two forms $\{hkl\}$ and $\{h\bar{k}l\}$ is therefore commonly designated as a **monoclinic hemi-pyramid**. It is termed a prism of the fourth order by von Groth and von Fedorow, because it is not parallel to any one of the three axes; and it is this general, or fourth order, prism which gives the name "prismatic" to the class. A monoclinic bipyramid thus composed of the two complementary hemi-pyramids or fourth order prisms is shown in Fig. 213, which represents it considered as the primary one. The four faces of the first form, $\{hkl\}$, situated in the obtuse angles of the inclined axes, two faces vertically over each other in the projection in each case, are usually referred to as forming the **negative hemi-pyramid** or negative fourth order prism; while the form $\{h\bar{k}l\}$ shown in the projection, Fig. 211, with poles in the acute axial angles, is regarded as the **positive hemi-pyramid** or positive prism of the fourth order.

Thus in the monoclinic system, for the first time (taking the systems as we are doing, in descending order), we have the holohedral pyramidal solid (the bipyramid) divided into two separate forms. In the case of the parametral bipyramid, shown in Fig. 213, we have the two forms $\{111\}$ and $\{1\bar{1}1\}$.

There are three further types of forms, corresponding to special cases of the general

form $\{hkl\}$, afforded when the latter migrates on to one of the sides of the fundamental spherical triangle (Fig. 211) formed by the poles (100), (010), and (001). In

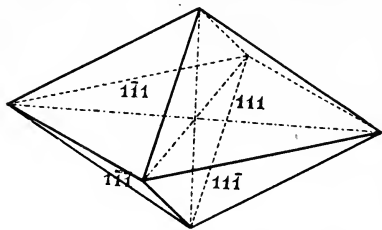


FIG. 213.—The Primary Monoclinic Bipyramid (2 Forms).

each case the faces of the form produced are parallel to one or other of the three crystallographic axes, and are therefore respectively said to be of the "first," "second," or "third" order by von Groth and von Fedorow.

In the first case, when the pole lies on the side (100):(010), the symmetry conditions demand that there shall be four faces in the form; one pole is situated immediately beneath the first pole, on the lower hemisphere, owing to the presence of the symmetry plane, and a similar pair are situated above and below each other on the same horizontal-diametral arc at the other side of the centre, owing to the operation of the digonal symmetry axis. These four faces, opposite pairs of which are parallel, thus form an open prism of rhombic section, all the faces and edges of which are parallel to the vertical axis c , and the prism is therefore of the third order according to the Groth-Fedorow classification. It is generally termed a **monoclinic prism**, and the symbol of the most general form is $\{hk0\}$ and that of the primary prism $\{110\}$. The latter is shown in Fig. 214 in combination with and closed by its most fundamental termination, the basal pinakoid $\{001\}$.

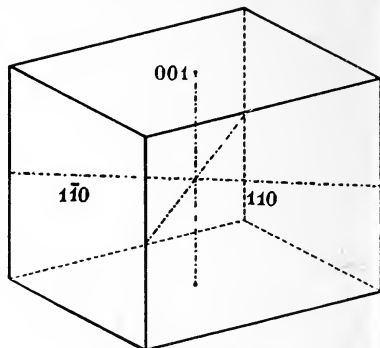


FIG. 214.—The Primary Monoclinic Prism with Basal Pinakoid.

In the second case, when the pole migrates on to the arc (010):(001), a similar four-faced open prism is produced, likewise symmetrical to the symmetry plane and to the centre, but the faces and edges of which are parallel to the clino-diagonal, the axis a . It is, therefore, a prism

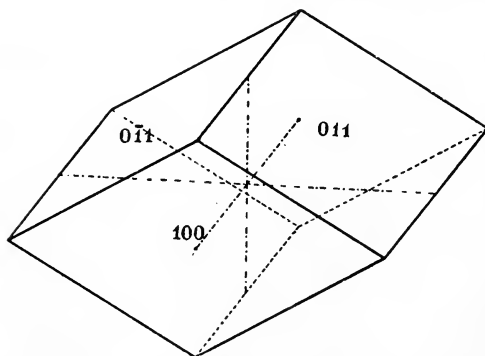


FIG. 215.—The Primary Clinodomal Prism with Ortho-Pinakoid.

of the first order, and is generally termed a **clino-prism** or **clinodomal prism**, and has the general symbol $\{0kl\}$, the primary clino-prism being $\{011\}$. The latter is shown in Fig. 215, in combination with and closed by the ortho-pinakoid $\{100\}$.

The third case is when the pole has migrated to the side (100):(001), that is, on to the primitive circle and therefore lies in the plane of symmetry; the symmetry axis is thus parallel to the faces and edges of all such forms, being their zone axis,

and the zone will include also the faces of the ortho-pinakoid and of the basal pinakoid. Now the operation of the only element of symmetry here able to act, the digonal axis,

only requires that any face the pole of which lies on this primitive zone-circle should have another face parallel to it. Hence, as in the case of the ortho-pinakoid, the form consists of two faces only, parallel to each other, and the two other complementary faces having similar indices, the poles of which lie in the acute angle of the inclined axes, belong to a separate pinakoidal form, termed the positive, the original form with poles in the obtuse angle being the negative. The two forms together make up an open prism parallel to the ortho-diagonal, the symmetry axis b , and is called, analogously to the clino-prism, an ortho-prism or orthodomal prism. Hence the two separate forms are hemi-prisms, or pinakoids of the second order, and are generally known as the **negative hemi-ortho-prism** or **orthodomal prism** $\{h0\bar{l}\}$, and the **positive hemi-ortho-prism** or **orthodomal prism** $\{h0l\}$. Fig. 216 represents such a combination of two ortho-prisms, in this case the primary ones $\{10l\}$ and $\{10\bar{l}\}$.

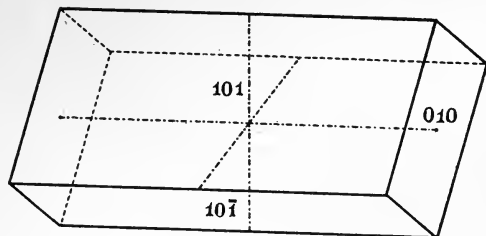


FIG. 216.—Combination of the two Primary Orthodomal Prisms and Clinopinakoid.

The holohedral class of the monoclinic system comprises, therefore, the forms given in the following list; besides the descriptive name referred to in the foregoing description, each form is also labelled with the name indicative of the character of the form as to whether it is a "first," "second" or "third" pinakoid, or a prism or pinakoid of a particular "order," in accordance with the scheme outlined in connection with the rhombic system (Chapter XV. p. 230).

List of Forms in Class 5.

- $\{001\}$ Basal pinakoid; or third pinakoid. 2 faces.
- $\{100\}$ Ortho-pinakoid; or first pinakoid. 2 faces.
- $\{010\}$ Clino-pinakoid; or second pinakoid. 2 faces.
- $\{hk0\}$ Monoclinic prism, or monoclinic prism of the third order, including the primary prism $\{110\}$. 4 faces.
- $\{h0l\}$ Negative hemi-ortho-prism or hemi-orthodomal prism, or monoclinic pinakoid of the second order, including the primary one $\{10l\}$; $\{h0\bar{l}\}$ positive hemi-ortho-prism, or -domal prism, or second order pinakoid, including the primary $\{10\bar{l}\}$. Each 2 faces.
- $\{0kl\}$ Clino-prism or clinodomal prism, or monoclinic prism of the first order, including the primary form $\{011\}$. 4 faces.
- $\{hkl\}$ Negative monoclinic hemi-pyramid, or prism of the fourth order, including the

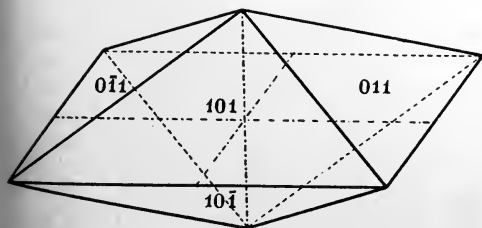


FIG. 217.—Combination of Clino- and Orthodomal Prisms forming Oblique Rectangular Pyramid.

used to be distinguished as the "oblique rectangular pyramid," being comparable to the

parametral hemi-pyramid or prism $\{111\}$; $\{hkl\}$ positive monoclinic hemi-pyramid, or fourth order prism, including the complementary primary hemi-pyramid or prism $\{11\bar{1}\}$. Each 4 faces.

The two last cases but one, the clinodomal prism and the two hemi-orthodomal prisms, when in combination produce a solid of pyramidal form, which

rectangular rhombic pyramid but with an inclined equatorial plane. Such a combination is shown in Fig. 217, but it very rarely indeed happens that the three forms are present in the exact equipoise shown in the figure. Usually the three are developed to different extents, frequently one or other of them largely preponderating.

Excellent examples of substances crystallising according to holohedral monoclinic symmetry are formed by the well-known series of isomorphous double sulphates and selenates the generic formula of which is $R_2M(S_{Se}O_4)_2 \cdot 6H_2O$, in which R may be potassium, rubidium, caesium, thallium, or ammonium, and M may be magnesium, zinc, iron, nickel, cobalt, manganese, copper, or cadmium. These salts have been investigated in great detail by the author, and the collected results have recently been published in book form.¹ One of these salts will be considered in detail in the next chapter, as a practical example of the goniometry of this system, so that it is unnecessary here to refer further to them, except in passing to give an illustration of their general type of combination in Fig. 218, which represents a crystal of ammonium magnesium sulphate, showing as its forms: the clino-pinakoid $b = \{010\}$, the basal pinakoid $c = \{001\}$, the prisms $p = \{110\}$ and $p''' = \{130\}$, the clino-prism $q = \{011\}$,

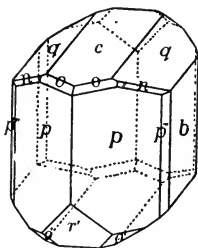


FIG. 218.
Crystal of Ammonium Magnesium Sulphate.

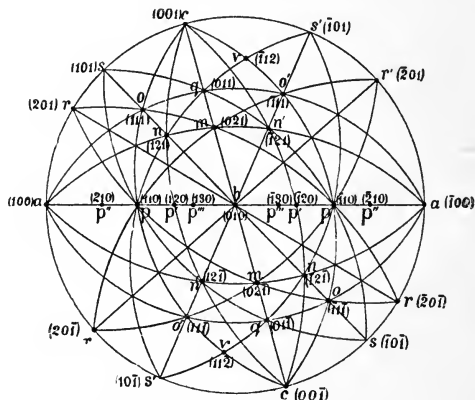


FIG. 219.—Stereographic Projection of Monoclinic Series
 $R_2M(S_{Se}O_4)_2 \cdot 6H_2O$.

the positive hemi-ortho-prism $r' = \{201\}$, the primary hemi-pyramids $o = \{111\}$ and $o' = \{\bar{1}\bar{1}1\}$, and the hemi-pyramid $n = \{121\}$. On other salts of the series still more forms are developed, and the general stereographic projection of the series is here given in Fig. 219 (taken from the author's memoir on the double sulphates²) as a practical example of a monoclinic projection showing the poles of the whole of the types of forms enumerated in the list of class 5 forms.

The recognition of monoclinic symmetry in a new substance will usually be a fairly simple matter after a few measurements have been

¹ *Crystalline Structure and Chemical Constitution*, Macmillan and Co., Ltd., 1909.

² *Journ. Chem. Soc.*, 1893, 63, 343.

made, and the stereographic projection has been constructed. For the plane of symmetry will soon make itself apparent, and the fact that no symmetry is shown along any other plane at right angles to the one recognised as a plane of symmetry, that is, the fact of the absence of other than centro-symmetry among the poles lying in that recognised symmetry plane, will show that the system cannot be orthorhombic, or at any rate cannot be holohedral-orthorhombic, and certainly nothing higher still.

As regards the calculation of the elements, general directions for the case of the monoclinic system have already been given in Chapter VII., and they will be practically employed in the next chapter dealing with the concrete example just referred to.

Class 4.—Monoclinic-Sphenoidal Class. Monoclinic-Hemimorphic Class. Type, Digonal Polar.

The symmetry plane is eliminated in this class, and the only element of symmetry present is the digonal axis, as shown in the stereographic projection in Fig. 220, which also gives the positions of the poles of one of the varieties of the general form $\{hkl\}$.

It will be at once apparent that the latter consists now of only two faces, both of which are at the same end of the symmetry axis, that is, on the same side of the crystallographic axial plane ac , which is no longer a plane of symmetry. There will be two modifications of the form at the two different ends of the axis. The pair of faces in the first form have the indices (hkl) and $(\bar{h}kl)$; the pair comprising the other enantiomorphous modification will be $(h\bar{k}l)$ and $(\bar{h}\bar{k}l)$, hence the form symbols are $\{hkl\}$ and $\{h\bar{k}l\}$, the former being the right variety and the latter the left modification, both negative, because the poles are situated in the obtuse axial angle β . There will also be two other similar but positive forms, when the poles lie inside the acute instead of the obtuse angles of the axes a and c . The right variety will consist of the two faces $(\bar{h}kl)$ and $(h\bar{k}l)$, and its primary representative is the one the poles of which are shown in Fig. 220. The left variety will consist of the pair of faces $(\bar{h}\bar{k}l)$ and (hkl) . The symbols of the forms will thus be $\{\bar{h}kl\}$ and $\{\bar{h}\bar{k}l\}$. The four primary forms will consequently be $\{111\}$, $\{\bar{1}\bar{1}1\}$, $\{\bar{1}11\}$, and $\{1\bar{1}1\}$. All these forms, general and primary, are termed **monoclinic sphenoids**, from the Greek word $\sigma\phi\eta\nu$, meaning "wedge," being pairs of faces forming a wedge in each case, and which, together with the variety of opposite sign at the other end of the symmetry axis in each case, make up a closed so-called "oblique sphenoid," which is the monoclinic representative of the tetrahedron.

Such sphenoids are of the fourth order according to the Groth-Fedorow classification, as they are not parallel to any one of the three crystallographic axes.

The usual six special cases are obtained when the pole lies on one of the sides of the fundamental spherical triangle formed by the poles (100) , (010) , and (001) (see Figs. 211 and 219), or at one of the angles. The same forms result as regards the side

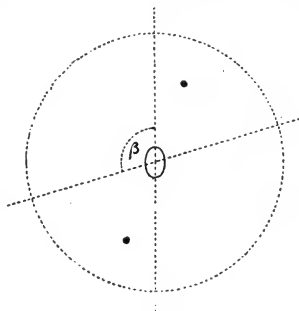


FIG. 220.—Symmetry Elements and General Form of Class 4.

(100):(001), that is, when the poles lie on the primitive circle, as in the holohedral class. As in that class, the ortho-pinakoid (first pinakoid) $\{100\}$, basal pinakoid (third pinakoid) $\{001\}$, and the hemi-ortho-prism or hemi-orthodomal prism (pinakoid of the second order) $\{h0l\}$, each consist of a pair of faces parallel to each other and to the symmetry axis. On the other hand the forms the poles of which lie on the other sides of the triangle, the clino-prisms or domes (first order prisms) $\{0kl\}$ and the monoclinic prisms (third order prisms) $\{hk0\}$, will consist of two faces only, forming a sphenoid of the first or third order, instead of four, the four holohedral faces being divided into two separate pairs, one at each end of the symmetry axis, that is, a right and a left form, the right form having the same symbol as the holohedral form, and the left variety having a negative k index number.

Moreover, the clino-pinakoid, consisting of two faces parallel to each other and to the symmetry plane in the holohedral class, now divides into two single-face or pedial forms, the right clino-pedion (second pedion) $\{010\}$ and the left clino-pedion $\{0\bar{1}0\}$, employing the term "pedion" proposed by Professor von Groth for such single-face forms.

To summarise, we have the following forms in this class :

List of Forms in Class 4.

- $\{001\}$ Basal pinakoid, or third pinakoid. 2 faces.
- $\{100\}$ Ortho-pinakoid, or first pinakoid. 2 faces.
- $\{010\}$ Right clino-pedion, or right second pedion ; $\{0\bar{1}0\}$ left clino-pedion, or left second pedion. Each 1 face.
- $\{hk0\}$ Right monoclinic prism, or sphenoid of the third order, including the primary prism or sphenoid $\{110\}$; $\{h\bar{k}0\}$ left form of same, including the primary $\{1\bar{1}0\}$. Each 2 faces.
- $\{h0l\}$ Negative hemi-ortho-prism or hemi-orthodomal prism, or negative pinakoid of the second order, including the primary $\{101\}$; $\{h0\bar{l}\}$ positive hemi-ortho-prism or orthodomal prism, or positive pinakoid of the second order, including the primary $\{10\bar{1}\}$. Each 2 faces.
- $\{0kl\}$ Right clino-prism or clino-dome, or right sphenoid of the first order, including the primary $\{011\}$; $\{0\bar{k}l\}$ left clino-prism or dome, or left sphenoid of the first order, including the primary $\{0\bar{1}\bar{1}\}$. Each 2 faces.
- $\{hkl\}$ Right negative monoclinic sphenoid (of the fourth order), including the primary $\{111\}$; $\{h\bar{k}l\}$ left negative form, including the primary $\{1\bar{1}\bar{1}\}$; $\{\bar{h}kl\}$ right positive form, including the primary $\{\bar{1}11\}$; $\{h\bar{k}\bar{l}\}$ left positive form, including the primary $\{\bar{1}\bar{1}1\}$. Each 2 faces.

The best example of this class of symmetry is afforded by tartaric

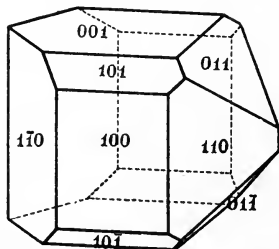


FIG. 221.—Dextro (Ordinary) Tartaric Acid.

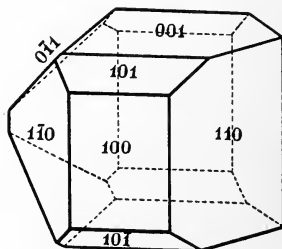


FIG. 222.—Lævo Tartaric Acid.

acid (Figs. 221 and 222), $C_4H_6O_6$, which exhibits combinations of $\{100\}$, $\{001\}$, $\{101\}$, $\{10\bar{1}\}$, $\{110\}$, $\{1\bar{1}0\}$, $\{011\}$, and $\{0\bar{1}\bar{1}\}$. Two optically

active and enantiomorphous varieties are known, which were investigated by Pasteur in a research which attracted great attention at the time, and will ever be of highly historic interest as being the pioneer of such researches, having for their object the separation and identification of the two enantiomorphous and oppositely optically active forms of the same chemical substance. The two modifications are shown in Figs. 221 and 222. The right-handed and dextro-rotatory variety, Fig. 221, is obviously the mirror-image of the left-handed or lævo-rotatory variety exhibited in Fig. 222, and the two are respectively distinguished by the presence of only one of the two clinó-prisms, the dextro variety (Fig. 221) exhibiting the right form $\{011\}$ and the lævo (Fig. 222) variety showing only the left form $\{0\bar{1}1\}$. The two ends of the symmetry axis also exhibit opposite pyro-electric properties, as is usual with hemimorphic crystals. The ratio of the axes, as determined by De la Provostaye, who reinvestigated the substance after Pasteur, is $a:b:c = 1.2747:1:1.0266$, and the axial angle $\beta = 100^\circ 17'$.

Class 3.—Monoclinic Domal Class. Monoclinic-Hemihedral Class.

Type, symmetrical about a Plane.

Of the two elements of monoclinic symmetry the plane of symmetry is here alone operative, while the digonal axis is dormant. The symmetry conditions and the poles of one variety of the general form $\{hkl\}$ are shown in Fig. 223.

As in the last class, this form consists of only a pair of faces, which, however, are now symmetrical to the plane of symmetry, instead of being both on one side of the latter as in class 4. There will thus be four varieties of this general form, corresponding to the four angles made by the intersection of the axes a and c , a pair of faces with poles above and below each other in the same angle being comprised in each. The symbols will be the following: $\{hkl\}$ the upper negative form, with poles in the upper obtuse axial angle, and consisting of the faces (hkl) and $(\bar{h}\bar{k}l)$; $\{h\bar{k}l\}$ the lower positive form, with poles in the lower obtuse axial angle, consisting of $(h\bar{k}l)$ and $(\bar{h}kl)$; $\{\bar{h}kl\}$ the lower negative form, with poles in the lower obtuse angle, consisting of $(\bar{h}kl)$ and $(\bar{h}\bar{k}l)$; and $\{\bar{h}\bar{k}l\}$ the upper positive variety, the one actually shown in Fig. 223, with poles in the upper acute axial angle, consisting of $(\bar{h}kl)$ and $(\bar{h}\bar{k}l)$. These four forms are termed domes, and they are domes of the fourth order, according to the classification of von Groth and von Fedorow. The domal edge always lies in the symmetry plane. The four primary domes are thus $\{111\}$, $\{1\bar{1}\bar{1}\}$, $\{\bar{1}11\}$, and $\{\bar{1}\bar{1}1\}$.

When the pole has migrated on to the sides $(100):(010)$ and $(010):(001)$ of the fundamental triangle, the hemihedral representative of the monoclinic prism or the clino-prism respectively produced will also consist of only two faces symmetrical to the plane of symmetry, instead of the holohedral four faces. In the case of the former we shall have two hemi-prisms or domes of the third order with the symbols $\{hko\}$ and $\{\bar{h}k0\}$, the first consisting of (hko) and $(\bar{h}\bar{k}0)$, and the second of $(\bar{h}k0)$ and $(\bar{h}\bar{k}0)$.

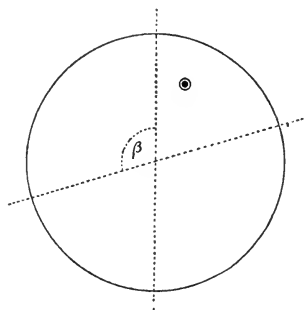


FIG. 223.—Symmetry Elements and General Form of Class 3.

In the case of the latter we have two clino-domes, which are domes of the first order, having the symbols $\{0kl\}$ and $\{0\bar{k}l\}$; the first will consist of $(0kl)$ and $(0\bar{k}l)$ and the second of $(0k\bar{l})$ and $(0\bar{k}\bar{l})$.

All forms with poles on the primitive circle will consist of one face only, as the diagonal axis is absent. Hence we have two pedions together making up the holohedral basal pinakoid, ortho-pinakoid, and hemi-ortho-prism respectively. The clino-pinakoid, however, continues to be represented by both its faces, one above and one below the symmetry plane.

In résumé, we have consequently the following forms to distinguish in this class :

List of Forms in Class 3.

- $\{001\}$ Upper basal plane or pedion, or third positive pedion; $\{00\bar{1}\}$ lower basal plane or pedion, or third negative pedion. Each 1 face.
- $\{100\}$ Front ortho-pedion, or first positive pedion; $\{\bar{1}00\}$ back ortho-pedion, or first negative pedion. Each 1 face.
- $\{010\}$ Clino-pinakoid or second pinakoid. 2 faces.
- $\{hk0\}$ Front hemi-prism, or dome of the third order, including the primary prism $\{110\}$; $\{\bar{h}k0\}$ back hemi-prism, or dome of the third order, including the primary $\{\bar{1}10\}$. Each 2 faces.
- $\{h0l\}$ Upper negative ortho-pedion, or pedion of the second order, including the primary $\{101\}$; $\{h0\bar{l}\}$ lower positive ortho- or second order pedion, including the primary $\{10\bar{1}\}$; $\{\bar{h}0l\}$ lower negative ortho- or second order pedion, including the primary $\{\bar{1}01\}$; $\{\bar{h}0\bar{l}\}$ upper positive ortho- or second order pedion, including the primary $\{\bar{1}0\bar{1}\}$. 1 face each.
- $\{0kl\}$ Upper clino-dome, or dome of the first order, including the primary $\{011\}$; $\{0k\bar{l}\}$ lower form of same, including the primary $\{0\bar{1}\bar{1}\}$. Each 2 faces.
- $\{hkl\}$ Upper negative dome, or dome of the fourth order, including the parametral primary dome $\{111\}$; $\{hk\bar{l}\}$ lower positive form, including the primary $\{11\bar{1}\}$; $\{\bar{h}kl\}$ lower negative form, including the primary $\{\bar{1}11\}$; $\{\bar{h}k\bar{l}\}$ upper positive form, including the primary $\{\bar{1}\bar{1}1\}$. Each 2 faces.

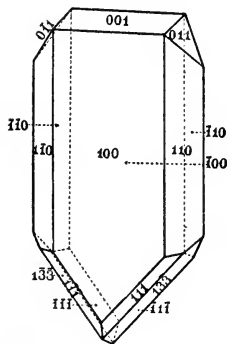


FIG. 224.—Crystal of Potassium Tetrathionate.

Potassium tetrathionate, $K_2S_4O_6$, is one of the few substances crystallising in this class of symmetry. Fig. 224 shows a crystal of this salt, on which are developed the forms $\{100\}$, $\{\bar{1}00\}$ (behind), $\{110\}$, $\{\bar{1}10\}$ (behind), $\{001\}$, $\{011\}$, $\{11\bar{1}\}$, $\{133\}$, and $\{\bar{1}11\}$ (behind). The indices attached to dotted lines and arrows in the figure refer to the back faces of the crystal. The ratio of the axes is: $a:b:c = 0.9302:1:1.2666$, and the axial angle $\beta = 104^\circ 16'$ according to the measurements of Fock.

Generally it is only by the nature of the etched figures on the faces of the primary forms, particularly those of the primary prisms, that full evidence can be obtained that a monoclinic crystal belongs to this class, as it may happen that the complementary hemi-forms may both be developed, and the symmetry thus simulate that of the holohedral class 5.

CHAPTER XVIII

PRACTICAL EXAMPLE OF A MONOCLINIC CRYSTAL.

Ammonium Magnesium Sulphate, $(\text{NH}_4)_2\text{Mg}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$.

Class 5, Monoclinic-Holohedral.

THIS salt is chosen for the monoclinic example to be worked through in detail because it can so readily be obtained in excellent crystals, from a solution containing the molecularly equivalent weights of magnesium sulphate and ammonium sulphate, taking the precautions mentioned in Chapter II. ; and also because the series of double salts, of which it forms one of the most convenient members for the purpose, exhibits very clearly typical holohedral monoclinic symmetry, and is one of the most important series of double salts known to us.

An illustration of the individual crystal chosen is given in Fig. 225,

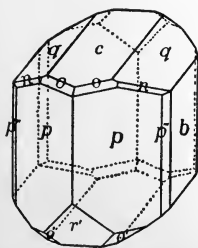


FIG. 225.—The measured Crystal
of Ammonium Magnesium
Sulphate.

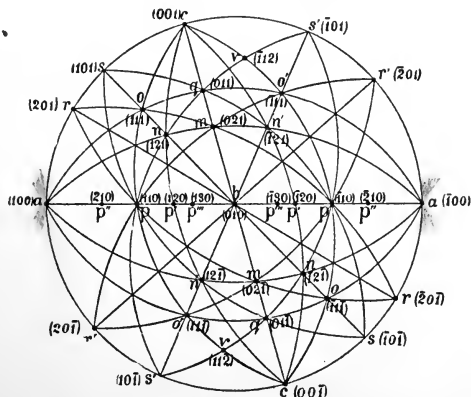


FIG. 226.—Stereographic Projection of Ammonium Magnesium Sulphate.

drawn to scale by the conventional parallel-lined method to be fully explained, with constructional drawing, Fig. 388, page 410, in Chapter XXV., from a careful freehand drawing made at the time of the measurement of the crystal. Its stereographic projection is also given in

Fig. 226, and the mode of constructing it will be described as we proceed with the measurement of the various zones.

The crystal was a brilliant-faced perfectly transparent and colourless one, about three millimetres in length along the prism axis, and two millimetres thick.

The prism zone was obviously the one first inviting measurement. The crystal was set on the wax of the crystal holder with the prism axis parallel to the goniometer axis, and with the broad plane b arranged parallel to one of the adjusting movements, and was then adjusted with the aid of the opposite screw, that of the other rectangularly arranged adjusting segment, so that the image of the signal-slit reflected from the plane b was symmetrical to the horizontal spider line of the eyepiece; then afterwards one of the adjacent prism faces was similarly adjusted with the aid of the other screw and movement, and a final touch given to the adjustment of b which had been thereby slightly disturbed. When this was achieved the whole of the faces of the prism zone were in adjustment, and gave the measurements detailed in the following table. The two p'' faces were only narrow strips, and one of them did not give trustworthy reflections, but the other gave quite good ones.

PRISM ZONE [bp].

Circle Readings.	Angles.
b 360° 0' A	bp 54° 40'
p 305 20	pp 70 35
p 234 45 A	pb 54 44 A
b 180 1 A	bp 54 46 A
p 125 15 A	pp 70 31 A
p 54 44 A	pp'' 29 26
p''' 25 18	$p''b$ 25 18 pb 54° 44' A
b 0 0 A	

An inspection of these angular values undoubtedly indicates that the same angle of fifty-four degrees and about forty minutes occurs symmetrically repeated four times in the circle, once in each quadrant. We shall see this very clearly if we plot out the values along a primitive circle, as usual for a prism zone in the first instance, whether it is permanently to form the primitive-circle-zone or not. That the four values of bp are really intended to be identical is clear from the closeness of the three "A" values, which only differ by 2'. It was on account of this symmetry that the four p -faces were all marked with the same letter p ; and the two b -faces were obviously parallel and hence were both distinguished by the same letter.

We can, therefore, legitimately take the mean of these three values of bp , and even include the fourth value as it is also so close, and the value was all but an "A" value. The resulting mean value of bp is 54° 44'. The mean of the two values of pp is similarly 70° 33'. The face of the other prismatic form is labelled p''' rather than p' because in the cases of other salts of the same isomorphous series p' and p'' are both appropriated to denote other forms developed in the prism zone.

The next zone to measure is obviously that containing the b -faces.

and those marked *q* and *c*. It will inform us incidentally whether or not the *c*-faces are perpendicular to the *b*-faces, and if so, from the appearance of the crystal, it would be probable that a third zone, containing the *c* and *r'* faces, which should next afterwards be measured, would be found to be perpendicular to both the zones [*bp*] and [*bqc*].

The crystal was reset on the wax for the purpose, with the prism this time nearly horizontal, and the *b*-faces parallel to one of the adjusting movements, and then arranged at the exact tilt required to bring the zone into perfect adjustment with the aid of both adjusting movements ; when the adjustment had been achieved for a *b*-face and a *c*-face, it was found that the *q*-faces were also in adjustment, as was to be expected if *b*, *c* and *q* faces all belonged to the same zone. The following measurements were then obtained :

MEASUREMENTS FOR THE ZONE [*bqc*].

Circle Readings.	Angles.	
<i>b</i> 360° 0' A	<i>bq</i> 64° 47' A	<i>bc</i> 90° 0' A
<i>q</i> 295 13 A	<i>qc</i> 25 13 A	
<i>c</i> 270 0 A	<i>cq</i> 25 11 A	<i>cb</i> 89 59 A
<i>q</i> 244 49 A	<i>qb</i> 64 48 A	
<i>b</i> 180 1 A	<i>bq</i> 64 48 A	<i>bc</i> 90 0 A
<i>q</i> 115 13 A	<i>qc</i> 25 12 A	
<i>c</i> 90 1 A	<i>cq</i> 25 12 A	<i>cb</i> 90 0 A
<i>q</i> 64 49 A	<i>qb</i> 64 48 A	
<i>b</i> 0 1 A		

Thus the measurements indicate that the *c*-faces are indeed perpendicular to the *b*-faces, and that the *q*-faces are symmetrically situated with respect to the *b*- and *c*-faces, on either side of them in the zone [*bqc*], for the angle *bq* or *cq* in each quadrant of the zone is the same, the mean values being 64° 48' and 25° 12' respectively. The four individual values in each case are within 1' of these mean values, so that no ambiguity is possible. We have thus found two zones on the crystal in which the angles are **symmetrically arranged**, and **with respect to a plane**, that of the *b*-faces, common to the two zones, and which is obviously therefore perpendicular to the two zone planes (a zone plane, it will be remembered, being the imaginary plane perpendicular to the faces of the zone). The symmetrical angles in each of the zones are not 45°, nor are they 60° or 30°, so that there is no immediate evidence of the crystal being either tetragonal or hexagonal. It might be rhombic, if we were also to find that the faces the poles of which lie in the common plane of symmetry just referred to (their zone plane) at right angles to the two zones just measured were also symmetrically arranged among themselves, and also with respect to the intersections of this third zone plane with the other two. We should next, therefore, measure the zone in question, which is the one containing the *c* and *r'* faces. We cannot construct the stereographic projection until we have done so ; all that we can at present do is to draw a circle, quite independent of that already drawn to represent the prism zone, and arrange the poles *b*, *q*, and *c* symmetrically along it. But the

relationship of this zone circle to the prism zone we do not yet know. In the zone in question the only faces developed on the crystal were those marked c and r' in Fig. 225.

The crystal was now reset on the wax, on a b -face, which is perpendicular to the zone of faces in question, and with a c -face parallel to one of the movements. After adjustment of both c and r' faces, and centring as usual, the following measurements were obtained :

MEASUREMENTS FOR THE ZONE $[cr']$.

Circle Readings.	Angles.
c 360° 0'A	cr' 115° 33'
r' 244 27	$r'c$ 64 27
c 180 0 A	cr' 115 35 A
r' 64 25 A	$r'c$ 64 25 A
c 0 0 A	

It was fairly obvious from a mere inspection of the crystal, that the angle cr' of 115° 34' (the mean of the two values) is not bisected by the pole, that of a possible face, at the intersection of the zone $[bp]$ with the zone $[cr']$. The possible face in question is actually developed on many of the salts of this isomorphous series, particularly on potassium magnesium sulphate, on the crystals of which it is often quite prominent, as shown in Fig. 227, which is taken from the author's memoir¹ on the subject of these salts.

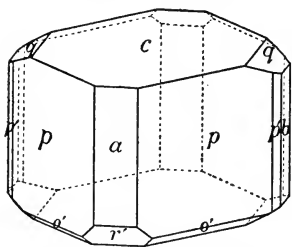


FIG. 227.—Crystal of Potassium Magnesium Sulphate.

It is marked with the letter a , both in this illustration of the potassium salt and in the stereographic projection common to the whole series given in Fig. 226. When the crystal now under consideration was set up with the prism axis, and therefore with the faces b and p , vertical, it was clear that the faces c and r' were not inclined equally to the vertical, the inclination of c being markedly greater (nearer to 90°) than that of r' . If the face a had been present on the crystal of ammonium magnesium sulphate before us, we could at once have verified this fact and determined the actual inclinations. It would appear certain, however, that only centrosymmetry, causing the presence of pairs of parallel faces, is present in the cr' zone just measured, and we can, therefore, at any rate preliminarily, assume that the crystal is not orthorhombic, but monoclinic, the symmetry plane being that already found, namely, parallel to the b -faces, and which is the zone-plane of, and therefore perpendicular to, the zone cr' .

As in the stereographic projection of monoclinic crystals we take the symmetry plane for the plane of projection, we can at once draw in the real primitive circle, on

¹ Journ. Chem. Soc. Trans., 1893, 63, 345.

which the poles c and r' are to appear. The two zones $[bp]$ and $[bqc]$ previously measured will appear in the projection as diameters, intersecting at the centre, which will be the pole of the face b . The prism zone $[bp]$ is to be the horizontal diameter, for its faces and edges are parallel to the axis which we take for the vertical one. We can, therefore, at once place on it the poles p and p'' , finding them by the method already several times explained, of marking off the angles on the primitive circle, and joining the points thus marked off to the pole of the zone, which is the opposite extremity of the diameter (in this case only constructional and afterwards erased) at right angles to the zonal diameter on which we are finding the poles. The extremities of this bp -zonal diameter, where it meets the primitive circle, would be the poles of the a -faces if they were developed.

In order to find the position of the other measured zone $[bqc]$, on which we as yet only know the position of the pole b at the centre, we require to know the position of the extremity-poles c , at the intersection of the zone with the primitive circle. The angle ac would at once fix it, but the a -faces being absent we cannot measure the angle directly, and our knowledge of the angle cr' , in the third zone measured, does not help us, because c and r' may still be anywhere, separated at this angle, on the primitive circle. We know, however, that the two p -faces shown on the projection are accompanied by two others symmetrical to the plane of projection, and therefore situated immediately underneath them and represented by the same dot in each case, or by a ring round each dot. If, therefore, we draw a circular arc to pass through the three points c (assuming for the moment an approximate position for c), p , and the parallel c , the arc will really represent two zones, passing through the dot-pole p and the ring-pole, p respectively, and intersecting at each of the two c poles. Now if our supposition be correct that the plane of projection is really a plane of symmetry perpendicular to the zone cr' , the two shorter arcs cp , as well as of course the two longer supplementary arcs cp , will be equal; translated into actual facts on the crystal this means that the angle between the face of basal-pinakoidal character c and the prism face on its right front, and the similar angle between the same c -face and the prism face on its left front, will be equal. This we can at once test by measurement, and when we have determined the actual magnitude of the angle we can use it in conjunction with our knowledge of the angle ap , for this is the complement of the actually measured bp , to calculate the required angle ac . Our next task, therefore, is to measure the two zones $[cp]$.

The same setting of the crystal on the wax as for the zone cr' can be retained, and first one and then the other zone may be adjusted and measured, by use chiefly of one only of the adjusting movements, the other being only required to perfect the adjustment of the c -face, which had been set approximately perpendicularly to it. When this had been actually done for the crystal under investigation, and the c and p faces were in precise adjustment parallel to the goniometer axis, it was found that besides the two pairs of parallel c and p faces there were also in automatic adjustment, as belonging to the same zone, several faces of the forms o and o' . The actual measurements were as follows :—

already commenced, by inserting the c -poles on the primitive circle at $72^\circ 54'$ from the a -poles, and joining them by the diameter cbc . On this diameter, which will form the projection of the zone $[cqb]$, the positions of the q -poles should be found, similarly to the p -poles on the zone $[bp]$, by the usual process already alluded to. We can then construct accurately the two arc-zones $[cpc]$, one on each side of the diameter. We have next to find the positions on these arc-zones of the poles o and o' . This we are enabled to do very easily, for if we adjust on the goniometer, parallel to one of the movements, the face r' instead of the c -face, and again the same p -face parallel to the other movement, we find that in the complete arc-zone $[r'p]$ there are also automatically adjusted two o' faces. In other words, the poles of the o' faces are at the intersections of the arc-zones $[cpc]$ and $[r'pr']$. If we now construct a further pair of arc-zones on each side of the horizontal diameter, to pass through the three points a , q , and a , we shall find that they also pass through the o' -poles, and where the arcs intersect the shorter cp arcs (the arcs $76^\circ 7'$) will be the positions of the o -poles, if they are, as seems likely from the appearance of the crystal and as will shortly be proved to be the fact, the complementary hemi-pyramid faces to the o' -faces.

Having thus fixed the position of all the principal facial poles, leaving in fact only the faces n still to be located, we may now decide on what further zones require measurement. We should draw diameters through the b and o poles, and through the b and o' poles, and then measure first those two further diametral zones. On measuring the former, we shall discover the n poles in the zone, and can mark off their positions by the usual process already referred to. Then we ought certainly to measure the important pair of zones $[r'o'pr']$, which have enabled us to locate so many poles. And finally, there are two pairs of zones between the well-developed p and q faces, and one pair of these, having the shorter pq arc, will be found to contain also the n -faces, as will be obvious from the fact that these narrow n -faces modify the edges qp in the drawing, Fig. 225. The intersection of this pair of zones with the $[bno]$ zone thus finally confirms the position of the n -poles, and completes the stereographic projection.

The actual circle readings and angular values obtained for one of each of these five pairs of zones will now be given. Similar angles were afforded by the complementary five zones, the details of which it is unnecessary to quote.

In the case of the first zone $[bno]$ only one value, that of the angle bo , was sufficiently trustworthy to record, the angles bn and no being only determinable approximately, owing to the poor quality of the reflection from the n -faces; the images were sufficiently clearly visible, however, to render it certain that the n -faces belonged to the zone. In the zone $[pmq]$ the images afforded by the n -faces were much sharper, and consequently trustworthy.

Zone $[bno]$.		Zone $[bo']$.	
Readings.	Angle.	Readings.	Angles.
$\left\{ \begin{array}{l} b \ 360^\circ \ 0' A \\ o \ 289 \ 9 \end{array} \right.$	$bo \ 70^\circ \ 51'$	$\left\{ \begin{array}{l} b \ 360^\circ \ 0' A \\ o' \ 294 \ 36 \ A \\ o' \ 245 \ 19 \ A \\ b \ 180 \ 0 \ A \\ o' \ 114 \ 40 \ A \\ o' \ 65 \ 24 \ A \\ b \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} bo' \ 65^\circ \ 24' A \\ o'o' \ 49 \ 17 \ A \\ o'b \ 65 \ 19 \ A \\ bo' \ 65 \ 20 \ A \\ o'o' \ 49 \ 16 \ A \\ o'b \ 65 \ 24 \ A \end{array} \right.$

Zone [$r'o'p$].		Zone [pq].		Zone [pq].	
Readings.	Angles.	Readings.	Angles.	Readings.	Angles.
$\left\{ \begin{array}{l} p \ 360^\circ \ 0'A \\ o' \ 268 \ 26 \ A \\ r' \ 233 \ 10 \ A \\ p \ 180 \ 3 \ A \\ o' \ 88 \ 29 \\ r' \ 53 \ 9 \\ p \ 0 \ 1 \ A \end{array} \right.$	$\left\{ \begin{array}{l} p'o' \ 91^\circ \ 34'A \\ o'r' \ 35 \ 16 \ A \\ r'p' \ 53 \ 7 \ A \\ p'o' \ 91 \ 34 \\ o'r' \ 35 \cdot 20 \\ r'p' \ 53 \ 8 \end{array} \right.$	$\left\{ \begin{array}{l} p \ 360^\circ \ 0'A \\ q \ 271 \ 39 \ A \\ p \ 180 \ 0 \ A \\ q \ 91 \ 38 \ A \\ p \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} pq \ 88^\circ \ 21'A \\ qp \ 91 \ 39 \ A \\ pq \ 88 \ 22 \ A \\ qp \ 91 \ 38 \ A \end{array} \right.$	$\left\{ \begin{array}{l} q \ 360^\circ \ 0'A \\ n \ 234 \ 2 \\ p \ 297 \ 34 \ A \\ q \ 180 \ 2 \ A \\ p \ 117 \ 35 \ A \\ q \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} qn \ 25^\circ \ 58' \\ np \ 36 \ 28 \\ qp \ 62 \ 26 \ A \\ pq \ 117 \ 32 \ A \\ qp \ 62 \ 27 \ A \\ pq \ 117 \ 35 \ A \end{array} \right.$

The mean values derived from the five pairs of zones were: $bo=70^\circ \ 51'$; $bo'=65^\circ \ 22'$, $o'o'=49^\circ \ 16'$; $po'=91^\circ \ 36'$, $o'r'=35^\circ \ 18'$, $r'p=53^\circ \ 6'$; $pq=88^\circ \ 21'$, $qp=91^\circ \ 39'$; $qp=62^\circ \ 25'$, $qn=25^\circ \ 58'$, $np=36^\circ \ 28'$, $pq=117^\circ \ 35'$.

This completes the measurements. A final list of the mean measured angles, in their proper order in the various zones, will be found at the end of the chapter, alongside the values calculated from the three most reliably measured basal angles, three being required for the calculation of monoclinic crystals, as stated in Chapter VII.

Before passing to the calculations, a word or two must be said about the symmetry, and the indices of the faces. It has been fully proved that the plane parallel to the b -faces is a symmetry plane, and no other symmetry plane has been discovered with respect to the crystal. Hence, presumably, the symmetry is monoclinic. The disposition of angles in the zone of the primitive circle confirms this view, for it is not similar to the zones $[apb]$ and $[bqc]$ in exhibiting angles symmetrical about a and about c , each face being only accompanied by its parallel companion and not by a fellow symmetrically arranged with respect to either of the primary faces just mentioned. The symmetry exhibited in the zones $[apb]$ and $[bqc]$ is demanded if b be really a symmetry plane, for if any p or q face in the top hemisphere be considered, for instance, a second face is required, immediately underneath it, in the lower hemisphere of the projection, and might be indicated by a ring round the dot representing p or q in the upper hemisphere. Similarly, any pole whatsoever on the projection, other than one on the primitive circle, must be accompanied by a symmetrical fellow in the lower hemisphere, so that if the lower hemisphere poles were actually drawn in they would all appear as rings round the dots representing in the usual manner the poles on the upper hemisphere. Our measurements have fully confirmed this, every are on the upper hemisphere having its equal companion on the lower. The great advantage will now also be apparent of choosing the plane of symmetry for the plane of projection.

It will be shown in a later chapter that the conclusion as to the monoclinic nature of the symmetry is fully borne out by the optical properties of the crystal.

As regards the facial indices. The b -faces parallel to the plane of symmetry are chosen as, and are in fact obviously, those of the

clino-pinakoid $\{010\}$, the indices of the upper face in the projection, the right face in Fig. 225, being (010) , and those of the lower $(\bar{0}10)$. As is customary in drawing the projection of monoclinic crystals, the possible front face of the orthopinakoid $a = \{100\}$ is placed at the left end of the horizontal diameter in the projection. The position of this possible face will be rendered clear by the illustration of the crystal of potassium magnesium sulphate given in Fig. 227, which shows the face (100) well developed in the foreground on the front of the crystal. The parallel ortho-pinakoid face at the back of the crystal, the pole of which is placed at the right end of the horizontal diameter in the projection, is $(\bar{1}00)$. The edges of intersection of a and b faces, or of b and p faces, all of which edges are parallel to each other, are parallel to the vertical axis c of the crystal, indeed these edges are chosen as the direction of the vertical axis. The faces c , situated as we have seen at $72^\circ 54'$ from the a -faces, are chosen as those of the basal pinakoid $\{001\}$, the upper one being (001) and the lower parallel one $(00\bar{1})$, and their edges of intersection with the b -faces, or with the q -faces (all these edges being mutually parallel) mark the direction of the inclined crystallographic axis a . The symmetry (digonal) axis is the normal to the clino-pinakoid $b = \{010\}$ and is the crystallographic axis b , indicated by the central dot of the projection which also marks the pole of $b = (010)$.

The most prominent faces of prismatic character, p and q , are taken respectively as those of the primary monoclinic prism $\{110\}$ and the primary clino-prism $\{011\}$. Now it has been shown that the shorter zonal arcs cp and aq intersect at the poles of the o -faces, and therefore the latter are those of the parametral form $\{111\}$, the negative monoclinic hemi-pyramid. If any doubt arise, or the fact be not sufficiently obvious, it can immediately be proved by cross-multiplication of the symbols of the two zones, as under :

$$\begin{array}{rcl}
 001001 & & 100100 \\
 \times \times \times & = & \bar{1}10 \\
 110110 & & 011011 \\
 & & \bar{1}10\bar{1}10 \\
 & \times \times \times & = 111. \\
 & 0\bar{1}10\bar{1}1 &
 \end{array}$$

Similarly, the longer arcs cp and aq intersect at the o' poles, and these are consequently those of the faces of the form $\{11\bar{1}\}$, the positive monoclinic hemi-pyramid. The indices of the forms r' , p''' , and n will be determined during the course of the calculations. It may be preliminarily stated that they will be shown to be those of the positive hemi-ortho-prism or pinakoid of the second order $\{20\bar{1}\}$, the monoclinic prism of the third order $\{130\}$, and the negative monoclinic hemi-pyramid $\{121\}$.

Having now fully indicated the nature of the symmetry and the disposition and character of the faces present on the crystal, as well as having completely constructed the stereographic projection, we are

in a position to proceed at once to the calculations of angles and elements.

Calculation of Angles and Elements.

Basal angles: $bp = (010):(110) = 54^\circ 44'$,
 $cp = (001):(110) = 76^\circ 7'$,
 $bq = (010):(011) = 64^\circ 49'$.

These three were chosen as basal angles because of their being most trustworthy, as furnished by uniformly excellent faces, and as being the most frequently measured of all the principal angles, not only on the individual crystal taken as our example, but on all the eleven crystals employed during the exhaustive investigation of this salt which has been referred to, and of which our example was one of the most perfect. The mean values of 39 and 32 measurements respectively of bp and cp are identical with the mean values furnished by our example. The mean value of 40 measurements of bq is 1' larger than was afforded by our example, which was $64^\circ 48'$.

From the values of bp and bq we can at once find the values of $ap = (100):(110)$ and $cq = (001):(011)$ by difference from 90° , ap being $35^\circ 16'$ and cq being $25^\circ 11'$. The former was unmeasurable on account of the absence of any a -faces, but the angle $pp = (110):(\bar{1}\bar{1}0)$, its double, was measured 19 times, and its mean value found to be $70^\circ 34'$. The mean of the two values yielded by our example has been shown to be $70^\circ 33'$. The half of these is $35^\circ 17'$, which corresponds to ap . The mean of the four values of cq yielded by our crystal was $25^\circ 12'$, and of 40 values derived from the whole eleven crystals $25^\circ 11'$.

The Axial Angle $\beta = ac = (100):(001)$ has already been calculated, for the purpose of drawing the projection, from the mean values of the angles bp and cp derived from the measurements of the crystal before us, and as these values are identical with the above given basal angular values, the result, namely $ac = 72^\circ 54'$, stands valid for the final value of ac . This is the acute angle between the vertical and inclined axes, and the obtuse angle usually quoted as the axial angle, being the angle in the upper right front octant, is obviously its supplement, $107^\circ 6'$.

The Axial Ratios.—We can find the axial ratios at once very conveniently by the general method given on page 98 in Chapter VII. In connection with Fig. 41 there given, which shows the general arrangement of the primary triangle formed by the poles of $a = (100)$, $b = (010)$, and $c = (001)$, and the parametral triangles lying within it formed by arcs from the corners to the poles of (110) , (011) , and (101) , and intersecting at the pole of (111) , it was shown that:

$$\frac{a}{b} = \frac{\sin \theta}{\sin \phi}, \quad \frac{c}{b} = \frac{\sin \psi}{\sin \chi}.$$

The general conditions thus defined for triclinic crystals in Fig. 41, and these values thence derived for the ratios a/b and c/b , when applied to the case of the monoclinic crystal before us simplify considerably. The primary triangle $a = (100):b = (010):c = (001)$ is equally well given by the stereographic projection,

and that part of the latter comprising this triangle and the parametral triangles contained within it are separately given for the sake of clearness in Fig. 229, and the same Greek letters for the various angles are inserted as in Fig. 41. The angles θ and ϕ together make up 90° , and so also do χ and ψ , so that the values for the said ratios simplify as under:

$$\frac{a}{b} = \frac{\sin \theta}{\sin \phi} = \frac{\sin \theta}{\cos \theta} = \tan \theta;$$

$$\frac{c}{b} = \frac{\sin \psi}{\sin \chi} = \frac{\sin \psi}{\cos \psi} = \tan \psi.$$

Now θ is the angle at c in the triangle acp , which has a right angle at a , and in which we know the sides $ac = 72^\circ 54'$, $cp = 76^\circ 7'$, and $ap = 35^\circ 16'$.

Drawing the Napierian diagram, Fig. 230, we deduce by the rules that:

$$\cos c = \cot 76^\circ 7' \tan 72^\circ 54'. \quad c = 36^\circ 30' = \theta.$$

$$\frac{a}{b} = \tan 36^\circ 30' = 0.7400.$$

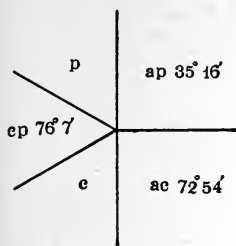


FIG. 230.

Similarly, the angle ψ is the angle at a in the triangle acq , which has a right angle at c , and in which we know the sides $ac = 72^\circ 54'$ and $cq = 25^\circ 11'$. We can at once find its value, as in the case of θ , by constructing a diagram, Fig. 231, and applying Napier's rules, as follows:

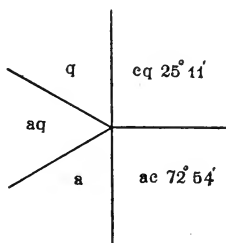


FIG. 231.

$$\cot a = \sin 72^\circ 54' \cot 25^\circ 11'. \quad a = 26^\circ 11\frac{1}{2}' = \psi.$$

$$\frac{c}{b} = \tan 26^\circ 11\frac{1}{2}' = 0.4918.$$

Hence:

$$a : b : c = 0.7400 : 1 : 0.4918.$$

Calculation of Angles (Stereographic projection, Fig. 226).—We can begin most conveniently by finding $aq = (100) : (011)$, for this angle can be calculated from the triangle just considered for which Fig. 231 is the Napierian diagram, from which we derive:

$$\cos aq = \cos 25^\circ 11' \cos 72^\circ 54'. \quad aq = 74^\circ 34'.$$

This angle was not measured, as no a -faces were developed.

To find $pq = (110) : (011)$. This angle can next be found from the triangle $bqp = (010) : (011) : (110)$. This triangle is not right-angled, but we cannot expect to find right-angled triangles so plentiful in monoclinic crystals as we did in the case of rhombic crystals. We must employ therefore, the formulæ of (b) on page 96 of Chapter VII. In the triangle in question we know that the angle at b is equal to $ac = 72^\circ 54'$, and we also know the sides including this angle, namely, $bp = 54^\circ 44'$ and $bq = 64^\circ 49'$. Applying the formulæ we have:

$$\begin{aligned} \tan \theta &= \tan bp \cos ac, & \cos pq &= \frac{\cos bp \cos (bq - \theta)}{\cos \theta} \\ \tan \theta &= \tan 54^\circ 44' \cos 72^\circ 54', & \cos pq &= \frac{\cos 54^\circ 44' \cos (64^\circ 49' - \theta)}{\cos \theta} \\ \theta &= 22^\circ 34'. & pq &= 62^\circ 26'. \end{aligned}$$

The mean measured value of pq was $62^\circ 25'$, as derived from the crystal now under consideration, and $62^\circ 26'$ for all eleven crystals.

To find $po' = (110) : (11\bar{1})$ and $ao' = (100) : (11\bar{1})$.

We can calculate these two angles simultaneously by the formulæ at (c) on the same page of Chapter VII. just quoted, for finding the two sides when the two angles and the included side are known. In the triangle $a = (100) : p = (110) : o' = (11\bar{1})$ we know the side $ap = 35^\circ 16'$, and the angle at a is the complement of the angle at a in the triangle acq , which has already been calculated for the purpose of finding the ratio of the axes and found to be $26^\circ 11\frac{1}{2}'$, so that the angle at a now required is $63^\circ 48\frac{1}{2}'$. The angle at p also required is the supplement of the angle denoted by that letter in the triangle acp , the Napierian diagram corresponding to which is shown in Fig. 228, the triangle having been employed for the calculation of the arc ac on page 268. Proceeding first to calculate this angle at p by Napier's rules we have:

$$\cos p = \tan 35^\circ 16' \cot 76^\circ 7' \\ p = 79^\circ 56'. \text{ Supplement} = 100^\circ 4'.$$

Our conditions are now all fulfilled for the employment of formulæ (c), for we know the two angles $a = 63^\circ 48\frac{1}{2}'$ and $p = 100^\circ 4'$ and the side included by them $ap = 35^\circ 16'$, in the triangle under consideration apo' . Denoting the larger angle at p by the letter A of the formulæ, and the angle at a by B, we have:

$$A + B = 100^\circ 4' + 63^\circ 48\frac{1}{2}' = 163^\circ 52\frac{1}{2}', \text{ and } \frac{1}{2}(A + B) = 81^\circ 56', \\ A - B = 100^\circ 4' - 63^\circ 48\frac{1}{2}' = 36^\circ 15\frac{1}{2}', \text{ and } \frac{1}{2}(A - B) = 18^\circ 8'.$$

Inserting these values in the formulæ we get:

$$\tan \frac{1}{2}(ao' + po') = \frac{\cos 18^\circ 8'}{\cos 81^\circ 56'} \tan \frac{1}{2}(35^\circ 16') \\ \tan \frac{1}{2}(ao' - po') = \frac{\sin 18^\circ 8'}{\sin 81^\circ 56'} \tan 17^\circ 38'$$

$$\frac{1}{2}(ao' + po') = 65^\circ 5'; \quad \frac{1}{2}(ao' - po') = 5^\circ 42\frac{1}{2}'. \\ ao' + po' = 130^\circ 10', \quad 2ao' = 141^\circ 35', \quad 2po' = 118^\circ 45' \text{ (by addition and subtraction),} \\ ao' - po' = 11^\circ 25', \quad ao' = 70^\circ 48', \quad po' = 59^\circ 22'.$$

The mean measured value of po' derived from the crystal under consideration was $59^\circ 21'$, and from all eleven crystals $59^\circ 19'$.

Two further angles at once follow from our knowledge of ao' and po' , namely,

$$o'q = (11\bar{1}) : (01\bar{1}) = 180^\circ - (aq + ao') = 180^\circ - (74^\circ 34' + 70^\circ 48') = 34^\circ 38'. \\ o'c = (11\bar{1}) : (00\bar{1}) = 180^\circ - (cp + po') = 180^\circ - (76^\circ 7' + 59^\circ 22') = 44^\circ 31'.$$

The mean measured value of the latter angle on the crystal which forms our example was $44^\circ 32'$, and for the whole eleven crystals $44^\circ 34'$.

To find $bo' = (010) : (11\bar{1})$ and $o'o' = (11\bar{1}) : (1\bar{1}\bar{1})$.

This we can do from the right-angled triangle $abo' = (100) : (010) : (11\bar{1})$, the side ab being the right angle, the angle at a having been shown to be $63^\circ 48\frac{1}{2}'$, and the side ao' having just been found to be $70^\circ 48'$. Constructing the Napierian diagram, Fig. 232, we deduce:

$$\cos bo' = \sin 70^\circ 48' \cos 63^\circ 48\frac{1}{2}'. \quad bo' = 65^\circ 22'. \\ o'o' = 2(90^\circ - 65^\circ 22') = 49^\circ 16'.$$

These values are identical with the mean of the measured values of bo' and $o'o'$ respectively.

While dealing with this triangle, it will be convenient to calculate the angle at b , for it is the angle on the primitive circle between $a = (100)$ and the possible face $(10\bar{1})$, the pole of which is situated on that circle at the end of the diameter bo' , and which has been found developed on several of the salts of the series, the form having been allotted the letter s' .

$$\sin b = \tan 63^\circ 48\frac{1}{2}' \cot 65^\circ 22'. \quad b = as' = 68^\circ 47'.$$

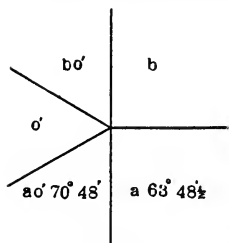


Fig. 232.

From this it follows that:

$$cs' = 180^\circ - (ac + as') = 180^\circ - (72^\circ 54' + 68^\circ 47') = 38^\circ 19'.$$

To find $cr' = (001) : (\bar{2}01)$ and $ar' = (100) : (\bar{2}0\bar{1})$.

For this calculation we can conveniently employ the anharmonic ratio of the four poles in the zone $c = (001)$, $s' = (\bar{1}01)$, $r' = (\bar{2}01)$, $a = (\bar{1}00)$. A preliminary use of the ratio, employing the measured value of cr' and the value of the total angle ac , the supplement of the already calculated angle ac , indicated that the indices of the form r' were $\{\bar{2}01\}$, and we now proceed to calculate the angles cr' and ar' , assuming these indices for r' . The conditions are shown in Fig 233, for we have just found the values of as' and cs' .

$$\begin{aligned} & \frac{\sin cs'}{\sin cr'} \cdot \frac{\sin ar'}{\sin as'} = \frac{001 \cdot \bar{1}01}{001 \cdot \bar{1}00} = \frac{\bar{1}01 \cdot \bar{2}01}{\bar{2}01 \cdot \bar{1}01} \\ & \frac{\sin 38^\circ 19'}{\sin cr'} \cdot \frac{\sin ar'}{\sin 68^\circ 47'} = \frac{1}{2} \cdot \frac{1}{1} \\ & \frac{\sin ar'}{\sin cr'} = \frac{\sin 68^\circ 47'}{2 \sin 38^\circ 19'} = \tan \theta \end{aligned}$$

in accordance with the method of treatment of p. 78, Chapter VI.

$$\theta = 36^\circ 56'.$$

Then, as shown on p. 79 of Chapter VI.,

$$\begin{aligned} \tan \frac{1}{2}(cr' - ar') &= \tan \frac{1}{2}(cr' + ar') \cdot \tan (45^\circ - \theta) \\ &= \tan \frac{1}{2}(107^\circ 6') \cdot \tan 8^\circ 4' \\ &= \tan 53^\circ 33' \tan 8^\circ 4' \\ \frac{1}{2}(cr' - ar') &= 10^\circ 52'. \end{aligned}$$

We already know :
and we now find :

$$\begin{aligned} cr' + ar' &= 107^\circ 6', \\ cr' - ar' &= 21^\circ 44'. \end{aligned}$$

By adding: $2cr' = 128^\circ 50'$, and therefore $cr' = 64^\circ 25'$;
and by subtracting: $2ar' = 85^\circ 22'$, and thus $ar' = 42^\circ 41'$.

The former of these two angles was measured, and the mean of the two values yielded by the excellent crystal which forms our example was $64^\circ 26'$, the two independent values being $64^\circ 27'$ and $64^\circ 25'$. For the whole eleven crystals the mean was $64^\circ 29'$. This satisfactory agreement of the calculated and measured angle cr' finally confirms the indices $\{\bar{2}01\}$ for the form r' .

To find $pr' = (110) : (\bar{2}0\bar{1})$ and $or' = (11\bar{1}) : (\bar{2}0\bar{1})$.

These two angles can next be found from the two right-angled triangles $a = (100) : p = (110) : r' = (\bar{2}0\bar{1})$, and $r' = (\bar{2}0\bar{1}) : s' = (\bar{1}0\bar{1}) : o' = (11\bar{1})$. For in the first we know the sides $ap = 35^\circ 16'$ and $ar' = 42^\circ 41'$, and in the second also two sides, $s'o' = 24^\circ 38'$ and $r's' = 26^\circ 6'$. The two Napierian diagrams are given in Figs. 234 and 235 respectively, and from them we have :

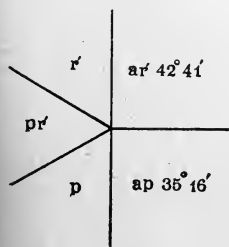


FIG. 234.

$$\begin{aligned} \cos pr' &= \cos 42^\circ 41' \cos 35^\circ 16', \\ pr' &= 53^\circ 7'. \\ \cos r'o' &= \cos 24^\circ 38' \cos 26^\circ 6', \\ r'o' &= 35^\circ 17'. \end{aligned}$$

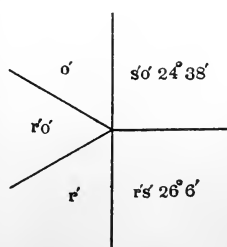


FIG. 235.

These two angles pr' and $r'o'$, together with one other, $po' = (110) : (\bar{1}11)$, make up

the zone $[r'po'r']$, hence we can at once get the value of this third angle in the zone by difference, as follows :

$$po' = 180^\circ - (pr' + r'o') = 180^\circ - 88^\circ 24' = 91^\circ 36'.$$

The mean of the four measured values of each of these three angles on our example was almost identical with the calculated value just found, the three values being $53^\circ 6'$, $35^\circ 18'$, and $91^\circ 36'$. For the whole eleven crystals the mean values were $53^\circ 5'$, $35^\circ 19'$, and $91^\circ 37'$.

It is next necessary to calculate the position of the possible primary hemi-ortho-prism $s=(101)$, and this can be done in the same manner as the complementary hemi-ortho-prism $s'=(\bar{1}01)$ was calculated, namely, by means of the anharmonic ratio of four poles in a zone.

To find $as=(100):(101)$ and $cs=(001):(101)$. The four poles in question are $a=(100)$, $s=(101)$, $c=(001)$, and $s'=(\bar{1}01)$, and the conditions, showing the angles known, are defined in Fig. 236. The ratio is :

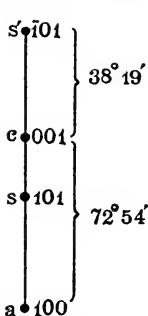


FIG. 236.

$$\frac{\sin s'c}{\sin s's} \cdot \frac{\sin as}{\sin ac} = \frac{\bar{1}01}{101} \cdot \frac{100}{001}$$

$$\frac{\sin s'c}{\sin s's} \cdot \frac{\sin as}{\sin ac} = \frac{\bar{1}}{2} \cdot \frac{1}{\bar{1}}$$

$$\frac{\sin as}{\sin ss'} = \frac{\sin ac}{2 \sin cs'} = \tan \theta$$

$$\theta = 37^\circ 37\frac{1}{2}'$$

$$\tan \frac{1}{2}(ss' - as) = \tan \frac{1}{2}(ss' + as) \cdot \tan (45^\circ - \theta) \\ = \tan 55^\circ 37' \tan 7^\circ 22\frac{1}{2}'.$$

$$\frac{1}{2}(ss' - as) = 10^\circ 43'.$$

$$ss' - as = 21^\circ 26'$$

$$ss' + as = 111^\circ 13'.$$

$$2as = 89^\circ 47', \text{ and consequently } as = 44^\circ 54'.$$

$$cs = ac - as = 72^\circ 54' - 44^\circ 54' = 28^\circ 0'.$$

The position of the possible pole $s=(101)$ is thus now defined between the poles $a=(100)$ and $c=(001)$.

We may next conveniently determine the position of the primary pyramid $o=\{111\}$.

To find $ao=(100):(111)$ and $oq=(111):(011)$.

This can be done by utilising the right-angled triangle $a=(100)$, $o=(111)$, $s=(101)$, in which we know, besides the right angle at s , the angle at a , which has been shown to be $26^\circ 11\frac{1}{2}'$, and the side $as=44^\circ 54'$ just calculated. From the Napierian diagram of Fig. 237 we derive :

$$\cos 26^\circ 11\frac{1}{2}' = \cot ao \tan 44^\circ 54',$$

$$\cot ao = \cos 26^\circ 11\frac{1}{2}' \cot 44^\circ 54'. \quad ao = 48^\circ 0'.$$

$$\text{Then: } oq = aq - ao = 74^\circ 34' - 48^\circ 0' = 26^\circ 34'.$$

To find $os=(111):(101)$ and $ob=(111):(010)$.

This can be done from the same triangle, and from Fig. 237 we deduce :

$$\sin so = \sin 26^\circ 11\frac{1}{2}' \sin 48^\circ 0'. \quad so = 19^\circ 9'.$$

Then $bo = 90^\circ - so = 90^\circ - 19^\circ 9' = 70^\circ 51'$. This value is identical with the result of the only trustworthy measurement of bo .

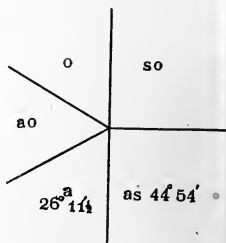


FIG. 237.

To find $co = (001) : (111)$ and $op = (111) : (110)$.

We can find co from the triangle cos , for which Fig. 238 represents the Napierian diagram. By Napier's rules we have:

$$\cos co = \cos 19^\circ 9' \cos 28^\circ 0'. \quad co = 33^\circ 29'.$$

$$\text{Then } op = cp - co = 76^\circ 7' - 33^\circ 29' = 42^\circ 38'.$$

The mean measured values were, for co $33^\circ 32'$, and for op $42^\circ 35'$.

We may next find the position of the pole n , assuming the indices to be (121) , which were indicated by preliminary use of the measured angles bn and no in the anharmonic ratio of the four poles in the 90° -zone $[bnos]$. We shall now use the reverse process to find the angles bn and no , assuming the indices. Fig. 239 shows the conditions, and the ratio derived therefrom is:

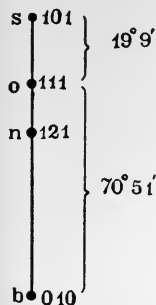


FIG. 239.

$$\begin{array}{r} \begin{array}{cc} 101 & 010 \\ \times & \times \\ \sin so & \sin bn \\ \hline \sin sn & \sin bo \end{array} = \frac{111}{101} \cdot \frac{121}{010} \\ \begin{array}{cc} \times & \times \\ 121 & 111 \end{array} \\ \frac{\sin 19^\circ 9'}{\cos bn} \cdot \frac{\sin bn}{\cos 19^\circ 9'} = \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2}. \end{array}$$

$$\tan bn = \frac{1}{2} \cot 19^\circ 19'. \quad bn = 55^\circ 13'.$$

$$\text{Then } no = bo - bn = 70^\circ 51' - 55^\circ 13' = 15^\circ 38'.$$

To find the angles $pn = (110) : (121)$ and $nq = (121) : (011)$.

This we can do from the triangle $b = (010)$, $n = (121)$, $p = (110)$ in which we know the angle at b , for it is the same as the angle $as = 44^\circ 54'$, and the two sides including it, namely, $bn = 55^\circ 13'$ and $bp = 54^\circ 44'$. Employing the formulæ (b) on page 96 of Chapter VII. we have:

$$\tan \theta = \tan 54^\circ 44' \cos 44^\circ 54', \quad \cos pn = \frac{\cos 54^\circ 44' \cos (55^\circ 13' - \theta)}{\cos \theta}$$

$$\theta = 45^\circ 3'. \quad pn = 36^\circ 27'.$$

$$\text{Then } nq = (121) : (011) = pq - pn = 62^\circ 26' - 36^\circ 27' = 25^\circ 59'.$$

The measured values of pn and nq were respectively $36^\circ 28'$ and $25^\circ 58'$, only $1'$ different from the calculated values.

To find $pq = (110) : (011)$.

This angle can be most conveniently calculated by determining the two other angles in the same zone, $sp = (101) : (110)$ and $sq = (101) : (011)$, which together form the supplement of pq . We can find sp from the right-angled triangle aps , the Napierian diagram for which is given in Fig. 240, and the angle sq may be similarly found from the right-angled triangle eqs , the diagram for which is given in Fig. 241. From the diagrams we at once derive:

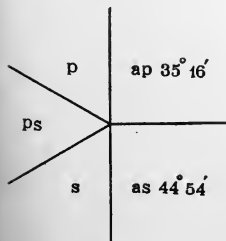


FIG. 240.

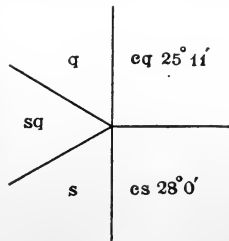


FIG. 241.

$$\cos sp = \cos 35^\circ 16' \cos 44^\circ 54';$$

$$sp = 54^\circ 40'$$

$$\cos sq = \cos 25^\circ 11' \cos 28^\circ 0'.$$

$$sq = 36^\circ 58'.$$

Then $pq = 180^\circ - (54^\circ 40' + 36^\circ 58') = 88^\circ 22'$.

The mean of the four measured values of pq on the crystal under consideration was $88^\circ 21'$, and the same value was obtained as the mean of the 40 measurements on the whole 11 crystals.

To find $ps' = (110) : (10\bar{1})$ and $qs' = (011) : (\bar{1}01)$.

The latter can be found from the right-angled triangle cqs' , of which the Napierian diagram is given in Fig. 242, the sides cq and cs' being known. From it we derive :

$$\cos qs' = \cos 25^\circ 11' \cos 38^\circ 19'. \quad qs' = 44^\circ 46'.$$

Then $ps' = 180^\circ - (62^\circ 26' + 44^\circ 46') = 72^\circ 48'$.

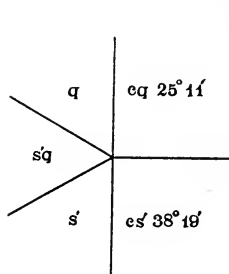


FIG. 242.

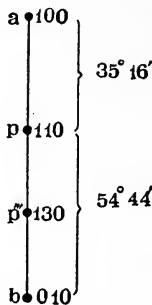


FIG. 243.

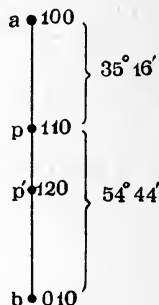


FIG. 244.

To find the positions of the prismatic forms $p' = \{120\}$ and $p''' = \{130\}$.

The latter form alone has been found developed on the crystals of this salt, but the former is quite a common form on several other salts of the series, so that it was desirable to calculate its position for purposes of comparison. Both positions can be found from the anharmonic ratio of four poles in the right-angled zone $[apb]$, the pole p' or p''' being the fourth pole. The conditions are defined in the two diagrams, Figs. 243 and 244, and we can at once set down with their aid the ratios as follows :

$$\frac{\sin ap}{\sin ap'''} \cdot \frac{\sin bp'''}{\sin bp} = \frac{100}{100} \cdot \frac{010}{010} = \frac{130}{110}$$

$$\frac{\sin ap}{\cos ap} \cdot \frac{\sin bp'''}{\cos bp'''} = \frac{1}{3} \cdot \frac{1}{1}$$

$$\tan bp''' = \frac{1}{3} \cot ap;$$

$$bp''' = 25^\circ 14'.$$

$$\frac{\sin ap}{\sin ap'} \cdot \frac{\sin bp'}{\sin bp} = \frac{100}{100} \cdot \frac{120}{010} = \frac{120}{110}$$

$$\frac{\sin ap}{\cos ap} \cdot \frac{\sin bp'}{\cos bp'} = \frac{1}{2} \cdot \frac{1}{1}$$

$$\tan bp = \frac{1}{2} \cot ap;$$

$$bp = 35^\circ 16'.$$

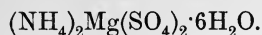
Then $p'''p = 54^\circ 44' - 25^\circ 14' = 29^\circ 30'$.

$p'p = 54^\circ 44' - 35^\circ 16' = 19^\circ 28'$.

The only measured values of bp''' and $p'''p$ obtained were respectively $25^\circ 18'$ and $29^\circ 26'$.

This completes the calculations of angles and elements, and it only remains now to present the summary of results.

TABLE OF RESULTS FOR AMMONIUM MAGNESIUM SULPHATE



Crystal-system : Monoclinic. Class 5, holohedral, prismatic.

Habit : Short prismatic.

Axial angle : $= 107^\circ 6'$.

Ratio of axes : $a : b : c = 0.7400 : 1 : 0.4918$.

Forms observed : $b = \{010\}$, $c = \{001\}$, $p = \{110\}$, $p''' = \{130\}$,
 $q = \{011\}$, $r' = \{201\}$, $o = \{111\}$, $o' = \{1\bar{1}1\}$, $n = \{121\}$.

Table of Angles : In the table of angles which follows, the mean observed values are those derived from the whole eleven crystals included in the investigation, of which the crystal employed as an example in this chapter was the most perfect, and the mean values derived from which, where they differed at all from the mean given in the table, were even closer to the calculated values. Generally speaking, however, the mean values derived from the example are identical with those derived from the whole eleven crystals; but the advantage of having employed also the other ten crystals lies in the fact that proof is afforded that the values derived from the example are not exceptional, but do really represent the truth as regards the angular magnitudes of this substance. The three angles used as basal angles in the calculations are marked with an asterisk.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$ac = (100) : (001)$	$72^\circ 54'$...
$as = (100) : (101)$	$44 \ 54$...
$sc = (101) : (001)$	$23 \ 0$...
$cr = (001) : (201)$	13	$64^\circ 25' - 64^\circ 32'$	$64^\circ 29'$	$64 \ 25$	$4'$
$cs' = (001) : (101)$	$38 \ 19$...
$s'r' = (101) : (201)$	$26 \ 6$...
$r'a = (201) : (100)$	$42 \ 41$...
$r'c = (201) : (001)$	12	$115 \ 26 - 115 \ 34$	$115 \ 31$	$115 \ 35$	4
$ap = (100) : (110)$	$35 \ 16$...
$pp' = (110) : (120)$	$19 \ 28$...
$p'b = (120) : (010)$	$35 \ 16$...
$pp''' = (110) : (130)$	1	...	$29 \ 26$	$29 \ 30$	4
$p''b = (130) : (010)$	1	...	$25 \ 18$	$25 \ 14$	4
$pb = (110) : (010)$	39	$54 \ 35 - 54 \ 50$	$54 \ 44$	*	...
$pp = (110) : (1\bar{1}0)$	19	$70 \ 25 - 70 \ 46$	$70 \ 34$	$70 \ 32$	2
$cq = (001) : (011)$	40	$25 \ 4 - 25 \ 16$	$25 \ 11$	$25 \ 11$	0
$qb = (011) : (010)$	40	$64 \ 43 - 64 \ 56$	$64 \ 49$	*	...
$ao = (100) : (111)$	$48 \ 0$...
$oq = (111) : (011)$	$26 \ 34$...
$aq = (100) : (011)$	$74 \ 34$..
$qo' = (011) : (1\bar{1}1)$	$34 \ 38$...
$o'a = (1\bar{1}1) : (100)$	$70 \ 48$...

TABLE OF ANGLES (*continued*).

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$\left\{ \begin{array}{l} co = (001) : (111) \\ op = (111) : (110) \\ cp = (001) : (110) \\ po' = (110) : (\bar{1}\bar{1}\bar{1}) \\ o'c = (\bar{1}\bar{1}\bar{1}) : (00\bar{1}) \\ pc = (110) : (00\bar{1}) \end{array} \right.$	$\left\{ \begin{array}{l} 4 \\ 4 \\ 32 \\ 27 \\ 25 \\ 31 \end{array} \right.$	$\left\{ \begin{array}{l} 33^\circ 25' - 33^\circ 39' \\ 42 23 - 42 43 \\ 76 1 - 76 17 \\ 59 12 - 59 24 \\ 44 25 - 44 43 \\ 103 44 - 104 2 \end{array} \right.$	$\left\{ \begin{array}{l} 33^\circ 32' \\ 42 35 \\ 76 7 \\ 59 19 \\ 44 34 \\ 103 52 \end{array} \right.$	$\left\{ \begin{array}{l} 33^\circ 29' \\ 42 38 \\ * \\ 59 22 \\ 44 31 \\ 103 53 \end{array} \right.$	$\left\{ \begin{array}{l} 3' \\ 3 \\ ... \\ 3 \\ 3 \\ 1 \end{array} \right.$
$\left\{ \begin{array}{l} bn = (010) : (121) \\ no = (121) : (111) \\ bo = (010) : (111) \\ os = (111) : (101) \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ ... \\ 1 \\ ... \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ ... \\ ... \\ ... \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ ... \\ 70 51 \\ ... \end{array} \right.$	$\left\{ \begin{array}{l} 55 13 \\ 15 38 \\ 70 51 \\ 19 9 \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ ... \\ 0 \\ ... \end{array} \right.$
$\left\{ \begin{array}{l} bo' = (010) : (\bar{1}\bar{1}\bar{1}) \\ o's' = (\bar{1}\bar{1}\bar{1}) : (\bar{1}\bar{0}\bar{1}) \\ o'o' = (\bar{1}\bar{1}\bar{1}) : (\bar{1}\bar{1}\bar{1}) \end{array} \right.$	$\left\{ \begin{array}{l} 7 \\ ... \\ 3 \end{array} \right.$	$\left\{ \begin{array}{l} 65 19 - 65 28 \\ ... \\ 49 9 - 49 21 \end{array} \right.$	$\left\{ \begin{array}{l} 65 22 \\ ... \\ 49 16 \end{array} \right.$	$\left\{ \begin{array}{l} 65 22 \\ 24 38 \\ 49 16 \end{array} \right.$	$\left\{ \begin{array}{l} 0 \\ ... \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} sq = (101) : (011) \\ qp = (011) : (\bar{1}\bar{1}\bar{0}) \\ ps = (\bar{1}\bar{1}\bar{0}) : (\bar{1}\bar{0}\bar{1}) \\ pq = (\bar{1}\bar{1}\bar{0}) : (0\bar{1}\bar{1}) \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ 40 \\ ... \\ 40 \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ 88 11 - 88 29 \\ ... \\ 91 33 - 91 49 \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ 88 21 \\ ... \\ 91 39 \end{array} \right.$	$\left\{ \begin{array}{l} 36 58 \\ 88 22 \\ 54 40 \\ 91 38 \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ 1 \\ ... \\ 1 \end{array} \right.$
$\left\{ \begin{array}{l} s'q = (\bar{1}\bar{0}\bar{1}) : (011) \\ qn = (011) : (121) \\ np = (121) : (110) \\ qp = (011) : (110) \\ ps' = (110) : (10\bar{1}) \\ pq = (110) : (0\bar{1}\bar{1}) \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ 1 \\ 1 \\ 40 \\ ... \\ 40 \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ ... \\ ... \\ 62 18 - 62 38 \\ ... \\ 117 23 - 117 44 \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ 25 58 \\ 36 28 \\ 62 26 \\ ... \\ 117 34 \end{array} \right.$	$\left\{ \begin{array}{l} 44 46 \\ 25 59 \\ 36 27 \\ 62 26 \\ 72 48 \\ 117 34 \end{array} \right.$	$\left\{ \begin{array}{l} ... \\ 1 \\ 1 \\ 0 \\ ... \\ 0 \end{array} \right.$
$\left\{ \begin{array}{l} r'o' = (\bar{2}01) : (\bar{1}\bar{1}\bar{1}) \\ o'p = (\bar{1}\bar{1}\bar{1}) : (110) \\ pr' = (110) : (20\bar{1}) \\ r'p = (\bar{2}01) : (110) \end{array} \right.$	$\left\{ \begin{array}{l} 27 \\ 28 \\ 33 \\ 32 \end{array} \right.$	$\left\{ \begin{array}{l} 35 11 - 35 28 \\ 91 28 - 91 43 \\ 52 53 - 53 14 \\ 126 45 - 127 5 \end{array} \right.$	$\left\{ \begin{array}{l} 35 19 \\ 91 37 \\ 53 5 \\ 126 55 \end{array} \right.$	$\left\{ \begin{array}{l} 35 17 \\ 91 36 \\ 53 7 \\ 126 53 \end{array} \right.$	$\left\{ \begin{array}{l} 2 \\ 1 \\ 2 \\ 2 \end{array} \right.$

Total number of measurements : 581.

This table embodies over a hundred further measurements, on the same eleven crystals, than were given in the author's memoir in the *Journal of the Chemical Society* for 1905, pages 1134-35, of the angles po' and $o'c$, and $r'o'$ and $o'p$. Also the angles in the zone $[bno]$ have been revised, only one measured value in this zone, that of the angle bo , having been found sufficiently trustworthy, owing to the poor quality of the images from the n -faces when adjusted in this zone, as compared with their much clearer images when adjusted in the zone $[sqnp]$.

The general close agreement of the observed and calculated values, the maximum difference being only $4'$, may be taken as evidence both of accurate measurement and correct calculation.

CHAPTER XIX

TRICLINIC OR ANORTHIC SYSTEM.

Three unequal crystallographic axes all inclined to each other at angles other than 90° . Characterised by no symmetry at all or only centro-symmetry.

THERE can obviously be only two classes of symmetry fulfilling these conditions, namely, one, class 2, in which there is centro-symmetry, and another, class 1, in which there is not. The stereographic projections of the two are given in Fig. 245 (class 1) and Fig. 246 (class 2). Each

face in class 1 is a separate form, hence one pole only is shown in Fig. 245 as representative of $\{hkl\}$. In class 2 each form consists of a pair of parallel faces, so that one pole appears in each hemisphere in Fig. 246. In

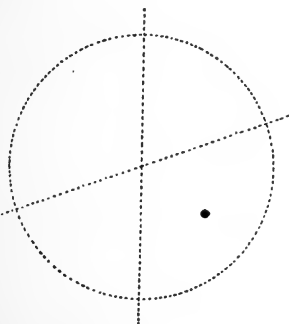


FIG. 245.—Class 1.

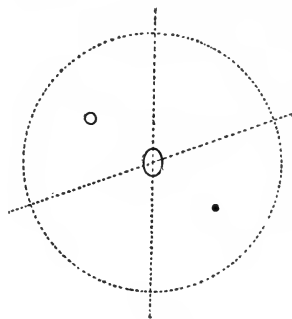


FIG. 246.—Class 2.

the triclinic system the ordinary mode of projection is reverted to, the axes shown in dotted lines being the a and b axes, the plane of these axes being the plane of projection; the monoclinic system is alone exceptional in having the axial plane ac as the plane of projection.

Class 2.—Triclinic-Pinakoidal or Holohedral Class. Type, Centro-Symmetrical or Digonal Alternating.

The nature of the symmetry of this holohedral class of the triclinic system indicated in stereographic projection in Fig. 246, has already been fully discussed in Chapter IX. and illustrated in Fig. 68, p. 141, as it forms the first of the three cases of alternating symmetry there analysed. It was clearly shown that the centro-symmetry, which has hitherto been regarded as the only symmetry present in the class, is really the effect in

a specific case of the simultaneous operation of a plane of compound or reflection-symmetry and of a digonal axis perpendicular thereto. Similar cases of rotation about an axis and simultaneous reflection over a plane were investigated for tetragonal and hexagonal axes, and shown to lead to a similar result in the case of a hexagonal axis, centro-symmetry being again the accompaniment; while the tetragonal case did not lead to centro-symmetry at all, and the trigonal case also referred to was shown to be from its very nature an impossible one. Hence we may equally correctly describe the symmetry of this class 2 as being either "centro-symmetrical" or "digonal alternating."

The result of the presence of nothing but centro-symmetry is that the whole of the forms of this holohedral class of the triclinic system consist in each case of two parallel faces only, one on each side of the centre, hence the class is called "pinakoidal." It is scarcely convenient for the purposes of ready identification to term them all merely pinakoids of various orders, in accordance with the scheme of von Fedorow, but better to retain also the nomenclature of the orthorhombic system, in conjunction with the word "pinakoid," in labelling the seven specific types of forms. If the order of the pinakoid be given in addition to this, the advantages of both the new and the older familiar nomenclature are combined.

Three well-developed faces, which exhibit mutual inclinations as near 90° as possible, are chosen from the most prominent faces developed on the crystal, as the directions

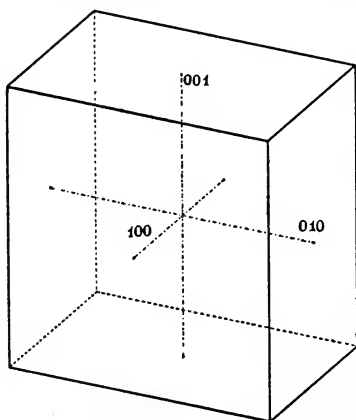


Fig. 247.—Triclinic Prism, formed by the three Primary Pinakoids.

of the axial planes and for the fundamental pinakoids (100), (010), and (001), and their edges of intersection as the directions of the three crystallographic axes. The closed doubly-oblique prismatic six-faced solid formed by these three faces and their parallel fellows, shown in Fig. 247, may be regarded as a rectangular orthorhombic prism closed by basal plane, which has been subjected to deformation in two directions, and thus converted into a parallelepipedon with the same edges but now inclined at angles other than 90° .

A fourth prominent face inclined to all three axial planes, that is, replacing a solid angle (see Fig. 27, p. 62, Chapter V., the face *o*), is next chosen as the primary parametral plane (111), determining the relative lengths of the axes, which will always be unequal (for if all three were equal

trigonal symmetry would be produced). Failing the presence of such a plane, the axial lengths may be fixed by means of two faces each of which is inclined to two of the axes, that is, each of which replaces an edge of the fundamental doubly-oblique prism.

Maintaining the analogy of the orthorhombic nomenclature, the pair of faces intersected by the axis chosen as the vertical one *c*, but parallel to the two other axes, may be termed the **basal pinakoid**, or **third pinakoid**, their indices being (001) and (00 $\bar{1}$) and the form symbol {001}. The longer of the two other axes is arranged laterally right-and-left and made the *b* axis; it is termed the **macro-diagonal**, and the pair of faces parallel to it and to the vertical axis but intersected by the shorter front-and-back axis *a*, the **brachy-diagonal**, are called the **macro-pinakoid** or **first pinakoid**. Their indices are (100) and ($\bar{1}00$), and the form symbol is {100}. Similarly,

the pair of faces parallel to the brachy-diagonal and to the vertical axis c , but intersected by the macro-diagonal, are named the **brachy-pinakoid** or **second pinakoid**, their indices being $\{010\}$ and $\{0\bar{1}0\}$, and the form symbol $\{010\}$.

Other forms parallel to the vertical axis, $\{hk0\}$, may analogously be termed **prismatic pinakoids**, or pinakoids of the third order, two such complementary forms $\{hk0\}$ and $\{h\bar{k}0\}$ being together the doubly-oblique equivalent of an orthorhombic prism.

Forms parallel to the macro-diagonal $\{h0l\}$, are **macro-domal pinakoids** or pinakoids of the second order, two complementary ones $\{h0l\}$ and $\{h\bar{0}l\}$ being together analogous to the orthorhombic macro-domal prism.

Forms parallel to the brachy-diagonal $\{0kl\}$ and $\{0\bar{k}l\}$ are similarly **brachy-domal pinakoids**, or pinakoids of the first order.

The general form $\{hkl\}$ intersected by all three axes is still by analogy termed pyramidal, although it takes four such forms to make up the equivalent of the rhombic bipyramid, as shown in Fig. 248, namely, $\{hkl\}$ consisting of the faces (hkl) and $(h\bar{k}l)$, $\{h\bar{k}l\}$ made up of $(h\bar{k}l)$ and (hkl) , $\{\bar{h}kl\}$ consisting of $(\bar{h}kl)$ and $(h\bar{k}l)$, and $\{\bar{h}\bar{k}l\}$ composed of $(\bar{h}\bar{k}l)$ and $(\bar{h}kl)$. Each form may be most fittingly described as **bipyramidal-pinakoidal**, or as a pinakoid of the fourth order, that is, not parallel to any of the three axes; the former term may be taken as indicating that while each form consists of only a pair of parallel pinakoids the *tout ensemble* of the faces having similar index numbers makes up a triclinic bipyramid. This latter solid was formerly known as a "doubly-oblique rhombic pyramid." Its primary representative is shown in Fig. 248. The only other term which expresses the nature of these bipyramidal-pinakoids is "tetarto-bipyramid," but this term is unsuitable, as it would appear to indicate a tetartohedral type of form, whereas these pinakoids are fully holohedral or holosymmetric, that is to say, they exhibit the whole symmetry (such as it is, being only centro-symmetry) of the system.

The parametral face (111) chosen to determine the lengths of the axes is one of the two faces of the first primary bipyramidal-pinakoid. The pole shown in the upper hemisphere (the solid dot) in Fig. 246 is the (111) face of a real triclinic crystal measured by the author, of the substance methyl triphenylpyrrholone, $(6_6\text{H}_5)_2\text{C}-\text{CH}$. The actual crystal is shown in Fig. 34 and its stereographic projection in Fig. 35, p. 85. It is a particularly characteristic example of a triclinic crystal fairly rich in faces. The ratio of the axes is $a:b:c=0.9059:1:0.8695$; the axial angles are $\alpha=79^\circ 52'$, $\beta=86^\circ 3'$, and $\gamma=70^\circ 26'$. The forms developed are: $a=\{100\}$, $b=\{010\}$, $c=\{001\}$, $p=\{110\}$, $p'=\{1\bar{1}0\}$, $o=\{111\}$, $r=\{1\bar{1}1\}$, $o'=\{1\bar{1}\bar{1}\}$, $s=\{1\bar{1}\bar{1}\}$, $m=\{101\}$, $n=\{1\bar{0}1\}$, $q=\{011\}$, $q'=\{0\bar{1}1\}$, and $r'=\{1\bar{1}2\}$.

List of Forms in Class 2, each of 2 Faces.

- $\{001\}$ Basal pinakoid or third pinakoid.
- $\{100\}$ Macro-pinakoid or first pinakoid.
- $\{010\}$ Brachy-pinakoid or second pinakoid.
- $\{hk0\}$ and $\{h\bar{k}0\}$ Prismatic pinakoids or pinakoids of the third order, including the primary $\{110\}$ and $\{1\bar{1}0\}$.
- $\{h0l\}$ and $\{h\bar{0}l\}$ Macro-domal pinakoids or pinakoids of the second order, including the primary ones $\{101\}$ and $\{1\bar{0}1\}$.
- $\{0kl\}$ and $\{0\bar{k}l\}$ Brachy-domal pinakoids or pinakoids of the first order, including the primary ones $\{011\}$ and $\{0\bar{1}1\}$.

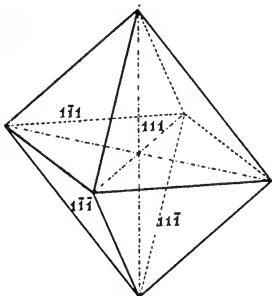
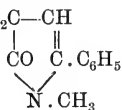


FIG. 248.—Triclinic Bipyramid, formed by four primary fourth order Pinakoids.



$\{hkl\}$, $\{\bar{h}kl\}$, $\{\bar{h}\bar{k}l\}$, and $\{h\bar{k}l\}$ Bipyramidal pinakoids or pinakoids of the fourth order, including the primary ones $\{111\}$ (parametral), $\{\bar{1}11\}$, $\{\bar{1}\bar{1}1\}$, and $\{1\bar{1}1\}$.

It will be observed that the whole of the forms in this list are represented by their primary representatives on the crystal of methyl triphenylpyrrholone, Fig. 34, hence its peculiar suitability as an example of triclinic symmetry.

Copper sulphate, $\text{CuSO}_4 \cdot 5\text{H}_2\text{O}$, is one of the best-known examples of holohedral triclinic symmetry, and a crystal of this salt will be worked through in detail in the next chapter, and all its elements found by calculation from direct measurements of the interfacial angles.

Class 1.—Asymmetric Class. No Symmetry. Triclinic-Hemihedral Class. Type, Pedial.

This class, represented in stereographic projection in Fig. 249, has the unique distinction of exhibiting no symmetry whatever. It merely obeys the fundamental crystallographic law of rational indices.

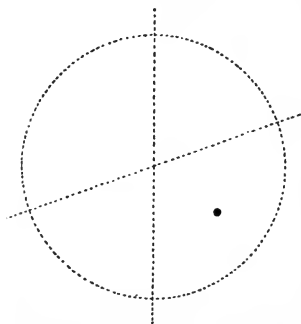


FIG. 249.—General Form of Class 1 (no Symmetry).

Every face developed on the crystal is a separate form, a pedion (from $\pi\epsilon\delta\iota\omicron\nu$, a plane), hence the character of this class is pedial, just as that of class 2 is pinakoidal. Sometimes, however, two parallel faces are developed on crystals of class 1, and this may happen as regards several, or even all, of the forms present, so that on preliminary inspection it may appear to belong to the holohedral class 2 just referred to. But closer examination will reveal the fact that the two faces of each parallel pair are of unequal brilliancy, or exhibit different markings, or afford different etched figures with solvents, which distinguish each face as a separate form. Moreover, it is an interesting fact, which has doubtless to do with the most favourable conditions for the simultaneous growth of the various faces, that both the complementary forms, that is, the two faces which in class 2 make up a single form, are more frequently found simultaneously developed when the indices are low numbers, than when they are more complicated. Hence, we most frequently find the complementary primary forms both present, while in the cases of forms with higher indices, the higher the latter are, the more likely are we to find only a single face-form represented, its parallel face-form being absent. Such a fact alone is of great value in enabling a decision to be arrived at as to whether a crystal belongs to class 1 or to class 2, and the conclusion can be confirmed by etched-figure tests with a minute quantity of a solvent.

The nomenclature of these pedial forms of class 1 is a somewhat difficult matter, there being two courses open, neither of which in the author's opinion is alone satisfactory, but which together are fully so. The first possibility is to retain the nomenclature of class 2, and this ought at any rate to be done for reasons of consistency, and to distinguish the two individual faces, which are together a form in class 2, by the term positive or negative in the case of class 1, the sign being determined by that of the index number which is positive in the one form and negative in the other. While this course is both logical and consistent, there is very considerable ground for accepting the suggestion of von Fedorow, already referred to on several occasions, that the forms parallel to the three axial planes should be termed respectively the first, second,

and third forms; and that those parallel to an axis should be called forms of the first, second, and third order respectively, according as the axis is a , b or c ; and that forms parallel to no axis at all should be distinguished as forms of the fourth order. In the following list of forms belonging to class 1, therefore, both systems of nomenclature will be given, the former because it at once shows the relation of the two class 1 forms to the holohedral class 2 form comprising them, and also the analogy to the orthorhombic forms; and the latter as being a completely scientific mode of identification of the various single-face forms among themselves, totally irrespective of the idea of hemihedrism, and as descriptive of the actual structural conditions of no symmetry which prevail in this particular class, without any reference to any other.

List of Forms in Class 1, each of 1 Face only.

- $\{001\}$ Positive basal plane, or third positive pedion ; $\{00\bar{1}\}$ negative basal plane, or third negative pedion.
 $\{100\}$ Positive hemi-macro-pinakoid, or first positive pedion ; $\{\bar{1}00\}$ negative hemi-macro-pinakoid, or first negative pedion.
 $\{010\}$ Positive hemi-brachy-pinakoid, or second positive pedion : $\{0\bar{1}0\}$ negative hemi-brachy-pinakoid, or second negative pedion.
 $\{hk0\}$ and $\{h\bar{k}0\}$ Positive prismatic hemi-pinakoids, or positive pedions of the third order ; $\{\bar{h}k0\}$ and $\{\bar{h}\bar{k}0\}$ negative prismatic hemi-pinakoids, or negative pedions of the third order. Including the four primary forms $\{110\}$, $\{1\bar{1}0\}$, $\{\bar{1}\bar{1}0\}$, and $\{\bar{1}10\}$.
 $\{h0l\}$ and $\{h\bar{0}l\}$ Positive macro-domal hemi-pinakoids, or positive pedions of the second order ; $\{\bar{h}0l\}$ and $\{h0\bar{l}\}$ negative macro-domal hemi-pinakoids, or negative pedions of the second order. Including the four primary forms $\{101\}$, $\{\bar{1}01\}$, $\{\bar{1}0\bar{1}\}$, $\{10\bar{1}\}$.
 $\{0kl\}$ and $\{0\bar{k}l\}$ Positive brachy-domal hemi-pinakoids, or positive pedions of the first order ; $\{0\bar{k}\bar{l}\}$ and $\{0k\bar{l}\}$ negative brachy-domal hemi-pinakoids, or negative pedions of the first order. Including the four primary forms $\{011\}$, $\{0\bar{1}1\}$, $\{0\bar{1}\bar{1}\}$, and $\{01\bar{1}\}$.
 $\{hkl\}$, $\{\bar{h}kl\}$, $\{\bar{h}\bar{k}l\}$, $\{h\bar{k}l\}$ Positive bipyramidal hemi-pinakoids, or positive pedions of the fourth order ; $\{\bar{h}\bar{k}\bar{l}\}$, $\{h\bar{k}\bar{l}\}$, $\{h\bar{k}l\}$, and $\{\bar{h}kl\}$ negative bipyramidal hemi-pinakoids, or negative pedions of the fourth order. Including the eight primary forms $\{111\}$ (parametral), $\{\bar{1}\bar{1}1\}$, $\{\bar{1}\bar{1}\bar{1}\}$, $\{1\bar{1}\bar{1}\}$, $\{1\bar{1}1\}$, and $\{\bar{1}1\bar{1}\}$.

An example of this asymmetric class is afforded by calcium thiosulphate, $\text{CaS}_2\text{O}_3 \cdot 6\text{H}_2\text{O}$, a crystal of which, measured by G. Wulff, is shown in Fig. 250. It exhibits the forms: $\{100\}$, $\{\bar{1}00\}$, $\{010\}$, $\{0\bar{1}0\}$, $\{001\}$, $\{00\bar{1}\}$, $\{101\}$, $\{\bar{1}01\}$, $\{\bar{1}0\bar{1}\}$, $\{110\}$, $\{\bar{1}\bar{1}0\}$, $\{\bar{1}\bar{1}0\}$, $\{\bar{1}10\}$, $\{111\}$, $\{\bar{1}\bar{1}\bar{1}\}$, $\{\bar{1}11\}$, $\{\bar{1}\bar{1}2\}$, and $\{\bar{1}02\}$. As each face on the crystal is a separate form, the indices of the back faces are inserted in the drawing, in broken figures, as well as those of the front faces (continuous figures).

The ratio of the axes is: $a:b:c =$ Crystal of Calcium Thiosulphate.
 $0.7847:1:1.5179$; and the three axial angles are: $\alpha = 72^\circ 24'$,
 $\beta = 98^\circ 32'$, $\gamma = 92^\circ 40'$, referring, as usual for triclinic crystals in
this book, to the axial angles in the right-front-upper octant.

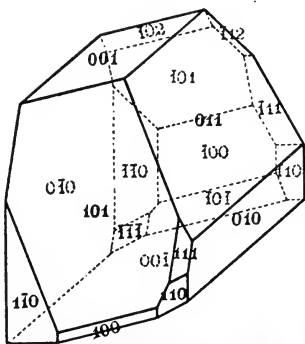


FIG. 250.
Crystal of Calcium Thiosulphate.

CHAPTER XX

PRACTICAL EXAMPLE OF A TRICLINIC CRYSTAL.

Copper Sulphate, $\text{CuSO}_4 \cdot 5\text{H}_2\text{O}$.

Class 2.

SULPHATE of copper is well known to crystallise from its solution in water in magnificent blue crystals, commonly termed "blue vitriol," which contain five molecules of water of crystallisation. These crystals belong to the holohedral class (class 2) of the triclinic system, and as they are so easily procurable, by the simple evaporation of only very slightly supersaturated solutions, in adequately perfect form for measurement, they afford perhaps the best example of triclinic symmetry that could be chosen for the purposes of this book. Although the substance is so cheap and common, it is somewhat singular that the literature concerning the crystallography of copper sulphate is in a most unsatisfactory state. The hitherto accepted elements and angles of the crystals are those of Kupffer,¹ made in the year 1826. These are quoted at some length by Rammelsberg in his *Krystallographisch-physikalischen Chemie*, but the values and their order are quite different in the earlier 1855 edition from those given in the later 1881 edition. In fact the values given by Kupffer, and quoted by Rammelsberg, are so confused, the acute and obtuse angles being almost inextricably mixed up, that a reinvestigation of the crystals of the salt was imperative. The values of Kupffer quoted by von Groth in his *Physikalische Krystallographie* as late as the third, 1895, edition, include the following for the elements :

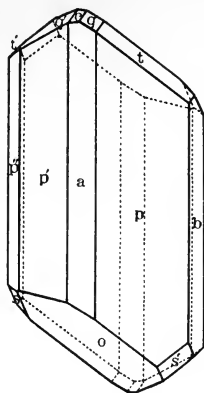
$$a : b : c = 0.5656 : 1 : 0.5499 ; \alpha = 97^\circ 39', \beta = 106^\circ 49', \gamma = 77^\circ 37'.$$

Now it will be shown in this chapter that if the axial ratio just quoted be approximately correct, then the values for a and γ are the supplements of the true values, while the β value is correctly given. In the fourth, 1905, edition von Groth corrects this inversion and gives the following altogether revised elements, derived from unpublished measurements made by Barker in his (von Groth's) laboratory at Munich :

$$a : b : c = 0.5721 : 1 : 0.5554 ; \alpha = 82^\circ 5', \beta = 107^\circ 8', \gamma = 102^\circ 41'.$$

¹ *Pogg. Ann.*, 1826, 8, 217.

The measurements on which these elements were based remain as yet unpublished, and the author has, therefore, felt it incumbent to carry out a complete morphological investigation of the crystals of copper sulphate, and its results are embodied in this chapter, and are expressed in concrete form in those derived from the particularly excellent crystal chosen as the typical example of a triclinic crystal. They agree quite well with the elements of Barker, so that it is certain that he and the author are unanimous as to the mode of setting up the crystal and the disposition of the axes, a conclusion which has been further confirmed by a friendly communication from Mr. Barker.



this face appeared at first sight to be the only one at each end, although examination with a lens revealed smaller ones.

The prismatically developed zone is arranged for descriptive purposes vertically, that is, its zone-axis is chosen as the direction of the vertical axis c . The two faces in it marked respectively a and b and their parallel fellows are chosen as the pinakoids $\{100\}$ and $\{010\}$; the two parallel a -faces are not so broad as the two b -faces. The intermediate broader faces on each side of a are considered as the two primary-prism forms $p = \{110\}$ and $p' = \{1\bar{1}0\}$, each consisting of a pair of parallel faces. In addition, the prism zone also includes a pair of parallel faces p'' of the form $\{1\bar{2}0\}$. The stereographic projection is so arranged that this prism zone affords the primitive circle, that is, its zone plane is the plane of projection.

The mode of measuring a crystal has now been so thoroughly dealt with in the examples taken in previous chapters to illustrate the other crystal-systems, that we may here pass on to give at once a concise record of the measurements obtained with the various zones on this typical triclinic crystal. The relation of the zones to each other will be obvious from the drawing of the crystal in Fig. 251 and its stereographic projection in Fig. 252.

THE PRISM ZONE.

Circle Readings.	Angles.
$\left\{ \begin{array}{l} p \quad 360^\circ \quad 0'A \\ b \quad 306 \quad 56 A \\ p'' \quad 259 \quad 36 \\ p' \quad 237 \quad 4 A \\ a \quad 205 \quad 56 \\ p \quad 179 \quad 55 A \\ b \quad 126 \quad 55 \\ p'' \quad 79 \quad 45 \\ p' \quad 57 \quad 9 A \\ a \quad 26 \quad 3 \\ p \quad 0 \quad 0 A \end{array} \right.$	$\left\{ \begin{array}{l} pb \quad 53^\circ \quad 4'A \\ bp'' \quad 47 \quad 20 \quad \left. \vphantom{\begin{array}{l} pb \\ bp'' \end{array}} \right\} bp' \quad 69^\circ \quad 52' \\ p''p' \quad 22 \quad 32 \quad \left. \vphantom{\begin{array}{l} p''p' \\ p'a \end{array}} \right\} p'p \quad 57 \quad 9 A \\ p'a \quad 31 \quad 8 \quad \left. \vphantom{\begin{array}{l} p'a \\ ap \end{array}} \right\} ab \quad 79 \quad 1 \\ ap \quad 26 \quad 1 \quad \left. \vphantom{\begin{array}{l} p'b \\ bp'' \end{array}} \right\} \\ p'b \quad 53 \quad 0 \quad \left. \vphantom{\begin{array}{l} p'b \\ bp'' \end{array}} \right\} \\ bp'' \quad 47 \quad 10 \quad \left. \vphantom{\begin{array}{l} p'b \\ bp'' \end{array}} \right\} bp' \quad 69 \quad 46 \\ p'p' \quad 22 \quad 36 \quad \left. \vphantom{\begin{array}{l} p'a \\ ap \end{array}} \right\} \\ p'a \quad 31 \quad 6 \quad \left. \vphantom{\begin{array}{l} p'a \\ ap \end{array}} \right\} p'p \quad 57 \quad 9 A \\ ap \quad 26 \quad 3 \quad \left. \vphantom{\begin{array}{l} p'a \\ ap \end{array}} \right\} ab \quad 79 \quad 7 \end{array} \right.$

ZONE $[bc]$.

Circle Readings.	Angles.
$\left\{ \begin{array}{l} b \quad 360^\circ \quad 0'A \\ t \quad 315 \quad 35 A \\ q \quad 295 \quad 21 A \\ c \quad 265 \quad 43 \\ q' \quad 238 \quad 12 \\ t' \quad 220 \quad 47 \\ b \quad 180 \quad 0 A \\ t \quad 135 \quad 30 A \\ q \quad 115 \quad 14 \\ c \quad 85 \quad 37 \\ q' \quad 58 \quad 7 \\ t' \quad 40 \quad 34 \\ b \quad 0 \quad 0 A \end{array} \right.$	$\left\{ \begin{array}{l} bt \quad 44^\circ \quad 25'A \\ tq \quad 20 \quad 14 A \quad \left. \vphantom{\begin{array}{l} bt \\ tq \end{array}} \right\} bq \quad 64^\circ \quad 39'A \\ qc \quad 29 \quad 38 \quad \left. \vphantom{\begin{array}{l} qc \\ cq' \end{array}} \right\} qq' \quad 57 \quad 9 \\ cq' \quad 27 \quad 31 \quad \left. \vphantom{\begin{array}{l} qc \\ cq' \end{array}} \right\} \\ q't' \quad 17 \quad 25 \quad \left. \vphantom{\begin{array}{l} q't' \\ t'b \end{array}} \right\} q'b \quad 58 \quad 12 \\ t'b \quad 40 \quad 47 \quad \left. \vphantom{\begin{array}{l} q't' \\ t'b \end{array}} \right\} \\ bt \quad 44 \quad 30 A \quad \left. \vphantom{\begin{array}{l} bt \\ tq \end{array}} \right\} bq \quad 64 \quad 46 \\ tq \quad 20 \quad 16 \quad \left. \vphantom{\begin{array}{l} bt \\ tq \end{array}} \right\} \\ qc \quad 29 \quad 37 \quad \left. \vphantom{\begin{array}{l} qc \\ cq' \end{array}} \right\} qq' \quad 57 \quad 7 \\ cq' \quad 27 \quad 30 \quad \left. \vphantom{\begin{array}{l} qc \\ cq' \end{array}} \right\} \\ q't' \quad 17 \quad 33 \quad \left. \vphantom{\begin{array}{l} q't' \\ t'b \end{array}} \right\} q'b \quad 58 \quad 7 \\ t'b \quad 40 \quad 34 \quad \left. \vphantom{\begin{array}{l} q't' \\ t'b \end{array}} \right\} \end{array} \right.$

Circle Readings.	Angles.	Circle Readings.	Angles.
Zone [aoq]		Zone [ast]	
$\left\{ \begin{array}{ll} a & 360^\circ \text{ O'A} \\ o & 300 \text{ 25 A} \\ q & 249 \text{ 43 A} \\ a & 180 \text{ 0} \\ o & 120 \text{ 23 A} \\ q & 69 \text{ 43 A} \\ a & 0 \text{ 0 A} \end{array} \right.$	$\left\{ \begin{array}{ll} ao & 59^\circ 35' \text{ A} \\ oq & 50 \text{ 42 A} \\ qa & 69 \text{ 43} \\ ao & 59 \text{ 37} \\ oq & 50 \text{ 40 A} \\ qa & 69 \text{ 43 A} \end{array} \right.$	$\left\{ \begin{array}{ll} a & 360^\circ \text{ 0'} \\ s & 290 \text{ 47} \\ t & 250 \text{ 9} \\ a & 180 \text{ 0} \\ s & 110 \text{ 42} \\ t & 70 \text{ 6 A} \\ a & 0 \text{ 0} \end{array} \right.$	$\left\{ \begin{array}{ll} as & 69^\circ 13' \\ st & 40 \text{ 38} \\ ta & 70 \text{ 9} \\ as & 69 \text{ 18} \\ st & 40 \text{ 36} \\ ta & 70 \text{ 6} \end{array} \right.$
Zone [at's']		Zone [bs'o]	
$\left\{ \begin{array}{ll} a & 360^\circ \text{ 0'} \\ t' & 272 \text{ 45 A} \\ s' & 238 \text{ 59 A} \\ a & 180 \text{ 0} \\ t' & 92 \text{ 49 A} \\ s' & 59 \text{ 8 A} \\ a & 0 \text{ 0} \end{array} \right.$	$\left\{ \begin{array}{ll} at' & 87^\circ 15' \\ t's' & 33 \text{ 46 A} \\ s'a & 58 \text{ 59} \\ at' & 87 \text{ 11} \\ t's' & 33 \text{ 41 A} \\ s'a & 59 \text{ 8} \end{array} \right.$	$\left\{ \begin{array}{ll} b & 360^\circ \text{ 0'A} \\ s' & 319 \text{ 8 A} \\ o & 256 \text{ 24} \\ s & 234 \text{ 37 A} \\ v & 220 \text{ 22} \\ b & 180 \text{ 0} \\ s' & 139 \text{ 8 A} \\ o & 76 \text{ 19 A} \\ s & 54 \text{ 39 A} \\ v & 40 \text{ 23 A} \\ b & 0 \text{ 1 A} \end{array} \right.$	$\left\{ \begin{array}{ll} bs' & 40^\circ 52' \text{ A} \\ s'o & 62 \text{ 44} \\ os & 21 \text{ 47} \\ sv & 14 \text{ 15} \\ vb & 40 \text{ 22} \\ bs' & 40 \text{ 52} \\ s'o & 62 \text{ 49 A} \\ os & 21 \text{ 40 A} \\ sv & 14 \text{ 16 A} \\ vb & 40 \text{ 22 A} \end{array} \right.$
Zone [pot]		Zone [p'qs]	
$\left\{ \begin{array}{ll} p & 360^\circ \text{ 0'} \\ o & 287 \text{ 58 A} \\ t & 233 \text{ 29 A} \\ p & 179 \text{ 58 A} \\ o & 107 \text{ 58 A} \\ t & 53 \text{ 23 A} \\ p & 0 \text{ 0} \end{array} \right.$	$\left\{ \begin{array}{ll} po & 72^\circ \text{ 2'} \\ ot & 54 \text{ 29 A} \\ tp & 53 \text{ 31 A} \\ po & 72 \text{ 0 A} \\ ot & 54 \text{ 35 A} \\ tp & 53 \text{ 23} \end{array} \right.$	$\left\{ \begin{array}{ll} p' & 360^\circ \text{ 0'} \\ q & 276 \text{ 2} \\ s & 229 \text{ 57 A} \\ p' & 180 \text{ 0 A} \end{array} \right.$	$\left\{ \begin{array}{ll} p'q & 83^\circ 58' \\ qs & 46 \text{ 5} \\ sp' & 49 \text{ 57 A} \end{array} \right.$

The two faces of the basal pinakoid $c = \{001\}$ were so small and poorly reflecting that a satisfactory adjustment of the zone $[ac]$, in which no other faces besides a and c were present, was not possible, and consequently trustworthy measurements of the angle ac were not obtained. As the zone $[bc]$ did not depend for its adjustment on the c -faces, excellent other faces being present, reliable measurements were obtained for the angles involving c in that zone. The faces of all forms other than c afforded excellent reflections, the chosen crystal being almost free from striation, so common in the prism zone of copper sulphate.

In order to assist in following the relationship of these several zones an approximate stereographic projection had been constructed, but having now completed the measurements we are in a position to construct it accurately. The procedure for the construction of the projection for a triclinic crystal, which is the most general case that could be proposed owing to the entire absence of right angles, has been clearly set forth in Chapter VI., and an example there given. We shall proceed, therefore, on those lines.

We first draw the primitive circle. Its radius in the author's original drawing from which Fig. 252 was reproduced approximately half-size, was 55 millimetres. The brachy-pinakoid poles $b = (010)$ and $\bar{b} = (0\bar{1}0)$ are inserted at the extremities of the horizontal diameter, which latter, however, should only be drawn in dotted line, as it is not the great circle of a zone, all the zones being obliquely inclined to each other. The poles of the succeeding faces in the prism zone, p , a , p' , and p'' , are next inserted at their proper angular distances around the primitive circle, as indicated by the prism-zone measurements. We have next to find the position of some pole situated

not far from the centre, in order to be able to proceed to other zones. Usually the basal pinakoid $c=(001)$ is the most convenient pole to find as it fulfils the *desideratum* of being near the centre, and is obviously the representative in the triclinic system of the basal plane which in the rectangular systems is situated at the centre itself. That course has been pursued in the example, methyl triphenylpyrrolone, worked out in Chapter VI. But in the case of copper sulphate it has just been pointed out that a satisfactory measurement of the primary angle between $a=(100)$, and $c=(001)$ was impossible. We shall subsequently show how the angle can be calculated from other trustworthy measurements, but we have not yet proceeded to the description of the calculations. Moreover, the angle bc , of which good measurements were obtained, and which would be the other angle required to enable us to fix the position of the pole $c=(001)$, is greater than 90° , namely, $94^\circ 20'$, and is not, therefore, quite so convenient for the purposes of the calculation of the point B and the radius Bc by the simple formulæ of page 85, Chapter VI. We shall, therefore, choose the pole $q=(011)$ as the more convenient one for determination.

Following the instructions given in Chapter VI., we have the following data :

$$r=55\text{mm.}; \theta=aq=69^\circ 43'; \phi=bq=64^\circ 43'.$$

We then draw the produced radii OA and OB through O and $a=(100)$ and through O and $b=(010)$, such that

$$OA = \frac{r}{\cos \theta} = 158.7, \text{ and } OB = \frac{r}{\cos \phi} = 128.8.$$

Having thus found the points A and B, the calculated distances OA and OB being in millimetres, we next find the radii Aq and Bq of two little arcs, constructed from A and B as centres, which intersect at q and thus define the position of that pole.

By the formulæ of Chapter VI.,

$$Aq = r \tan \theta = 148.8, \text{ and } Bq = r \tan \phi = 116.4.$$

We then draw the two little arcs, with radius Aq (in millimetres) from A and with radius Bq from B, and insert the pole $q=(011)$ at their point of intersection.

We have then three points $b=(010)$, $q=(011)$, and $\bar{b}=(0\bar{1}0)$ on the zone $[bqc]$, and can at once construct the zonal arc by the usual method of drawing a circular arc to pass through three points. The other poles in the zone, $t=(021)$, $c=(001)$, $q'=(0\bar{1}1)$, and $t'=(0\bar{2}1)$ are then to be inserted at their proper positions corresponding to the measured angles. In order to do this, we first find the pole of the zone, by joining $b=(010)$ to the point of intersection of the zone circle with the vertical diameter (drawn in dotted line exactly at right angles to the diameter $b\bar{b}$) and producing the line till it cuts the primitive circle, then marking off another point on that circle 90° from this, and joining this last point to $b=(010)$. Where this line cuts the vertical diameter is the zonal pole P required. To find the positions of t , q , c , q' , and t' we mark off the observed angles along the lower half of the primitive circle, and join the points thus marked off to the pole P; these lines will cut the zonal arc at the facial poles required.

We can next draw zonal arcs terminating at $a=(100)$ and its parallel fellow $\bar{a}=(\bar{1}00)$ to pass through t , q , c , q' , and t' , respectively as a third point. Only one further piece of construction is then needed in order to complete the projection, namely, to fix the position of the zonal arc $[bsos'b]$, by determining the position of one of the poles on it, for the other poles will then also be fixed by the intersections of this arc with the arcs just drawn. The pole s' was taken for the purpose, the pole P' of the zone $[at's'a]$ being first found in a similar manner to P, the angle as' marked off from $\bar{a}=(\bar{1}00)$ along the left half of the primitive circle and the point marked off joined to P', cutting the zonal arc at the facial pole s' required. We then have three points $\bar{b}=(0\bar{1}0)$, $s'=(0\bar{1}21)$, and $b=(010)$ on the zonal arc and can construct it, the poles $o=(\bar{1}11)$, and $s=(\bar{1}21)$ being determined by the intersections with the arcs from a

through q and t . If our construction has been an accurate one, it will then be found that another arc drawn to pass through the two p -poles and t will also pass through $o = (\bar{1}11)$; also that arcs drawn to terminate at the two p' -poles and to pass respectively through $c = (001)$, $q = (011)$, and $t = (021)$ will also pass through $o = (\bar{1}11)$, $s = (\bar{1}21)$ and $v = (\bar{1}31)$ in the several cases, and further, that the zone $[p''qp'']$ also passes through v . The zone $[ac]$ contains no other poles than $a = (100)$, $c = (001)$, and $a = (\bar{1}00)$, but it should be drawn, being a primary zone. Having drawn all these zonal arcs and labelled the facial poles with their proper indices, the stereographic projection is completed, and what appeared at first sight to be a difficult geometrical problem is found in reality to be remarkably simple.

The v -faces $(\bar{1}31)$ and $(\bar{1}\bar{3}\bar{1})$ are not shown on the drawing of the crystal, Fig. 251, as it was found that they would only appear as a line in each case truncating the little edges bs at the top right and lower left corners of the crystal, the line of sight being parallel to the faces; they were very small, but gave excellent reflections of the signal.

The indices of the face v are readily found by cross-multiplication of any two of the three zones at the intersection of which it lies, for instance, the zones $[tp']$ and $[bs]$.

$$\begin{array}{rcl} \begin{array}{c} 021021 \\ \times \times \times \\ \bar{1}10\bar{1}10 \\ \\ 010010 \\ \times \times \times \\ \bar{1}21\bar{1}21 \end{array} & \left. \begin{array}{l} [\bar{1}\bar{1}2] \\ \\ [101] \end{array} \right\} & \begin{array}{c} \bar{1}\bar{1}2\bar{1}\bar{1}2 \\ \times \times \times \\ 101101 \end{array} \quad (\bar{1}31) \end{array}$$

The indices of p'' , t , t' , s , and s' will all be found to be correct as given, when we come to the use of the anharmonic ratio of four poles in a zone involving each of them, for the calculation of the angles, assuming the indices in the first place, as these forms have been previously observed by former investigators and their descriptions obviously tallied. The indices $\{\bar{1}31\}$ of the form v above found by cross-multiplication will be shown to be further confirmed by the calculation of the angles involving it from the anharmonic ratio, in the case of one of the zones. The prism $\{130\}$ observed by Barker was not developed on any of the crystals obtained by the author, nor was the form $\bar{1}31$ which had been observed in the year 1905 by Boeris.

The results of the measurements given in detail in the foregoing description of the practical work on this typical crystal of copper sulphate will be found tabulated at the end of the chapter, the mean value of the two individual values obtained for each angle, derived from the centro-symmetrically parallel faces, being taken as the observed angle. The calculated values are given alongside, and to these calculations we will now proceed. Although they are the most difficult that have to be tackled, right angles being entirely absent, they will all be found to be quite readily carried out with the aid of the few formulæ of Chapter VII., and the anharmonic ratio of Chapter VI.

The Calculations—Basal Angles.—In accordance with the explanation in Chapter VII., five basal angles are required for the calculation of the elements and the remaining angles, in the case (the general one) of a triclinic crystal. The following five were selected as having been measured with particularly good faces :

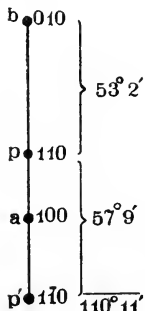
$$\begin{aligned} bp' &= (0\bar{1}0) : (1\bar{1}0) = 69^\circ 49', \\ p'p &= (1\bar{1}0) : (110) = 57^\circ 9', & \text{By difference from } 180^\circ, bp &= 53^\circ 2'. \\ bq &= (010) : (011) = 64^\circ 43', \\ bq' &= (0\bar{1}0) : (0\bar{1}1) = 58^\circ 10', & \text{,,} & \text{,,} & qq' &= 57^\circ 7'. \\ aq &= (100) : (011) = 69^\circ 43'. \end{aligned}$$

The procedure will be, to begin by employing the first two basal angles to find all the other angles in the primitive-circle zone, then to use the third and fourth to calculate the angles in the zone $[bqc]$; for both zones the method will be the use of the anharmonic ratio of four poles. Having calculated all the angles in these two important zones we can employ the fifth basal angle along with two of those already determined for the calculation of the angle at b between the two zones, which is the element β , and with the aid of that we can further connect the two zones by the determination of the arc ac . Knowing then the three primary angles ab , bc , and ac , as well as β the angle at b , we can readily calculate the two other angular elements a and γ , which are the angles respectively at a and at c . The data then accumulated will also be found to be ample to enable us to calculate the ratio of the axes, $a : b : c$.

A few other calculations on similar lines will finally enable us to determine all the other angles which have been measured.

Primitive circle zone. To find $ab = (100) : (010)$.

We employ the anharmonic ratio of the four poles b , p , a , p' , which we can write down directly from the data given in Fig. 253.



$$\begin{aligned} & \begin{array}{cc} 010 & 1\bar{1}0 \\ \times & \times \\ \frac{\sin bp}{\sin ba} \cdot \frac{\sin p'a}{\sin p'p} = & \frac{110}{010} \cdot \frac{100}{1\bar{1}0} \\ & \times \quad \times \\ & 100 \quad 110 \end{array} \\ & \frac{\sin 53^\circ 2'}{\sin ba} \cdot \frac{\sin p'a}{\sin 57^\circ 9'} = \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2} \\ & \frac{\sin p'a}{\sin ba} = \frac{\sin 57^\circ 9'}{2 \sin 53^\circ 2'} = \tan \theta \\ & \theta = 27^\circ 44'. \end{aligned}$$

as suggested in formula (a) on page 78, Chapter VI.

Then, from formula (e) of Chapter VI., page 79, we have :

$$\begin{aligned} \tan \frac{ba - p'a}{2} &= \tan \frac{ba + p'a}{2} \cdot \tan (45^\circ - \theta) \\ &= \tan 55^\circ 5\frac{1}{2}' \cdot \tan 17^\circ 16' \\ \frac{ba - p'a}{2} &= 24^\circ 0\frac{1}{2}'; \quad ba - p'a = 48^\circ 1'. \end{aligned}$$

FIG. 253.

We have now, therefore, $ba + p'a = 110^\circ 11'$
 $ba - p'a = 48^\circ 1'$.

By addition $2ab = 158^\circ 12'$, and $ab = 79^\circ 6'$.

And by subtraction $2ap' = 62^\circ 10'$, and $ap' = 31^\circ 5'$.

By difference also $ap = pp' - ap' = 57^\circ 9' - 31^\circ 5' = 26^\circ 4'$.

To find $bp'' = (0\bar{1}0) : (1\bar{2}0)$. The four poles of which we use the anharmonic ratio are shown in Fig. 254, from which we get :

$$\begin{array}{c} 0\bar{1}0 \quad 100 \\ \times \quad \times \\ \sin bp'' \cdot \sin ap' = \frac{1\bar{2}0}{0\bar{1}0} \cdot \frac{1\bar{1}0}{100} = \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2} \\ \times \quad \times \\ 1\bar{1}0 \quad 1\bar{2}0 \end{array}$$

$$\frac{\sin bp''}{\sin ap'} = \frac{\sin bp'}{2 \sin ap'} = \frac{\sin 69^\circ 49'}{2 \sin 31^\circ 5'} = \tan \theta. \quad \theta = 42^\circ 16'.$$

$$\tan \frac{ap'' - bp''}{2} = \tan \frac{ap'' + bp''}{2} \cdot \tan (45^\circ - \theta)$$

$$= \tan 50^\circ 27' \cdot \tan 2^\circ 44'.$$

$$\frac{ap'' - bp''}{2} = 3^\circ 18\frac{1}{2}'; \quad ap'' - bp'' = 6^\circ 37'.$$

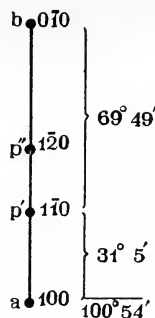


FIG. 254.

We have now

$$ap'' + bp'' = 100^\circ 54'$$

$$ap'' - bp'' = 6^\circ 37'.$$

By subtraction

$$2bp'' = 94^\circ 17', \text{ and } bp'' = 47^\circ 9'.$$

Then also

$$p''p' = bp' - bp'' = 69^\circ 49' - 47^\circ 9' = 22^\circ 40'.$$

This completes the zone of the primitive circle, the prism zone.

Zone $[bqc]$. To find $bt = (010) : (021)$.

The four poles are set out in Fig. 255, and their anharmonic ratio is :

$$\begin{array}{c} 010 \quad 0\bar{1}1 \\ \times \quad \times \\ \sin bt \cdot \sin q't = \frac{021}{010} \cdot \frac{011}{0\bar{1}1} = \frac{1}{1} \cdot \frac{2}{3} = \frac{2}{3}, \\ \times \quad \times \\ 011 \quad 021 \end{array}$$

$$\frac{\sin bt}{\sin q't} = \frac{2 \sin 64^\circ 43'}{3 \sin 57^\circ 7'} = \tan \theta. \quad \theta = 35^\circ 40'.$$

$$\tan \frac{q't - bt}{2} = \tan \frac{q't + bt}{2} \cdot \tan (45^\circ - \theta).$$

$$= \tan 60^\circ 55' \tan 9^\circ 20'.$$

$$\frac{q't - bt}{2} = 16^\circ 28', \quad q't - bt = 32^\circ 56'.$$

FIG. 255.

We now know that :

$$q't + bt = 121^\circ 50',$$

$$q't - bt = 32^\circ 56'.$$

By subtraction

$$2bt = 88^\circ 54', \text{ and } bt = 44^\circ 27'.$$

Then also :

$$tq = bq - bt = 64^\circ 43' - 44^\circ 27' = 20^\circ 16'.$$

To find $bc = (010) : (001)$.

The four poles the anharmonic ratio of which we use are diagrammatically set out in Fig. 256. The ratio is as follows :

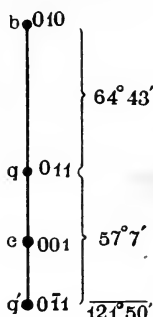


FIG. 256.

By addition
and by subtraction
Moreover,

The following are now, therefore, known :

$$\begin{aligned} bc + q'c &= 121^\circ 50' \\ bc - q'c &= 66^\circ 40'. \\ 2bc &= 188^\circ 30', \text{ and } bc = 94^\circ 15', \\ 2q'c &= 55^\circ 10', \text{ and } q'c = 27^\circ 35'. \\ cq = qq' - cq' &= 57^\circ 7' - 27^\circ 35' = 29^\circ 32'. \end{aligned}$$

To find $bt' = (0\bar{1}0) : (0\bar{2}1)$.

The anharmonic ratio to be used will be clear from Fig. 257.

$$\begin{aligned} \frac{\sin bt'}{\sin 58^\circ 10'} \cdot \frac{\sin 27^\circ 35'}{\sin ct'} &= \frac{0\bar{1}0}{0\bar{1}0} \cdot \frac{001}{001} = \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2} \\ \frac{\sin bt'}{\sin ct'} &= \frac{\sin 58^\circ 10'}{2 \sin 27^\circ 35'} = \tan \theta. \quad \theta = 42^\circ 32'. \\ \tan \frac{ct' - bt'}{2} &= \tan \frac{ct' + bt'}{2} \cdot \tan (45^\circ - \theta) = \tan 42^\circ 52\frac{1}{2}' \cdot \tan 2^\circ 28'. \\ \frac{1}{2}(ct' - bt') &= 2^\circ 17\frac{1}{2}'. \end{aligned}$$

The known data are now :

$$\begin{aligned} ct' + bt' &= 85^\circ 45' \\ ct' - bt' &= 4^\circ 35'. \end{aligned}$$

Subtracting :

$$2bt' = 81^\circ 10', \text{ and } bt' = 40^\circ 35'.$$

Also :

$$t'q' = bq' - bt' = 58^\circ 10' - 40^\circ 35' = 17^\circ 35'.$$

This completes the calculation of the second important zone [bqc].

To find the angle at b between these two zones, that is, β , knowing the three sides of the triangle aqb containing it.

We employ formula (α) on page 96, Chapter VII. The three sides are :

$$\begin{aligned} aq &= 69^\circ 43' \\ bq &= 64^\circ 43' \\ ab &= 79^\circ 6' \end{aligned}$$

$$\text{sum} = 213^\circ 32' \text{ and } s = \frac{1}{2} \text{ sum} = 106^\circ 46'$$

$$\sin^2 \frac{\beta}{2} = \frac{\sin (106^\circ 46' - 64^\circ 43') \cdot \sin (106^\circ 46' - 79^\circ 6')}{\sin 64^\circ 43' \cdot \sin 79^\circ 6'}$$

$$\frac{\beta}{2} = 36^\circ 17', \text{ and } \beta = 72^\circ 34'.$$

The upper front angle between the axes a and c is the supplementary axial angle $\beta = 107^\circ 26'$.

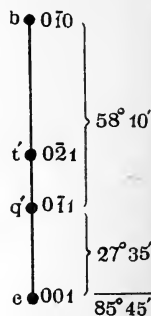


FIG. 257.

To find $ac=(100):(001)$ in the triangle abc , knowing the two sides $bc=94^\circ 15'$ and $ab=79^\circ 6'$, and the included angle at $b=72^\circ 34'$, by the formulæ (b) of page 96, Chapter VII.

$$\tan \theta = \tan 79^\circ 6' \cdot \cos 72^\circ 34'; \quad \cos ac = \frac{\cos 79^\circ 6' \cdot \cos (94^\circ 15' - \theta')}{\cos \theta}$$

$$\theta = 57^\circ 16'. \quad ac = 73^\circ 47'.$$

We are now in a position to be able to calculate the other two axial angles α and γ , for we know all three sides of the primary triangle abc , namely, $ab=79^\circ 6'$, $bc=94^\circ 15'$, and $ac=73^\circ 47'$.

$$\begin{array}{r} ab = 79^\circ 6' \\ bc = 94 \quad 15 \\ ac = 73 \quad 47 \\ \hline \end{array}$$

To find α .

By formula (a) of Chapter VII. :

$$\sin^2 \frac{\alpha}{2} = \frac{\sin (123^\circ 34' - 73^\circ 47') \cdot \sin (123^\circ 34' - 79^\circ 6')}{\sin 73^\circ 47' \cdot \sin 79^\circ 6'} \quad \frac{1}{2} \text{ sum} = 123^\circ 34'$$

$$\frac{\alpha}{2} = 48^\circ 52', \text{ and } \alpha = 97^\circ 44'.$$

The upper right-hand angle between the axes b and c is the supplementary axial angle $\alpha = 82^\circ 16'$.

To find γ .

$$\sin^2 \frac{\gamma}{2} = \frac{\sin (123^\circ 34' - 94^\circ 15') \cdot \sin (123^\circ 34' - 73^\circ 47')}{\sin 94^\circ 15' \cdot \sin 73^\circ 47'}$$

$$\text{or } \cos 4^\circ 15'.$$

$$\frac{\gamma}{2} = 38^\circ 40', \text{ and } \gamma = 77^\circ 20'.$$

The front right-hand angle between the axes a and b is the supplementary axial angle $\gamma = 102^\circ 40'$.

We have thus completed the calculation of the three axial angles α , β , and γ .

To find the axial ratio we require to find first the four angles, χ , ψ , θ , and ϕ in accordance with Chapter VII. They are shown in Fig. 41, page 98.

We can at once find χ and ψ knowing, as we do, the three sides of each of the triangles abq and acq .

In the triangle abq half the sum of the sides has already been shown to be $106^\circ 46'$, and the three sides are $aq=69^\circ 43'$, $bq=64^\circ 43'$, $ab=79^\circ 6'$.

$$\sin^2 \frac{\chi}{2} = \frac{\sin (106^\circ 46' - 69^\circ 43') \cdot \sin (106^\circ 46' - 79^\circ 6')}{\sin 69^\circ 43' \cdot \sin 79^\circ 6'}$$

$$\frac{\chi}{2} = 33^\circ 27', \text{ and } \chi = 66^\circ 54'.$$

In the triangle acq the three sides are $ac=73^\circ 47'$, $cq=29^\circ 32'$, and $aq=69^\circ 43'$. The sum of these is $173^\circ 2'$ and half the sum $=86^\circ 31'$.

$$\sin^2 \frac{\psi}{2} = \frac{\sin (86^\circ 31' - 69^\circ 43') \cdot \sin (86^\circ 31' - 73^\circ 47')}{\sin 69^\circ 43' \cdot \sin 73^\circ 47'}$$

$$\frac{\psi}{2} = 15^\circ 25\frac{1}{2}', \text{ and } \psi = 30^\circ 51'.$$

Then by page 98, Chapter VII. : $\frac{c}{b} = \frac{\sin \psi}{\sin \chi} = \frac{\sin 30^\circ 51'}{\sin 66^\circ 54'}$. $c/b = 0.5575$.

Before we can calculate θ and ϕ , and so a/b (which $= \frac{\sin \theta}{\sin \phi}$), we must find $cp=(001):(110)$ from the triangle bcp , knowing the two sides $bc=94^\circ 15'$ and $bp=53^\circ 2'$, and also the included angle $\beta=72^\circ 34'$.

By formula (b) page 96, Chapter VII. :

$$\tan \theta = \tan 53^\circ 2' \cdot \cos 72^\circ 34'; \quad \cos cp = \frac{\cos 53^\circ 2' \cdot \cos (94^\circ 15' - \theta)}{\cos \theta}$$

$$\theta = 21^\circ 42'. \quad cp = 78^\circ 48'.$$

Now to find θ and ϕ knowing the three sides of each of the triangles acp and bcp .

<i>acp.</i>	<i>bcp.</i>
$ac = 73^\circ 47'$	$bc = 94^\circ 15'$
$cp = 78^\circ 48'$	$cp = 78^\circ 48'$
$ap = 26^\circ 4'$	$bp = 53^\circ 2'$
sum = 178 39	sum = 226 5
$\frac{1}{2}$ sum = 89 20	$\frac{1}{2}$ sum = 113 3
$\sin^2 \frac{\theta}{2} = \frac{\sin (89^\circ 20' - 73^\circ 47') \cdot \sin (89^\circ 20' - 78^\circ 48')}{\sin 73^\circ 47' \cdot \sin 78^\circ 48'}$	
$\frac{\theta}{2} = 13^\circ 11', \text{ and } \theta = 26^\circ 22'.$	
$\sin^2 \frac{\phi}{2} = \frac{\sin (113^\circ 3' - 94^\circ 15') \cdot \sin (113^\circ 3' - 78^\circ 48')}{\sin 94^\circ 15' \cdot \sin 78^\circ 48'}$	
$\text{or } \cos 4^\circ 15'$	
$\frac{\phi}{2} = 25^\circ 30', \text{ and } \phi = 51^\circ 0'.$	

Then from page 98, Chapter VII., we have: $\frac{a}{b} = \frac{\sin \theta}{\sin \phi} = \frac{\sin 26^\circ 22'}{\sin 51^\circ 0'}$
 $a/b = 0.5715.$

Hence the ratio of the axes is: $a:b:c = 0.5715:1:0.5575.$

Having now calculated all the elements of the crystal, as well as all the angles in the three primary zones $[ab]$, $[bc]$, and $[ac]$, we have only to calculate the angles in the other zones in order to complete the work on the crystal. We shall take the various angles now in the order of convenience of calculation, not necessarily a complete zone at a time.

To find $pt = (110):(021)$, from the triangle bpt , knowing the two sides $bt = 44^\circ 27'$ and $bp = 53^\circ 2'$, and the included angle $\beta = 72^\circ 34'$.

$$\tan \theta = \tan 44^\circ 27' \cdot \cos 72^\circ 34'; \quad \cos pt = \frac{\cos 44^\circ 27' \cdot \cos (53^\circ 2' - \theta)}{\cos \theta}$$

$$\theta = 16^\circ 23'. \quad pt = 53^\circ 21'.$$

To find $at = (100):(021)$, from the triangle abt , knowing the two sides $ab = 79^\circ 6'$, and $bt = 44^\circ 27'$, and the included angle $\beta = 72^\circ 34'$.

$$\tan \theta = \tan 44^\circ 27' \cdot \cos 72^\circ 34'; \quad \cos at = \frac{\cos 44^\circ 27' \cdot \cos (79^\circ 6' - \theta)}{\cos \theta}$$

$$\theta = 16^\circ 23' \text{ as above calculated. } at = 70^\circ 4'.$$

To find $at' = (100):(0\bar{2}1)$, from the triangle $b=(0\bar{1}0):t'=(0\bar{2}1):a=(100)$, knowing the two sides $bt' = 40^\circ 35'$ and $ab = 100^\circ 54'$, and the included angle $\beta = 72^\circ 34'$ [the angle at $b=(0\bar{1}0)$, which is of course equal to that at $b=(010)$].

$$\tan \theta = \tan 40^\circ 35' \cdot \cos 72^\circ 34'; \quad \cos at' = \frac{\cos 40^\circ 35' \cdot \cos (100^\circ 54' - \theta)}{\cos \theta}$$

$$\theta = 14^\circ 24'. \quad at' = 87^\circ 15'.$$

To find $p'q = (1\bar{1}0):(011)$, from the triangle $b=(0\bar{1}0):q=(011):p'=(1\bar{1}0)$, knowing $bq = 115^\circ 17'$, $bp' = 69^\circ 49'$, and $\beta = 72^\circ 34'$.

$$\tan \theta = \tan 69^\circ 49' \cdot \cos 72^\circ 34'; \cos p'q = \frac{\cos 69^\circ 49' \cdot \cos (115^\circ 17' - \theta)}{\cos \theta}$$

$$\theta = 39^\circ 11'. \quad p'q = 83^\circ 52'.$$

We now come to the more difficult part of this calculation. So far, we have been able to proceed straightforwardly, triangles with three known elements, or anharmonic ratios, lying clearly before us. It will be found that the next step may most conveniently be the calculation of the two sides qs and st in the triangle $q = (011) : s = (\bar{1}21) : t = (021)$, although we at present only know one element, the side $qt = 20^\circ 16'$. But the angle at q is equal to the angle at q in the last triangle considered, bqp' , and the angle at t is similarly the same as that at t in the triangle $b = (010) : t = (021) : a = (100)$. Now these angles at q and t can be readily calculated from the two triangles referred to, in which more than an adequate number of elements are known, so we proceed first to these two calculations.

To find the angle at $q = (011)$ in the triangle $b = (0\bar{1}0) : q = (011) : p' = (0\bar{1}1)$, from the three following elements summed in the left column, and to find the angle at $t = (021)$ in the triangle abt from the three known elements summed in the right column below.

$bq = 115^\circ 17'$	$ab = 79^\circ 6'$
$bp' = 69 49$	$bt = 44 27$
$qp' = 83 52$	$at = 70 4$
sum 268 58	sum 193 37
$\frac{1}{2}$ sum 134 29	$\frac{1}{2}$ sum 96 49
$\sin^2 \frac{q}{2} = \frac{\sin (134^\circ 29' - 115^\circ 17') \cdot \sin (134^\circ 29' - 83^\circ 52')}{\sin 115^\circ 17' \cdot \sin 83^\circ 52'}$	$\frac{\sin (96^\circ 49' - 44^\circ 27') \cdot \sin (96^\circ 49' - 70^\circ 4')}{\sin 44^\circ 27' \cdot \sin 70^\circ 4'}$
$\text{or } \cos 25^\circ 17'$	
$\frac{q}{2} = 32^\circ 7', \text{ and } q = 64^\circ 14'.$	
$\sin^2 \frac{t}{2} = \frac{\sin (96^\circ 49' - 44^\circ 27') \cdot \sin (96^\circ 49' - 70^\circ 4')}{\sin 44^\circ 27' \cdot \sin 70^\circ 4'}$	
$\frac{t}{2} = 47^\circ 23', \text{ and } t = 94^\circ 46'.$	

To now find $qs = (011) : (\bar{1}21)$ and $st = (\bar{1}21) : (021)$, knowing the side $qt = 20^\circ 16'$ and the two angles adjacent to it $q = 64^\circ 14'$ and $t = 94^\circ 46'$.

We here for the first time employ the formula (c) of page 96, Chapter VII.

$$\tan \frac{1}{2}(qs + st) = \frac{\cos \frac{1}{2}(94^\circ 46' - 64^\circ 14')}{\cos \frac{1}{2}(94^\circ 46' + 64^\circ 14')} \cdot \tan 10^\circ 8'$$

$$\frac{1}{2}(qs + st) = 43^\circ 25'; \frac{1}{2}(qs - st) = 2^\circ 44\frac{1}{2}'.$$

$$\tan \frac{1}{2}(qs - st) = \frac{\sin 15^\circ 16'}{\sin 79^\circ 30'} \cdot \tan 10^\circ 8'.$$

$$\frac{1}{2}(qs + st) = 43^\circ 25'; \frac{1}{2}(qs - st) = 2^\circ 44\frac{1}{2}'.$$

$$qs + st = 86^\circ 50'$$

$$qs - st = 5^\circ 29'.$$

By addition

$$2qs = 92^\circ 19', \text{ and } qs = 46^\circ 9'.$$

By subtraction

$$2st = 81^\circ 21', \text{ and } st = 40^\circ 40'.$$

Then further:

$$sp' = 180^\circ - (p'q + qs) = 180^\circ - (83^\circ 52' + 46^\circ 9') = 49^\circ 59'.$$

Also:

$$as = 180^\circ - (at + ts) = 180^\circ - (70^\circ 4' + 40^\circ 40') = 69^\circ 16'.$$

To find $bs = (010) : (\bar{1}21)$, from the triangle $sp'b$, knowing the two sides $sp' = 49^\circ 59'$, and $bp' = 69^\circ 49'$, and first finding the included angle at p' from the triangle app' in which all three sides are known as follows:

$$ap' = 31^\circ 5'$$

$$p'q = 83 \ 52$$

$$aq = 69 \ 43$$

$$\text{sum} = 184 \ 40 \quad \frac{1}{2} \text{ sum} = 92^\circ 20'.$$

$$\sin^2 \frac{p'}{2} = \frac{\sin(92^\circ 20' - 83^\circ 52') \cdot \sin(92^\circ 20' - 31^\circ 5')}{\sin 83^\circ 52' \cdot \sin 31^\circ 5'}$$

$$\frac{p'}{2} = 30^\circ 6', \text{ and } p' = 60^\circ 12'.$$

$$\text{Then:} \quad \tan \theta = \tan 49^\circ 59' \cdot \cos 60^\circ 12'; \quad \cos bs = \frac{\cos 49^\circ 59' \cdot \cos(69^\circ 49' - \theta)}{\cos \theta}$$

$$\theta = 30^\circ 37'.$$

$$bs = 54^\circ 37'.$$

To find bs' and as' should be our next task, from the triangle $a = (\bar{1}00) : b = (0\bar{1}0) : s' = (\bar{1}\bar{2}1)$, but at present we only know one element in it, the side $ab = 79^\circ 6'$. But we can find the adjacent angles at a and b , the former from the triangle abt' and the latter from the triangle bsp' , knowing the three sides in each case. We shall proceed, therefore, first to find these two angles, the three sides being summed as under in the two cases:

Triangle abt' .

$$ab = 100^\circ 54'$$

$$at' = 87 \ 15$$

$$bt' = 40 \ 35$$

$$\text{sum} = 228 \ 44$$

$$\frac{1}{2} \text{ sum} = 114 \ 22$$

Triangle bsp' .

$$bs = 54^\circ 37'$$

$$sp' = 49 \ 59$$

$$bp' = 69 \ 49$$

$$\text{sum} = 174 \ 25$$

$$\frac{1}{2} \text{ sum} = 87 \ 13$$

$$\sin^2 \frac{a}{2} = \frac{\sin(114^\circ 22' - 100^\circ 54') \cdot \sin(114^\circ 22' - 87^\circ 15')}{\sin 100^\circ 54' \cdot \sin 87^\circ 15'}$$

$$\frac{a}{2} = 19^\circ 12', \text{ and } a = 38^\circ 24'.$$

$$\sin^2 \frac{b}{2} = \frac{\sin(87^\circ 13' - 54^\circ 37') \cdot \sin(87^\circ 13' - 69^\circ 49')}{\sin 54^\circ 37' \cdot \sin 69^\circ 49'}$$

$$\frac{b}{2} = 27^\circ 19', \text{ and } b = 54^\circ 38'.$$

We are now in a position to proceed with the calculation of $bs' = (0\bar{1}0) : (\bar{1}\bar{2}1)$ and $as' = (\bar{1}00) : (\bar{1}\bar{2}1)$ from the triangle $a = (\bar{1}00) : b = (0\bar{1}0) : s' = (\bar{1}\bar{2}1)$, knowing the side $ab = 79^\circ 6'$ and the two adjacent angles $a = 38^\circ 24'$ and $b = 54^\circ 38'$, employing the formula (c) of p. 96, Chapter VII.

$$\tan \frac{1}{2} (as' + bs') = \frac{\cos \frac{1}{2} (54^\circ 38' - 38^\circ 24')}{\cos \frac{1}{2} (54^\circ 38' + 38^\circ 24')} \cdot \tan 39^\circ 33'$$

$$\tan \frac{1}{2} (as' - bs') = \frac{\sin 8^\circ 7'}{\sin 46^\circ 31'} \cdot \tan 39^\circ 33'.$$

$$\begin{aligned} \text{Hence,} \quad & \frac{1}{2}(as' + bs') = 49^\circ 55'. & \frac{1}{2}(as - bs') = 9^\circ 8'. \\ \text{and} \quad & as' + bs' = 99^\circ 50' \\ & as' - bs' = 18^\circ 16'. \\ \text{Adding,} \quad & 2as' = 118^\circ 6', \text{ and } as' = 59^\circ 3'. \\ \text{Subtracting,} \quad & 2bs' = 81^\circ 34', \text{ and } bs' = 40^\circ 47'. \\ \text{Then:} \quad & s't' = 180^\circ - (as' + at') = 180^\circ - (59^\circ 3' + 87^\circ 15') = 33^\circ 42'. \\ \text{Also:} \quad & s's = 180^\circ - (bs' + bs) = 180^\circ - (40^\circ 47' + 54^\circ 37') = 84^\circ 36'. \end{aligned}$$

We are now in a position to complete the calculation of the angles in the zone $[bso]$ by determining the position of the primary pyramid $o = (\bar{1}11)$, employing our knowledge of bs' and $s's$ in the anharmonic ratio of the four poles $b = (010)$, $s' = (\bar{1}21)$, $o = (\bar{1}11)$, and $s = (\bar{1}21)$. The conditions are shown in Fig. 258, and the ratio runs as under:

$$\begin{aligned} & \begin{array}{ccc} 010 & \bar{1}21 & \\ & \times & \times \\ & \bar{1}\bar{2}1 & \bar{1}11 \\ \sin bs' & \cdot \sin so & = \frac{\bar{1}\bar{2}1}{010} \cdot \frac{\bar{1}11}{\bar{1}21} = \frac{\bar{1}}{1} \cdot \frac{1}{4}, \\ & \times & \times \\ & \bar{1}11 & \bar{1}\bar{2}1 \end{array} \\ & \frac{\sin so}{\sin bo} = \frac{\sin 84^\circ 36'}{4 \sin 40^\circ 47'} = \tan \theta. \\ & \theta = 20^\circ 51\frac{1}{2}'. \end{aligned}$$

$$\begin{aligned} \text{Then:} \quad & \tan \frac{1}{2}(bo - so) = \tan \frac{1}{2}(bo + so) \cdot \tan 45^\circ - \theta. \\ & = \tan 62^\circ 41\frac{1}{2}' \cdot \tan 24^\circ 8\frac{1}{2}'. \end{aligned}$$

$$\begin{aligned} & \frac{1}{2}(bo - so) = 40^\circ 57\frac{1}{2}'. \\ \text{We then have} \quad & bo + so = 125^\circ 23', \\ \text{and} \quad & bo - so = 81^\circ 55'. \end{aligned}$$

$$\begin{aligned} \text{By addition,} \quad & 2bo = 207^\circ 18' \text{ and } bo = 103^\circ 39'; \\ \text{By subtraction} \quad & 2so = 43^\circ 28' \text{ and } so = 21^\circ 44'. \end{aligned}$$

$$\text{Then} \quad s'o = bo - bs' = 103^\circ 39' - 40^\circ 47' = 62^\circ 52'.$$

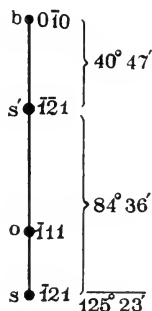


FIG. 258.

To find $bv = (010) : (\bar{1}31)$ and $vs = (\bar{1}31) : (\bar{1}21)$. The indices of the form v have been shown to be $\{\bar{1}31\}$, from its position at the intersection of the zones $[bso]$ and $[p't]$. We can find the position of either of the two parallel v -faces in the zone $[bso]$ from the anharmonic ratio, the conditions for which are set forth in Fig. 259. The ratio is as under:

$$\begin{aligned} & \begin{array}{ccc} 010 & \bar{1}11 & \\ & \times & \times \\ & \bar{1}31 & \bar{1}21 \\ \sin bv & \cdot \sin os & = \frac{\bar{1}31}{010} \cdot \frac{\bar{1}21}{\bar{1}11} = \frac{1}{1} \cdot \frac{\bar{1}}{2} = \frac{1}{2}, \\ & \times & \times \\ & \bar{1}21 & \bar{1}31 \end{array} \end{aligned}$$

$$\frac{\sin bv}{\sin ov} = \frac{\sin 54^\circ 37'}{2 \sin 21^\circ 44'} = \tan \theta. \quad \theta = 47^\circ 45'.$$

$$\begin{aligned} \tan \frac{1}{2}(ov - bv) &= \tan \frac{1}{2}(ov + bv) \cdot \tan (45^\circ - \theta) \\ &= \tan 38^\circ 10\frac{1}{2}' \cdot \tan (-2^\circ 45') \\ &= -(\tan 38^\circ 10\frac{1}{2}' \cdot \tan 2^\circ 45') \end{aligned}$$

$$\frac{1}{2}(ov - bv) = -2^\circ 10',$$

$$ov - bv = -4^\circ 20',$$

$$bv - ov = 4^\circ 20'.$$

$$bv + ov = 76^\circ 21'$$

and
or
We also have

$$2bv = 80^\circ 41', \text{ and } bv = 40^\circ 21'.$$

$$2ov = 72^\circ 1', \text{ and } ov = 36^\circ 0'.$$

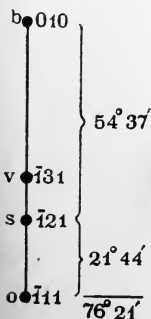


FIG. 259.

Adding
Subtracting,

Then also :

$$vs = ov - os = 36^\circ 0' - 21^\circ 44' = 14^\circ 16'.$$

This completes the important zone $[bso]$.

To find $ao = (\bar{1}00) : (\bar{1}11)$, knowing two sides and the included angle of the triangle $a = (\bar{1}00) : b = (010) : o = (\bar{1}11)$, namely, $ab = 100^\circ 54'$, $bo = 76^\circ 21'$, and the angle at $b = 54^\circ 38'$. The angle bo is the supplement of the angle bo just found.

$$\tan \theta = \tan 76^\circ 21' \cdot \cos 54^\circ 38' ; \cos ao = \frac{\cos 76^\circ 21' \cdot \cos (100^\circ 54' - \theta)}{\cos \theta}.$$

$$\theta = 67^\circ 14' \quad ao = 59^\circ 30'.$$

Then :

$$oq = 180^\circ - (ao + aq) = 180^\circ - (59^\circ 30' + 69^\circ 43') = 50^\circ 47'.$$

There is one more measured angle **to calculate**, namely, $po = (\bar{1}\bar{1}0) : (\bar{1}11)$, from the triangle $p = (\bar{1}\bar{1}0) : o = (\bar{1}11) : b = (0\bar{1}0)$, of which we know the two sides $pb = 53^\circ 2'$ and $bo = 103^\circ 39'$, and the included angle at $b = 54^\circ 38'$.

$$\tan \theta = \tan 53^\circ 2' \cdot \cos 54^\circ 38' ; \cos po = \frac{\cos 53^\circ 2' \cdot \cos (103^\circ 39' - \theta)}{\cos \theta}$$

$$\theta = 37^\circ 34' \quad po = 72^\circ 5'.$$

$$\text{Then also } ot = 180^\circ - (po + qt) = 180^\circ - (72^\circ 5' + 53^\circ 21') = 54^\circ 34'.$$

This finally completes the calculations for copper sulphate.

They have been longer than in the cases of the typical crystals of the systems of higher symmetry, and their relative difficulty has been about as great as will ever be met with, except perhaps in the cases of crystals exceptionally rich in faces belonging to forms of higher indices. Yet they have all proved perfectly straightforward, and the few formulæ given in Chapter VII. and the anharmonic ratio of Chapter VI. have sufficed to carry them out completely. This should prove a source of satisfaction to all who endeavour to follow the lines of progress in crystallography laid down in this book, for it demonstrates that if the few principles and the small amount of simple mathematics indicated in those two chapters have been mastered, all difficulties will have passed away. Even in the cases of the investigations of such exceptionally rich crystals as have just been alluded to, any additional formulæ will only be required for the calculation of one or two angles, and will be one or other of the formulæ of spherical trigonometry not included in Chapter VII., but which are given in every elementary text-book on that subject.

The mean angular values derived from the whole twelve crystals measured were so nearly identical with those derived from the typical crystal alone, that only the latter are included in the final table of angles and elements which will now be given. For instance, 12 independent irreproachable values of the angle $bt = (010) : (021)$ were obtained, all within the limits $44^\circ 24'$ to $44^\circ 32'$, and the mean was $44^\circ 28'$, identical with the mean value of the two measurements on the typical crystal. We can thus rely with full confidence on the observed values derived from the best individual crystal, which in this case is the example chosen to represent typical triclinic symmetry.

SUMMARY OF RESULTS FOR CRYSTAL OF COPPER SULPHATE $\text{CuSO}_4 \cdot 5\text{H}_2\text{O}$.

Crystal-system : Triclinic. Class : pinakoidal-holohedral.

Habit : prismatic.

Axial Angles : $\alpha = 82^\circ 16'$, $\beta = 107^\circ 26'$, $\gamma = 102^\circ 40'$.

Ratio of Axes : $a : b : c = 0.5715 : 1 : 0.5575$.

Forms observed : $a = \{100\}$, $b = \{010\}$, $c = \{001\}$, $p = \{110\}$, $p' = \{1\bar{1}0\}$, $p'' = \{1\bar{2}0\}$, $q = \{011\}$, $q' = \{0\bar{1}1\}$, $t = \{021\}$, $t' = \{0\bar{2}1\}$, $o = \{111\}$, $s = \{121\}$, $s' = \{1\bar{2}1\}$, $v = \{131\}$.

Each of these forms consists of a pair of parallel faces, or pinakoid, only, in accordance with holohedral triclinic symmetry.

The prism zone was always the most prominently developed, on all the crystals of the numerous crops examined by the author, and usually the two pairs of pinakoids $p = \{110\}$ and $p' = \{1\bar{1}0\}$ were represented by the largest faces in it, although the brachy-pinakoid faces $b = (010)$ and $b = (0\bar{1}0)$ were occasionally quite as broad. The two faces of the macro-pinakoid $a = \{100\}$ were generally narrower. The faces of the secondary prismatic pinakoid $p'' = \{1\bar{2}0\}$ were generally well-developed, although sometimes narrow. The prismatic pinakoid $\{130\}$ observed by Barker was never met with on the dozen crystals measured, of which the typical crystal here described was the most perfect. Of the end-forms the two parallel faces of the primary pyramidal pinakoid $o = \{111\}$ were usually predominating, but the crystals of certain crops showed instead a more prominent development of the two forms $t = \{021\}$ and $s = \{121\}$, or t and $s' = \{1\bar{2}1\}$. The basal pinakoid $c = \{001\}$ was very small, and in fact was only found with measurable faces on one crystal, the typical one chosen as example.

The elements found by Barker, as given by von Groth on page 349 of the fourth edition of his *Physikalische Krystallographie*, and also on page 419 of his *Chemische Krystallographie*, Part II. 1908, the details being unpublished, are : $\alpha = 82^\circ 5'$, $\beta = 107^\circ 8'$, $\gamma = 102^\circ 41'$; and $a : b : c = 0.5721 : 1 : 0.5554$. If we assume that Kupffer's values for α and γ are the supplements of the real axial angles, while his value for β is the real value, then his corrected real values would stand as follows : $\alpha = 82^\circ 21'$, $\beta = 106^\circ 49'$, $\gamma = 102^\circ 23'$. The values of all three observers are then not far removed from each other, considering the difficulty of obtaining absolutely perfect crystals, and the general agreement between the values now given by the author and those of Barker is fairly satisfactory, and shows that no error has been made in regard to the recognition of the various faces in the two descriptions.

The interfacial angles are given in the accompanying table, in which the mean observed values derived from the typical perfect crystal are set down alongside the calculated values. The agreement of the two columns affords the final proof of the trustworthiness of the results.

CRYSTAL ANGLES OF COPPER SULPHATE.

Angle.	No. of Measurements.	Limits.	Observed.	Calculated.	Difference.
$ab = 100 : 010$	2	79° 1' - 79° 7'	79° 4'	79° 6'	2'
$bp = 010 : 110$	2	53 0 - 53 4	53 2	53 2	0
$pa : 110 : 100$	2	26 1 - 26 3	26 2	26 4	2
$ap' : 100 : 110$	2	31 6 - 31 8	31 7	31 5	2
$pp' = 110 : 110$	2	57 9 - 57 9	57 9	*	...
$p'p'' = 110 : 120$	2	22 32 - 22 36	22 34	22 40	6
$p''b = 120 : 010$	2	47 10 - 47 20	47 15	47 9	6
$p'b = 110 : 010$	2	69 46 - 69 52	69 49	*	...
$bc = 010 : 001$	2	94 17 - 94 23	94 20	94 15	5
$bt = 010 : 021$	2	44 25 - 44 30	44 28	44 27	1
$tq = 021 : 011$	2	20 14 - 20 16	20 15	20 16	1
$bq = 010 : 011$	2	64 39 - 64 46	64 43	*	...
$qc = 011 : 001$	2	29 37 - 29 38	29 37	29 32	5
$cq' = 001 : 011$	2	27 30 - 27 31	27 31	27 35	4
$qq' = 011 : 011$	2	57 7 - 57 9	57 8	57 7	1
$q't' = 011 : 021$	2	17 25 - 17 33	17 29	17 35	6
$t'b = 021 : 010$	2	40 34 - 40 47	40 41	40 35	6
$q'b = 011 : 010$	2	58 7 - 58 12	58 10	*	...
$ac = 100 : 001$	73 47	...
$aq = 100 : 011$	2	69 43 - 69 43	69 43	*	...
$qo = 011 : 111$	2	50 40 - 50 42	50 41	50 47	6
$oa = 111 : 100$	2	59 35 - 59 37	59 36	59 30	6
$at = 100 : 021$	2	70 6 - 70 9	70 7	70 4	3
$ts = 021 : 121$	2	40 36 - 40 38	40 37	40 40	3
$sa = 121 : 100$	2	69 13 - 69 18	69 16	69 16	0
$at' = 100 : 021$	2	87 11 - 87 15	87 13	87 15	2
$t's' = 021 : 121$	2	33 41 - 33 46	33 43	33 42	1
$s'a = 121 : 100$	2	58 59 - 59 8	59 4	59 3	1
$bv = 010 : 131$	2	40 22 - 40 23	40 23	40 21	2
$vs = 131 : 121$	2	14 15 - 14 16	14 15	14 16	1
$bs = 010 : 121$	2	54 37 - 54 38	54 38	54 37	1
$so = 121 : 111$	2	21 40 - 21 47	21 43	21 44	1
$os' = 111 : 121$	2	62 44 - 62 49	62 47	62 52	5
$s'b = 121 : 010$	2	40 52 - 40 52	40 52	40 47	5
$pc = 110 : 001$	78 48	...
$pt = 110 : 021$	2	53 23 - 53 31	53 27	53 21	6
$to = 021 : 111$	2	54 29 - 54 35	54 32	54 34	2
$op = 111 : 110$	2	72 0 - 72 2	72 1	72 5	4
$p'q = 110 : 011$	1	...	83 58	83 52	6
$qs = 011 : 121$	1	...	46 5	46 9	4
$sp' = 121 : 110$	1	...	49 57	49 59	2

The angles marked with an asterisk are the five angles required by triclinic symmetry as the basis of the calculations.

CHAPTER XXI

HEXAGONAL SYSTEM.

Three equal crystallographic axes lying in the same (horizontal) plane and inclined at 60° to each other, and a fourth (principal or vertical) axis perpendicular to them and of different length.

Characterised by a hexagonal axis of symmetry, with or without also 6 digonal axes of symmetry perpendicular to it, an equatorial plane of symmetry which contains the latter if present, and 6 symmetry planes intersecting in the hexagonal axis which then becomes dihexagonal.

THE hexagonal system bears the unique distinction of having its crystals referred to four instead of three morphological axes. It is analogous to the tetragonal system in possessing a vertical axis of special character, to which the like term of "principal axis" is given. But instead of the two equal rectangular horizontal axes lying in the equatorial plane in the case of the tetragonal system, we have now in the hexagonal system three equal axes inclined at 60° . The natural result of there being four axes is that there are four index numbers in the symbol of each form, namely $\{hkl\}$, l referring as usual to the vertical axis.

The arrangement of the three horizontal axes is shown in Fig. 260. They are conveniently lettered A_1 , A_2 , and A_3 . The axis A_1 is the one which emerges in front somewhat to the left, and its back half is of negative sign, the front half being positive; the axis A_2 runs right-and-left parallel to the edge of the paper, and its right half is the positive one; the A_3 axis runs between the A_1 and A_2 axes, at 60° to both, and its positive part is the back half, emerging towards the back-left corner of the page. The vertical axis is represented in plan by the central point. This particular arrangement of axes and of the signs of their two halves is chosen because it bears a definite and interesting relationship to the three axes of the

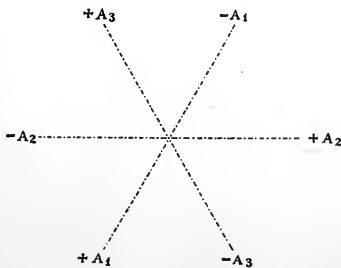


FIG. 260.

trigonal system, which has formerly been regarded as merely a hemihedral form of the hexagonal system. The relationship will be discussed in Chapter XXIII. in connection with the trigonal system.

Great confusion has been introduced into the description of hexagonal forms owing to the fact that no less than five different kinds of axes have from time to time been proposed and employed. The purely Millerian method referred to three axes is admirably suited to the description of trigonal forms, and in this book will be employed with and reserved for such forms, while the four-axial method above defined, suggested by Bravais and commonly known as the Bravais-Miller method, will be used for the fully hexagonal forms.

Although two out of the three equatorial axes inclined at 60° , together with the vertical axis, are alone adequate for the purpose of defining the position of a face, still if we refer the faces to these three axes only we are confronted with the grave disadvantage that for the three primary faces, and in general for the different faces of any one and the same form, we obtain symbols of a totally different character, incapable of being represented by a single form symbol. Thus the three primary faces would have the indices (111) , $(\bar{1}01)$, and $(0\bar{1}1)$. It was only to be expected that this should be so, for the two axes would have to be quite arbitrarily chosen from amongst the three, as the latter have all the same value and are all equally entitled to be chosen. Hence, it is only by considering all three horizontal equatorial axes, together with the vertical axis, making four altogether, as the crystallographic axes, that we can obtain a form symbol capable of representing the whole of the faces of that form, the same index numbers applying to the cases of all the faces in the form. As it is highly important that faces having an equal value with reference to the symmetry of the crystal should be so represented by a single form symbol, the employment of all four Bravais-Miller axes is essential.

It is worthy of note that the sum of the three Bravais-Miller indices h , i , k , referring to the three horizontal axes, is always equal to zero; that is, $h + i + k = 0$. For instance, the hexagonal prism of the kind shortly to be described as one of the second order has the symbol $\{11\bar{2}0\}$, which obviously follows the rule, the sum of two positive units and *minus* two being zero. This fact is an important aid in checking the indices given to any face. Further, if two of the three indices can be found by the usual rules regarding zones, or from any other obvious considerations, the third is also given by the rule just enunciated; for it must be that number which, with the two numbers first found, makes the sum of all three $= 0$. A further consequence is that the signs of all these three index numbers cannot be the same.

There are five classes in the hexagonal system, four corresponding to the possible modes of combination of the symmetry elements specified at the head of the chapter, and the remaining one to the presence of the essential hexagonal axis alone. These five modes are illustrated by the five stereographic projections of the symmetry elements and of the general form $\{hikl\}$ given in Figs. 261 to 265. In the first class repre-

sented in Fig. 261 (class 23) the hexagonal axis of symmetry operates alone, and this is the only one of the five classes with only a single element of symmetry. The general form here consists of a simple

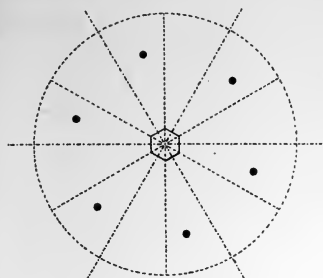


FIG. 261.—Class 23.

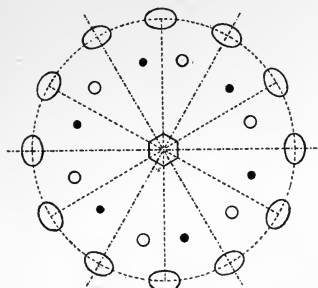


FIG. 262.—Class 24.

hexagonal pyramid. In the second class shown in Fig. 262 (class 24) the six digonal axes also operate, causing a repetition of the facial poles of the first class in the lower hemisphere, but not immediately under

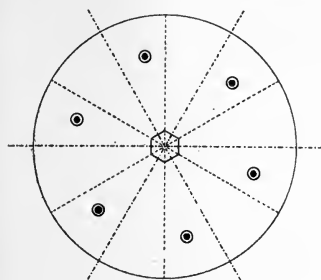


FIG. 263.—Class 25.

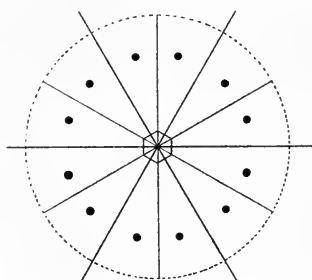


FIG. 264.—Class 26.

those of that class, being brought to the other side of the digonal axis by the rotation for 180° , producing a bipyramidal form of trapezohedral character. The third class is distinguished by the joint operation of the hexagonal axis and the equatorial plane of symmetry, as indicated in Fig. 263 (class 25). The general form produced thereby is the hexagonal bipyramid. In the fourth class the hexagonal axis acts dihexagonally, that is, simultaneously as a hexagonal axis and in combination with six planes of symmetry intersecting in it, producing a dihexagonal pyramid, as shown in Fig. 264 (class 26). Lastly, the fifth class (class 27) combines all the symmetry elements of the system, including of course the equatorial plane, as represented in Fig. 265;

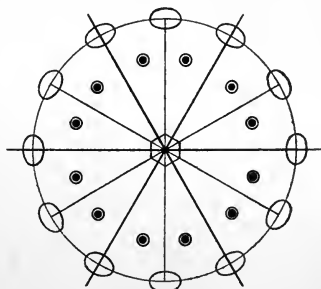


FIG. 265.—Class 27.

and the general form resulting has consequently poles in both hemispheres, immediately over each other, in all the positions in which poles have been shown in the other classes. This general form is the dihexagonal bipyramid.

We may now pass to the detailed consideration of these five classes.

Class 27.—Dihexagonal-Bipyramidal Class. Beryl Class.

Hexagonal-Holohedral Class. Type, Dihexagonal Equatorial.

This class, just shown in stereographic projection in Fig. 265, embodies all the elements of symmetry enumerated at the beginning of the chapter, and includes all crystals which exhibit the full symmetry of the hexagonal system.

The general form, as just explained, is a 24-faced **dihexagonal bipyramid**. Its symbol is $\{hikl\}$, and the indices of its various faces are obtained by making all the possible interchanges of the index numbers h , i , and k , both positive and negative, and by making l successively positive and negative also. This solid is shown in Fig. 266, the particular representative being the dihexagonal bipyramid $\{2\bar{1}31\}$ of beryl, drawn to scale to the axial dimensions of the mineral given at the end of the description of this class. It is characterised by three different kinds of edges, namely, the equatorial edges forming the dihexagonal base, and two kinds, occurring alternately, of pyramidal (polar) edges. Of the two latter kinds, the angle over those lying in the planes which contain the vertical axis and the three respective horizontal crystallographical axes, is either acuter or more obtuse than that over those lying in the intermediate planes containing the vertical axis and the digonal interaxes, the magnitude of the angle depending on the position of the pole, in the triangle formed by two of the axes (a crystallographic one and an interaxis) radiating from the centre and the segment of the primitive circle which they cut off. When the position is as given

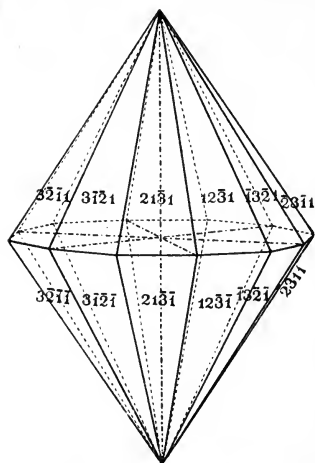


FIG. 266.—Dihexagonal Bipyramid.

in the stereographic projection in Fig. 265, the angle over the axial-plane polar edges is the more acute, but if the pole were placed nearer to the crystallographic than to the interaxis, as in the case of the dihexagonal pyramid $\{2\bar{1}31\}$ of beryl shown in Fig. 266, the case would be reversed. The special case in which the pole would be exactly midway between the crystallographic axis and the interaxis is an impossible one, as it would render the hexagonal axis then an axis of twelve-fold symmetry, which is not possible; the indices hik , moreover, would be irrational. The only cases, therefore, corresponding to rational indices are when the pole is nearer to one or other of the axes.

When the general pole moves towards the 30° -angle of the triangle, that is, towards the centre, the dihexagonal bipyramid produced becomes flatter. Fig. 267 shows a combination of such a flatter bipyramid $\{2132\}$ with the first one $\{2\bar{1}31\}$ represented in Fig. 266, the h , i , and k indices being the same but the l index number different. The limiting case of the shortening of the vertical intercept (and corresponding increase of l) is the **basal plane or pinakoid** $\{0001\}$. This primary form consists of

two parallel faces, which are also shown in Fig. 267 forming the blunted ends of the bipyramid.

When the pole migrates from within the triangle on to the intersection of the primitive circle and the interaxis, we have each pair of general pyramid faces and the similar pair in the other hemisphere coalescing into a single face, parallel to the

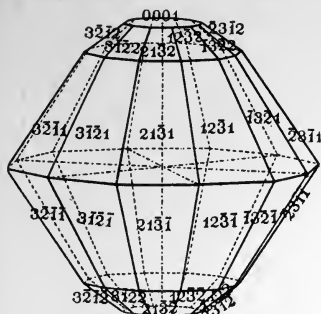


FIG. 267.—Combination of two Dihexagonal Bipyramids and Basal Pinakoid.

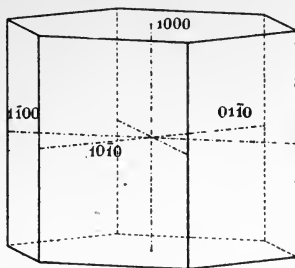


FIG. 268.—First Order Hexagonal Prism.

vertical axis and to one of the horizontal-crystallographic axes, thus producing a hexagonal prism. Its symbol will be $\{10\bar{1}0\}$, and it is termed the **hexagonal prism of the first order**. It is shown in Fig. 268, closed by the two faces of the basal pinakoid.

When the pole still lies on the interaxis but not at its extremity, the corresponding **hexagonal bipyramid of the first order** is produced, $\{h0\bar{h}l\}$, the primary representative of which $\{10\bar{1}1\}$ is shown in Fig. 269; the h and k indices being equal, a second h is written in the general symbol instead of k , to indicate the fact.

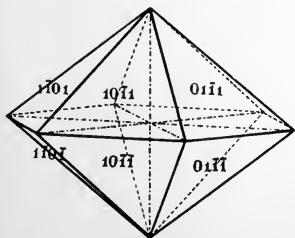


FIG. 269.—Primary First Order Hexagonal Bipyramid.

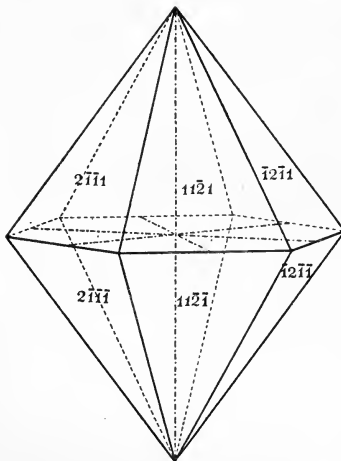


FIG. 270.—Primary Second Order Hexagonal Bipyramid.

Similarly, when the pole migrates on to a crystallographic axis, another hexagonal bipyramid is produced, and in the limiting case of the pole being situated at the intersection of that axis with the primitive circle the two faces of the bipyramid in the two hemispheres become fused into a single face of prismatic character, thus producing a second hexagonal prism. These two forms are respectively called the **hexagonal bipyramid of the second order**, and the **hexagonal prism of the second order**. They are shown in Figs. 270 and 271. The symbol of the pyramid is $\{h.h.\bar{2}h.l\}$ (that shown in Fig. 270 being $\{11\bar{2}1\}$ of apatite) and of the prism $\{11\bar{2}0\}$.

It will be obvious from the projection that the two first horizontal intercepts corresponding to the indices h and i are equal, and twice as long as the third k ; hence the indices corresponding are written as $h \cdot h \cdot 2h$.

There remains only one further holohedral hexagonal form to describe, namely, the **dihexagonal prism**, $\{h\bar{i}k0\}$, which is produced when the general dihexagonal pyramidal

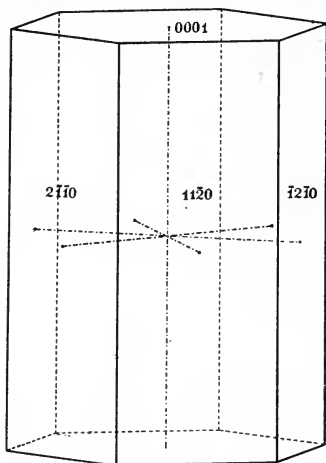


FIG. 271.—Second Order Hexagonal Prism.

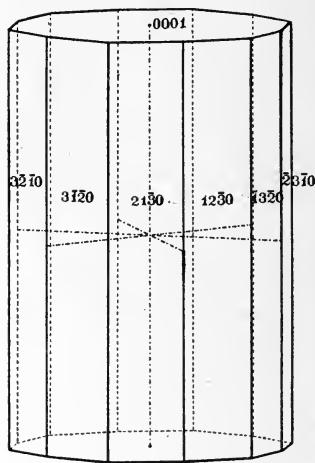


FIG. 272.—Dihexagonal Prism.

pole migrates on to the primitive circle, while still remaining in an intermediate position between the horizontal crystallographic axes and interaxes. Its section is the same as that of the dihexagonal bipyramid having the same h , i , and k indices. Its $\{21\bar{3}0\}$ representative is shown in Fig. 272, closed by the basal pinakoids.

These seven holohedral forms of the hexagonal system are tabulated in the following list :

List of Forms in Class 27.

- $\{0001\}$ Basal plane or pinakoid. 2 faces.
- $\{10\bar{1}0\}$ Hexagonal prism of the first order. 6 faces.
- $\{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.
- $\{h\bar{i}k0\}$ Dihexagonal prism. 12 faces.
- $\{h0\bar{h}l\}$ Hexagonal bipyramid of the first order, including the primary one $\{10\bar{1}1\}$. 12 faces.
- $\{h \cdot h \cdot 2\bar{h} \cdot l\}$ Hexagonal bipyramid of the second order, including the primary one $\{11\bar{2}1\}$. 12 faces.
- $\{h\bar{i}k\bar{l}\}$ Dihexagonal bipyramid. 24 faces.

The poles of all these forms, including several representatives of each of the last three, are shown in the stereographic projection given in Fig. 273. The axial ratio used in drawing this projection was that of calcite, $a : c = 1 : 0.8545$. Calcite is really of trigonal symmetry, belonging to the class formerly known as the rhombohedral hemihedral division of the hexagonal system. In Chapter XXIII, describing trigonal crystals, a second stereographic projection of calcite will also be given however, drawn according to the Millerian system of trigonal axes, and it will prove very useful to have these two projections of the same substance on the Bravais-Miller and the Millerian axial systems, in order

that the corresponding two sets of indices for any one particular form may at once be compared or read off.

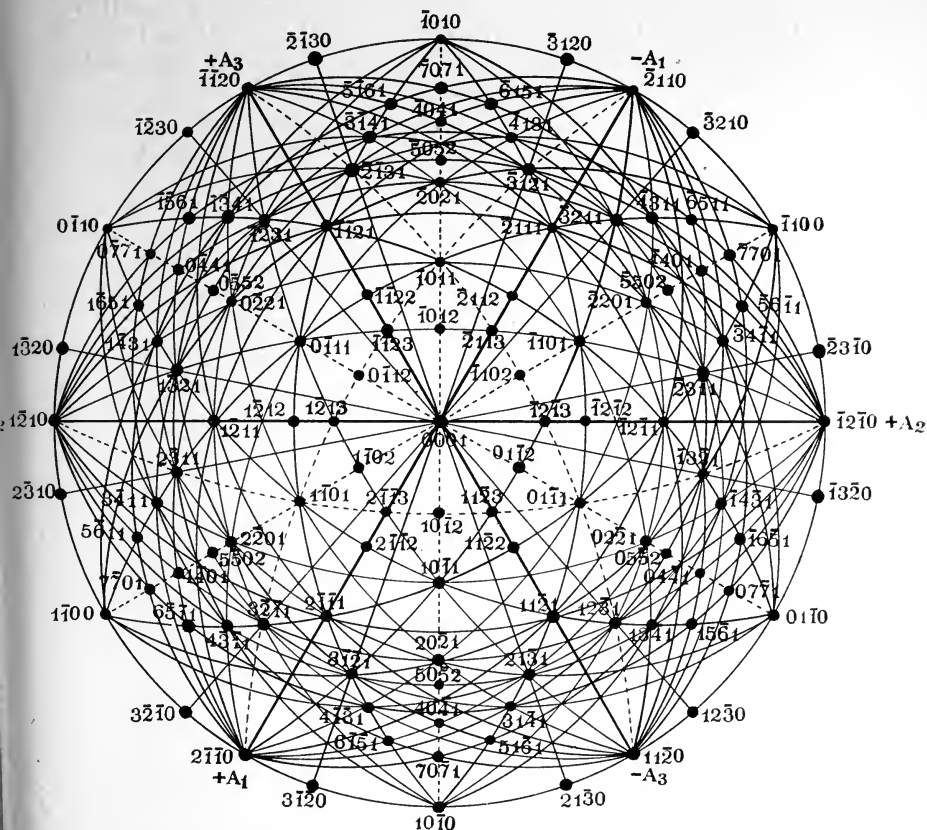


FIG. 273.—Stereographic Projection of Hexagonal Forms.

As regards the crystal elements, the axial angles are all known to be 60° or 90° by the conditions of the symmetry, and the three horizontal axes are of equal length. Hence the only element requiring to be found, and to be stated in connection with the description of any hexagonal crystal, is the axial ratio $a:c$, that of any one of the three equal horizontal axes a to the vertical axis c . This is determined by the situation of any of the pyramidal forms, and an actual example of the mode of calculation, on the lines of the general instructions already given in Chapter VII., will be worked out in detail in the next chapter.

One of the best-known instances of a substance

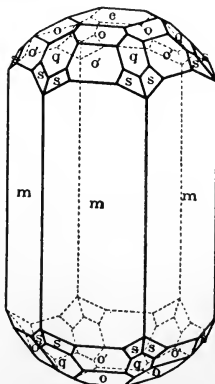


FIG. 274.—Crystal of Beryl.

crystallising in the holohedral class of the hexagonal system is beryl, $\text{Be}_3\text{Al}_2(\text{SiO}_3)_6$, a typical doubly terminated crystal of which mineral is represented in Fig. 274. The forms present are $m = \{10\bar{1}0\}$, $o = \{10\bar{1}1\}$, $o' = \{20\bar{2}1\}$, $q = \{11\bar{2}1\}$, $s = \{21\bar{3}1\}$, and $c = \{0001\}$. All these forms are marked on the stereographic projection Fig. 273. The axial ratio is $a : c = 1 : 0.4989$ (von Kokscharow).

Class 26.—Dihexagonal Pyramidal Class. Hexagonal-Hemimorphic Class. Type, Dihexagonal Polar.

This class is distinguished by the intersection of six planes of symmetry mutually inclined at 30° , in the hexagonal (here consequently dihexagonal) symmetry axis, the vertical or principal morphological axis, as stated at the opening of the chapter, and illustrated in Fig. 275.

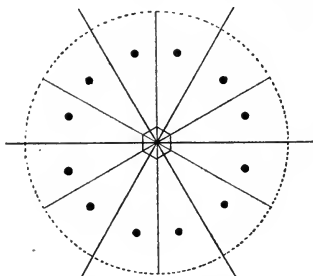


FIG. 275.—Symmetry Elements and General Form of Class 26.

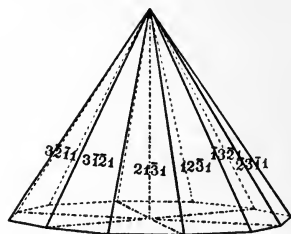


FIG. 276.—Dihexagonal Pyramid.

The operation of these symmetry elements causes the repetition of the general pole (hkl), by reflection over the symmetry planes, eleven more times, all in the same hemisphere, thus producing the twelve-faced **dihexagonal pyramid**, shown in Fig. 276. It resembles one polar half of the dihexagonal bipyramid described in the holohedral class 27 and shown in Fig. 266, and may be closed by the basal plane.

The remarks as to the two kinds of edges and their dihedral angles, and as to the impossibility, by reason of irrationality of the indices, of a form with twelve equal dihedral angles over the polar edges and a regular twelve-sided polygon as base or section of the pyramid, apply here equally as to class 27. There are two such dihexagonal pyramids, an upper one (pointing upwards) with a positive value of the index l , and a lower one (pointing downwards) with a negative value of l .

The two basal planes are separate pedial forms in this class, the upper or positive pedion, $\{0001\}$, and the lower or negative pedion, $\{00\bar{0}1\}$.

Similarly, when the general pole migrates on to one of the axial radii of the projection, two **hexagonal pyramids** are produced, an **upper or positive** one and a **lower or negative** one, and of the first or second order according as the radius on which the pole falls is an interaxis or a crystallographic axis. The sign is determined by that of the index l relating to the vertical axis.

The dihexagonal prism and the hexagonal prisms of the first and second orders are identical with those of class 27, for they are the limiting cases of both the positive and negative pyramidal forms, when the poles have migrated on to the primitive circle.

There are to be distinguished in this class, therefore, the following forms :

List of Forms in Class 26.

- $\{0001\}$ Upper or positive basal plane or pedion; $\{000\bar{1}\}$ lower or negative basal plane or pedion. 1 face each.
- $\{10\bar{1}0\}$ Hexagonal prism of the first order. 6 faces.
- $\{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.
- $\{h\bar{h}k0\}$ Dihexagonal prism. 12 faces.
- $\{h0\bar{h}l\}$ Upper or positive hexagonal pyramid of the first order; $\{h0\bar{h}\bar{l}\}$ lower or negative pyramid of the first order. Including the primary ones $\{10\bar{1}1\}$ and $\{10\bar{1}\bar{1}\}$. Each 6 faces.
- $\{h.h.\bar{2}h.l\}$ Upper or positive hexagonal pyramid of the second order; $\{h.h.\bar{2}h.\bar{l}\}$ lower or negative hexagonal pyramid of the second order. Including the primary ones $\{11\bar{2}1\}$ and $\{11\bar{2}\bar{1}\}$. Each 6 faces.
- $\{h\bar{h}kl\}$ Upper or positive dihexagonal pyramid; $\{h\bar{h}k\bar{l}\}$ lower or negative dihexagonal pyramid. Each 12 faces.

An instance of a substance crystallising according to this class of symmetry is the naturally occurring sulphide of cadmium, greenockite, CdS. Fig. 277 shows a typical crystal of greenockite consisting of the hexagonal prism $\{10\bar{1}0\}$, the two basal planes $\{0001\}$ and $\{000\bar{1}\}$ developed to very different extents, and the four pyramids (two only of which are complementary forms) $\{20\bar{2}1\}$, $\{10\bar{1}1\}$ and $\{10\bar{1}\bar{1}\}$, and $\{10\bar{1}2\}$. The axial ratio is $a:c=1:0.8252$. The poles of all these forms are indicated on the stereographic projection in Fig. 273, and although for a crystal of the axial dimensions of calcite, the relative positions of the forms are similar.

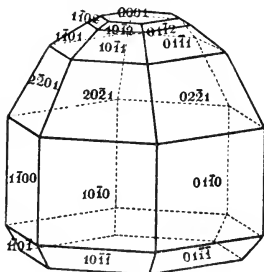


FIG. 277.—Crystal of Greenockite.

Class 25.—Hexagonal Bipyramidal Class. Apatite Class. Pyramidal-Hemihedral Class. Type, Hexagonal Equatorial.

In this class the operating symmetry elements are the essential hexagonal axis and the equatorial plane of symmetry.

From the stereographic projection, Fig. 278, it will be clearly apparent that the general pole $\{h\bar{h}kl\}$ is repeated five times owing to the operation of the hexagonal axis, making a hexagonal pyramid, and each of these six poles in the upper hemisphere is repeated vertically underneath in the lower hemisphere, by virtue of the operation of the equatorial plane of symmetry, converting the pyramid into a bipyramid. To distinguish this hexagonal bipyramid from the holohedral ones of the first and second orders, it is termed a **hexagonal bipyramid of the third order**, as in the case of the analogous tetragonal bipyramid of the third order belonging to class 12. This third order hexagonal bipyramid is shown in

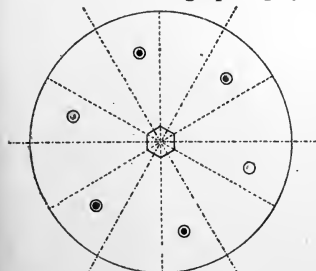


FIG. 278.—Symmetry Elements and General Form of Class 25.

Fig. 279. The actual representative shown is the form $\{21\bar{3}1\}$ constructed to scale

to the axial ratio of apatite $a:c=1:0.7345$, a mineral which shows the form characteristically. There will be a second similar form possible, having the same index numbers, corresponding to the other six poles shown in the holohedral Fig. 265. The first form $\{h\bar{k}l\}$ (Fig. 279) is distinguished as the **right hexagonal bipyramid of the third order**, and the second form, indistinguishable as a separate solid from the first, has the indices $\{k\bar{h}l\}$ and is termed the **left hexagonal bipyramid of the third order**.

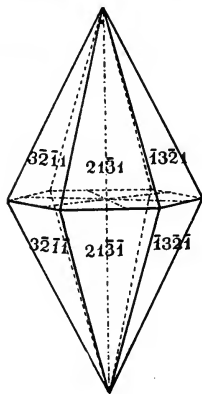


FIG. 279.
Right Third Order Hexagonal Bipyramid.

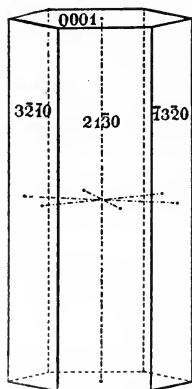


FIG. 280.
Right Third Order Hexagonal Prism.

corresponding to the right bipyramid $\{21\bar{3}1\}$ of Fig. 279, is shown in Fig. 280, drawn to scale in equipoise and terminated by the basal pinakoid $\{0001\}$. This latter form, the basal pinakoid, remains as in the holohedral class 27.

There are, consequently, to be distinguished the following forms in this class :

List of Forms in Class 25.

- $\{0001\}$ Basal pinakoid. 2 faces.
- $\{10\bar{1}0\}$ Hexagonal prism of the first order. 6 faces.
- $\{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.
- $\{h\bar{k}l0\}$ Right hexagonal prism of the third order ; $\{k\bar{h}l0\}$ left hexagonal prism of the third order. Each 6 faces.
- $\{h0\bar{h}l\}$ Hexagonal bipyramid of the first order, including the primary one, $\{10\bar{1}1\}$, 12 faces.
- $\{h.h.\bar{2}h.l\}$ Hexagonal bipyramid of the second order, including the primary one $\{11\bar{2}1\}$. 12 faces.
- $\{h\bar{k}l\}$ Right hexagonal bipyramid of the third order ; $\{k\bar{h}l\}$ left hexagonal bipyramid of the third order. Each 12 faces.

An excellent example of a substance crystallising in class 25 is the natural mineral apatite, $\text{Ca}_5\text{F}(\text{PO}_4)_3$ or $\text{Ca}_5\text{Cl}(\text{PO}_4)_3$. A typical crystal of apatite will be worked through goniometrically in the next chapter, its axial ratio calculated, and the mode of carrying out hexagonal calculations in general thereby exemplified.

Class 24.—Hexagonal Trapezohedral Class. Trapezohedral-Hemihedral Class. Type, Hexagonal Holoaxial.

The symmetry elements present in this class, as the type implies, are the full number of axes of symmetry belonging to the system,

namely, the hexagonal axis and the 6 digonal axes perpendicular thereto lying in the equatorial plane. But this latter plane is no longer a plane of symmetry, nor are any planes of symmetry whatever present.

The mode of repetition of the general pole $\{hikl\}$, by passing from one hemisphere to the other alternately, will be clear from Fig. 281. The hexagonal axis causes the repetition of the pole five more times at 60° -intervals round the circle in the same hemisphere, and the digonal character of each of the 6 axes of symmetry in the equatorial plane causes each pole to be again repeated in the other hemisphere, but not vertically under the pole in the upper hemisphere, but instead, at the same distance on the other side of the axis, as projected. It is as if alternate poles in the holohedral projection in Fig. 265 had been suppressed, the dot being removed from one double pole and the ring from the adjoining one. The effect is to produce a hexagonal bipyramid of the third order, but of which the two halves (single pyramids), upper and lower, belong respectively to the right and the left forms described in Class 25. That is, the two pyramids are not congruent, one being rotated round the vertical axis somewhat with respect to the other, so that each of the edges of the usual hexagonal basal plane section is replaced by a pair of unequal shorter edges, so arranged that the twelve form a zig-zag around the wide middle belt of the bipyramid. Each face of the solid is thus bounded by four edges, two equal polar and two unequal equatorial, rendering it a trapezohedron.

It will be obvious that there must be two enantiomorphous **hexagonal trapezohedra** corresponding to the same index numbers, a **right** one $\{hikl\}$, so called because the

faces of the right upper hexagonal pyramid of the third order form the upper half, and which is represented in Fig. 282 and by its poles in Fig. 281; and a **left** one $\{k\bar{i}hl\}$, shown in Fig. 283, and the upper faces of which are formed by the left upper hexagonal pyramid of the third order. The pair shown in Figs. 282 and 283 are the right and left trapezohedra $\{2\bar{1}\bar{3}1\}$ and $\{3\bar{1}\bar{2}1\}$, the axial ratio employed in the drawing being that of apatite. These two forms distinguish the class, all the remaining forms being identical with those of the holohedral class 27. For all these other forms correspond to the six special cases, in which the pole is no longer a general one within the spherical triangle formed by two radii and a twelfth-part segment

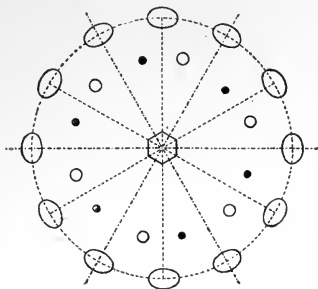


FIG. 281.—Symmetry Elements and General Form of Class 24.

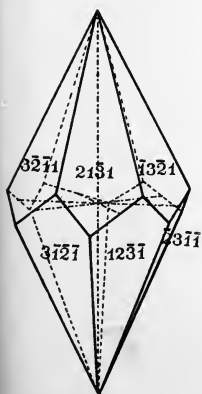


FIG. 282.

Right and Left Hexagonal Trapezohedra.

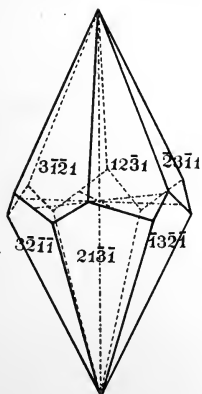


FIG. 283.

of the primitive circle, but lies on one of the six elements of the triangle, that is either on one of the arc-sides or at one of the corner-angles formed by their intersection. In all these cases the effect of the axial elements of symmetry alone is to produce the holohedral forms. When the pole is at the central corner the basal pinakoid is produced, when at the end of an interaxis the first order hexagonal prism is formed, and when at the end of a crystallographic axis the second order prism is developed.

Its position on the primitive circle segmental side-arc of the triangle gives rise to the dihexagonal prism, when located on an interaxial arc (radius in the projection) the first order hexagonal bipyramid is arrived at, and when on a crystallographic axial arc the second order bipyramid is the form brought about.

Hence there are the following forms comprised in this class :

List of Forms in Class 24.

- {0001} Basal pinakoid. 2 faces.
- {10 $\bar{1}$ 0} Hexagonal prism of the first order. 6 faces.
- {11 $\bar{2}$ 0} Hexagonal prism of the second order. 6 faces.
- { $h\bar{k}l$ 0} Dihexagonal prism. 12 faces.
- { $h0\bar{h}l$ } Hexagonal bipyramid of the first order, including the primary one {10 $\bar{1}$ 1}. 12 faces.
- { $h.h.\bar{2}h.l$ } Hexagonal bipyramid of the second order, including the primary one {11 $\bar{2}$ 1}. 12 faces.
- { $h\bar{k}l$ } Right hexagonal trapezohedron; { $h\bar{k}l$ } left hexagonal trapezohedron. Each 12 faces.

Examples of this class are uncommon. The best-known is that of the complex double salt barium antimonyl tartrate with potassium nitrate, $\text{Ba}(\text{SbO})_2(\text{C}_4\text{H}_4\text{O}_6)_2 \cdot \text{KNO}_3$. The forms exhibited by this salt, however, do not display the trapezohedral character of the symmetry, being the holohedral forms {10 $\bar{1}$ 0}, {0001}, {10 $\bar{1}$ 1}, and {20 $\bar{2}$ 1}. But the etched figures afforded by small quantities of water exhibit very clearly the presence of digonal axes and the absence of a symmetry plane.

Class 23.—Hexagonal Pyramidal Class. Hexagonal-Hemimorphic-Hemihedral Class. Type, Hexagonal Polar.

This class exhibits the minimum of hexagonal symmetry, namely, nothing but the essential hexagonal axis of symmetry. This, the vertical and crystallographically principal axis, is consequently one of hemimorphism, for the forms developed at its two ends may be different, there being neither equatorial plane of symmetry nor digonal axes of symmetry lying in it.

The pole of any face whatsoever is simply repeated at every 60° symmetrically round the hexagonal vertical axis, so that in general,—the starting pole not lying

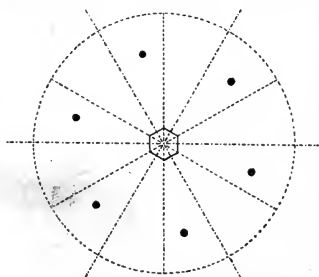


FIG. 284.—Symmetry Elements and General Form of Class 23.

on any of the six elements of the spherical triangle (formed in the projection by an interaxial radius, an adjacent crystallographic radius, and the segment cut off by them on the primitive circle) but within the triangle,—the form produced is a simple hexagonal pyramid of the third order, upper or lower according to the hemisphere. Its projection is shown in Fig. 284, and its appearance, as closed by the basal plane, is represented

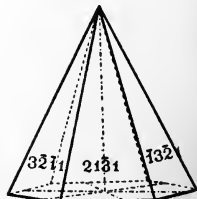


FIG. 285.—Right Upper Third-Order Hexagonal Pyramid.

in Fig. 285, the actual one drawn being the right upper hexagonal pyramid of the third order {21 $\bar{3}$ 1}, corresponding to the upper half of Fig. 279. For, besides the two forms, upper and lower, corresponding to the two ends of the hexagonal axis, there will be another pair of forms enantiomorphous with them, the two pairs being right and

left respectively and having the same index numbers; the poles of the second form will lie on the other side of the interaxis in each case, and at the same distance from it as the poles of the first form (shown in the projection and drawing). The latter is a right form, and the enantiomorphous second form is the left one. Hence, four general forms are altogether to be distinguished in this class, as together making up the 24-faced holohedral dihexagonal bipyramid of class 27.

Of the six special cases, the only ones which remain as in the holohedral class are the first and second order hexagonal prisms, produced when the poles fall at the ends of the two kinds of axes. The two pyramids, first and second order, are only simple pyramids, either upper or lower, and not bipyramids, so that each of the holohedral bipyramids is now divided into an upper and a lower form, differing by the sign of the l index. Similarly, the holohedral basal pinakoid is now divided into two basal planes or pedions, an upper or positive $\{0001\}$ and a lower or negative $\{000\bar{1}\}$. When the general pole falls on the primitive-circle-segment side of the triangle two varieties of hexagonal prism of the third order are produced, just as in class 25, a right and a left modification, according as the pole is to the right or the same distance to the left of the interaxis.

To summarise, we have to distinguish the following forms in the class :

List of Forms in Class 23.

- $\{0001\}$ Upper or positive basal plane or pedion; $\{000\bar{1}\}$ lower or negative basal plane or pedion. Each 1 face.
- $\{10\bar{1}0\}$ Hexagonal prism of the first order. 6 faces.
- $\{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.
- $\{h\bar{k}l0\}$ Right hexagonal prism of the third order; $\{k\bar{h}l0\}$ left hexagonal prism of the third order. Each 6 faces.
- $\{h0\bar{h}l\}$ Upper or positive hexagonal pyramid of the first order; $\{h0\bar{h}\bar{l}\}$ lower or negative hexagonal pyramid of the first order. Including the primary pyramids $\{10\bar{1}1\}$ and $\{10\bar{1}\bar{1}\}$. Each 6 faces.
- $\{h \cdot h \cdot 2\bar{h} \cdot l\}$ Upper or positive hexagonal pyramid of the second order; $\{h \cdot h \cdot 2\bar{h} \cdot \bar{l}\}$ lower or negative hexagonal pyramid of the second order. Including the primary pyramids $\{11\bar{2}1\}$ and $\{11\bar{2}\bar{1}\}$. Each 6 faces.
- $\{h\bar{k}l\}$ Right upper or positive hexagonal pyramid of the third order; $\{k\bar{h}l\}$ left upper or positive form of same; $\{h\bar{k}\bar{l}\}$ right lower or negative hexagonal pyramid of the third order; $\{k\bar{h}\bar{l}\}$ left lower or negative form of same. Each 6 faces.

An excellent example of this class is afforded by the salt strontium antimonyl tartrate, $\text{Sr}(\text{SbO})_2(\text{C}_4\text{H}_4\text{O}_6)_2$, investigated by Traube, a crystal of which is shown in Fig. 286. The hemimorphic pyramidal character is at once displayed in the different forms developed at the two ends of the hexagonal prism $\{10\bar{1}0\}$ of the first order, the predominating form. At the upper end is the primary positive hexagonal pyramid of the first order $\{10\bar{1}1\}$, while the lower termination is formed by the entirely different first order pyramid $\{20\bar{2}1\}$. The etched figures on the prism faces confirm the fact that the crystal belongs to this class, as does also the circumstance that the two poles exhibit strongly opposite pyroelectric properties. The upper pole is the antilogous pole and the lower one the analogous.

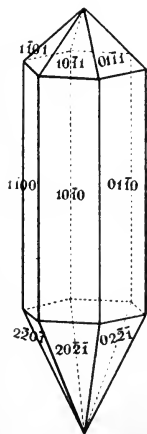


FIG. 286.
Crystal of Strontium Antimonyl Tartrate.

CHAPTER XXII

PRACTICAL EXAMPLE OF A HEXAGONAL CRYSTAL.

Class 25.—Apatite, $\text{Ca}_5\text{F}(\text{PO}_4)_3$ or $\text{Ca}_5\text{Cl}(\text{PO}_4)_3$.

THE example chosen as illustrative of the mode of dealing with crystals of the hexagonal system is a beautiful little crystal of apatite of prismatic habit, perfectly transparent and colourless, and very rich in faces. It is about three millimetres long and just over a millimetre in the diameter of the prism. One end only is fully developed, the crystal having been attached to a cavity wall by the other end. But on the free end twenty-four faces are developed, making thirty-six faces altogether, including the prism faces.

Investigation showed that the basal pinakoid was prominently represented by the broad flat-end plane (0001), marked *c* on the reproduced drawing given in Fig. 287; the crystal was so evenly developed that this basal plane was almost a regular hexagon. Its six edges were parallel to the three Bravais-Miller horizontal axes, and to the faces of the hexagonal prism of the first order, $m = \{10\bar{1}0\}$, and to those of the first order hexagonal pyramid $\sigma' = \{20\bar{2}1\}$, which were the two other most prominent forms developed on the crystal, the pyramid faces intervening as replacing or sharpening facets between the prism and the basal plane. The prism was the most largely developed, conferring on the crystal its prismatic habit. But besides these three predominating forms there were four others present, three of them forming numerous additional little facets of great brilliancy about the termination.

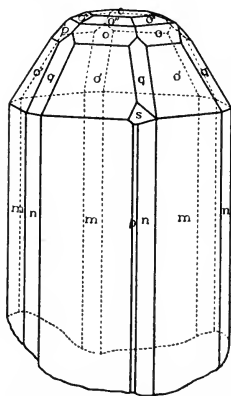


FIG. 287.—Measured Crystal of Apatite.

They consisted of faces of the primary first order pyramid $\sigma = \{10\bar{1}1\}$, of the first order pyramid $\sigma'' = \{10\bar{1}2\}$, and of the primary pyramid of the second order $q = \{11\bar{2}1\}$. The fourth of these forms was the hexagonal prism of the second order, $n = \{11\bar{2}0\}$, narrower faces of which replaced the edges of intersection of the broader faces of the prism of the first order *m*.

The whole of these seven forms, practically all the faces of which

were present and afforded brilliant reflections in spite of the very minute size of some of them, are identical with the holohedral forms of class 27 of the hexagonal system, the class which exhibits the full symmetry of the system. There were also present, however, two very small faces of the forms $s = \{2131\}$, a right hexagonal pyramid of the third order, and $p = \{21\bar{3}0\}$, a right hexagonal prism of the third order, both of which are illustrated on page 312, Figs. 279 and 280. The light reflected from them, however, was too feeble to permit of satisfactory measurements, although adequate for the purpose of identification and the determination of their approximate positions. It is the frequent presence of either or both of these forms on crystals of the mineral which determines that apatite belongs to class 25, the so-called pyramidal-hemihedral class of the hexagonal system according to the older method of classification, or according to our new and better-based structural method, the hexagonal bipyramidal or hexagonal equatorial class, fully dealt with in the preceding chapter. The chief points of difference from the holohedral class 27 are that the dihexagonal prism and dihexagonal pyramid are each divided into two forms, a right and a left hexagonal prism or bipyramid of the third order, which are not necessarily or even usually present on the same crystal. The right bipyramid $s = \{21\bar{3}1\}$ is characteristic of apatite, and the corresponding right prism $p = \{21\bar{3}0\}$ is also not infrequently present on crystals of the mineral.

The prism zone is obviously the one with which to commence the goniometrical measurements.

Fixing the crystal upright on the wax of the crystal holder, on adjusting two faces of the zone to the axis of the goniometer, the whole twelve faces constituting the zone are found to be also automatically adjusted in accordance with the chief property of a zone. There are six broad faces of the first order prism m , and one broad and five narrow faces of the second order prism n . All seven broad faces gave excellent "A" reflections of the signal-slit of the goniometer, and all but one of the narrower n faces also gave good images of the signal, although not so brilliant as the seven just mentioned. The actual measurements obtained on the Fuess horizontal circle goniometer were as follows :

PRISM ZONE OF APATITE.

Circle Readings.	Angles.	
	<i>mn.</i>	<i>mm.</i>
<i>m</i> 360° 0' A	29° 56'	
<i>n</i> 330. 4	29 59	59° 55' A
<i>m</i> 300 5 A	30 5	
<i>n</i> 270 0	30 0	60 5 A
<i>m</i> 240 0 A	29 58	
<i>n</i> 210 2	29 56	59 54 A
<i>m</i> 180 6 A	30 4 A	
<i>n</i> 150 2 A	29 59 A	60 3 A
<i>m</i> 120 3 A	29 56	
<i>n</i> 90 7	30 4	60 0 A
<i>m</i> 60 3 A	29 58 A	
<i>n</i> 30 5 A	30 3 A	60 1 A
<i>m</i> 0 2 A		

A careful inspection of this table of measurements convinces us that the zone is that of a hexagonal or trigonal crystal. The whole of the values of the angles between adjacent m and n faces are within $5'$ of 30° , and those of the angles of the first order prism mm are within $6'$ of 60° . Now this is a case where it is useless and positively misleading to take the mean of all the mm , mn or nm values, for they would obviously be bound to work out exactly to 60° , 30° , and 60° respectively, as in each case it is equivalent to dividing 360° by 6 or 12. When the whole of the angles in a zone are nearly alike, the method of taking the mean is not legitimate, for the round figure for the aliquot part of the circle is bound to be the result of the operation. The only safe thing to do is to rely on the best "A" individual values as indicating the truth, and also to consider these in connection with the measurements of other zones, which will aid in arriving at the true symmetry, and also finally, if the substance be a transparent one, in connection with the optical investigation of the crystal, which is sure to afford additional and unmistakable evidence of a decisive character as to the system of symmetry to which the crystal belongs, and which will be fully dealt with in the later chapters of this book. If this crystal of apatite were not of truly hexagonal symmetry, which would correspond to the possession of a single optic axis along which there would be no double refraction, and which would be identical with the direction of the principal, hexagonal, axis of symmetry, it would belong to one of the systems, probably the rhombic, which possess two optic axes; and as optically uniaxial and biaxial crystals exhibit very clearly their single axis or their two optic axes, surrounded by circular interference rings in the one case, and by rings and loop-like or elliptical lemniscates in the other, when examined in convergent polarised light, there could be no possible ambiguity.

An excellent example of a rhombic crystal which exhibits in the prism zone a very close approximation to hexagonal symmetry, the prism angles being within a quarter of a degree of 30° , has already been discussed in the case of the typical crystal of potassium sulphate, in Chapters IV. and VIII. In that case, however, which is one of the best known and nearest to perfect hexagonality, the true exact (final mean) differences from 30° were $12'$, the angles in each quadrant being $30^\circ 12'$, $30^\circ 0'$, and $29^\circ 48'$ respectively, an amount quite outside possible experimental error. On the other hand, in the case of apatite, the maximum apparent difference is only $6'$ irregularly distributed, and the difference almost entirely disappears if we take the mean of the angular values derived from parallel faces, which is quite legitimate, as there is absolutely no doubt that these faces were intended to be truly parallel. That is, if we take the mean of the first and fourth values of mm (which gives $59^\circ 59'$), of the second and fifth ($60^\circ 2'$), and of the third and sixth ($59^\circ 58'$), we have three legitimate values which lie at a maximum of $2'$ on both sides of 60° , and which therefore indicate that the endeavour of the growing crystal was to produce faces inclined truly at $60^\circ 0'$, the minute difference being undoubtedly due to slight disturbance during growth. It would not be legitimate, however, to take the mean

of the first three values of mm , or of the last three, or of the three mean values just given, as evidence of hexagonal symmetry; it is rather the fact of the closeness to within $2'$ of the three individual values in each of the two parallel sets which indicates the probability of such hexagonal symmetry.

Similarly, if we take the mean of the mn angles formed by parallel m and parallel n faces, beginning with the mean of the first and seventh and finishing with the mean of the sixth and twelfth, we obtain six legitimate mean values for mn , namely, $30^\circ 0'$, $29^\circ 59'$, $30^\circ 1'$, $30^\circ 2'$, $29^\circ 58'$, and $30^\circ 0'$, the maximum difference among which is again only $2'$, which is within the limits of error or malformation for the very best of crystals. The true angle mn can, therefore, be accepted as $30^\circ 0'$. The difference of $12'$ in the case of the rhombic potassium sulphate is fairly comparable to this, for it was the mean result derived from numerous strictly legitimate "A" values afforded by eleven different excellent crystals, each of such values corresponding to the specific angle in the quadrant only, that is, to the proper one of the three. There can, therefore, be no hesitation in concluding that the symmetry of apatite is truly hexagonal, while that of potassium sulphate is only pseudo-hexagonal, and in reality rhombic, the angle $30^\circ 12'$ being repeated four times in such positions in the zone as are consonant with the symmetry of that system. These conclusions are fully borne out by the optical properties, apatite showing a single optic axis in the direction of the vertical axis, the axis of the zone in question and that of hexagonal symmetry, while potassium sulphate is optically biaxial.

The point which has just been discussed at some length is of supreme importance, or so much space would not have been devoted to it. It is perhaps the chief pitfall into which a beginner in crystallographic measurement may precipitate himself. It is so easy, when a zone of angles verging on 30° or 60° is discovered on a crystal, to take the mean and obtain exactly $30^\circ 0'$ or $60^\circ 0'$ as the result, and to jump immediately to the conclusion that a hexagonal crystal is in question. For pseudo-hexagonal crystals are by no means uncommon, indeed von Fedorow has recently concluded that all crystals are either pseudo-hexagonal or pseudo-cubic (but within considerably wider limits than the above) when they are not truly hexagonal or cubic. A word of emphatic caution is thus necessary against arriving at hasty conclusions as to the system of symmetry to which a crystallised substance belongs. An investigator of ripe experience almost completes his exhaustive series of goniometrical and optical measurements and observations before he feels justified in finally stating a definite conclusion as to the system and class of symmetry.

We may now pass on to the measurement of other zones on the crystal of apatite. The next in order of importance are those perpendicular to the one already measured. This latter, the prism zone, may at once be graphically exhibited on the primitive circle of the stereographic projection, the poles being arranged at 30° from each other, as shown in Fig. 288. The centre of the circle represents the vertical hexagonal axis in plan, and the pole situated there is that of the basal plane $c=(0001)$. If we now draw diameters joining opposite prism poles mm and nn , we have the projection of six zones perpendicular to the prism zone, three similar ones being represented by those

diameters which indicate the directions of the Bravais-Miller axes, marked in Fig. 288 by greater strength of line, and three other zones different from the above but like each other, represented by the interaxial diameters.

It will be convenient first to proceed to the measurement of the three latter zones. For the crystal is set up, both in the drawing of Fig. 287 and in the stereographic projection, Fig. 288, with the broad *m*-faces arranged as is usual for the prism of the first order, namely, so that one of these *m*-faces is in front, parallel to the right-and-

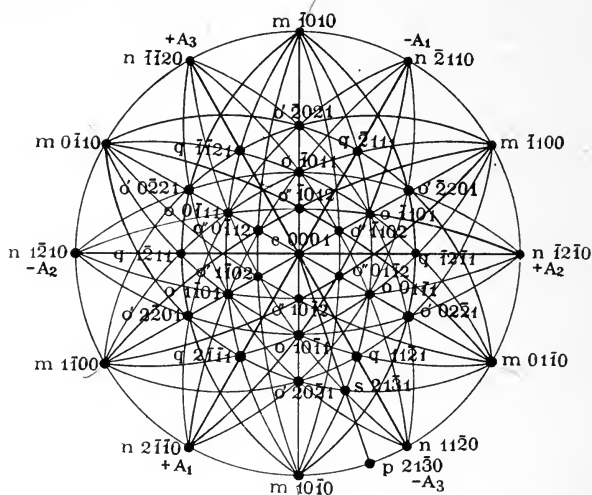


Fig. 288.

left Bravais-Miller axis (horizontal in the projection) and perpendicular to the back-and-front interaxis (vertical in the projection). The larger pyramidal faces are those in the zone [*mcm*], and it is these three similar [*mcm*] zones indicated in the projection by the interaxial diameters that we shall next proceed to measure. The crystal is set up for the purpose horizontally on the wax of the crystal holder, and with an *m*-face and a *c*-face respectively arranged approximately parallel to the two tangent screws of the adjusting movements of the goniometer. On effecting the fine adjustment of these two faces to the axis of the goniometer in the usual manner, with the aid of the two adjusting and two centring movements, there are found to be also automatically adjusted, as belonging to the zone, the other parallel *m*-face, and between *m* and *c* on each side of the latter, a face of each of the pyramids *o*, *o'*, and *o''*.

The following measurements were obtained from the three such zones present on the crystal :

3 ZONES [*mcm*].

Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 360^\circ \ 0' A \\ o' \ 329 \ 29 \\ o \ 310 \ 15 \\ c \ 270 \ 0 A \\ o'' \ 247 \ 2 A \\ o \ 229 \ 40 A \\ o' \ 210 \ 30 A \\ m \ 179 \ 59 A \end{array} \right.$	$\left\{ \begin{array}{l} mo' \ 30^\circ \ 31' \\ o'o \ 19 \ 14 \\ oc \ 40 \ 15 \\ co'' \ 22 \ 58 A \\ o'o \ 17 \ 22 A \\ oo' \ 19 \ 10 A \\ o'm \ 30 \ 31 A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 360^\circ \ 0' A \\ o' \ 329 \ 29 A \\ o \ 310 \ 20 A \\ c \ 270 \ 2 A \\ o'' \ 247 \ 1 A \\ o \ 229 \ 46 \\ o' \ 210 \ 32 A \\ m \ 180 \ 0 A \end{array} \right.$	$\left\{ \begin{array}{l} mo' \ 30^\circ \ 31' A \\ o'o \ 19 \ 9 A \\ oc \ 40 \ 18 A \\ co'' \ 23 \ 1 A \\ o'o \ 17 \ 15 \\ oo' \ 19 \ 14 \\ o'm \ 30 \ 32 A \end{array} \right.$
	$co \ 40^\circ \ 20' A$		$co \ 40^\circ \ 16'$

Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 360^\circ \ 0'A \\ o' \ 329 \ 30 \ A \\ o \ 310 \ 20 \ A \\ o'' \ 292 \ 56 \\ c \ 270 \ 0 \ A \\ o'' \ 247 \ 1 \ A \\ o \ 229 \ 41 \\ o' \ 210 \ 30 \ A \\ m \ 179 \ 59 \ A \end{array} \right.$	$\left\{ \begin{array}{l} mo' \ 30^\circ \ 30'A \\ o'o \ 19 \ 10 \ A \\ oo'' \ 17 \ 24 \\ o''c \ 22 \ 56 \\ co'' \ 22 \ 59 \ A \\ o'o \ 17 \ 20 \\ oo' \ 19 \ 11 \\ o'm \ 30 \ 31 \ A \end{array} \right. \quad \begin{array}{l} oc \ 40^\circ \ 20'A \\ co \ 40 \ 19 \end{array}$

It will be obvious that the angles are symmetrically arranged on each side of the basal plane *c*. The values of like angles may be collected together and the means taken. The proof of the symmetry lies, however, in the fact that the “A” values of similar angles on the two sides of *c* are so nearly absolutely identical, that the differences are within the range of error or malformation exhibited even by the best crystals; the values of *mo'*, for instance, are all identical within 2', a range of error which is certainly permissible in the values of angles intended to be actually identical. The mean values are: *mo'* = 30° 31', *o'o* = 19° 11', *oo''* = 17° 20', *o''c* = 22° 59', and *oc* = 40° 18'.

We pass next to the measurement of the three other equal zones, the projections of which are the Bravais-Miller axial diameters, namely, the zones [*ncn*], the crystal being again fixed horizontally on the wax, but rotated about its axis so as to get each of the three zones in turn adjusted, as in the case of the three zones just measured. When any *n*-face and the basal plane *c* are adjusted parallel to the movements and then finally to the axis, and properly centred, it is found that a second (the parallel) *n*-face is also adjusted, and also two *q*-faces, one on each side of *c*. The result of the measurements actually obtained is given below.

3 ZONES [*ncn*].

Circle Readings.	Angles.	Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} n \ 360^\circ \ 0'A \\ q \ 325 \ 43 \ A \\ c \ 269 \ 59 \\ q \ 214 \ 11 \ A \\ n \ 179 \ 57 \ A \end{array} \right.$	$\left\{ \begin{array}{l} nq \ 34^\circ \ 17'A \\ qc \ 55 \ 44 \\ cq \ 55 \ 48 \\ qn \ 34 \ 14 \ A \end{array} \right.$	$\left\{ \begin{array}{l} n \ 360^\circ \ 0'A \\ q \ 325 \ 44 \ A \\ c \ 270 \ 0 \\ n \ 180 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} nq \ 34^\circ \ 16'A \\ qc \ 55 \ 44 \\ cn \ 90 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} n \ 360^\circ \ 0'A \\ q \ 325 \ 45 \ A \\ c \ 269 \ 59 \ A \\ n \ 180 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} nq \ 34^\circ \ 15'A \\ qc \ 55 \ 46 \ A \\ cn \ 89 \ 59 \end{array} \right.$

We may collect the four values of each of the angles *nq* and *qc*, and extract their mean values, which prove to be *nq* = 34° 15' and *qc* = 55° 45'. For we are now satisfied that the angles on the two sides of the basal plane *c* are intended to be truly equal, a supposition which is still further strengthened by the near equality of the values on each side of *c* in the first of these three zones; the other two zones each exhibited a second *q*-face, but it was too minute to give trustworthy readings.

We have now completed the measurement of the zones indicated on

the projection by the primitive circle and the diameters, and should next consider those indicated on the projection by the circular arcs terminating at parallel m and n poles. There are three such arcs on each side of the mcm diameters, symmetrical to the latter, and four symmetrically on each side of the ncn diameters; each arc-zone occurs thus in duplicate on the two sides of a diameter, and as there are three similar diameters mcm or ncn the same zone occurs six times altogether, and there are consequently seven different arc-zones repeated thus six times each. In every case the zones are only half-zones, or 180° zones, like the diametral zones, as the crystal is only singly terminated, but if it had been doubly terminated they would have been complete 360° zones. Many of the angles are repeated twice in each semi-zone, that is, they occur once in each quadrant, hence it is possible that twelve independent values of the same angle may be obtained from this single crystal, derived from absolutely different faces, symmetrically situated however.

In the measurement of these arc-zones, or cross-zones as they are sometimes called, much labour in resetting the crystal may be saved if those about each diameter are measured while the crystal is set for the measurement of the diametral zone itself. It is then only necessary that the adjustment for the latter zone shall be done fairly approximately by the hand-setting on the wax, that is, so that the crystal is approximately adjusted for the diametral zone when the adjusting movements are all in their normal zero positions; only a little use of them is required then for the fine adjustment of that zone, and practically the whole of the circular movement of each of the two adjusting segments is available for the adjustment of the arc-zones about that diameter, without any resetting of the crystal on the wax. After a little experience in goniometrical measurement the importance of this point will be realised, both as regards the saving of trouble and the chance, ever present during hand-movement of the crystal on the wax, of losing, by its becoming detached, a valuable and possibly half-measured minute crystal. For this reason it is advisable to construct at once the stereographic projection of the first zones tackled, in this case say the primitive zone and a couple of diametral zones, an m and an n zone; so that it is possible after tentatively also drawing in the projections of the other similar diametral zones, to see early on in the work what arc-zones there are which ought to be measured, and then their measurement can be carried out as recommended, while each diametral zone in turn is *in situ* on the wax, so that none but the first will have to be reset. The mode of finding any facial pole on the projection has been fully described in Chapter VI. In the case of a pole on a diametral zone, the procedure is very simple. For instance, to find the pole of the face $o=(10\bar{1}1)$ on the diameter mcm , we mark off from $m=(10\bar{1}0)$ along the primitive circle on either side, say towards $m=(01\bar{1}0)$, an arc equal to the measured angle $mo=(10\bar{1}0):(10\bar{1}1)$ which was found to be $49^\circ 42'$, and we join the point thus marked off to the opposite pole of the diametral zone, which is identical with the pole $n=(1\bar{2}10)$. The point of intersection of this line and the diameter mcm is the position of the pole $o=(10\bar{1}1)$. The same distance co can then be marked off along the other five radii mc in order to get the positions of the other o poles. A similar procedure is then followed with regard to the other poles on these mcm diameters, and for those on the ncn diameters. The various arc-zones can then be readily drawn in with the compasses, to pass through the ends of the diameters and the various poles on the zones, the proper radius being found in each case by the ordinary method of elementary geometry, of constructing a circular arc to pass through three given points; in every case the centre of the arc will lie on a diameter, so that the radius can even be found by trial.

In the next tables are given the results of the measurements of the seven arc-zones which have been referred to. In only two cases where twelve independent values are possible did the whole of the twelve different faces yield good images; all doubtful cases have been ignored and do not appear in the tables, so that the angular values which do appear are either derived from "A" faces, or from such as left no doubt as to the trustworthiness of their signal images. From the very close equality of the values in each quadrant, or occurring in positions which should be symmetrical if the crystal be truly hexagonal, there can be no further doubt as to this latter being a real fact. Hence, the mean of like angles may with safety be taken, for in no case are all the angles in a zone equal, as was the case with the prism zone, where all were 30° . The well-nigh absolute identity of individual "A" values of symmetrically placed angles in each one and the same zone, is however, as cannot be too strongly or too frequently emphasised, the only real goniometrical test or proof of that symmetry.

3 ARC-ZONES IN DUPLICATE ABOUT A DIAMETER *mcm*.

Readings.	Angles.	Readings.	Angles.
$\left\{ \begin{array}{l} m \ 360^\circ \ 0' A \\ o'' \ 281 \ 17 \\ o'' \ 258 \ 41 A \\ m \ 179 \ 59 A \end{array} \right.$	$\left\{ \begin{array}{l} mo'' \ 78^\circ \ 43' \\ o''o'' \ 22 \ 36 \\ o''m \ 78 \ 42 A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 180^\circ \ 0' A \\ o'' \ 101 \ 16 \\ o'' \ 78 \ 49 \\ m \ 0 \ 0 A \end{array} \right.$	$\left\{ \begin{array}{l} mo'' \ 78^\circ \ 44' \\ o''o'' \ 22 \ 27 \\ o''m \ 78 \ 49 \end{array} \right.$
$\left\{ \begin{array}{l} m \ 360 \ 0 A \\ q \ 315 \ 42 A \\ o \ 288 \ 52 A \\ o \ 251 \ 6 A \\ q \ 224 \ 15 A \\ m \ 179 \ 58 A \end{array} \right.$	$\left\{ \begin{array}{l} mq \ 44 \ 18 A \\ qo \ 26 \ 50 A \\ oo \ 37 \ 46 A \\ oq \ 26 \ 51 A \\ qm \ 44 \ 17 A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 360 \ 0 A \\ q \ 315 \ 42 \\ o \ 288 \ 50 A \\ o \ 251 \ 5 A \\ q \ 224 \ 14 A \\ m \ 180 \ 0 A \end{array} \right.$	$\left\{ \begin{array}{l} mq \ 44 \ 18 \\ qo \ 26 \ 52 \\ oo \ 37 \ 45 A \\ oq \ 26 \ 51 A \\ qm \ 44 \ 14 A \end{array} \right.$
$\left\{ \begin{array}{l} m \ 360 \ 0 A \\ o' \ 295 \ 27 A \\ q \ 269 \ 57 A \\ o' \ 244 \ 26 A \\ m \ 180 \ 0 A \end{array} \right.$	$\left\{ \begin{array}{l} mo' \ 64 \ 33 A \\ o'q \ 25 \ 30 A \\ qo' \ 25 \ 31 A \\ o'm \ 64 \ 26 A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 180 \ 0 A \\ o' \ 115 \ 32 A \\ q \ 90 \ 1 A \\ o' \ 64 \ 29 A \\ m \ 0 \ 1 A \end{array} \right.$	$\left\{ \begin{array}{l} mo' \ 64 \ 28 A \\ o'q \ 25 \ 31 A \\ qo' \ 25 \ 32 A \\ o'm \ 64 \ 28 A \end{array} \right.$

A similar series of measurements were also made for the six analogous arc-zones about each of the other two *mcm* diameters, and as no advantage would be gained by transcribing the contents of the note-book for these precisely similar measurements, we can at once collect together the various values obtained for like angles, including those derived from the other two sets of zones as well as those set down above.

The mean values derived were :

$mo'' = 78^\circ \ 45'$, and $o''o'' = 22^\circ \ 32'$; $mq = 44^\circ \ 16'$, $qo = 26^\circ \ 51'$, $mo = 71^\circ \ 9'$, and $oo = 37^\circ \ 45'$; $mo' = 64^\circ \ 29'$, and $o'q = 25^\circ \ 31'$.

In a similar manner, the results of the measurements of one of the three sets of four zones in duplicate about an *ncn* diameter will next be given. The individual values of analogous angles were collected together, the similar values for the two other sets added to them, and the mean extracted for each angle.

4 ARC-ZONES IN DUPLICATE ABOUT A DIAMETER *nen*.

Readings.	Angles.	Readings.	Angles.
$\left\{ \begin{array}{l} n \ 180^\circ \ 0' A \\ o'' \ 109 \ 46 \ A \\ o'' \ 70 \ 15 \ A \\ n \ 0 \ 1 \ A \end{array} \right.$	$\left\{ \begin{array}{l} no'' \ 70^\circ \ 14' A \\ o''o'' \ 39 \ 31 \ A \\ o''n \ 70 \ 14 \ A \end{array} \right.$	$\left\{ \begin{array}{l} n \ 70^\circ \ 13' \\ o'' \ 0 \ 0 \ A \end{array} \right.$	$\left\{ \begin{array}{l} no'' \ 70^\circ \ 13' \end{array} \right.$
$\left\{ \begin{array}{l} n \ 360 \ 12 \ A \\ o \ 304 \ 18 \ A \\ o'' \ 270 \ 12 \ A \\ o \ 236 \ 8 \\ n \ 180 \ 7 \end{array} \right.$	$\left\{ \begin{array}{l} no \ 55 \ 54 \ A \\ oo'' \ 34 \ 6 \ A \\ o''o \ 34 \ 4 \\ on \ 56 \ 1 \end{array} \right.$	$\left\{ \begin{array}{l} n \ 360 \ 0 \ A \\ o \ 304 \ 5 \ A \\ o'' \ 270 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} no \ 55 \ 55 \ A \\ oo'' \ 34 \ 5 \end{array} \right.$
$\left\{ \begin{array}{l} n \ 360 \ 0 \ A \\ o' \ 318 \ 15 \ A \\ o \ 269 \ 59 \ A \\ o' \ 221 \ 45 \ A \\ n \ 180 \ 1 \ A \end{array} \right.$	$\left\{ \begin{array}{l} no' \ 41 \ 45 \ A \\ o'o \ 48 \ 16 \ A \\ oo' \ 48 \ 14 \ A \\ o'n \ 41 \ 44 \end{array} \right.$	$\left\{ \begin{array}{l} n \ 360 \ 0 \ A \\ o' \ 318 \ 15 \ A \\ o' \ 221 \ 45 \ A \\ n \ 180 \ 1 \end{array} \right.$	$\left\{ \begin{array}{l} no' \ 41 \ 45 \ A \\ o'o' \ 96 \ 30 \ A \\ o'n \ 41 \ 44 \end{array} \right.$
$\left\{ \begin{array}{l} n \ 180 \ 0 \ A \\ q \ 114 \ 24 \ A \\ q \ 65 \ 35 \ A \\ n \ 0 \ 1 \ A \end{array} \right.$	$\left\{ \begin{array}{l} nq \ 65 \ 36 \ A \\ qq \ 48 \ 49 \ A \\ qn \ 65 \ 34 \ A \end{array} \right.$	$\left\{ \begin{array}{l} n \ 180 \ 0 \\ q \ 114 \ 25 \ A \\ q \ 65 \ 34 \ A \\ n \ 0 \ 0 \end{array} \right.$	$\left\{ \begin{array}{l} nq \ 65 \ 35 \\ qq \ 48 \ 51 \ A \\ qn \ 65 \ 34 \end{array} \right.$

These mean values for the angles in the four arc-zones are :

$no'' = 70^\circ 15'$, and $o''o'' = 39^\circ 31'$; $no = 55^\circ 56'$, and $oo'' = 34^\circ 4'$; $no' = 41^\circ 44'$, and $o'o = 48^\circ 15'$; $nq = 65^\circ 34'$, and $qq = 48^\circ 51'$.

This completes the goniometrical work in connection with apatite, all the numerous zones on the crystal having now been measured, in all 49 zones, namely, the primitive-circle prism zone, six diametral zones, and 42 arc-zones, the diametral zones being of two different kinds and the arc-zones of seven different kinds. Hence, we have only to consider, as zones composed of different angles, (1) the prism zone, (2) two diametral zones, and (3) seven arc-zones, the rest being repetitions of these in accordance with the symmetry. The values of the angles in these 10 zones will be found tabulated at the conclusion of this chapter, after the necessary calculations of angles and elements have been made.

Calculation of angles and elements. Basal angle, $co = (0001) : (10\bar{1}1) = 40^\circ 18'$.

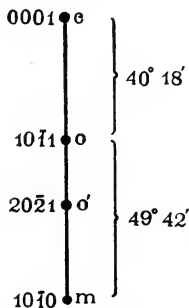


FIG. 289.

It is convenient to proceed first to find the position of the pole o' on the same radius moc .

To find $mo' = (10\bar{1}0) : (20\bar{2}1)$.

The pole o' may be considered the fourth pole in the 90° zone [moc], and the anharmonic ratio of four poles in a zone employed. The conditions are represented by the diagram in Fig. 289, from which we at once deduce, as explained in Chapter VI., that :

$$\frac{10\bar{1}0 \ 0001}{20\bar{2}1 \ 10\bar{1}1} = \frac{\sin mo'}{\sin 10\bar{1}0 \ 0001} = \frac{\sin 40^\circ 18'}{\sin 49^\circ 42'} \cdot \frac{\sin co'}{\sin co'}$$

$$\frac{1}{1} \cdot \frac{1}{2} = \frac{\sin mo'}{\cos mo'} \cdot \frac{\sin 40^\circ 18'}{\cos 40^\circ 18'} = \tan mo' \cdot \tan 40^\circ 18'$$

$$\tan mo' = \frac{1}{2} \cot 40^\circ 18'. \quad mo' = 30^\circ 31'.$$

This value is identical with the mean measured angle, five individual "A" values of which only varied one minute on either side of it.

We can then get oo' by the difference of this angle and mo , thus:

$$oo' = 49^\circ 42' - 30^\circ 31' = 19^\circ 11'.$$

This is again identical with the mean measured value.

We may next proceed conveniently to find $nq = (11\bar{2}0) : (11\bar{2}1)$ and $qo' = (11\bar{2}1) : (20\bar{2}1)$.

This we may do from the right-angled triangle $co'q$, $(0001) : (20\bar{2}1) : (11\bar{2}1)$, in which we know the side co' ($40^\circ 18' + 19^\circ 11' = 59^\circ 29'$), the angle at c (30°), and the right angle at q , and also that cq is the complement of nq . From the Napierian diagram, Fig. 290, we derive:

$$\sin o'q = \sin 30^\circ \sin 59^\circ 29'; \quad \sin cq = \cot 30^\circ \tan o'q.$$

$$o'q = 25^\circ 31'. \quad cq = 55^\circ 45'.$$

$$\text{Then} \quad nq = (11\bar{2}0) : (11\bar{2}1) = 90^\circ - 55^\circ 45' = 34^\circ 15'.$$

$$\text{Also} \quad o'm = (20\bar{2}1) : (1\bar{1}00) = 90^\circ - 25^\circ 31' = 64^\circ 29'.$$

These four values are all absolutely identical with the mean measured values, the whole of the individual values having also been remarkably close, showing the extraordinary accuracy with which, in a good crystal, the intention of nature is carried out.

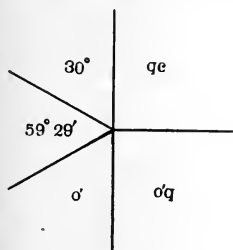


FIG. 290.

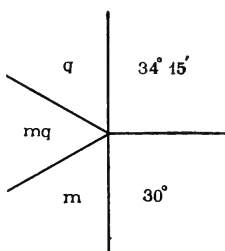


FIG. 291.

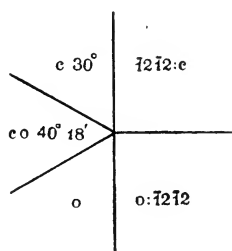


FIG. 292.

We may next address ourselves to the calculation of $nq = (10\bar{1}0) : (11\bar{2}1)$. We can again use a right-angled triangle, namely, mqn $(10\bar{1}0) : (11\bar{2}1) : (11\bar{2}0)$, in which the right angle is at n , the side mn is 30° , and the side nq we have just found to be $34^\circ 15'$. Setting out the usual Napierian diagram, Fig. 291, we at once deduce from it:

$$\cos nq = \cos 30^\circ \cos 34^\circ 15'. \quad nq = 44^\circ 17'.$$

The mean measured angle was $44^\circ 16'$.

Continuing the calculation of the angles in the same zone, we should next find $qo = (11\bar{2}1) : (01\bar{1}1)$, and $oo = (01\bar{1}1) : (1\bar{1}01)$.

Commencing with the latter, we can find the half of it from the right-angled triangle $c = (0001)$, $o = (01\bar{1}1)$, $(1\bar{2}\bar{1}2)$; the last-named possible face is not developed, but the symbol serves to mark the third corner of the triangle in question, in which we know the side $co = 40^\circ 18'$, the basal angle, and the angles at c and at $(1\bar{2}\bar{1}2)$, which are 30° and 90° respectively. Making up, therefore, the usual Napierian diagram, Fig. 292, we deduce:

$$[\sin (\frac{1}{2}oo) = \sin o : (1\bar{2}\bar{1}2) = \sin 30^\circ \sin 40^\circ 18'.$$

$$\frac{1}{2}oo = 18^\circ 52', \text{ and } oo = 37^\circ 44'.$$

$$\text{Then} \quad oq = (01\bar{1}1) : (11\bar{2}1) = 90^\circ - (nq + \frac{1}{2}oo) = 90^\circ - 63^\circ 9' = 26^\circ 51'.$$

$$44^\circ 17' + 18^\circ 52'$$

Also $mo = (10\bar{1}0) : (01\bar{1}1) = 44^\circ 17' + 26^\circ 51' = 71^\circ 8'$, which was also one of the well-measured angles.

The mean measured values of these three angles oo , oq , and mo in this zone were respectively $37^\circ 45'$, $26^\circ 51'$, and $71^\circ 9'$, the first and third being thus only one minute removed from, and the second identical with, the calculated value.

In order to complete the calculation of the angles in the diametral zones and those of the arc-zones terminating at m poles, it remains to find the position of the pole $o'' = (10\bar{1}2)$ in the zone $[moc]$, and the angle $mo'' = (1\bar{1}00) : (10\bar{1}2)$.

Commencing with the former, we can find $co'' = (0001) : (10\bar{1}2)$ from the anharmonic ratio of the four poles c , o'' , o , m , the conditions being defined diagrammatically in Fig. 293. The ratio is as follows:

$$\begin{array}{l} \begin{array}{c} 0001 \bullet c \\ 10\bar{1}2 \bullet o'' \\ 10\bar{1}1 \bullet o \\ 10\bar{1}0 \bullet m \end{array} \left\{ \begin{array}{l} 40^\circ 18' \\ 49^\circ 42' \end{array} \right. \end{array} \quad \begin{array}{l} \begin{array}{c} 10\bar{1}0 \quad 0001 \\ \times \quad \times \\ 10\bar{1}1 \quad 10\bar{1}2 \\ \times \quad \times \\ 10\bar{1}2 \quad 10\bar{1}1 \\ \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2} \end{array} \\ \frac{10\bar{1}1}{10\bar{1}0} \cdot \frac{10\bar{1}2}{0001} = \frac{\sin 49^\circ 42'}{\sin mo''} \cdot \frac{\sin co''}{\sin 40^\circ 18'} \\ \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2} = \frac{\sin co''}{\cos co''} \cdot \frac{\cos 40^\circ 18'}{\sin 40^\circ 18'} \\ \tan co'' = \frac{1}{2} \tan 40^\circ 18'. \quad co'' = 22^\circ 59'. \\ \text{Then} \quad oo'' = co - co'' = 40^\circ 18' - 22^\circ 59' = 17^\circ 19'. \end{array}$$

FIG. 293.

The former value is identical with the measured value, and the latter is only $1'$ removed.

To find $mo'' = (1\bar{1}00) : (10\bar{1}2)$.

We can readily find first the complement of mo'' from the right-angled triangle $c = (0001) : o'' = (10\bar{1}2) : (11\bar{2}3)$, the last pole being that of a possible but not of an actually developed face in the zone $[cqn]$, the symbol of which, however, serves to mark that corner of the triangle at which the right angle occurs. We know that the angle at c is 30° , and have just found the side co'' to be $22^\circ 59'$. From the Napierian diagram expressing the conditions, Fig. 294, we deduce:

$\sin o'' : (11\bar{2}3) = \sin 30^\circ \sin 22^\circ 59'$. $o'' : (11\bar{2}3) = 11^\circ 15'$.
Hence $o'o'' = 22^\circ 30'$, and $mo'' = 90^\circ - 11^\circ 15' = 78^\circ 45'$.

The mean measured value of mo'' was identical with this value, namely, $78^\circ 45'$, and that of $o'o''$ was $22^\circ 32'$, only two minutes removed from the calculated value.

We may now proceed to calculate the values of the angles in the arc-zones ending at n -poles.

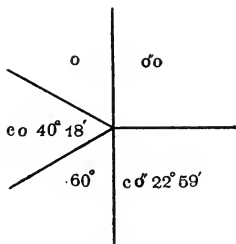


FIG. 295.

mean measured values.

Taking now the next arc n , o' , o , or its analogue ending at the axis $-A_3$, we can

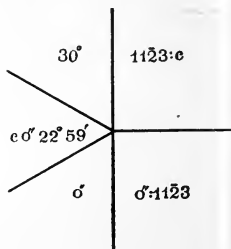


FIG. 294.

To find $no = (\bar{1}2\bar{1}0) : (01\bar{1}1)$ and $oo'' = (01\bar{1}1) : (10\bar{1}2)$.

We can readily find the latter from the right-angled triangle $co''o$, the particular o'' and o poles being those having the indices just stated. In this triangle the right angle is at $o'' = (10\bar{1}2)$, the angle at $c = (0001)$ is 60° , the side $co'' = (0001) : (10\bar{1}2)$ is $22^\circ 59'$, and the side $co = (0001) : (01\bar{1}1)$ is $40^\circ 18'$. Making the usual diagram, Fig. 295, we obtain:

$\sin o''o = \sin 60^\circ \sin 40^\circ 18'$. $o''o = 34^\circ 4'$.
Then $no = 90^\circ - 34^\circ 4' = 55^\circ 56'$.

Both these calculated values are identical with the

find $no' = (11\bar{2}0) : (20\bar{2}1)$ from the triangle $m = (10\bar{1}0) : n = (11\bar{2}0) : o' = (20\bar{2}1)$, in which the angle at m is a right angle, the side mn is 30° , and the side mo' is $30^\circ 31'$. From the diagram, Fig. 296, and Napier's rules we derive:

$$\cos no' = \cos 30^\circ \cos 30^\circ 31'. \quad no' = 41^\circ 45'.$$

Then

$$o'o = (20\bar{2}1) : (1\bar{1}01) = 90^\circ - 41^\circ 45' = 48^\circ 15'.$$

The latter is identical with the mean measured value, while that for no' was only $1'$ removed, namely, $41^\circ 44'$.

Taking now the next and most outward arc-zone terminating at n on the projection, to find $nq = (\bar{1}2\bar{1}0) : (11\bar{2}1)$.

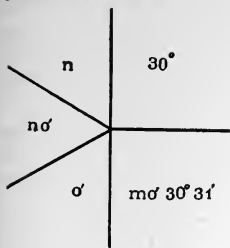


FIG. 296.

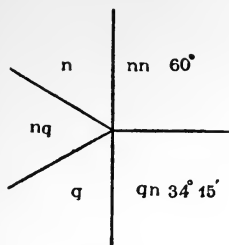


FIG. 297.

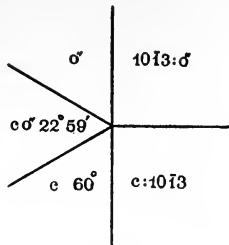


FIG. 298.

This angle can readily be found from the triangle $n = (\bar{1}2\bar{1}0) : q = (11\bar{2}1) : n = (11\bar{2}0)$, in which the angle at $n = (11\bar{2}0)$ is a right angle, nn is 60° , and $n = (11\bar{2}0) : q$ is $34^\circ 15'$. From the Napierian diagram, Fig. 297, we get:

$$\cos nq = \cos 60^\circ \cos 34^\circ 15'. \quad nq = 65^\circ 35'.$$

Then also

$$qq = (11\bar{2}1) : (2\bar{1}\bar{1}1) = 2(90^\circ - 65^\circ 35') = 48^\circ 50'.$$

The mean measured angles were $nq = 65^\circ 34'$ and $qq = 48^\circ 51'$, in each case only one minute removed from the calculated value.

There is still the innermost of these n -arc-zones remaining, in which we have to calculate $no'' = (\bar{1}2\bar{1}0) : (01\bar{1}2)$.

We can obtain the complement of this angle most conveniently, namely, half $o''o' = (01\bar{1}2) : (1\bar{1}02)$, for this angle forms one side of the triangle $c = (0001) : o'' = (01\bar{1}2) : (10\bar{1}3)$, the last pole being that one on the diametral zone $[moe]$ which marks the half of $o'o''$, and, while not being that of a developed face, is useful to insert in the projection for the purpose of marking the third corner of the triangle under consideration. In this triangle the angle at $(10\bar{1}3)$ is a right angle, the side co'' we have found to be $22^\circ 59'$, and the angle at c is 60° . Constructing our diagram, Fig. 298, we have, by Napier's rules,

$$\sin (10\bar{1}3) : o'' = \sin 60^\circ \sin 22^\circ 59'. \quad (10\bar{1}3) : o'' = 19^\circ 46'.$$

From which we get

$$no'' = 90^\circ - 19^\circ 46' = 70^\circ 14',$$

and

$$o''o' = 2(19^\circ 46') = 39^\circ 32'.$$

The mean measured value of no'' was $70^\circ 15'$, and of $o''o'$ $39^\circ 31'$, an agreement within one minute.

This completes the calculations of all the angles between faces actually measurable on the crystal, but it will be interesting to conclude by also calculating the positions of the two class-determining faces also present of the forms $s = \{21\bar{3}1\}$ and $p = \{21\bar{3}0\}$, which were too minute to afford trustworthy measurements.

To find the position of the right pyramid face $s = (21\bar{3}1)$ in the zone $[msqo]$, in other words, to find $ms = (10\bar{1}0) : (21\bar{3}1)$.

This can do by making use of the anharmonic ratio of the four poles $m = (10\bar{1}0)$, $s = (21\bar{3}1)$, $q = (11\bar{2}1)$, and $(\bar{1}2\bar{1}2)$ in the 90° -zone ending at the last-mentioned pole on the $+A_2$ diameter, which is not, however, represented by a developed face on the crystal. The conditions are shown in Fig. 299, from which we deduce:

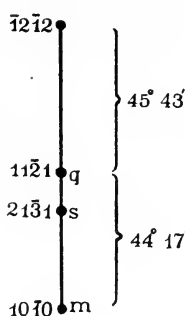


FIG. 299.

$$\begin{array}{r} 10\bar{1}0 \quad \bar{1}2\bar{1}2 \\ \times \quad \times \\ 21\bar{3}1 \quad 11\bar{2}1 \\ \hline 10\bar{1}0 \cdot \bar{1}2\bar{1}2 = \frac{\sin ms}{\sin 44^\circ 17'} \cdot \frac{\sin 45^\circ 43'}{\sin (\bar{1}2\bar{1}2):s} \\ \times \quad \times \\ 11\bar{2}1 \quad 21\bar{3}1 \\ \hline \frac{1}{1} \cdot \frac{3}{5} = \frac{\sin ms}{\cos ms} \cdot \frac{\sin 45^\circ 43'}{\cos 45^\circ 43'} \end{array}$$

$$\tan ms = \frac{3}{5} \cot 45^\circ 43'$$

$$ms = 30^\circ 20', \text{ and } sq = 44^\circ 17' - 30^\circ 20' = 13^\circ 57'.$$

Next, to find $ms = (01\bar{1}0) : (21\bar{3}1)$ in the zone $[ms'o'q]$.

The zone in question measures 90° from m to q , so that the anharmonic ratio can again be employed in its simpler form. The

conditions are given in Fig. 300, from which we derive :

$$\begin{array}{r} 01\bar{1}0 \quad 2\bar{1}\bar{1}1 \\ \times \quad \times \\ 21\bar{3}1 \quad 20\bar{2}1 \\ \hline 01\bar{1}0 \cdot 2\bar{1}\bar{1}1 = \frac{\sin ms}{\sin 64^\circ 29'} \cdot \frac{\sin 25^\circ 31'}{\sin qs} \\ \times \quad \times \\ 20\bar{2}1 \quad 21\bar{3}1 \\ \hline \frac{2}{2} \cdot \frac{2}{4} = \frac{\sin ms}{\cos ms} \cdot \frac{\sin 25^\circ 31'}{\cos 25^\circ 31'} \end{array}$$

$$\tan ms = \frac{1}{2} \cot 25^\circ 31'. \quad ms = 46^\circ 20'.$$

$$\text{Then } so' = (21\bar{3}1) : (20\bar{2}1) = 64^\circ 29' - 46^\circ 20' = 18^\circ 9'.$$

We have next to find the position of $p = (21\bar{3}0)$ in the prism zone, that is, to find $mp = (10\bar{1}0) : (21\bar{3}0)$. The conditions are shown in Fig. 301, and we derive therefrom :

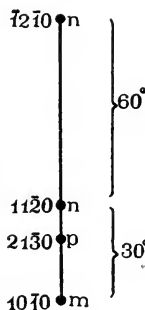


FIG. 301.

$$\begin{array}{r} 10\bar{1}0 \quad \bar{1}2\bar{1}0 \\ \times \quad \times \\ 21\bar{3}0 \quad 11\bar{2}0 \\ \hline 10\bar{1}0 \cdot \bar{1}2\bar{1}0 = \frac{\sin mp}{\sin 30^\circ} \cdot \frac{\sin 60^\circ}{\sin np} \\ \times \quad \times \\ 11\bar{2}0 \quad 21\bar{3}0 \\ \hline \frac{1}{1} \cdot \frac{3}{5} = \frac{\sin mp}{\cos mp} \cdot \frac{\cos 30^\circ}{\sin 30^\circ} \end{array}$$

$$\tan mp = \frac{3}{5} \tan 30^\circ. \quad mp = 19^\circ 6'.$$

$$pn = 30^\circ - 19^\circ 6' = 10^\circ 54'.$$

Finally, to find $ps = (21\bar{3}0) : (21\bar{3}1)$ on the radius csp . This we can do from the triangle mps , $(10\bar{1}0) : (21\bar{3}0) : (21\bar{3}1)$, which has a right angle at p , and in which we now know the sides $mp = 19^\circ 6'$ and $ms = 30^\circ 20'$. Constructing the Napierian diagram as in Fig. 302, we derive the equation :

$$\cos 30^\circ 20' = \cos ps \cos 19^\circ 6'$$

$$\text{or } \cos ps = \frac{\cos 30^\circ 20'}{\cos 19^\circ 6'}. \quad ps = 24^\circ 1'.$$

$$\text{Then } cs = 90^\circ - 24^\circ 1' = 65^\circ 59'.$$

This concludes all the calculations of interfacial angles.

The Ratio of the Axes.—This can at once be found from the position of any pole of the primary second order pyramid $q = \{11\bar{2}1\}$ in the axial diametral zone $[eqn]$; for each face of this form is

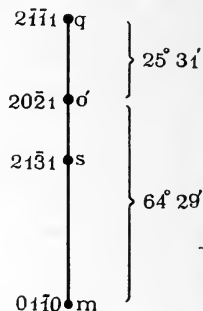


FIG. 300.

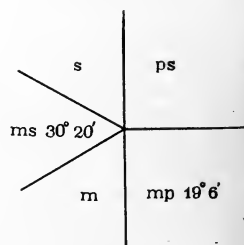


FIG. 302.

inclined both to the vertical axis c and to one of the three equal horizontal axes, while being also perpendicular to the plane containing those two axes. Suppose, for instance, we take the q face $(11\bar{2}1)$; it is inclined to the axis c and to the horizontal axis $-A_3$ as well as being perpendicular to their plane, the angle of inclination being indicated by $nq = 34^\circ 15'$ (both the measured and calculated values being identically $34^\circ 15'$). Now this angle is that between the normal to $n = (11\bar{2}0)$ (which normal is identical with the crystallographic axis $-A_3$) and the normal to $q = (11\bar{2}1)$, that is, it is the angle AON in Fig. 303.

But this angle is equal to ACO, indeed the side OC is parallel to the prism face n and the side AC parallel to the face q of the second order pyramid in question. Hence it is clear that the triangle AOC defines the axial relationships, and AO represents the intercept on the axis a and OC that on the axis c . The indices being $(11\bar{2}1)$ and the particular horizontal axis being A_3 , it will be obvious that the index relating to the horizontal axis is 2, while that relating to the vertical axis is 1, and therefore, reciprocally, that the length cut off, the intercept, OC is $2c$, while the intercept OA is a . Hence we have:

$$\frac{2c}{a} = \cot 34^\circ 15'.$$

Taking out the value of $\cot 34^\circ 15'$ from a table of natural cotangents, we get:

$$\frac{2c}{a} = 1.4687, \text{ and } \frac{c}{a} = 0.7344.$$

We can also find the ratio $a:c$ from the angle between any face of the primary first order pyramid $o = \{10\bar{1}1\}$ and the adjoining first order prism face m of the form $\{10\bar{1}0\}$, say between the faces $(10\bar{1}1)$ and $(10\bar{1}0)$, which has been shown to be $49^\circ 42'$. This method will be found developed in detail, with an explanatory illustration on page 372 of Chapter XXIV., in connection with the axial ratio of quartz. From the analysis there given, the combination of the two primary rhombohedra of quartz corresponding exactly to the hexagonal first order pyramid of apatite, it will be clear that:

$$\frac{c}{a} = \frac{\sqrt{3}}{2} \tan [\text{complement of angle } (10\bar{1}1) : (10\bar{1}0)]$$

and this in the case of apatite is:

$$\frac{c}{a} = \frac{\sqrt{3}}{2} \tan 40^\circ 18'.$$

As the angle $40^\circ 18'$ is the basal angle co , it is a most appropriate angle to take for the calculation of the elements. On calculating the axial ratio from this simple equation, we find that $\frac{c}{a} = 0.7345$.

The results of the two methods of calculation are in such close agreement that we can with confidence take the ratio of the axes to be:

$$a : c = 1 : 0.7345.$$

This result is within one unit in the fourth decimal place of the value derived from the measurements of v. Kokscharow.

It now only remains to collect together the results of the investigation

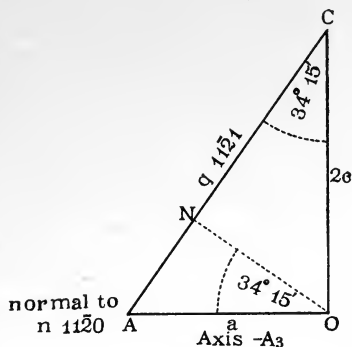


FIG. 303.

of this crystal of apatite—one of the most perfect little crystals, and one of the richest in faces, with which the author has ever had the pleasure to work—and to present them in concise tabular form. The agreement between the observed and calculated angles is most satisfactory, the maximum difference being only 2'.

TABLE OF RESULTS FOR CRYSTAL OF APATITE, $\text{Ca}_5\text{F}(\text{PO}_4)_3$.
Cl

Crystal-system: Hexagonal. Class: 25, bipyramidal.

Ratio of Axes: $a : c = 1 : 0.7345$.

Forms observed: $c = \{0001\}$, $m = \{10\bar{1}0\}$, $n = \{11\bar{2}0\}$, $p = \{21\bar{3}0\}$,
 $o = \{10\bar{1}1\}$, $o' = \{20\bar{2}1\}$, $o'' = \{10\bar{1}2\}$, $q = \{11\bar{2}1\}$,
 $s = \{21\bar{3}1\}$.

Interfacial angles: as given in the following table, the basal angle being marked with an asterisk, and the brackets indicating the zones.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated Value.	Difference.
$\left\{ \begin{array}{l} mn = (10\bar{1}0) : (11\bar{2}0) \\ mp = (10\bar{1}0) : (21\bar{3}0) \\ pn = (21\bar{3}0) : (11\bar{2}0) \\ nm = (10\bar{1}0) : (01\bar{1}0) \end{array} \right.$	12 6	29° 56'–30° 5' 59 54–60 5	30° 0' 60 0	30° 0' 19 6 10 54 60 0	0' 0
$\left\{ \begin{array}{l} mo' = (10\bar{1}0) : (20\bar{2}1) \\ o'o = (20\bar{2}1) : (10\bar{1}1) \\ oo'' = (10\bar{1}1) : (10\bar{1}2) \\ o''c = (10\bar{1}2) : (0001) \\ oc = (10\bar{1}1) : (0001) \end{array} \right.$	6 6 4 4 6	30 30–30 32 19 9–19 14 17 15–17 24 22 56–23 1 40 15–40 20	30 31 19 11 17 20 22 59 40 18	30 31 19 11 17 19 22 59 *	0 0 1 0 ...
$\left\{ \begin{array}{l} nq = (11\bar{2}0) : (11\bar{2}1) \\ qc = (11\bar{2}1) : (0001) \end{array} \right.$	4 4	34 14–34 17 55 44–55 48	34 15 55 45	34 15 55 45	0 0
$\left\{ \begin{array}{l} mo' = (10\bar{1}0) : (02\bar{2}1) \\ o'q = (02\bar{2}1) : (12\bar{1}1) \\ ms = (01\bar{1}0) : (21\bar{3}1) \\ so' = (21\bar{3}1) : (20\bar{2}1) \end{array} \right.$	12 8	64 24–64 33 25 30–25 32	64 29 25 31	64 29 25 31 46 20 18 9	0 0
$\left\{ \begin{array}{l} ms = (10\bar{1}0) : (21\bar{3}1) \\ sq = (21\bar{3}1) : (11\bar{2}1) \\ mq = (10\bar{1}0) : (11\bar{2}1) \\ qo = (11\bar{2}1) : (01\bar{1}1) \\ mo = (10\bar{1}0) : (01\bar{1}1) \\ oo = (01\bar{1}1) : (11\bar{2}1) \end{array} \right.$ 10 6 7 3 44 12–44 18 26 50–26 52 71 5–71 12 37 44–37 46 44 16 26 51 71 9 37 45	30 20 13 57 44 17 26 51 71 8 37 44 1 0 1 1
$\left\{ \begin{array}{l} mo'' = (10\bar{1}0) : (01\bar{1}2) \\ o''o'' = (01\bar{1}2) : (11\bar{2}2) \end{array} \right.$	5 2	78 42–78 49 22 27–22 36	78 45 22 32	78 45 22 30	0 2
$\left\{ \begin{array}{l} nq = (11\bar{2}0) : (21\bar{3}1) \\ qq = (21\bar{3}1) : (12\bar{1}1) \end{array} \right.$	8 3	65 33–65 36 48 49–48 53	65 34 48 51	65 35 48 50	1 1
$\left\{ \begin{array}{l} no' = (11\bar{2}0) : (20\bar{2}1) \\ o'o = (20\bar{2}1) : (11\bar{2}1) \end{array} \right.$	12 8	41 42–41 46 48 14–48 17	41 44 48 15	41 45 48 15	1 0
$\left\{ \begin{array}{l} no = (11\bar{2}0) : (10\bar{1}1) \\ oo'' = (10\bar{1}1) : (11\bar{2}2) \end{array} \right.$	8 8	55 53–56 1 34 1–34 6	55 56 34 4	55 56 34 4	0 0
$\left\{ \begin{array}{l} no'' = (11\bar{2}0) : (10\bar{1}2) \\ o''o'' = (10\bar{1}2) : (01\bar{1}2) \end{array} \right.$	7 2	70 12–70 18 39 31–39 31	70 15 39 31	70 14 39 32	1 1
$\left\{ \begin{array}{l} cs = (0001) : (21\bar{3}1) \\ sp = (21\bar{3}1) : (21\bar{3}0) \end{array} \right.$	65 59 24 1

CHAPTER XXIII

TRIGONAL SYSTEM.

Three equal crystallographic axes equally inclined at an angle other than 90° .

Characterised by a trigonal axis of symmetry.

THE classes which are here included in a separate trigonal system were formerly considered as hemihedral and tetartohedral classes of the hexagonal system. Although the idea of hemihedrism and tetartohedrism has now been displaced by the consideration of the various classes of crystals as distinct types of homogeneous structures, there is still considerable ground for considering the hexagonal and trigonal classes together; but on the whole, especially as when combined they are no less than twelve in number, it is undoubtedly better to consider the five clearly hexagonal classes as a separate hexagonal system, as was done in the

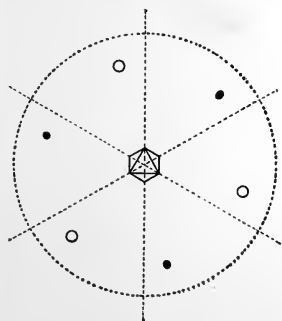


FIG. 304.—Class 17.

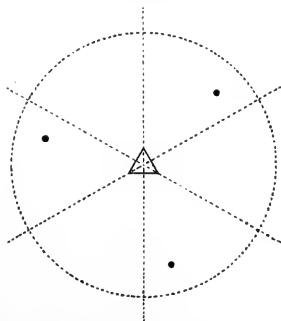


FIG. 305.—Class 16.

last chapter but one, and the seven classes distinguished by the presence of a merely trigonal and not hexagonal axis of symmetry as a separate trigonal system. An interesting link between the two is afforded, however, by class 17 of the trigonal system, in which we have a trigonal axis which is also a hexagonal axis of compound, alternating, or mirror-image symmetry. This case has already been considered in discussing the three instances of compound symmetry, on page 142 in Chapter X. Fig. 70 representing it in stereographic projection is repeated here in Fig. 304, for the sake of comparison with the purely trigonal axis

displayed by class 16 of the trigonal system, the projection of which is given alongside in Fig. 305. The difference between the two classes is that the purely trigonal axis only necessitates the presence of three faces, those of a trigonal pyramid of the third order in the most general case shown in Fig. 305, the poles of which lie on the same (upper) hemisphere, whereas on the other hand the hexagonal axis of compound symmetry furnishes three additional poles on the other (lower) hemisphere in alternate 60° positions from the first three, by rotation of the latter for 60° round the axis and then immediate reflection across the equatorial plane.

The same system of four crystallographic axes, a principal vertical axis and three axes mutually inclined at 60° in the equatorial plane perpendicular to the principal axis, may be employed with trigonal crystals as for hexagonal crystals. But there is another possible mode of description of purely trigonal crystals which is not only equally suitable, but which is also more in consonance with their structure. Moreover, it avoids the necessity for the use of a fourth axis, enabling us to return to the general Millerian system of three axes. It is consequently known as Miller's method of description, the four-axial method, as pointed out in Chapter XXI., being generally referred to as the Bravais-Miller method. The Millerian axes are derived in the following manner.

In the stereographic projection of the general form of the simplest trigonal class, 16, given in Fig. 305, each of the three pyramidal poles lies within one of the 60° -triangles, alternate triangles being left unoccupied, and the trigonal pyramid produced is therefore one of the third order on the analogy of the tetragonal system. But if the special case be considered in which the poles lie on three of the 60° -radii, as shown in Fig. 306, a trigonal pyramid of the first order is produced, employing the nomenclature of the hexagonal system. But three of the 60° -radii only are occupied by the poles, namely, those marked with a positive sign in Fig. 306; the other three radii marked with a negative sign, however, are the projections of the edges of the pyramid, for these latter lie in the planes formed by the vertical trigonal axis with each of the three diameters representing in the projection the result of the operation of that axis. In Miller's system

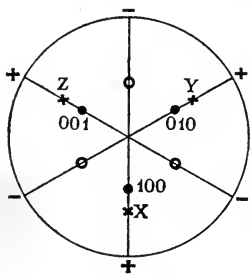


FIG. 306.

of axes these three edges of the primary first order trigonal pyramid are taken as the three axial directions. They are equal and equally inclined to each other, but the angle is not 90° . The actual edges themselves, as represented by the three negative radii, are considered as the negative halves of the axes, and their prolongations upwards, beyond the point of their intersection (the apex of the pyramid), are taken as the positive halves. The three faces of the pyramid, the poles of which fall on the positive halves, are the axial planes, and, just as in the case of the cube, the faces of which are the crystallographic axial

planes of the cubic system, they have symbols containing the same index numbers, namely, (100), (010), and (001). The trigonal solid formed by the three pyramidal faces thus chosen as axial planes, and their three parallel fellow-faces at the other side of the centre, is the rhombohedron. Its projection is obtained by placing rings on the negative radii at the same distance from the centre as the three solid-dot poles. It may be regarded as a cube more or less deformed by compression along a diagonal, the edges remaining the same and equal but the angles being deformed out of their 90° magnitudes. The cube can thus be legitimately taken as an illustration of the rhombohedron, and as a matter of fact the cube is a special case, crystallographically possible, of the rhombohedron. When a cube is thus set up on one of its solid angles, with a diagonal vertical, representing the trigonal axis, and is compressed along this axis, we have a very fair representation of the rhombohedron and of the Millerian axes (its edges) and axial planes (its faces). If several such rhombohedra of different diagonal dimensions are developed on a crystal, as frequently happens, that one which most nearly resembles a cube is chosen as the primary one, determining the axes, unless the cleavage directions correspond to another rhombohedron, when that would be chosen in preference, as being the most important structurally.

The trigonal (often called rhombohedral) Millerian axial system thus consists of three equal crystallographic axes intersecting at equal angles which are in general other than 90° , and disposed symmetrically about an axis of trigonal symmetry, which is arranged vertically just as is the hexagonal axis of the hexagonal system or the tetragonal axis of the tetragonal system, but which is not in Millerian trigonal notation a crystallographic axis, although it is so when the Bravais-Miller method is used. The stereographic projection of the poles of the axes is shown by the crosses in Fig. 306; they differ from those of the cubic system by not being identical in position with those of the axial facial-planes (100), (010), and (001). They are also given later in Fig. 325 (page 344), and their exact position is defined in connection with that figure.

The rhombohedron with its three crystallographic axes and the vertical trigonal symmetry axis are shown in Fig. 307. The parametral plane, defining the relative lengths of the axes, is the basal plane, which thus has the symbol (111) in Millerian trigonal notation. This plane is parallel to the equatorial plane and perpendicular to the trigonal axis, just as it is in the hexagonal system.

The elements in rhombohedral Millerian notation are consequently :

$$a : a : a = 1 : 1 : 1. \quad \alpha = \beta = \gamma = \text{any angle other than } 90^\circ.$$

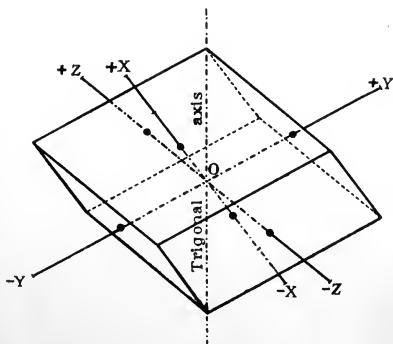


FIG. 307.

The value of this angle α is at once obtained by calculation from the goniometrical angle over any of the axial edges, that is between any pair of rhombohedral faces intersecting in one of the edges chosen as an axial direction (see page 371).

The three lower pyramidal faces of the rhombohedron, parallel to the upper three, have the Millerian symbols $(\bar{1}00)$, $(0\bar{1}0)$, and $(00\bar{1})$ respectively, as in the cases of the three faces of the cube parallel to the three chosen as axial planes. Indeed, in general in the trigonal system with Millerian axes, a parallel face has the same indices in the same order, but with all the signs reversed.

This Millerian system of axes is peculiarly suitable for the description of trigonal forms on account of the fact that the space-lattice systems of three of the classes included in the trigonal system are rhombohedral, that is, the elementary parallelepipedon is a rhombohedron. These classes are 17, 20 and 21. The other four classes have the simple trigonal prism for their elementary parallelepipedon, and in no class is the hexagonal prism the fundamental form. Hence there are very good structural grounds for the adoption of the Millerian axes for trigonal forms, and for the consideration of the trigonal as a separate system of symmetry.

The simple trigonal prism just referred to has its poles at the ends of the positive radii (see the more complete trigonal stereographic projection in Fig. 325), its faces being parallel to the trigonal vertical axis. It will be obvious that its facial indices must be $(2\bar{1}\bar{1})$, $(\bar{1}2\bar{1})$, and $(\bar{1}\bar{1}2)$, for each of the three faces, if imagined to be moved parallel to itself until it touches the ends of the two radii (projections of the axes) 60° on each side of it, will then bisect the radius at the end of which its pole lies, so that the intercept on that axis will be halved and the corresponding index number doubled relatively to the other two. This is the trigonal prism of the first order, which thus always has two of its index numbers equal and of the same sign.

The second order trigonal prism is indicated by the three poles on the primitive circle (Fig. 325) midway between the positive and negative radii; the indices will obviously be $(10\bar{1})$, $(\bar{1}10)$, and $(0\bar{1}1)$, for the symbol of each face must contain one 0 to indicate the parallelism of the face to one of the three axes, and the two other index numbers must be equal and of opposite sign, to indicate the fact that the pole lies midway between two axial radii of opposite sign, and they can consequently be expressed by a 1 and a $\bar{1}$.

Besides the rhombohedron of which the poles are given in Fig. 306, and which is shown in Fig. 307, it must be apparent that there is a second complementary one possible, the poles of which would be obtained by replacing the three dots representing the upper faces (100) , (010) , and (001) by rings, and the three rings representing the lower faces by dots. The two rhombohedra thus complementary to each other make up together the hexagonal bipyramid of the first order, hence the rhombohedron may be considered as a hemihedral form of the hexagonal system, and was entirely so considered until the development of the

theory of crystals as homogeneous structures. The indices of the three faces of the second rhombohedron in the upper hemisphere are $(2\bar{1}2)$ for the left face, $(22\bar{1})$ for the right face, and $(\bar{1}22)$ for the back face, the form symbol being $\{\bar{1}22\}$ (see Fig. 325). The fact, however, that the two complementary forms composing together the hexagonal bipyramid thus possess altogether different indices on Miller's system, and that the whole bipyramid, as a form, would therefore have to bear two symbols if Miller's indices were used to describe fully hexagonal forms, is an unanswerable argument against employing Miller's system for such purely hexagonal forms. The Bravais-Miller four-axes system, which describes the whole form by a single symbol, is consequently the only satisfactory one which can be employed with hexagonal forms, and it was for this reason that the four-axes method was employed exclusively in Chapter XXI. on the hexagonal system. Miller's axes are, however, perfect for the description of rhombohedral and other trigonal crystals, and, as we have seen, bear a definite relationship to their structure.

The first or primary rhombohedron $\{100\}$ is generally spoken of as the "direct" or positive one, and the secondary rhombohedron $\{122\}$ as the "inverse" or negative one. The two forms, and indeed any two trigonal direct and inverse forms in general, are connected by three important and most useful equations. If $\{pqr\}$ be the indices of the direct or positive form, and $\{p'q'r'\}$ represent those of the inverse or negative form (the letters hkl being reserved for use in the Bravais-Miller symbol $\{hkl\}$), then: $p' = -p + 2q + 2r$; $q' = 2p - q + 2r$; $r' = 2p + 2q - r$. The particular face the indices of which are afforded by the equations, and which are taken for the form symbol, is that represented by the pole lying on the same diameter in the projection, at an equal distance on the other side of the centre of the primitive circle, and on the same hemisphere as the pole represented by the symbol $\{pqr\}$.

We may now proceed to the discussion of the seven classes of the trigonal system. In the list of forms belonging to each class both Bravais-Miller and Millerian indices will be given, as both methods of description are in current use.

The seven classes of the trigonal system may be derived successively from class 16, the primitive class of the system possessing only the essential trigonal axis of symmetry, the projection of which is shown in Fig. 305. The general form of that class has been shown to be an upper or a lower trigonal (three-faced) pyramid, the poles of the upper one being exhibited in the figure.

Three of the other classes are obtained by adding to this upper pyramid three other similar lower faces in three different ways, to produce a simple closed form of higher symmetry. These three modes are indicated by Figs. 304 (given at the beginning of this chapter), 308 and 309. It is as if the lower pyramid were rotated round the vertical axis, with respect to the upper pyramid, and arrested in its rotation at three different positions.

In class 17 (Fig. 304) the poles (rings) appear in the projection equally removed, at 60° , from the three upper ones. This is the case already alluded to which occurs when the trigonal axis acts also as a

hexagonal axis of compound or reflective symmetry. The general form shown in the projection is a rhombohedron of the third order, which gives its name to the class. In class 18 (Fig. 308) the poles are at unequal distances between those of the upper poles, but analogously

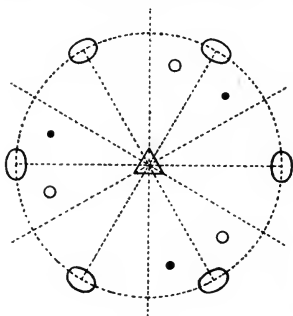


FIG. 308.—Class 18.

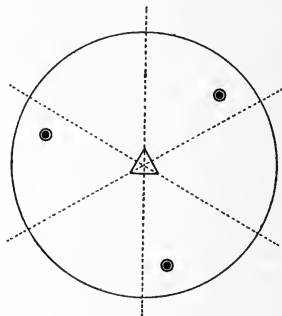


FIG. 309.—Class 19.

disposed as regards the crystallographic axes, so that they are represented by equal indices. These conditions are fulfilled when there are three digonal axes parallel to the basal plane, which are shown as diameters in the projection, intermediate between those representing the Millerian and identical with the Bravais-Millerian crystallographic axes. The trigonal bipyramid thus produced is of trapezohedral character. It is to this class 18 that quartz belongs. In class 19 (Fig. 309) the lower trigonal pyramid is precisely underneath the upper one, the poles being represented by rings around the dots. This obviously occurs when the basal plane is a plane of symmetry. The solid produced is the ordinary trigonal bipyramid.

A further class, 20, is derived directly from class 16 by the introduction of a symmetry plane parallel to the trigonal axis, and this involves, from the very nature of that axis, two others, making three planes of symmetry inclined at 120° to each other and all parallel to and intersecting in the trigonal axis. They are represented by the three diameters in the projection given in Fig. 310. The resulting solid is a ditrigonal pyramid.

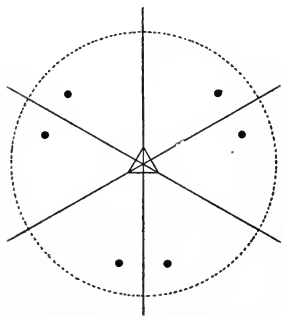


FIG. 310.—Class 20.

Lastly, we obtain two more classes of higher symmetry by adding to the symmetry elements of class 20 three digonal axes in two different ways. In class 21 the three digonal axes are the same as in class 18, namely, they bisect the angles between the projections of the Millerian crystallographic axes and thus also those between the three symmetry planes containing them. They are consequently identical in direction with the Bravais-Miller horizontal axes. This arrangement

gives six lower faces with poles corresponding to the six upper ones but at the alternate 60° positions, as shown in Fig. 311. It is like class 17, only with double poles, one symmetrically on each side of the Millerian crystallographic axes, rendering the solid a ditrigonal instead of merely a trigonal one. It is equivalent to considering the basal plane as a plane of compound (reflective) symmetry, combined with the

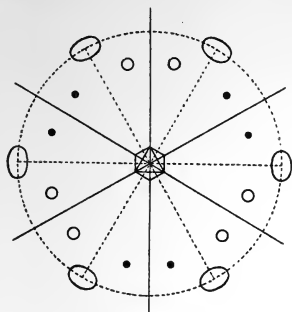


FIG. 311.—Class 21.

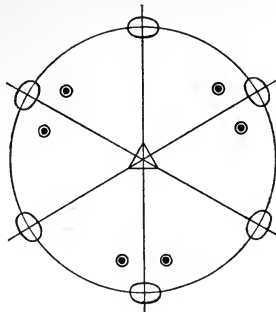


FIG. 312.—Class 22.

operation of a hexagonal axis instead of a trigonal one. The solid produced is the ditrigonal scalenohedron, which is thus of dihexagonal alternating type. This is the important class to which calcite belongs.

Class 22, the class of highest trigonal symmetry, is produced when the three digonal axes are parallel to the three symmetry planes, as shown in Fig. 312. The six lower poles then come immediately underneath the six upper ones, thus rendering the solid, the ditrigonal bipyramid, also symmetrical to the basal plane.

It will be obvious that of the foregoing seven classes of trigonal symmetry only classes 16 and 20 will exhibit hemimorphism, that is, polarity of the two ends of the trigonal axis, while classes 16 and 18 are the only two which afford enantiomorphous pairs of solids, not convertible one into the other by rotation.

Class 19, the trigonal bipyramidal class, is the only one of the 32 classes of symmetry which has never yet been observed in actual crystals.

Class 22.—Ditrigonal Bipyramidal Class. Holohedral-Trigonal Class, or Trigonal Hexagonal-Hemihedral Class. Type, Ditrigonal Equatorial.

This class exhibits the full trigonal symmetry, namely, a trigonal axis arranged vertically, and three symmetry planes mutually inclined at 120° intersecting in it, thus rendering it ditrigonal; together also with an equatorial plane of symmetry (horizontal, parallel to the basal plane), and three digonal axes lying in it at its intersections with the three vertical symmetry planes.

Only two examples of this class have yet been observed, namely, silver hydrogen phosphate, Ag_2HPO_4 , by Dufet in 1886, and benitoite, an acid titano-silicate of barium, $\text{BaTiSi}_3\text{O}_{10}$, by Louderback in 1907. The latter forms beautiful sapphire-blue crystals, which make excellent gem-stones.

The general form, the stereographic projection of which is given in Fig. 312, is a **ditrigrinal bipyramid**, which is shown in outline in Fig. 313; it is symmetrical to the equatorial basal plane in which the upper and lower pyramids unite, the shape of which is hexagonal with three more and three less obtuse angles, alternately arranged, corresponding to three larger and three smaller angles over the polar edges of the pyramid. The Millerian indices of this ditrigrinal bipyramid are $\{pqr\}$, and the Bravais-Miller indices $\{hkl\}$. The actual bipyramid shown in both Fig. 312 and Fig. 313 is the form $\{8\bar{1}4\}$ constructed to scale to the axial ratio of calcite. There is, however, a second form possible, having its poles arranged in pairs adjacent to the negative radii in the Millerian projection instead of the positive radii. The two forms are therefore distinguished as positive and negative respectively. The Millerian indices of the negative form are $\{p'q'r'\}$, where $p' = -p + 2q + 2r$, $q' = 2p - q + 2r$, and $r' = 2p + 2q - r$; and the Bravais-Miller symbol is $\{\bar{h}kl\}$. Thus the positive ditrigrinal bipyramid $\{20\bar{1}\}$ or $\{2\bar{1}3\}$ corresponds to the negative, complementary or inverse ditrigrinal bipyramid $\{\bar{4}25\}$ or $\{\bar{2}13\}$. The particular pole of which these latter inverse-form symbols are the facial indices, is the one at an equal distance on the other side of the centre, on the same diameter and in the same upper hemisphere as $(20\bar{1})$. (See Figs. 317 and 318 on pages 340 and 341.)

FIG. 313.—Positive Ditrigrinal Bipyramid.

It will be observed in Fig. 313 that the lower faces have altogether different indices to the upper faces; $(2\bar{1}2)$, for instance, in the case of the lower-right-front face corresponds to $(8\bar{1}4)$ as the indices of the face immediately above it, the face the indices of which are taken for the form symbol $\{8\bar{1}4\}$. It will be clear that this must be so, because the lower faces are parallel to the upper faces of the inverse form, and so will have the same numbers for indices as that inverse form, but with the signs all changed. Thus if we calculate the values of p' , q' , and r' (with the aid of the above formulæ) which correspond to p , q , r of the values $(8\bar{1}4)$, we shall find that $p' = -18$, $q' = 9$, and $r' = 18$, or, dividing by 9 throughout, $p' = -2$, $q' = 1$, and $r' = 2$; hence $(2\bar{1}2)$ are the indices of the inverse face, and if we change the signs we obtain $(2\bar{1}2)$ for the indices of the face parallel to that inverse face, that is, for the indices of the lower-right-front face in Fig. 313, just as they are marked on the face in the drawing.

The three Millerian axes and the vertical trigonal axis are shown in Fig. 313 in broken-and-dotted lines.

There are six special cases possible, giving rise to as many specific forms.

(1) When the poles lie in the three symmetry planes, each pair having coalesced into a single pole, a **trigrinal bipyramid of the first order** is produced. Two varieties are possible, convertible by 60° of rotation, according to which particular three of the six radii in the projection the pairs of poles approach and coalesce on. If they coalesce on the positive radii, as they appear to be about to do in Fig. 312, the positive form is produced; but if they coalesce on the other three, the negative radii, the negative or inverse form is developed. The Millerian indices of the positive form are $\{pqq\}$, and the Bravais-Miller indices $\{h0h\}$; in the case of the negative form the Millerian indices are $\{p'q'q'\}$, where $p' = -p + 4q$, and $q' = 2p + q$, and the Bravais-Miller indices are $\{\bar{h}0h\}$. Thus the inverse or negative form corresponding to the positive form $\{5\bar{1}1\}$ or $\{20\bar{2}1\}$ is $\{\bar{1}11\}$ or $\{\bar{2}021\}$, the 2 facial

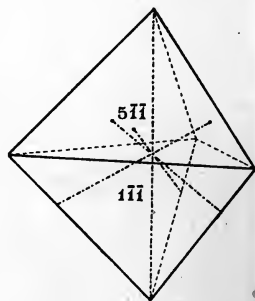


FIG. 314.—Positive First Order Trigrinal Bipyramid.

poles the indices of which are used in the form symbols being those on the back-and-front diameter, equidistant from the centre, in the upper hemisphere. (See Figs. 317 and 318.) The positive form $\{5\bar{1}\bar{1}\}$, drawn for the axial ratio of calcite, is shown in Fig. 314. The indices of the upper faces are $(5\bar{1}\bar{1})$, the front face marked in the drawing, $(\bar{1}5\bar{1})$, and $(\bar{1}\bar{1}5)$. The three lower faces, being parallel to the possible faces of the inverse form $\{\bar{1}11\}$, have the indices $(1\bar{1}\bar{1})$, the front face marked in Fig. 314, $(\bar{1}1\bar{1})$, and $(\bar{1}\bar{1}1)$.

(2) The second special case is when the pairs of poles leave the proximity of the symmetry planes and take up a more separated position exactly midway between those planes, that is, at 30° from them. The angles between the faces over all six polar edges then become equal and a **hexagonal bipyramid of the second order** is produced. The Millerian and Bravais-Miller symbols are respectively $\{pqr\}$ and $\{h.h.\bar{2}h.l\}$. In the Millerian symbol one index is always the mean of the other two.

(3) The third special case is afforded when the poles remain more or less adjacent to the symmetry planes but lie on the primitive circle. The solid produced is a **ditrigonal prism**, the six faces of which are parallel to the trigonal axis of symmetry. It is shown in Fig. 315, closed by the basal pinakoid. Its symbols are $\{pqr\}$, where $p+q+r=0$, and $\{hik0\}$. There are two such forms possible, the one for which the symbols have just been given being the positive form, having its poles adjacent to the positive radii in the projection. The negative form, with poles adjacent to the negative radii, has the symbols $\{p'q'r'\}$ or $\{hik0\}$, where p' , q' and r' have the values given on the previous page. The one actually shown, drawn to scale to the axial ratio of calcite, is the form $\{7\bar{2}5\}$, which is the ditrigonal prism corresponding to the bipyramid $\{8\bar{1}4\}$ shown in Fig. 313. The poles of both prisms and both bipyramids are shown in the stereographic projection Fig. 318.

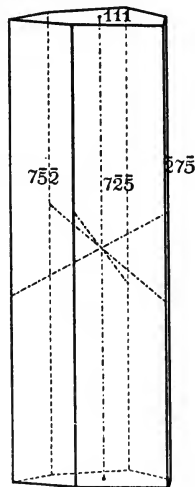


FIG. 315.—Positive Ditrigonal Prism.

(4) Just as case (3) corresponds to the ditrigonal bipyramid, so there is another special case of a prism corresponding to (1), namely, **the trigonal prism of the first order**, when the two poles near a symmetry plane coalesce into a single pole lying on the primitive circle at the end of the digonal symmetry axis which is also the projection of the symmetry plane. There are two varieties of this form, a positive and a negative, convertible one into the other by rotation for 60° about the trigonal axis, just as in the case of the pyramid. The symbols of the positive form, the poles of which occupy the positive radii, are $\{2\bar{1}1\}$ and $\{10\bar{1}0\}$. It is shown in Fig. 316, closed by the basal pinakoid. The symbols of the negative form are $\{\bar{2}11\}$ and $\{\bar{1}010\}$; its poles occupy the three negative radii.

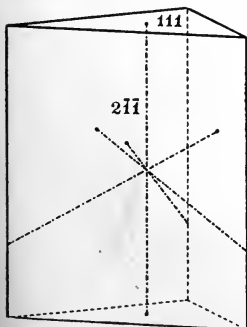


FIG. 316.—Positive First Order Trigonal Prism.

(5) This is the special case corresponding to (2), when the poles arranged at 30° from the digonal axes lie on the primitive circle. The result is the **hexagonal prism of the second order**, the symbols of which are $\{10\bar{1}\}$ and $\{11\bar{2}0\}$.

(6) The last special case is when all the six ditrigonal pyramid poles converge towards and coalesce in the centre of the projection, when as the final limiting form the **basal pinakoid** with pole at the centre itself is produced, which in Millerian notation is the parametral form $\{111\}$ and in Bravais-Miller notation is denoted by the symbol $\{0001\}$.

The net result of the above detailed discussion of the possible holohedral trigonal forms is embodied in the following list:

List of Forms in Class 22.

$\{111\} = \{0001\}$ Basal pinakoid. 2 faces.

$\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ Positive trigonal prism of the first order ; $\{211\} = \{\bar{1}010\}$ negative trigonal prism of the first order. Each 3 faces.

$\{10\bar{1}\} = \{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.

$\{pqr\}$ (where $p+q+r=0$) = $\{hik0\}$ Positive ditrigonal prism ; $\{p'q'r'\}$ (where $\left. \begin{matrix} p' = -p+2q+2r \\ q' = 2p-q+2r \\ r' = 2p+2q-r \end{matrix} \right\}$) = $\{\bar{h}ik0\}$ negative ditrigonal prism. Each 6 faces.

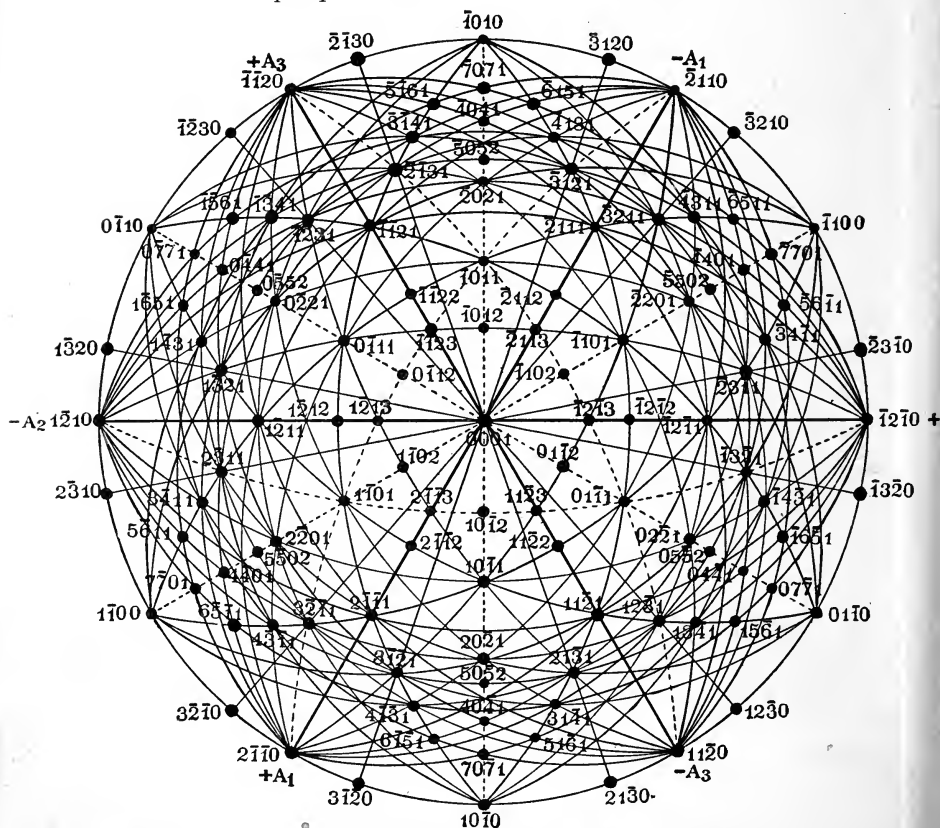


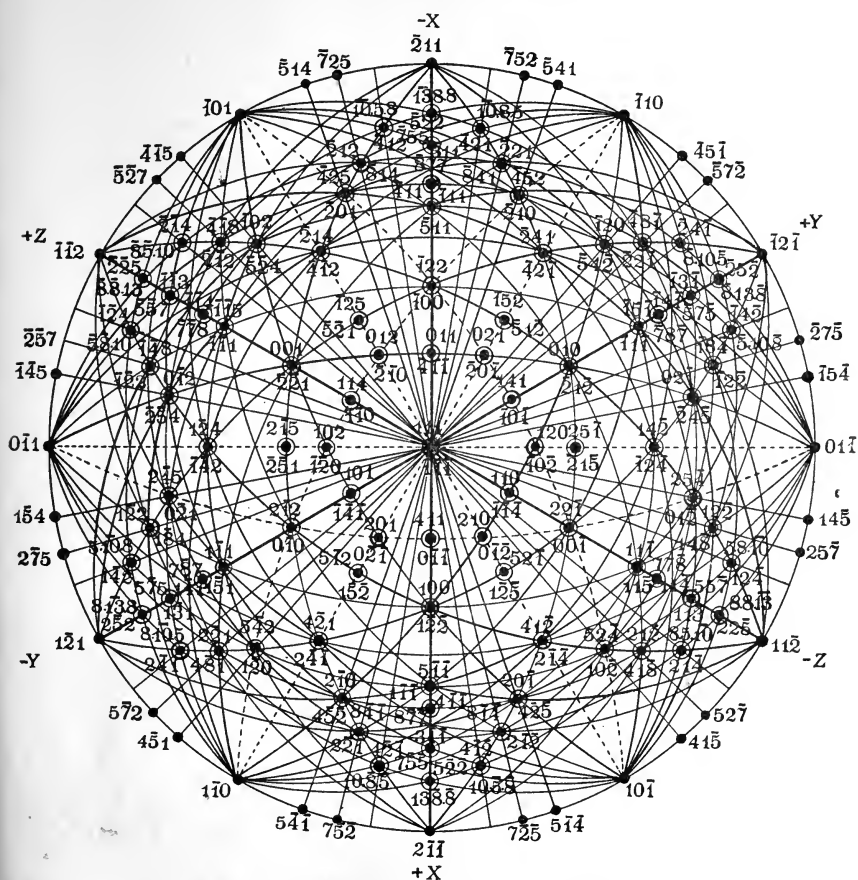
FIG. 317.—Bravais-Miller Stereographic Projection of Trigonal or Hexagonal Forms.

$\{pqq\}$ (two indices always equal) = $\{h0\bar{h}l\}$ Positive trigonal bipyramid of the first order, including the primary positive form $\{100\} = \{10\bar{1}1\}$; $\{p'q'q'\}$ (where $\left. \begin{matrix} p' = -p+4q \\ q' = 2p+q \end{matrix} \right\}$) = $\{\bar{h}0h\bar{l}\}$ negative trigonal first order bipyramid, including the complementary primary negative form $\{\bar{1}22\} = \{\bar{1}0\bar{1}1\}$. Each 6 faces.

$\{pqr\}$ (where one index = mean of other two) = $\{h.h.2h.l\}$ Hexagonal bipyramid of the second order. 12 faces.

$\{pqr\} = \{hikl\}$ Positive ditrigonal bipyramid ; $\{p'q'r'\}$ (where $\left. \begin{matrix} p' = -p+2q+2r \\ q' = 2p-q+2r \\ r' = 2p+2q-r \end{matrix} \right\}$) = $\{\bar{h}ikl\}$ negative ditrigonal bipyramid. Each 12 faces.

The necessity of being familiar with both the Bravais-Miller and the Millerian methods of description of trigonal crystals, owing to the employment of both in current crystallographic work, renders it essential to be perfectly clear as to the mode of deriving the indices of faces and symbols of forms in the two cases, and to have a ready means of converting those of the one into those of the other. The two com-



four, l) being negative instead of positive, the A_1 , A_2 , and A_3 indices being identical. In the case of the latter projection (Fig. 318) the poles of the lower hemisphere are actually indicated by rings, and as the two symbols are different they are both given, the one above the pole referring to the dot (the upper hemisphere pole) and the symbol below referring to the ring-pole, that of the lower hemisphere. An inspection of them will at once clear up any lingering ambiguity or difficulty arising from the perusal of the preceding description of class 22 forms, and they will be of use also in verifying the general or particular symbols given in connection with the other six classes of the trigonal system. The axial ratio of calcite was used in constructing both figures.

Three extremely useful equations connect the Bravais-Miller and Millerian indices of any face of a trigonal crystal, and these indispensable formulæ may be readily verified in the case of any pole shown on the two projections just given. As before, three different letters, namely, p , q , and r , are here employed to represent the Millerian index numbers, to those used for the Bravais-Miller indices, h , i , k , and l , in order that the two kinds of index numbers may at once be identified.

The three connecting equations are:—

$$p = h - k + l; \quad q = i - h + l; \quad r = k - i + l.$$

$$h = \frac{p - q}{3}; \quad i = \frac{q - r}{3}; \quad k = \frac{r - p}{3}; \quad l = \frac{p + q + r}{3}.$$

Given either the Millerian or the Bravais-Miller indices, the other indices can always be found with the aid of these equations.

Class 21.—Ditrigonal Scalenohedral Class. Calcite Class.

Rhombohedral Hexagonal-Hemihedral Class. Type, Dihexagonal Alternating.

This is one of the most important classes of the trigonal system, for it is the class to which calcite, calcium carbonate, CaCO_3 , and its allied minerals belong. The elements of symmetry are shown in Fig. 319. They are the essential trigonal axis, three symmetry planes intersecting in it and thus rendering it a ditrigonal axis, and three digonal axes lying in the equatorial plane midway between the symmetry planes, that is, identical with the three Bravais-Miller hexagonal crystallographic axes. It differs from the holohedral class 22 by having no equatorial plane of symmetry, and in the positions of the digonal axes, these lying in the three symmetry planes in class 22. The operation of the three digonal axes in class 21 has the same effect as if the ditrigonal axis were a hexagonal axis of compound symmetry, that is, a hexagonal axis combined with simultaneous

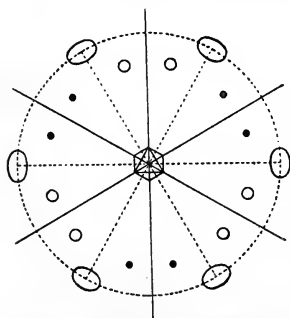


FIG. 319.—Symmetry Elements and General Form of Class 21.

reflection across the equatorial plane. The effect is also to produce centro-symmetry.

The general form the poles of which are shown in Fig. 319 is a twelve-faced solid consisting of two ditrigonal pyramids, an upper one and a lower one, the lower one of which is rotated for 60° with respect to the upper, round the vertical axis, so that the equatorial junction-edges of the two do not form a simple hexagon, lying in the equatorial plane and showing two alternating kinds of angles as in class 22, but is of a zigzag character, above and below the equator. The digonal axes emerge at the centres of the edges forming the zigzag. This general form is the **ditrigonal scalenohedron**, so called because each face is a scalene triangle. It is shown in Fig. 320, the particular representative being the common form of calcite $\{20\bar{1}\}$, drawn to scale to the correct axial ratio for calcite. The polar edges are of two kinds, the angles over them being alternately larger and smaller, as in the case of the ditrigonal bipyramid of class 22. The scalenohedron possesses in fact the faces of an upper ditrigonal pyramid $\{pqr\} = \{hikl\}$, and of the lower parallel ditrigonal pyramid $\{\bar{p}\bar{q}\bar{r}\} = \{\bar{h}\bar{i}\bar{k}\bar{l}\}$. The broken-

and -dotted lines indicate the Millerian axes and the vertical

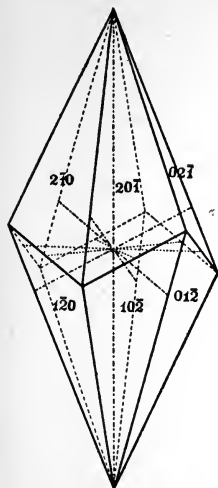


FIG. 320.—Positive Ditrigonal Scalenohedron.

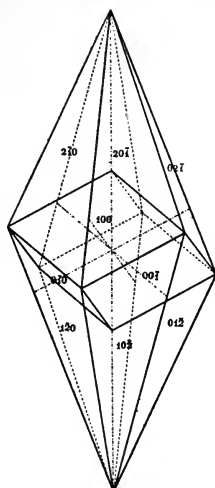


FIG. 321.—Primary Rhombohedron and Scalenohedron of Calcite.

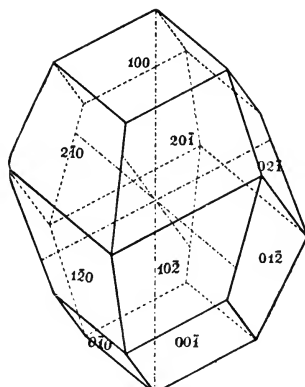


FIG. 322.—Combination of Rhombohedron and Scalenohedron.

ditrigonal axis, and the dotted lines emerging in the centres of the zigzag edges represent the digonal axes (the Bravais-Miller axes). It will be obvious, moreover, that there will be two such scalenohedra, namely, the one just described, which may be represented by the general symbol $\{pqr\} = \{hikl\}$ and termed the positive form, its upper poles being adjacent to the positive halves of the axes, and a second negative form $\{\bar{p}'\bar{q}'\bar{r}'\} = \{\bar{h}\bar{i}\bar{k}\bar{l}\}$, the upper poles of which are adjacent to the negative halves of the axis, and 60° from those of the positive form in the projection. The values of p' , q' , and r' are those for the inverse forms given in the introduction to this chapter and in the description of class 22, namely, $p' = -p + 2q + 2r$, $q' = 2p - q + 2r$, $r' = 2p + 2q - r$.

It is an interesting fact that the zigzag edges of the scalenohedron coincide with the six lateral edges of the rhombohedron, provided the indices bear a certain simple relationship. This occurs in the case of calcite, the common scalenohedron of which $\{20\bar{1}\}$, the predominating form of the well-known "dog-tooth" spar, and the primary rhombohedron $\{100\}$, so chosen because it is also a very common form and its facial planes are the cleavage planes of the mineral, have the same zigzag edges, as is shown in Fig. 321. The faces of this scalenohedron consequently bevel the equatorial edges of the primary rhombohedron, as will be still clearer from Fig. 322, which shows a combination

almost equally developed of $\{20\bar{1}\}$ and $\{100\}$. Moreover, the hexagonal prism of the second order $\{10\bar{1}\}$ truncates the edges of either, as is shown in the case of the scalenohedron $\{20\bar{1}\}$ in Fig. 323, and in the case of the rhombohedron $\{100\}$ in Fig. 324; in this latter figure the prism is largely developed, its ends being differently terminated by the two halves, upper and lower, of the rhombohedron.

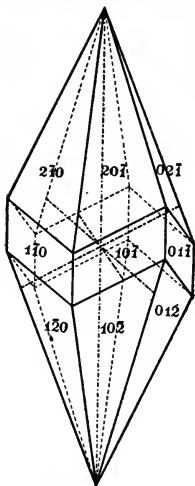


FIG. 323.—Combination of Scalenohedron and Second Order Hexagonal Prism.

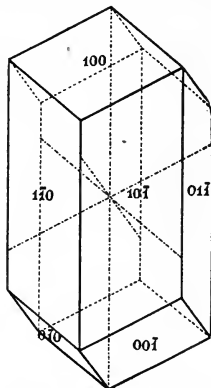


FIG. 324.—Combination of Rhombohedron and Second Order Hexagonal Prism.

The scalenohedron has one pole in each of the 30° triangles in the projection, which are formed by two radii and the twelfth part of the primitive circle. It will be obvious, therefore, that this general form will have the usual six special cases, in which the pole leaves the inside of the triangle and migrates on to one or other of the six elements of this spherical triangle. These cases will be considered in turn.

(1) When the pole falls on the crystallographic axial radius in the projection, as in Fig. 306 and as at r in the enlarged projection Fig. 325, the normal to the face thus lying in one of the three vertical symmetry planes, the six-faced solid produced, having three faces in the upper hemisphere, alternately arranged, is a **rhombohedron**. There is a positive (or direct) and a negative (or inverse) variety as in the case of the scalenohedron, according to which alternate three radii in the projection are occupied by the upper poles, the positive scalenohedron corresponding to the positive rhombohedron. The positive rhombohedron is the one which has one of its three upper faces fronting the observer; its projection has already been shown in Fig. 306, which represents the primary rhombohedron $\{100\}$, and its actual appearance in Fig. 307, in discussing the derivation of the Millerian axes. It is also indicated by the poles r in Fig. 325. It is reproduced in Fig. 326, the Millerian axes being indicated by the broken-and-dotted lines.

The negative rhombohedron, r' in Fig. 325, is obtained by rotating the positive

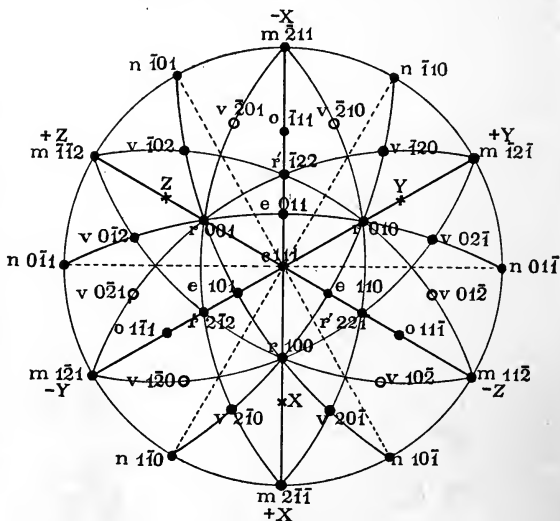


FIG. 325.—Stereographic Projection of Common Forms of Calcite.

one round the vertical trigonal axis for 60° or 180° . It is often called the inverse rhombohedron. Every rhombohedron consists of three pairs of parallel faces, each bounded by four equal sides when evenly developed, and in any case by two pairs of equal sides forming a parallelogram. It may be regarded as a cube deformed by either compression or elongation along a diagonal. If other rhombohedra are developed on a crystal, besides the primary one $\{100\}$ chosen as affording the axial planes, and which in the case of calcite is unmistakably the most

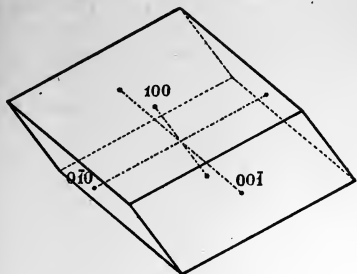


FIG. 326.—Positive Primary Rhombohedron.

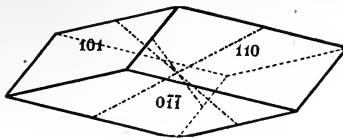


FIG. 327.—Obtuse Negative Rhombohedron.

important one as the mineral cleaves very readily along those planes, they are termed obtuse or acute according as the pole comes between the centre and the pole of $\{100\}$ or its inverse $\{2\bar{2}\bar{1}\}$, or between either of the latter and the pole $\{2\bar{1}\bar{1}\}$ or $\{1\bar{1}\bar{2}\}$ on the primitive circle. Thus $e = \{110\}$ in Fig. 325 is an obtuse and $o = \{1\bar{1}\bar{1}\}$ an acute inverse or negative rhombohedron. The obtuse negative rhombohedron $\{110\}$ is shown in Fig. 327.

The rhombohedron of this class 21 is really a rhombohedron of the first order, and as such is to be clearly distinguished from the more general rhombohedron of the third order which we shall have to consider in class 17; it has always two equal indices, by virtue of the fact that the normal to each face lies in one of the three trigonal planes of symmetry, which renders it symmetrical to the two crystallographic axes lying in the two other planes of symmetry. Indeed the rhombohedron may be considered as built up of two trigonal pyramids the faces of which are parallel to each other; the lower pyramid not being immediately under the upper so as to have a common base, but rotated 60° with respect to it, round the vertical trigonal axis, just as the scalenohedron has been shown to possess the faces of an upper ditrigonal pyramid and of the lower one having faces parallel to it, the parallelism having been brought about by the lower one being rotated 60° with respect to it. The general symbol is therefore that of the trigonal pyramid of the first order, namely, $\{pqq\}$ in Millerian notation.

The indices of the upper faces of the inverse or negative rhombohedron complementary to the primary one $\{100\}$ have already been shown in the introduction to this chapter to be $\{2\bar{1}\bar{2}\}$, $\{2\bar{2}\bar{1}\}$, and $\{\bar{1}22\}$, the form symbol being $\{\bar{1}22\}$. In general terms the Millerian symbol of the inverse form is $\{p'q'r'\}$, where the usual equations connecting trigonal direct and inverse forms apply, namely, $p' = -p + 2q + 2r$, $q' = 2p - q + 2r$, $r' = 2p + 2q - r$. In this special case, where q and r are equal, the inverse form is $\{p'q'q'\}$, where $p' = -p + 4q$ and $q' = 2p + q$. In Bravais notation the direct (positive) and inverse (negative) forms bear the symbols $\{h0\bar{h}l\}$ and $\{\bar{h}0hl\}$ respectively.

It is an interesting fact that the cube is the intermediate limiting form between the possible obtuse-angled and acute-angled rhombohedra. In this special case, which is crystallographically possible, the axial ratio on the Bravais-Miller system being $a : c = 1 : 1.2247$, the faces of the rhombohedron become squares, and the angles, both over the polar edges and the zigzag middle edges, all right angles; moreover, if the edges were taken for the Millerian axes the axial angles would of course also be right angles, and the lengths of the axes are already equal. But a rhombohedral cube belonging to the trigonal system would at once be distinguished from a true cube of the cubic, optically isotropic or singly-refracting, system by the fact that it would show double refraction, and the phenomenon in convergent polarised light of a single optic axis along a diagonal of the apparent cube. This is an exceptionally

striking case of the value of the optical properties, to be fully discussed in later chapters, in confirming or even determining the system to which a crystal belongs.

The frequently occurring combination of the primary direct rhombohedron $\{100\}$ with the hexagonal prism of the first order $\{2\bar{1}\bar{1}\}$ is shown in Fig. 328; the upper and lower halves of the rhombohedron form the two different terminations of the prism.

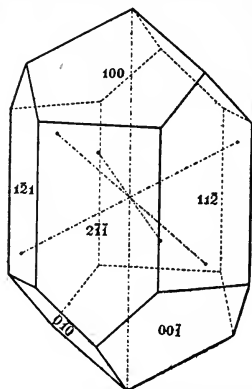


FIG. 328.—Combination of Rhombohedron and First Order Hexagonal Prism.

Fig. 329 gives a typical combination of the primary rhombohedron $r = \{100\}$ with the obtuse

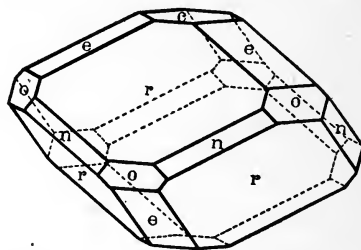


FIG. 329.—Combination of Three Rhombohedra, Second Order Hexagonal Prism, and Basal Pinakoid.

inverse rhombohedron $e = \{110\}$, the acute inverse rhombohedron $o = \{1\bar{1}\bar{1}\}$, the hexagonal prism $n = \{10\bar{1}\}$, and the basal pinakoid $c = \{111\}$. The poles of all these forms are shown in Figs. 325, 317, and 318; in Fig. 325 the poles of the primary inverse rhombohedron $r' = \{22\bar{1}\}$, the prism $m = \{2\bar{1}\bar{1}\}$, and the positive scalenohedron $v = \{20\bar{1}\}$ are also given, as well as the poles of emergence of the axes X, Y and Z, which are marked by a cross in each case. The axial ratio used in this projection is that of calcite, $a : c = 1 : 0.8545$. The position of these axial poles is readily calculated, for the angle mX , mY , or mZ is equal to that between the polar edge of the rhombohedron parallel to the axis in question and the equatorial plane, and the tangent of this angle is equal to the axial ratio divided by $\sqrt{3}$, that is, $\tan mX = \frac{c}{a\sqrt{3}}$. In the case of calcite this

gives $26^\circ 16'$ for mX . Now this is also the angle ce , between (111) and (011) . Hence the form e directly replaces the edges of r .

The predominating primary rhombohedron $\{100\}$ resembles closely a cube, and the other forms present on the crystal shown in Fig. 329 cause it to appear to be modified by the octahedron and rhombic dodecahedron, and if the first-mentioned form could be somewhat elongated parallel to the vertical axis the illusion would be perfect.

The rhombohedron is obviously, therefore, one of the most important of all crystallographic forms.

(2) The second special case occurs when the pole falls on the other kind of diameter in the projection, namely, on one of the digonal axes, at 30° from, and midway between, two crystallographic-axial-plane diameters. The operation of the three symmetry planes and the three digonal axes then determines that there shall be six such poles in each hemisphere, two immediately above and below each other on each of the digonal axial radii, and the result is consequently a **hexagonal bipyramid of the second order**, which is thus the limiting case of scalenohedra the two kinds of polar angles of which are becoming more and more nearly equal, and the equatorial zigzag edges of which are straightening out until they eventually become the hexagonal section of the hexagonal bipyramid, lying wholly in the equatorial plane. The Bravais-Miller symbol for this form has already been shown to be $\{h.h.\bar{2}h.l\}$.

The Millerian symbol is such that one index is always the mean of the other two, and the pole corresponding to the Bravais-Miller symbol just given will be $\{pqr\}$.

(3) When the pole lies anywhere on the primitive circle, between the ends of the two axial diameters, the form produced is the **dihexagonal prism**, the Bravais-Miller symbol of which is $\{hk\bar{0}\}$, and the Millerian symbol $\{pqr\}$, where the sum of p , q , and r is 0. The dihexagonal prism has invariably two different angles, alternately larger and smaller, for the particular one with equal angles, the section of which would be a regular twelve-sided polygon, is impossible on account of irrationality and the fact that impossible twelve-fold symmetry would thereby be introduced.

(4) When the pole migrates to the end of a crystallographic axial diameter, at its intersection with the primitive circle, the **hexagonal prism of the first order** is produced, the Bravais-Miller symbol being as in the hexagonal system $\{10\bar{1}0\}$, and the Millerian symbol being $\{2\bar{1}\bar{1}\}$.

(5) When the pole migrates to the end of the other kind of diameter, a digonal axis, the **second order hexagonal prism** is produced, $\{11\bar{2}0\}$ or $\{10\bar{1}\}$.

(6) Finally, when the pole migrates to the remaining element of the spherical triangle the intersection of two axial radii at the centre, that is, to that corner of the triangle which occupies the centre of the primitive circle and of the whole projection, the **basal pinakoid** is produced, $\{0001\}$ or $\{111\}$.

To summarise, we have the following forms in this class :

List of Forms in Class 21.

$\{111\} = \{0001\}$ Basal pinakoid. 2 faces.

$\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ Hexagonal prism of the first order. 6 faces.

$\{10\bar{1}\} = \{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.

$\{pqr\}$ (where $p+q+r=0$) = $\{hk\bar{0}\}$ Dihexagonal prism. 12 faces.

$\{pqg\}$ (two indices always equal) = $\{h0\bar{h}l\}$ Positive or direct rhombohedron of the first order, including the primary rhombohedron $\{100\} = \{10\bar{1}1\}$; $\{p'q'q'\}$ (where $\begin{cases} p' = -p+4q \\ q' = 2p+q \end{cases}$) = $\{\bar{h}0hl\}$ negative or inverse rhombohedron of the first order, including the complementary inverse primary one $\{\bar{1}22\} = \{\bar{1}011\}$. Each 6 faces.

$\{pqr\}$ (where one index = the mean of the other two) = $\{h.h.\bar{2}h.l\}$ Hexagonal bipyramid of the second order. 12 faces.

$\{pqr\} = \{h\bar{k}l\}$ Positive ditrigonal scalenohedron. $\{p'q'r'\}$ (where $p' = -p+2q+2r$, $q' = 2p-q+2r$, $r' = 2p+2q-r$) = $\{\bar{h}ikl\}$ negative ditrigonal scalenohedron. Each 12 faces.

By far the most important substance crystallising in this class of the trigonal system is calcite, calc spar, or Iceland spar, carbonate of lime, CaCO_3 . The name Iceland spar is usually confined to the primary rhombohedral variety (Fig. 326) of the clear transparent mineral, which has in the past been found in large pieces, the original discovery having occurred in the seventeenth century at Eskifjörður, a cavity of over 5000 cubic feet having been found filled with the calc spar. Large crystals are now becoming rare and exceedingly valuable on account of the use of the spar for polarising purposes in the Nicol prism. This well-known piece of optical apparatus is simply a cleavage rhomb of calc spar parallel to the faces of the primary rhombohedron $\{100\}$, cut along a certain diagonal direction into two halves, which are subsequently cemented together again by Canada balsam in such a manner as to eliminate by total reflection one of the two rays of

polarised light into which ordinary light is divided when it enters the spar, thus allowing the other ray to pass out alone as a beam of plane polarised light. A detailed description of it will find its proper place in Chapter XXXIX.

In the next chapter a crystal of calcite will be goniometrically described in detail, as an example of trigonal measurements and calculations. Figs. 320 (scalenohedron $\{20\bar{1}\}$), 322 (rhombohedral $\{100\}$ and scalenohedron $\{20\bar{1}\}$), 323 (scalenohedron $\{20\bar{1}\}$ and second order hexagonal prism $\{10\bar{1}\}$), 324 (rhombohedral $\{100\}$ and second order hexagonal prism $\{10\bar{1}\}$), 328 (rhombohedral $\{100\}$ and first order hexagonal prism $\{2\bar{1}\bar{1}\}$), and the rich rhombohedral combination shown in Fig. 329 all represent calcite crystals, drawn exactly in accordance with the correct axial ratio. The stereographic projections given in Figs. 317, 318, 325 and 360 (next chapter) all refer to calcite.

*Class 20.—Ditrigonal Pyramidal Class. Tourmaline Class.
Hemimorphic-Hemihedral Class. Type, Ditrigonal Polar.*

This class possesses only the three symmetry planes and the essential trigonal (here consequently ditrigonal) axis in which they intersect. Hence the two ends of this axis may exhibit quite different forms. Fig. 330 will render the nature of the symmetry clear.

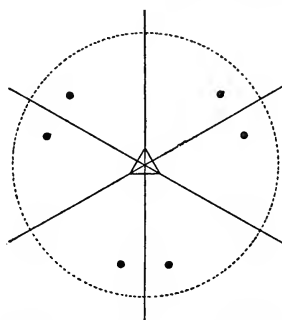


FIG. 330.—Symmetry Elements and General Form of Class 20.

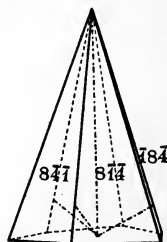


FIG. 331.—Positive Upper Ditrigonal Pyramid.

The general form is obviously a simple **ditrigonal pyramid**, the edges of which lie in the symmetry planes, exactly like the upper or lower half of either of the ditrigonal bipyramids of class 22 already described. It is shown in Fig. 331, closed below by the basal plane. The upper form represented in the figure has the symbols $\{pqr\}$, and $\{hikl\}$, and the lower one with faces parallel to it is denoted by $\{\bar{p}\bar{q}\bar{r}\}$ (that is, the signs of p , q and r are the opposite of what they are for the upper form) and $\{hikl\}$. This upper form having its pairs of poles adjacent to the positive radii in the projection, and the lower form parallel to it, are termed **positive**. There are also two negative forms of similar character parallel to each other, the poles of the upper of which lie in pairs adjacent to the negative radii. This negative upper form has the symbols $\{p'q'r'\}$ and $\{hikl\}$, and the lower negative form parallel to it has the symbols $\{\bar{p}'\bar{q}'\bar{r}'\}$ and $\{hikl\}$, where $p' = -p + 2q + 2r$, $q' = 2p - q + 2r$, $r' = 2p + 2q - r$. Thus there are four such pyramids possible altogether, having different symbols.

The special cases lead to the following forms:

(1) When the pairs of poles coalesce into single poles lying in the symmetry planes the 3-faced **trigonal pyramid of the first order** is produced, which occurs in four varieties corresponding to the four ditrigonal pyramids, and which are similarly distinguished as upper and lower positive and upper and lower negative. The positive upper form, closed by the basal plane, is shown in Fig. 332. In this form two of the indices are always equal, so that the general symbol is $\{pqq\}$, and although this is the same symbol as that of the rhombohedron of class 21, this form is not a rhombohedron, as only half the faces of the latter are comprised in the form. It may with some ground be regarded as a hemihedral form of the rhombohedron, and consequently as a tetartohedral form of the hexagonal system. There is, however, no necessity to consider it as either, but merely as the first special case of the general form conforming to that type of homogeneous structure which has the symmetry of class 20. Its faces are perpendicular to the three planes of symmetry. The indices of all four modifications are given in the list of forms.

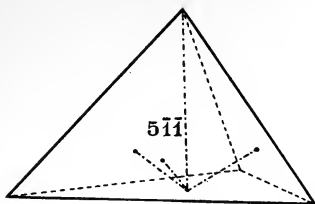


FIG. 332.—Positive Upper Trigonal Pyramid of the First Order.

(2) If each pair of poles adjacent to a symmetry plane separate further, until they reach the intermediate positions 30° from the neighbouring symmetry planes, when the two different alternate angles over the polar edges become equal, the **hexagonal pyramid of the second order** is produced. As explained under class 21, this occurs when one of the indices is the mean of the other two, so that the Millerian symbol is $\{pqr\}$ where p , q , and r have this relationship. There is an upper and a lower variety of this form, owing to the polarity of the vertical axis.

(3) When the poles of the general pyramid migrate on to the primitive circle we have the **ditrigonal prism** produced, already shown in Fig. 315, under class 22. It is a six-sided prism parallel to the trigonal axis, exhibiting two alternate kinds of angles. Its symbol is $\{pqr\}$ where $p+q+r=0$, or $\{hik0\}$. There are two such prisms possible, the one the symbols of which have just been given being derived from the upper positive or lower negative ditrigonal pyramids; the second variety, derived from the upper negative and lower positive pyramids, has its sharper angle in front and the more obtuse dihedral angle behind.

(4) When each pole is situated at the positive end of a diameter which forms the projection of a symmetry plane, the 3-faced **positive trigonal prism of the first order** is produced, which was described and illustrated in Fig. 316, under class 22. Its symbols are $\{2\bar{1}1\}$ and $\{10\bar{1}0\}$. When the poles are at the negative ends of the diameters, thus occupying the other alternate radial intersections with the primitive circle at 60° from the positive ones, the **negative trigonal prism** is produced, $\{21\bar{1}\}$ or $\{\bar{1}010\}$.

(5) When the poles on the primitive circle are intermediate between the symmetry-plane diameters, at 30° from each, the **hexagonal prism of the second order** is produced, $\{10\bar{1}\}$ or $\{11\bar{2}0\}$.

(6) When the pole occupies the centre of the projection, which is the common sixth element of all the spherical triangles under consideration, formed by the axial radial and the primitive circle, the **basal plane** is produced. Owing to the absence both of the equatorial plane of symmetry and of any digonal axes lying in it, this form is truly described as a plane and not as a pinakoid, for it consists of a single face, the two parallel faces being separate forms. The upper basal plane or pedion is denoted by $\{111\}$ or $\{0001\}$, and the lower one by $\{\bar{1}\bar{1}\bar{1}\}$ or $\{000\bar{1}\}$.

List of Forms in Class 20.

- $\{111\} = \{0001\}$ Upper basal plane or pedion ; $\{\bar{1}\bar{1}\bar{1}\} = \{000\bar{1}\}$ lower basal plane or pedion. Each 1 face.
- $\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ Positive trigonal prism of the first order ; $\{211\} = \{\bar{1}010\}$ negative trigonal prism. Each 3 faces.
- $\{10\bar{1}\} = \{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.
- $\{pqr\}$ (where $p+q+r=0\} = \{hik0\}$ Positive ditrigonal prism ; $\{p'q'r'\}$ where $\begin{cases} p' = -p+2q+2r \\ q' = 2p-q+2r \\ r' \neq 2p+2q-r \end{cases} = \{\bar{h}ik0\}$ negative ditrigonal prism. Each 6 faces.
- $\{pqq\}$ (two indices always equal) $= \{h0\bar{h}l\}$ Upper positive trigonal pyramid of the first order ; $\{\bar{p}\bar{q}\bar{q}\} = \{\bar{h}0hl\}$ lower (parallel to upper) positive trigonal pyramid ; $\{p'q'q'\} = \{h0hl\}$ upper negative trigonal pyramid ; $\{\bar{p}'\bar{q}'\bar{q}'\} = \{\bar{h}0\bar{h}l\}$ lower (parallel to upper) negative trigonal pyramid. Each 3 faces.
- $\{pqr\}$ (where one index = mean of other two) $= \{h.h.\bar{2}h.l\}$ Upper hexagonal pyramid of the second order ; $\{\bar{p}\bar{q}\bar{r}\} = \{\bar{h}.\bar{h}.2h.\bar{l}\}$ (or $\{h.h.\bar{2}h.l\}$ the symbol used for this form in the hexagonal system, both faces indicated by the symbols belonging to the form) lower form of same. Each 6 faces.
- $\{pqr\} = \{hikl\}$ Upper positive ditrigonal pyramid ; $\{\bar{p}\bar{q}\bar{r}\} = \{\bar{h}ik\bar{l}\}$ lower (parallel to upper) positive ditrigonal pyramid ; $\{p'q'r'\} = \{hikl\}$ upper negative ditrigonal pyramid ; $\{\bar{p}'\bar{q}'\bar{r}'\} = \{\bar{h}ik\bar{l}\}$ lower (parallel to upper) negative ditrigonal pyramid. Each 6 faces.

The mineral tourmaline, $H_6Na_2Fe_4B_6Al_3Si_{12}O_{63}$,—so well known from its property of transmitting only one of the two polarised rays into

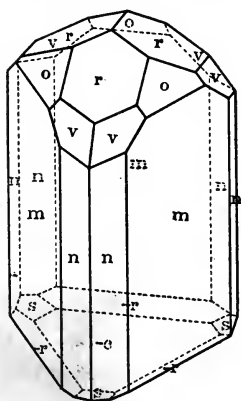


FIG. 333.
Crystal of Tourmaline.

which light traversing it is divided, and absorbing the other, so that a plate of the mineral may be employed for experiments requiring a beam of plane polarised light,—is an excellent example of a substance crystallising in this class. It exhibits prominently the phenomenon of oppositely polar pyro-electricity, and in fact was the first substance discovered to possess this remarkable property. Fig. 333 shows a typical crystal of tourmaline, as described by Kuppfer, on which are developed the forms $m = \{11\bar{2}\}$ (the negative trigonal prism of the first order), $n = \{10\bar{1}\}$ (the hexagonal prism of the second order), $r = \{100\}$, $-r = \{\bar{1}00\}$, $o = \{11\bar{1}\}$, $s = \{\bar{1}10\}$, $-c = \{\bar{1}\bar{1}1\}$, and $v = \{20\bar{1}\}$. The broken letters in the figure refer to the back faces indicated as usual by the dotted lines. The poles of all these faces are given on the stereographic projections in Figs. 317 and 318, and all but those of $s = \{\bar{1}10\}$ in Fig. 325. The upper end of the vertical trigonal axis, as the crystal is drawn in the illustration, is the antilogous pole. The two ends are clearly hemimorphic, certain distinct forms being characteristic of each.

The Millerian axial angle α of tourmaline is $113^\circ 58'$, and the Bravais-Miller axial ratio $a : c = 1 : 0.4474$.

On account of the importance of this mineral, optically and electrically as well as crystallographically, this class is generally known as the "tourmaline class."

Class 19.—Trigonal Bipyramidal Class. Trigonal-Tetartohedral Class. Type, Trigonal Equatorial.

This class is the only one of the 32 classes not yet known to be represented by any actual crystallised substance. It is the third of the possible trigonal classes obtained by adding to the simple trigonal pyramid of the third order,—the poles of which are shown in Fig. 305, and which is the open figure possessing the extreme minimum of trigonal symmetry, namely, a trigonal axis only,—the three faces of another similar pyramid but with faces directed with their apex downwards instead of upwards. In class 19 the lower pyramid is so arranged that the faces are precisely underneath those of the upper pyramid, the poles on the projection being ringed dots, and the two pyramids having a common equilaterally triangular base.

The poles of the general form of class 19 thus produced are shown in Fig. 334. The only other symmetry element present in the class, besides the essential trigonal axis, is thus the equatorial plane of symmetry.

It will be obvious that this general form is a **trigonal bipyramid of the third order**. It can occur in four varieties, only distinguishable, however, from each other or from the trigonal bipyramid of the first order shown in Fig. 314, so long as their positions of derivation are retained, and convertible into each other by rotation round the trigonal axis. These four varieties may be distinguished as (1) the **right positive**, the form the poles of which are given in Fig. 334; (2) the **left positive**, the poles of which

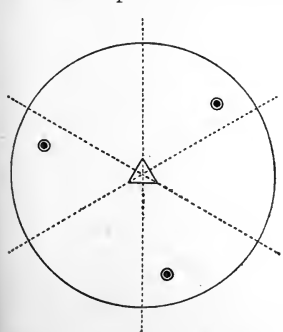


FIG. 334.—Symmetry Elements and General Form of Class 19.

are situated at an equal distance on the other side of the positive radii in each case, and which together with the right positive poles make up the positive ditrigonal bipyramid of the holohedral class 22; (3) the **right negative**, the faces of which are parallel to those of the right positive; its poles are on the right of the negative axial radii, that is, any particular pole will be on the right when the projection is rotated so as to bring the particular radius to which it is adjacent

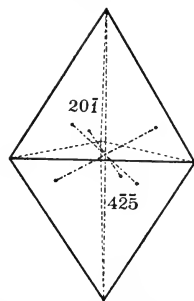


FIG. 335.—Right Positive Third Order Trigonal Bipyramid.

to front the observer; and finally, (4) the **left negative** modification with a pole on the other side of each negative half of the axes. Varieties (3) and (4) together make up the negative ditrigonal bipyramid. The symbols of these four forms are all given in the list which concludes the description of the class. The $\{201\}$ representative of the first of the four, the right positive, corresponding to the projection in Fig. 334, is shown in Fig. 335. The lower faces have the "inverse" symbols, the indices of the lower front face being (425).

Special cases.—(1) When the poles fall on the axial radii themselves the **two trigonal bipyramids of the first order** are produced, exactly as in the case of the holohedral

class 22; the right and left positive trigonal bipyramids of the third order yield the **positive trigonal bipyramid of the first order**, and the right and left negative ones the **negative trigonal bipyramid of the first order**.

(2) When the poles are on the diameters at 30° from the Millerian axial-trace diameters, that is, on the Bravais-Miller axes, the **trigonal bipyramid** becomes one of the **second order**. Two such are possible, a **right** form and a **left**, the right positive third order bipyramid or the left negative leading to the former, and the left positive or right negative to the latter.

(3) When the pole is on the primitive circle, in general a **trigonal prism of the third order** is produced, which may be of four kinds, corresponding to the four third order bipyramids, according as it is right or left of the positive or the negative axial radii. The one corresponding to the right positive bipyramid shown in Fig. 335 is given in Fig. 336, closed by the basal pinakoid. Its symbol is $\{5\bar{1}4\}$.

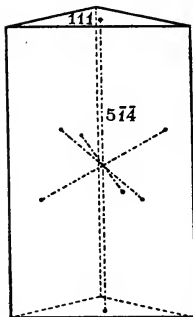


FIG. 336.—Right Positive Trigonal Prism of the Third Order.

(4) When the pole is on the primitive circle and at the end of a positive radius a **positive trigonal prism of the first order** is produced; if at the end of a negative radius the **negative** form is obtained.

(5) When the prismatic pole is midway between the two kinds of diameters, that is, at the end of a Bravais-Miller axial diameter, a **right or left trigonal prism of the second order** is produced.

(6) Finally, when the pole is at the inner corner of the fundamental spherical triangle, that is, at the centre of the whole projection, the **basal pinakoid** is produced.

The symbols of the whole of these forms are set forth in the following list:

List of Forms in Class 19.

- $\{111\} = \{0001\}$ Basal pinakoid. 2 faces.
 $\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ Positive trigonal prism of the first order; $\{211\} = \{\bar{1}010\}$ negative form of same. Each 3 faces.
 $\{10\bar{1}\} = \{11\bar{2}0\}$ Right trigonal prism of the second order; $\{1\bar{1}0\} = \{2\bar{1}\bar{1}0\}$ left form of same. Each 3 faces.
 $\{pqr\}$ (where $p+q+r=0$) = $\{hik0\}$ Right positive trigonal prism of the third order;
 $\{p'q'r'\}$ (same values) = $\{k\bar{i}\bar{h}0\}$ left positive form of same; $\{p'q'r'\}$ (where $\left. \begin{array}{l} p' = -p+2q+2r \\ q' = 2p-q+2r \\ r' = 2p+2q-r \end{array} \right\}$) = $\{\bar{h}ik0\}$ right negative form; $\{p'r'q'\}$ (same values) = $\{\bar{k}i\bar{h}0\}$ left negative form. Each 3 faces.
 $\{pqq\}$ (two indices always equal) = $\{h0\bar{h}l\}$ Positive trigonal bipyramid of the first order, including the primary positive $\{100\} = \{10\bar{1}1\}$; $\{p'q'q'\}$ (where $\left. \begin{array}{l} p' = -p+4q \\ q' = 2p+q \end{array} \right\}$) = $\{\bar{h}0hl\}$ negative form of same, including the complementary primary negative $\{\bar{1}22\} = \{\bar{1}011\}$. Each 6 faces.
 $\{pqr\}$ (where one index = mean of other two) = $\{h.h.\bar{2}h.l\}$ Right trigonal bipyramid of the second order; $\{p'q'q'\}$ (same values) = $\{2h.\bar{h}.h.l\}$ left form of same. Each 6 faces.
 $\{pqr\} = \{hikl\}$ Right positive trigonal bipyramid of the third order; $\{p'q'q'\}$ (same values) = $\{k\bar{i}\bar{h}l\}$ left positive form of same; $\{p'q'r'\}$ (where $\left. \begin{array}{l} p' = -p+2q+2r \\ q' = 2p-q+2r \\ r' = 2p+2q-r \end{array} \right\}$) = $\{\bar{h}ikl\}$ right negative form; $\{p'r'q'\}$ (same values) = $\{\bar{k}i\bar{h}l\}$ left negative form. Each 6 faces.

*Class 18.—Trigonal Trapezohedral Class. Quartz Class.**Hexagonal Trapezohedral-Tetartohedral Class. Type, Trigonal Holoaxial.*

This is another interesting class of the trigonal system, inasmuch as it is the type of symmetry exhibited by the very important and universally occurring mineral quartz, rock crystal, the crystallised form of silica, SiO_2 . The elements of symmetry are all axes, there being no plane of symmetry present. The essential trigonal axis is accompanied by three digonal axes lying in the equatorial plane, identical in position with the three horizontal Bravais-Miller axes. The result is that the trigonal pyramid, produced in the upper hemisphere from the general pole (pqr) by the operation of the trigonal axis, is accompanied owing to the operation of the digonal axes by a similar trigonal pyramid in the lower hemisphere, not immediately underneath, however, as in class 19 just considered, but rotated round the trigonal axis with respect to the upper one, so that the poles of these lower hemisphere faces are arranged in the projection symmetrically on the other side of the digonal axes. For it will be obvious that the operation (rotation for 180° round it) of each of the three digonal axes will cause a symmetrical repetition of any given neighbouring pole, in the other hemisphere.

If we start as just mentioned with the usual generally situated pole (pqr), namely, one within the fundamental spherical triangle to the right of the front positive axial radius, this repetition gives rise to the general form $\{pqr\}$ just described, and the projection of which is shown in Fig. 308 already given in the introduction to this

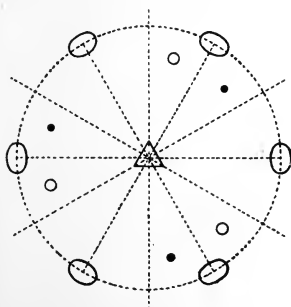


FIG. 337.

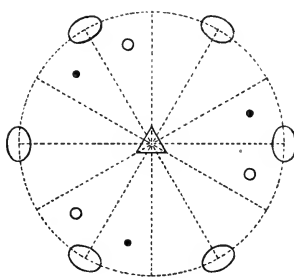


FIG. 338.

Symmetry Elements and Right and Left General Forms of Class 18.

chapter, and repeated here in Fig. 337 for the reader's convenience. The upper faces are those of the upper half of the right positive bipyramid of the third order of the last class, 19, and the three lower faces resemble the lower half of that solid, but rotated against the upper half round the vertical axis, so that the poles of the two halves lie in the projection symmetrically to the digonal axes, although one pole is a dot and the other a ring in each case. If we were to start instead with the upper half of the left positive third order trigonal bipyramid, we should get by a similar repetition in the lower hemisphere the six-faced solid the projection of which is shown in Fig. 338, and which is the enantiomorphous counterpart of Fig. 337. As the rotation involves the production of two unequal zigzag edges instead of each horizontal

basal edge of the trigonal bipyramid, a trapezoid shape is given to each face, and the solid is termed a **trigonal trapezohedron**. The one corresponding to Fig. 337 is shown in Fig. 339, and the other, the projection of which is given in Fig. 338, is shown in Fig. 340. The first is a **right positive trigonal trapezohedron**, and the second a **left positive**. The Millerian symbols of the two are $\{pqr\}$ and $\{\bar{p}q\bar{r}\}$, and the Bravais-Miller symbols $\{hikl\}$ and $\{\bar{h}\bar{i}\bar{k}l\}$, as in the case of the class 19 third order bipyramid the upper faces of which these trapezohedra possess. These two trigonal trapezohedra are truly enantiomorphous, not being mutually convertible by rotation. Besides these two positive trapezohedra, there are two similar negative ones, corresponding to the two negative trigonal pyramids. Their symbols are, for the **right negative** form $\{p'q'r'\}$ where p' , q' , and r' have the usual significations of inverse indices, or $\{\bar{h}\bar{i}\bar{k}l\}$; and for the **left negative** form $\{p'r'q'\}$ or $\{\bar{k}\bar{i}\bar{h}l\}$. It will be remembered that in order to calculate the inverse facial indices, we employ the equations:

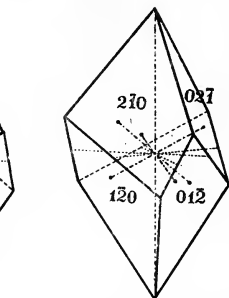


FIG. 339.—Right Positive Trigonal Trapezohedron. FIG. 340.—Left Positive Trigonal Trapezohedron.

have the usual significations of inverse indices, or $\{\bar{h}\bar{i}\bar{k}l\}$; and for the **left negative** form $\{p'r'q'\}$ or $\{\bar{k}\bar{i}\bar{h}l\}$. It will be remembered that in order to calculate the inverse facial indices, we employ the equations:

$$p' = -p + 2q + 2r, \quad q' = 2p - q + 2r, \quad r' = 2p + 2q - r,$$

The two negative forms are also truly enantiomorphous between themselves, but they may be converted into the two positive forms by rotation of 60° or 180° about the vertical trigonal axis. Hence there are not four enantiomorphous forms but two, a **right** and a **left**, although there are four distinct forms as derived *in situ*, having the four different symbols already just enumerated. It is an interesting fact that the optically active crystals of this class, of which quartz is an excellent and important example, do not exhibit both a right and a left trigonal trapezohedron on any one crystal, but only a right or a left form. In the case of quartz the type of crystal which rotates the plane of polarisation to the right bears also a face or faces of the right trapezohedron, while the left-handed optical variety bears the left trigonal trapezohedron.

Special cases.—(1) When the pole ceases to be a general one within the spherical triangle and migrates on to one of the Millerian crystallographic axial radii, which bisect the angles between the digonal axial radii, the result is the **first order rhombohedron** $\{pqq\} = \{h0\bar{h}l\}$, the same as is so characteristic of the calcite class. Both the right and left positive trigonal trapezohedra have for their limiting case the **positive or direct rhombohedron**, and similarly the right and left negative trapezohedra both coalesce into the **negative or inverse rhombohedron** $\{p'q'q'\} = \{\bar{h}0hl\}$, where p' , and q' have the usual inverse values. Consequently both rhombohedra may be developed on either right-handed or left-handed crystals of this class, as frequently happens in the case of quartz. With regard to the form symbols $\{pqq\}$ and $\{p'q'q'\}$, it should be pointed out that these are the symbols of the faces the poles of which lie on the two halves of the X Millerian axial diameter; the faces with poles on the Y diameter are $\{qqq\}$ and $\{q'p'q'\}$, and those with poles on the Z diameter are $\{qqp\}$ and $\{q'q'p'\}$. Owing to the equality of two indices, the formulæ above quoted for the calculation of the inverse forms in each case simplify to 2 equations each involving two quantities.

(2) When the general pole falls on the digonal axis, the other pole brought in by the operation of that axis falls also on it as a ring, so that a pair of poles vertically over each other occur on each of the three radii in question. The resulting solid is a **trigonal bipyramid of the second order**. The form derived from the right positive trapezohedron of Fig. 339 is the **right trigonal bipyramid**, and the left negative

trapezohedron gives rise to the same bipyramid. An analogous **left trigonal bipyramid of the second order** is yielded by the concentration on the three other digonal axial radii of the poles of the left positive and right negative trapezohedra. The symbols are the same as those of the last class (19), and are given in the list of forms. The right trigonal bipyramid of the second order $\{41\bar{2}\}$ is represented in Fig. 341. The Millerian and vertical axes are shown in broken-and-dotted lines, and the digonal axes in thin dotted lines.

(3) The limiting form of more and more acute trapezohedra is reached when the poles fall on the primitive circle, a **ditrigonal prism** resulting; the right positive and left negative trapezohedra both yielding the same prismatic form, which may be distinguished as the "**right**" one, while the left positive and right negative unite in producing a second ditrigonal prism, which is most conveniently called a "**left**" form. The right ditrigonal prism $\{51\bar{4}\}$, corresponding to the right positive trapezohedron $\{20\bar{1}\}$ (Fig. 339), is shown in Fig. 342. The Millerian and vertical axes and the digonal axes are discriminated as in Fig. 341. It is arranged with its alternately flatter and sharper angles in the correct positions, as derived. The complementary form would have these angles inverted, the sharper coming in front instead of the flatter. The relations of the two forms can at once be clearly understood by imagining the poles in

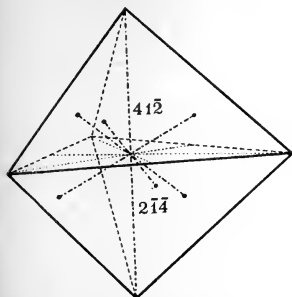


FIG. 341.—Right Trigonal Bipyramid of the Second Order.

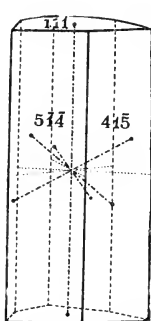


FIG. 342.—Right Ditrigonal Prism.

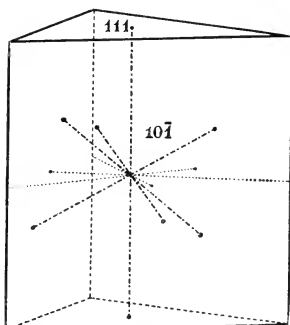


FIG. 343.—Right Trigonal Prism of the Second Order.

the two Figs. 337 and 338 carried out (moving along radii) on to the primitive circle. The one prism is convertible into the other by rotation round the vertical trigonal axis for 60° or 180° .

(4) When the poles of the ditrigonal prism separate until they lie at the ends of the Millerian axial radii, the **hexagonal prism of the first order** is produced; or the case may be stated in the form that when the general pole migrates to the end of the adjacent Millerian axis, the operation of the digonal axis at 30° from the latter requires a second pole to be situated at the end of the Millerian axial radius on the other side, also at 30° , from that digonal axis, and this occurring also at the two other pairs of Millerian radii, six prismatic faces are produced at regular intervals of 60° , the particular hexagonal prism produced being that of the first order. Both ditrigonal prisms naturally give rise to the same hexagonal prism of the first order, as there is but one such form possible. This form, consequently, may be equally developed on right-handed or left-handed crystals.

(5) If, however, the two poles of the ditrigonal prism approach each other and eventually concentrate and coalesce on the intersection of the digonal axis with the primitive circle, the prism produced is a trigonal one. The right positive and left negative trapezohedra, or the right ditrigonal prism, all yield the **right trigonal prism of the second order** $\{10\bar{1}\}$, which is represented in Fig. 343, closed by the

basal pinakoid $\{111\}$. The Millerian and vertical axes are drawn in thick broken-and-dotted lines, and the digonal axes in thin dotted lines. The left positive and right negative trapezohedra both yield, as does also the left ditrigonal prism, the **left second order trigonal prism** $\{1\bar{1}0\}$, the facial poles of which occupy the other ends of the three digonal axes inverse to those occupied by the right form. These two forms never appear on the same right-handed or left-handed crystal, but only the right form on the right-handed and the left form on the left-handed variety.

(6) When the general pole migrates to the centre of the projection the **basal pinakoid** $\{111\}$ is produced, both faces (111) and $(\bar{1}\bar{1}\bar{1})$ being in the form, on account of the operation of one or all three of the digonal axes.

The forms now shown to be possible to the class are summarised in the following list :

List of Forms in Class 18.

- $\{111\} = \{0001\}$ Basal pinakoid. 2 faces.
 $\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ Hexagonal prism of the first order. 6 faces.
 $\{10\bar{1}\} = \{11\bar{2}0\}$ Right trigonal prism of the second order ; $\{1\bar{1}0\} = \{2\bar{1}\bar{1}0\}$ Left form of same. Each 3 faces.
 $\{pqr\}$ (where $p+q+r=0$) = $\{h\bar{k}0\}$ Right ditrigonal prism ; $\{prq\}$ (same values) = $\{k\bar{h}0\}$ left ditrigonal prism. Each 6 faces.
 $\{pqq\}$ (two indices always equal) = $\{h0\bar{h}l\}$ Positive or direct rhombohedron of the first order, including the primary one $\{100\} = \{10\bar{1}1\}$; $\{p'q'q'\}$ (where $\left. \begin{matrix} p' = -p+4q \\ q' = 2p+q \end{matrix} \right\}$) = $\{h0h\bar{l}\}$ negative or inverse rhombohedron of the first order, including the complementary-primary inverse one $\{\bar{1}22\} = \{\bar{1}011\}$. Each 6 faces.
 $\{pqr\}$ (where one index = mean of other two) = $\{h . h . 2\bar{h} . l\}$ Right trigonal bipyramid of the second order ; $\{prq\}$ (same values) = $\{2h . \bar{h} . h . l\}$ left form of same. Each 6 faces.
 $\{pqr\} = \{h\bar{k}l\}$ Right positive trigonal trapezohedron ; $\{prq\}$ (same values) = $\{k\bar{h}l\}$ left positive form of same ; $\{p'q'r'\}$ (where $\left. \begin{matrix} p' = -p+2q+2r \\ q' = 2p-q+2r \\ r' = 2p+2q-r \end{matrix} \right\}$) = $\{\bar{h}k\bar{l}\}$ right negative form ; $\{p'r'q'\}$ (same values) = $\{\bar{k}h\bar{l}\}$ left negative form. Each 6 faces.

By far the most important substance crystallising in this class is the universally distributed mineral quartz, the common crystallised form of silica, SiO_2 . The two varieties, right- and left-handed, of this mineral, frequently called rock crystal, afford excellent examples of opposite optical activity ; the former rotates the plane of polarisation, of a plane-polarised beam of light travelling along the trigonal axis, to the right (in the same sense as the morphology), and the latter rotates it to the left. They are consequently very largely employed in experiments on the circular polarisation of light. The two enantiomorphous varieties, moreover, are both found in large quantity in nature, and the absence of any well-defined cleavage, together with the beautifully clear transparency of the mineral, render it peculiarly valuable in optical work. A typical enantiomorphous pair of quartz crystals are represented in Figs. 344 and 345, the former being a left-handed crystal and the latter a right-handed one. Both varieties exhibit usually the hexagonal prism $m = \{2\bar{1}1\}$ and the primary direct and complementary-primary inverse rhombohedra $r = \{100\}$ and $r' = \{\bar{1}22\}$. The two rhombohedra

are often so equally developed as to resemble a truly hexagonal terminal pyramid at each end of the prism. But in quartz crystals from certain particular localities, such as Schemnitz in Hungary, three faces of one rhombohedron largely predominate at each end, and crystals of quartz have also been found near Bristol as simple rhombohedra resembling cubes, for the rhombohedron angle is $85^{\circ} 46'$. But the fact that the symmetry is trapezohedral of class 18 is amply proved by the characteristic presence on optically right-handed crystals of more or less small faces of the right trigonal pyramid of the second order $s = \{41\bar{2}\}$, and of the right positive trapezohedron $x = \{4\bar{1}\bar{2}\}$, which give a distinctly right-handed appearance to the crystal, as is clearly shown in Fig. 345; and by the presence on optically left-handed crystals of

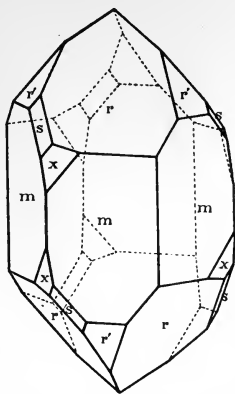


FIG. 344.—Left-Handed Crystal of Quartz.

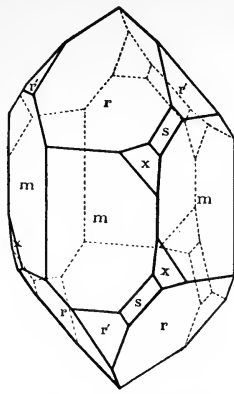


FIG. 345.—Right-Handed Crystal of Quartz.

the complementary enantiomorphous left trigonal pyramid of the second order $s = \{4\bar{2}1\}$ and of the left positive trigonal trapezohedron $x = \{4\bar{2}\bar{1}\}$, which endow the crystal morphologically also with a left-handed appearance, as illustrated in Fig. 344. It is usually possible to distinguish at once a crystal of quartz as a right-handed or a left-handed one by the presence of one or both of these forms, the two varieties of each or either of which never occur together on the same crystal, unless the latter be a twin and not a single individual. Moreover, when the x -face is absent, and the rhombohedral faces are badly formed and untrustworthy, it often happens that a good little s -face (the trigonal pyramid) is present, and it almost invariably exhibits striæ which are parallel to the edge sr , the edge of intersection of the primary rhombohedron and the trigonal pyramid, and these striæ consequently at once locate the position of both faces and of the zones $[r'sxm]$, to the direction or zone-plane of which the striæ are parallel, and of the zone $[rsm]$ to the axis of which they are parallel but to the direction or zone-plane of which they are transverse. The s -faces lie at the intersection of these two zones. As regards surface markings, moreover, the faces of the direct rhombohedron r are generally brilliantly reflective, while those of the inverse rhombohedron r' are dull, affording much poorer reflections of the goniometer signal.

The holoaxial symmetry of class 18 is strikingly brought out on quartz crystals both by the figures obtained on the faces by etching with hydrofluoric acid, and by the pyro-electric phenomena displayed. The etched figures obtained on right-handed and left-handed crystals

respectively are always oppositely enantiomorphous, that is, they are the mirror-images of each other. The figures on alternate faces of both the prism and the rhombohedra are characteristically different, exhibiting clearly the trigonal and not hexagonal nature of the symmetry. As regards the pyro-electric phenomena, any two opposite edges of the prism $m = \{2\bar{1}1\}$ are oppositely pyro-electric, so that when a section-plate of the hexagonal prism, parallel to the basal pinakoid, is heated and the Kundt mixture of powdered sulphur and red lead, usually employed in pyro-electric tests, is scattered over it, the negatively electrified sulphur is attracted to three alternate corners of the hexagonal plate as the latter cools, showing that the ends of the three digonal axes which emerge at these corners become positively electrified by the molecular movements occurring during cooling; while the positively electrified red lead accumulates at the other three corners, the three other ends of the digonal axes, which in this case are those corresponding to the prism edges carrying the s and x faces, showing that these ends of the digonal symmetry axes become negatively electrified. The pyro-electric poles are thus the digonal axes, and the investigation of a hexagonal section of a crystal, perpendicular to the trigonal vertical axis, by Kundt's method is an excellent mode of determining the right- or left-handed nature of a crystal the faces of which may not be good enough for goniometrical examination.

The morphology of quartz is so instructive and important that a crystal of this mineral will be worked through goniometrically in the next chapter, as an example of trigonal symmetry.

Class 17.—Rhombohedral Class. Diopase Class.

Rhombohedral Hexagonal-Tetartohedral Class. Type, Hexagonal Alternating.

As already stated in the introduction to this chapter, class 17 forms a link between the truly hexagonal classes and those of definitely trigonal symmetry. It may be almost equally well described as characterised by an alternating axis of hexagonal symmetry, or by a trigonal axis and a centre of symmetry. The symmetry is, however, most strictly described as consisting of the presence of a trigonal axis which is also a hexagonal axis of compound (mirror-image) symmetry. Fig. 346 represents this in stereographic projection, together with the poles of the general form.

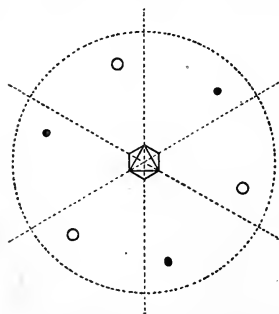


FIG. 346.—Symmetry Elements and General Form of Class 17.

This general form is one of those three (of classes 17, 18, and 19) already alluded to as consisting of an upper trigonal pyramid with a lower one below it arranged with respect to it either directly below (class 19) or rotated more or less about the trigonal axis. In the case of class 17, the operation of the hexagonal axis of compound symmetry causes the rotation to be exactly 60° , the reflection across the equatorial plane (not a plane of ordinary but only of reflective symmetry), immediately after the

rotation, causing the three poles thus arranged at 60° on the projection, to those of the upper trigonal pyramid started with, to be in the lower hemisphere. It will be at once apparent that the arrangement of poles thus brought about is similar to that of the calcite or quartz rhombohedron, the only difference being that the **rhombohedron** is here one of the **third order**, that is, its poles do not lie, as do those of the calcite or quartz rhombohedron, on crystallographic axial radii, but within the spherical triangles produced by those radii and their intercepts on the primitive circle. Such a third order rhombohedron is shown in Fig. 347, the edges no longer lying in the axial planes. The actual one drawn is the rhombohedron $\{20\bar{1}\}$ of dolomite, $\text{CaMg}(\text{CO}_3)_2$, the Bravais-Miller axial ratio of which is $a:c=1:0.8322$, and the Millerian axial angle $\alpha=102^\circ 53'$, the Millerian axes being shown in broken-and-dotted lines. The indices of the upper faces are $(20\bar{1})$, $(\bar{1}20)$, and $(0\bar{1}2)$; and of the lower faces $(25\bar{4})$, $(\bar{4}25)$, and $(5\bar{4}2)$.

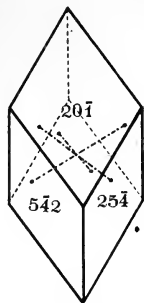


FIG. 347.
Third Order
Rhombohedron.

It will also be apparent that the rhombohedron the poles of which are shown in Fig. 346 is only one of four such rhombohedra, all, however, mutually convertible by rotation, having the same rhombohedral angle. For the upper poles with which we start may be either right (as in Fig. 346) of the positive axial radii, or left of them; and they may be also right or left of the negative radii. The symbols will be $\{pqr\}$, $\{prq\}$, $\{p'q'r'\}$, and $\{p'r'q'\}$, where p' , q' , and r' have the usual inverse values, as definitely stated in the list of forms on the next page.

Special Cases.—This class is well named the rhombohedral class, for in the first two special cases, when the poles fall on the Millerian axial radii themselves, or on the Bravais-Miller axial radii, **rhombohedra** are also produced, of the **first and second orders** respectively.

(1) When the poles lie on the Millerian axial radii and two of the indices are consequently equal, the **rhombohedron of the first order** is produced, the same as the calcite or quartz rhombohedron. It is a **positive or direct** one if the poles occupy the positive radii, or a **negative or inverse** one when the negative radii are occupied.

(2) When the poles fall on the Bravais-Miller axial radii, and one index becomes equal to the mean of the other two, a **right or a left rhombohedron of the second order** is produced, mutually convertible by rotation, as indeed are all three orders of rhombohedra, and each kind of each order. No enantiomorphism is developed at all in this class.

Just as the pyramidal forms of this class are all rhombohedra, so also the prismatic forms with faces parallel to the vertical trigonal axis are all hexagonal prisms. This will be obvious from the two facts that the poles are now all projected on to the primitive circle, and that the hexagonal axis of compound symmetry acts as if it were a truly hexagonal axis with respect to poles on the primitive circle, as all the poles lie equally in the plane of reflection.

(3) When the general poles of the third order rhombohedron are projected out on to the primitive circle, the **hexagonal prism** produced is of the **third order** also.

(4) If the poles lie at the ends of the Millerian axial radii, that is, when the poles of the first order rhombohedron are projected out on to the primitive circle, the **hexagonal prism** produced is one of the **first order**.

(5) When the second order rhombohedron is projected on to the primitive circle, or in other words, if the poles lie at the ends of the Bravais-Miller axial radii, the **hexagonal prism** is one of the **second order**.

(6) By the reflective operation of the hexagonal axis of compound symmetry the **basal pinakoid**, formed when the poles are concentrated at the centre of the projection, has both its faces in the same form, so that it is truly here a pinakoid.

A complete list of the forms in this class follows.

List of Forms in Class 17.

$\{111\} = \{0001\}$ Basal pinakoid. 2 faces.

$\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ Hexagonal prism of the first order. 6 faces.

$\{10\bar{1}\} = \{11\bar{2}0\}$ Hexagonal prism of the second order. 6 faces.

$\{pqr\}$ (where $p+q+r=0$) = $\{hik0\}$ Hexagonal prism of the third order. 6 faces.

$\{pgq\}$ (two indices always equal) = $\{h0hl\}$ Positive or direct rhombohedron of the first order, including the primary one $\{100\} = \{10\bar{1}1\}$;

$\{p'q'q'\}$ (where $\begin{cases} p' = -p + 4q \\ q' = 2p + q \end{cases}$) = $\{\bar{h}0hl\}$ negative or inverse rhombohedron of the first order, including the complementary-primary inverse one $\{\bar{1}22\} = \{\bar{1}011\}$. Each 6 faces.

$\{pqr\}$ (where one index = mean of other two) = $\{h.h.\bar{2}h.l\}$ Right rhombohedron of the second order ;

$\{prq\}$ (same values) = $\{2h.\bar{h}.\bar{h}.l\}$ left rhombohedron of the second order. Each 6 faces.

$\{pqr\} = \{hikl\}$ Right positive rhombohedron of the third order ;

$\{prq\} = \{k\bar{i}hl\}$ left positive rhombohedron of the third order ;

$\{p'q'r'\}$ (where $\begin{cases} p' = -p + 2q + 2r \\ q' = 2p - q + 2r \\ r' = 2p + 2q - r \end{cases}$) = $\{\bar{h}ikl\}$ right negative rhombohedron of the third order ;

$\{p'r'q'\}$ (same values) = $\{\bar{k}ihl\}$ left negative rhombohedron of the third order. Each 6 faces.

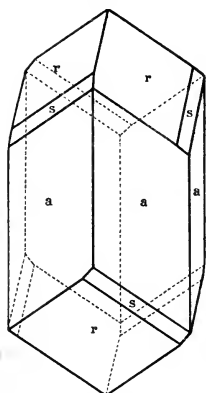


FIG. 348.
Crystal of Diopside.

A good example of class 17 symmetry is afforded by the emerald-like mineral diopside, acid silicate of copper, H_2CuSiO_4 , which has given its name to the class. Fig. 348 represents a crystal of diopside, the predominating prism faces a being those of the second order hexagonal prism $\{10\bar{1}\}$; the rhombohedron r is the negative or inverse rhombohedron $\{11\bar{1}\}$, and s is the third order rhombohedron $\{7\bar{7}6\}$. The presence of this latter form indicates the class conclusively. The mineral cleaves admirably parallel to the faces of the possible primary rhombohedron $\{100\}$, which latter form, however, is rarely developed in natural faces. The beautiful shimmer which the green and not perfectly transparent crystals exhibit is due to internal reflections from the cleavage cracks of $\{100\}$. According to Breithaupt the Bravais-Miller axial ratio $a:c = 1:0.5342$. The Millerian axial angle $\alpha = 111^\circ 42'$.

Class 16.—Trigonal Pyramidal Class. Hemimorphic Tetartohedral Class.
Type, Trigonal Polar.

We now come to the last of the 32 classes of symmetry, and the seventh and last class of the trigonal system. It is characterised by the minimum of symmetry essential to that system, namely, a trigonal axis of symmetry only. It has been shown in the introduction to this

chapter that the simple general form produced is a single (upper or lower) trigonal (three-faced) pyramid, as shown in the stereographic projection Fig. 349, which gives the poles of the upper pyramid.

The form in either case is a **trigonal pyramid of the third order**. There are, however, four such upper pyramids possible, the poles of which are respectively right and left of either the positive or negative Millerian axial radii. The one shown in Fig. 349 is the **upper right positive** form $\{pqr\} = \{hikl\}$. The **upper left positive** form is $\{prq\} = \{\bar{k}i\bar{h}l\}$, the Millerian index numbers of the right form being interchanged without any change of sign; its poles are as far on the left side of the positive radii as the right form poles were on the right. Similarly the two upper forms with poles equally far from the negative radii are called the **upper right negative trigonal pyramid** $\{p'q'r'\} = \{\bar{h}i\bar{k}l\}$, and the **upper left negative** form $\{p'r'q'\} = \{\bar{k}i\bar{h}l\}$, where the values p' , q' , and r' bear the relation to the values p , q , and r which has been shown to be the general relation of the indices of an inverse face to that of a direct face, namely,

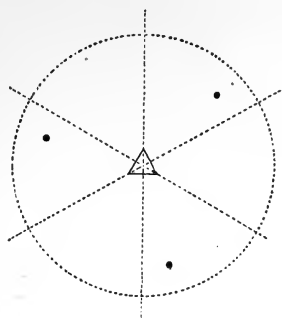


FIG. 349.—Symmetry Elements and General Form of Class 16.

$$p' = -p + 2q + 2r, \quad q' = 2p - q + 2r, \quad r' = 2p + 2q - r.$$

Besides these four upper trigonal pyramids, there are four analogous lower pyramids, the symbols of which (both Millerian and Bravais-Millerian) are the same as those of the upper pyramids, but with all the signs changed, the convention being that the lower right positive form is that one the faces of which are parallel to those of the upper right positive, and so on, the terms being similar as regards right or left, positive or negative, for such pairs of upper and lower forms as are parallel.

Hence there are no less than eight separate open forms, each consisting of three faces meeting pyramidally in a common apex, in the general case of class 16.

Special Cases.—(1) When two of the indices are equal, and the poles fall on the Millerian axial radii in the projection, the **trigonal pyramid** becomes one of the **first order**, and there can be four such, a **positive** and a **negative upper** form, according as the three poles occupy the positive or the negative radii, and two analogous **lower** forms parallel to the upper ones. It is a particular **upper positive primary trigonal pyramid** the faces of which are chosen for the Millerian axial planes, and its edges for the directions of the axes. Its symbol is, of course, $\{100\}$, and its poles generally occupy positions not far removed from half-way between the centre and the primitive circle. The form in general is usually expressed by the symbol $\{pqq\}$, as two of the indices are always equal. Considering the particular faces the poles of which fall on the back-and-front diameter, one of which is (100) , and the indices of which are taken for the form symbols, those between the primary face (100) and the basal plane (111) are of the nature (pqq) , as regards the signs of the indices, while those between (100) and the prism face $(2\bar{1}\bar{1})$ are of the type $(p\bar{q}\bar{q})$. This will be clear from Fig. 318. An important negative form is $\{011\}$, the faces of which replace the edges of the primary pyramid $\{100\}$, each face lying in the same zone with two adjacent faces of the latter form.

(2) When the poles of the trigonal pyramid occupy the Bravais-Miller axial radii intermediate between the Millerian, one of four possible **trigonal pyramids of the second order** is produced, which is distinguished as **upper right** when the right front sextant of the projection is occupied by one of the three poles in the upper hemisphere, or **upper left** when the left sextant is so occupied, and as **lower right** or

lower left when the faces are in the lower hemisphere and parallel to the directions of the faces of the two upper forms. The symbols of these four second order trigonal pyramids are given in the list of forms.

(3) When the poles migrate on to the primitive circle, in general a **trigonal prism of the third order** parallel to the vertical trigonal axis is produced. There are four such prismatic forms possible, each corresponding to the limiting case of a pair of third order pyramids, namely, a **right positive** and **left positive**, with poles on the right or the left of the positive radii respectively, and a **right negative** and **left negative**, with poles to right or left of the three negative radii. Their symbols are given in the list below.

(4) When the poles move along the primitive circle until they fall on the ends of the Millerian axial radii, the four forms are reduced to two, a **positive** and a **negative trigonal prism of the first order** respectively, according as the positive or the negative ends are occupied. The positive form is $\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$, and the negative form $\{211\} = \{\bar{1}010\}$. They are the limiting cases of pairs of pyramids of the first order.

(5) When the movement is on to the ends of the Bravais-Miller axial radii the **trigonal prism is of the second order**, and may be the **right** $\{10\bar{1}\} = \{11\bar{2}0\}$ or the **left** $\{\bar{1}\bar{1}0\} = \{2\bar{1}\bar{1}0\}$ of the two possible forms of the same, which are the limiting cases of pairs of second order pyramids.

(6) The **basal plane**, the limiting form of all trigonal pyramids,—when the poles concentrate and coalesce in the centre of the projection, the inner corner of each segmental triangle,—is divided into two separate forms, the **upper basal plane or pedion** $\{111\} = \{0001\}$, and the **lower basal plane or pedion** $\{\bar{1}\bar{1}\bar{1}\} = \{000\bar{1}\}$.

The following list gives the symbols of all these forms :

List of Forms in Class 16.

$\{111\} = \{0001\}$ Upper basal plane or pedion ; $\{\bar{1}\bar{1}\bar{1}\} = \{000\bar{1}\}$ lower basal plane or pedion. Each 1 face.

$\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ Positive trigonal prism of the first order ;

$\{211\} = \{\bar{1}010\}$ negative form of same. Each 3 faces.

$\{10\bar{1}\} = \{11\bar{2}0\}$ Right trigonal prism of the second order ;

$\{\bar{1}\bar{1}0\} = \{2\bar{1}\bar{1}0\}$ left form of same. Each 3 faces.

$\{pqr\}$ (where $p+q+r=0$) = $\{hik0\}$ Right positive trigonal prism of the third order ;

$\{p'q'r'\} = \{k\bar{i}\bar{h}0\}$ left positive form of same ;

$\{p'q'r'\}$ (where $\begin{cases} p' = -p+2q+2r \\ q' = 2p-q+2r \\ r' = 2p+2q-r \end{cases}$) = $\{\bar{h}ik0\}$ right negative form ;

$\{p'r'q'\} = \{k\bar{i}h0\}$ left negative form. Each 3 faces.

$\{pqq\}$ (where two indices are always equal) = $\{h0\bar{h}\bar{l}\}$ Upper positive trigonal pyramid of the first order ; including the primary form $\{100\} = \{10\bar{1}\bar{1}\}$;

$\{p'q'q'\}$ (where $\begin{cases} p' = -p+4q \\ q' = 2p+q \end{cases}$) = $\{\bar{h}0h\bar{l}\}$ upper negative form ;

$\{\bar{p}\bar{q}\bar{q}\} = \{\bar{h}0h\bar{l}\}$ lower positive form, parallel to upper positive ;

$\{\bar{p}'\bar{q}'\bar{q}'\} = \{h0\bar{h}\bar{l}\}$ lower negative form, parallel to upper negative. Each 3 faces.

$\{pqr\}$ (where one index = mean of other two) = $\{h.h.\bar{2}h.l\}$ Upper right trigonal pyramid of the second order ;

$\{prq\}$ (same values) = $\{2h.\bar{h}.h.l\}$ upper left form of same ;

$\{\bar{p}\bar{q}\bar{r}\} = \{\bar{h}.\bar{h}.2h.\bar{l}\}$ lower right form, parallel to upper right form ;

$\{\bar{p}\bar{r}\bar{q}\} = \{2h.h.h.\bar{l}\}$ lower left form, parallel to upper left form. Each 3 faces.

$\{pqr\} = \{hik\bar{l}\}$ Upper right positive trigonal pyramid of the third order ;

$\{prq\} = \{k\bar{i}\bar{h}\bar{l}\}$ upper left positive form of same ;

$\{p'q'r'\}$ (where $\begin{cases} p' = -p + 2q + 2r \\ q' = 2p - q + 2r \\ r' = 2p + 2q - r \end{cases}$) = $\{hikl\}$ upper right negative form ;

$\{p'r'q'\} = \{\bar{k}ihl\}$ upper left negative form ;

$\{\bar{p}\bar{q}\bar{r}\} = \{\bar{h}ik\bar{l}\}$ lower right positive form ;

$\{\bar{p}\bar{r}\bar{q}\} = \{\bar{k}ih\bar{l}\}$ lower left positive form ;

$\{\bar{p}'\bar{q}'\bar{r}'\} = \{\bar{h}ik\bar{l}\}$ lower right negative form ;

$\{\bar{p}'\bar{r}'\bar{q}'\} = \{\bar{k}ih\bar{l}\}$ lower left negative form. Each 3 faces.

The best example of this class of symmetry is afforded by sodium periodate, $\text{NaIO}_4 \cdot 3\text{H}_2\text{O}$, a crystal of which is represented in Fig. 350. It exhibits predominatingly the primary upper positive trigonal pyramid of the first order $r = \{100\}$ and the lower basal plane $c = \{\bar{1}\bar{1}\bar{1}\}$, also an upper negative trigonal pyramid of the first order $e = \{11\bar{1}\}$ in fair-sized faces, and as a subsidiary but very important form (as indicating the class) $s = \{201\}$ an upper left trigonal pyramid of the second order, the faces of which occur as narrow and oblique replacements of the edges of the primary predominating first order pyramid $\{100\}$. These latter edges are sometimes replaced instead, however, by symmetrical (not oblique) strips of the upper negative first order pyramid $\{011\}$, in which case still narrower strips of the third order right negative trigonal pyramid $\{504\}$ accompany them on one side only, thus proving conclusively that the symmetry is that of class 16. Such crystals rotate the plane of polarisation to the left. Other crystals show more or less of these forms, but with strips of the left negative trigonal pyramid of the third order developed instead of those of the right negative form, and are thus enantiomorphous to the first kind ; these rotate the plane of polarisation to the right. This substance thus shows unmistakably the enantiomorphism of class 16, and the opposite optical activity corresponding thereto. The Bravais-Miller ratio of the axes is $a : c = 1 : 1.094$, and the Millerian axial angle $\alpha = 94^\circ 9'$.

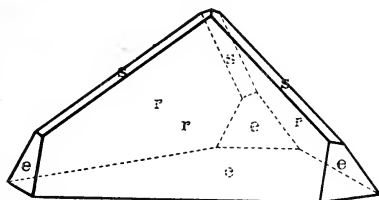


FIG. 350.—Crystal of Sodium Periodate.

CHAPTER XXIV

TWO PRACTICAL EXAMPLES OF TRIGONAL CRYSTALS, QUARTZ AND CALCITE.

(1) *Example of Class 18. Quartz, SiO_2 .*

THIS first example of a trigonal crystal is a particularly well formed left-handed crystal of quartz, about a centimetre long in the direction of the prism and half a centimetre thick across the prism. It is only singly terminated, the other end having been attached to the cavity wall in which it had grown. It forms an excellent example of class 18 symmetry, and also shows the characteristics of this exceptionally interesting and important mineral very clearly, as described and illustrated on pages 356 to 358 in the last chapter, particularly as regards the development of those faces which determine its right- or left-handedness.

As usual, a careful freehand drawing was made of the crystal. The decision as to the correct mode of arranging it offered no difficulty in this case, as the disposition of the faces was obviously that of the usual first order hexagonal prism of quartz $m = \{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$, terminated by the complementary primary direct and inverse rhombohedra $r = \{100\} = \{10\bar{1}1\}$ and $r' = \{\bar{1}22\} = \{\bar{1}011\}$. The cleavage directions of quartz are parallel to the faces of the primary rhombohedron $r = \{100\}$, but the cleavage is a very imperfect one, and no trace of it was visible on the crystal in question.

There were also two brilliant small faces present which appeared to be those of the forms $s = \{4\bar{2}1\} = \{2\bar{1}\bar{1}1\}$ and $x = \{4\bar{2}\bar{1}\} = \{6\bar{1}\bar{5}1\}$, the left trigonal pyramid of the second order and the left positive trapezohedron; the two actual faces were apparently those the indices of which are the same as the two form symbols just quoted, and were clearly indicative of the left-handed nature of the crystal. Another pair of these small faces were also present, although not so well developed as the first pair; the third pair, possible from the nature of trigonal symmetry, were absent. The freehand drawing was subsequently, after the completion of the measurements and consequent verification of the nature of the faces, elaborated into the correct representation to scale reproduced in Fig. 351, in accordance with the usual convention which ensures that

lines actually parallel on the crystal shall remain so in the drawing. The actual construction for this quartz crystal will be found described in Chapter XXV., page 403.

Obviously the first measurements to make were those of the angles in the prism zone. The crystal was therefore adjusted on the goniometer with the vertical axis, the axis of the prism, parallel to that of the goniometer, and the zone measured. Five of the six faces gave excellent single reflections of the signal-slit, and are accordingly marked with the letter A. The following results were obtained :

Prism Zone Readings.

m	$335^{\circ} 16'$	A
m	275	8
m	215	15 A
m	155	18 A
m	95	17 A
m	35	16 A
m	335	16 A

Interfacial Angles.

mm	$60^{\circ} 8'$
mm	$59 53$
mm	$59 57$ A value
mm	$60 1$ A „
mm	$60 1$ A „
mm	$60 0$ A „

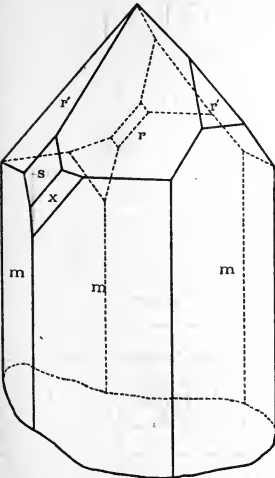


FIG. 351.—The Measured Crystal of Quartz.

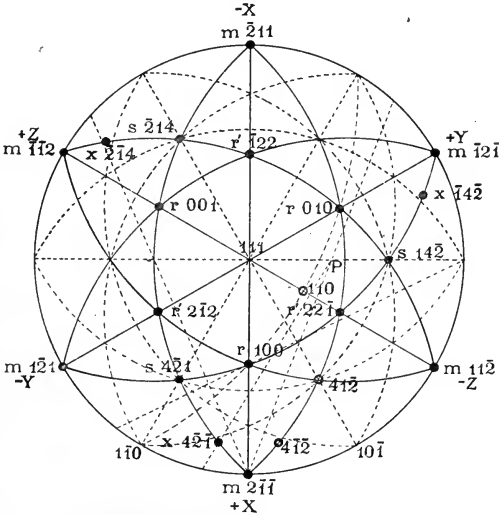


FIG. 352.—Stereographic Projection of Quartz.

It will be clear that the true values of all these six angles between successive faces of the prism zone are intended to be 60° , the four "A" values, each afforded by a pair of perfect faces yielding irreproachable "A" images, being within $3'$ of $60^{\circ} 0'$, that is, within the usual limits of error observed with excellent faces. It would be useless here, and if the symmetry of quartz were not so thoroughly established it might be very misleading, to take the arithmetical mean of the six values of the prism angle. For, as they together make up 360° the mean is bound to be $60^{\circ} 0'$, whereas it might have been possible that a pseudo-hexagonal crystal of orthorhombic symmetry were under investigation, with 3 angles of nearly but not quite 60° in each semicircle, and the mean here could have afforded us no information at all, and even worse, it would have caused us incorrectly to attribute real hexagonal symmetry

to the crystal. The prism is, however, in the case of quartz a truly hexagonal one, but the proof lies in the practical identity of adjacent "A" values of the angle, and not in any mean value. The six poles lie on the primitive circle of the stereographic projection, which we can now begin to construct, and which is shown in Fig. 352.

The next measurement in importance is that between any prism face and a face of either rhombohedron, that is, the determination of the angles mr and mr' . They belong to three similar zones, represented in the projection by the three diameters $mrr'm$, obtained by drawing straight lines to join opposite poles on the primitive circle, representing parallel prism faces, such straight lines obviously passing through the centre of the circle. For, on adjusting parallel to the goniometer axis the edge between one of the prism faces and the rhombohedral end face situated immediately above it, say the one marked r in Fig. 351, measuring the angle between these two faces, which are represented in the projection Fig. 352 by the poles $m=(2\bar{1}\bar{1})$ and $r=(100)$ on the front half of the diameter running vertically in the page, and then continuing to rotate the crystal on the goniometer, two other faces were found in the zone, namely, another rhombohedral face $r'=(\bar{1}22)$ in the projection, and $m=(\bar{2}11)$ the prism face parallel to the one started with. The r and r' faces were those of the direct and inverse rhombohedra. The angle between these two further faces in the zone proved to be the same as that between the first two faces, so that the angles $mr=(2\bar{1}\bar{1}) : (100)$ and $mr'=(\bar{2}11) : (\bar{1}22)$ appeared to be identical. Before being certain, however, it was necessary to measure also the two other similar zones, represented by the other two diameters. For according to trigonal symmetry these three zones, the planes of which (projected in the diameters) contain the Millerian crystallographic axes, should be exactly similar. The actual measurements derived from all three zones are appended.

Circle Readings.	Angles.	Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 225^\circ \ 10' \\ r \ 186 \ 54 \\ r' \ 83 \ 26 \ A \\ m \ 45 \ 10 \ A \end{array} \right.$	$\left\{ \begin{array}{l} mr \ 38^\circ \ 16' \\ rr' \ 103 \ 28 \\ r'm \ 38 \ 16 \ A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 239^\circ \ 24' \\ r' \ 201 \ 9 \\ r \ 97 \ 37 \ A \\ m \ 59 \ 24 \ A \end{array} \right.$	$\left\{ \begin{array}{l} mr' \ 38^\circ \ 15' \\ r'r \ 103 \ 32 \\ rm \ 38 \ 13 \ A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 226^\circ \ 44' \\ r \ 188 \ 33 \\ r' \ 84 \ 54 \ A \\ m \ 46 \ 44 \ A \end{array} \right.$	$\left\{ \begin{array}{l} mr \ 38^\circ \ 11' \\ rr' \ 103 \ 39 \\ r'm \ 38 \ 10 \ A \end{array} \right.$

It will be clearly apparent that the three values of mr and the three of mr' are all six so nearly equal that it can be safely accepted that the true values of mr and mr' are identical, the differences from equality being only such as are quite accounted for by the varying nature of the reflected signal-images.

The mean of the three values of mr is $38^\circ \ 13'$, and that of the three values of mr' is $38^\circ \ 14'$. The single A-value of mr is $38^\circ \ 13'$, and the mean of the two A-values of mr' is also $38^\circ \ 13'$. Hence there is no doubt that the real value of both angles is $38^\circ \ 13'$. This value is identical with that generally accepted for this angle on quartz crystals (Miers' *Mineralogy*, p. 367). Now the knowledge of this one angle suffices to enable us to calculate all the other angles between the crystal faces, one known basal angle only being required for hexagonal, trigonal, or tetragonal crystals, as explained in Chapter VII. This angle mr is a very convenient one to choose as basal angle, for it forms one of the elements of a right-angled spherical triangle, $m=(2\bar{1}\bar{1})$, $r=(100)$, $m=(11\bar{2})$, the side mm of which is also known, being 60° by reason of

the symmetry, a fact confirmed by the measurement of the prism zone. Hence we shall be able to use the simple Napierian rules for right-angled triangles in commencing the calculations.

We can now insert the six poles r and r' in their proper places on the three diameters of the projection, Fig. 352. It is only necessary to find the position of one of them by the proper construction, given on page 56 in Chapter IV., and then to mark off with the compasses similar distances from the centre in the cases of the other five. Constructing for the r face (100), we set off with the protractor an arc of $38^\circ 13'$ from m , the extremity of the diameter, along the primitive circle, say to the left, and join this point thus set off to the pole of the zone circle represented by the diameter, such pole being at the right end of the horizontal dotted diameter at right angles to that on which $r=(100)$ is situated. This junction line passes through the pole $r=(100)$, intersecting the diameter at it.

There are six other important zones to be measured, namely, the six represented by the circular arcs ending at the m -faces in the projection, which have been constructed to pass through an r and an r' face in each case. One of these zones is the very important one most obviously inviting measurement, in which occur the small faces s and x determinative of the right- or left-handed nature of the crystal. This zone consists of the front prism face m , then the trapezohedral face x , next the trigonal pyramid face s , then the inverse rhombohedral face r' ; we afterwards come to the faces which are behind in the drawing, Fig. 351, namely, the direct rhombohedral face r , the small second face of the trigonal pyramid s , and finally in completion of the zone the prism face m parallel to the one started with. The measurements obtained with this zone are set forth below:

Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 289^\circ \ 25' \\ x \ 277 \ 24 \\ s \ 251 \ 30 \ A \\ r' \ 222 \ 35 \ A \\ r \ 176 \ 18 \ A \\ s \ 147 \ 30 \\ m \ 109 \ 31 \end{array} \right.$	$\left. \begin{array}{l} mx \ 12^\circ \ 1' \\ xs \ 25 \ 54 \\ sr' \ 28 \ 55 \ A \\ r'r \ 46 \ 17 \ A \\ rs \ 28 \ 48 \\ sm \ 37 \ 59 \end{array} \right\} \begin{array}{l} ms \ 37^\circ \ 55' \\ mr' \ 66^\circ \ 50' \\ rm \ 66 \ 47 \end{array}$

The six zones, of which this is the most completely developed on the crystal before us, may be subdivided into two sets of three each, namely three on which the poles s and x occur, and three on which they do not occur. As already stated, two faces corresponding to each of these s and x forms were actually developed on the crystal, the third face, however, in each case being absent. Both the s faces were on the zone just measured, and gave trustworthy reflections, one affording an "A" image. Each of them also occurred on another of the three zones (a separate zone in each case) just referred to, at its intersection with the zone just measured, and these two further zones were next measured and gave the values set out on the next page. The second x -face was

present on one of them, but the reflection was not sufficiently good to afford a trustworthy value, although adequate to enable one to identify the face as belonging to the form x .

The fact of the s -poles being thus situated at the intersection of two of these zones indicated by the circular arcs defines their positions on the stereographic projection without any further construction. It also enables us to determine the indices of the form s , by cross-multiplication of the two zone symbols, and thus to verify that the symbol is $\{4\bar{2}1\}$.

Thus, to verify the indices of the pole $(4\bar{2}1)$ on the left of the radius $(2\bar{1}\bar{1}) : (111)$. It is on the zones $(2\bar{1}\bar{1}) : (2\bar{1}2)$ and $(\bar{1}21) : (100)$, and of these we find the zone symbols and cross-multiply.

$$\begin{array}{rcl} 2\bar{1}\bar{1}2\bar{1}\bar{1} & & 1\bar{2}11\bar{2}1 \\ \times \times \times & = & \times \times \times \\ 2\bar{1}2\bar{1}2 & = & 100100 \end{array}$$

$$\begin{array}{rcl} 120120 & & \\ \times \times \times & = & (4\bar{2}1). \\ 012012 & & \end{array}$$

Hence $(4\bar{2}1)$ is in fact the correct symbol.

In order to locate the position of the facial pole x on the projection, the pole of the arc $m \ x \ s \ r \ m$ is first found on the dotted horizontal diameter, by the construction of page 83, Chapter VI. A straight line is first drawn from $m = (2\bar{1}\bar{1})$ to the left upper part of the primitive circle, passing through the intersection of the arc and the horizontal diameter. An arc of 90° is then marked off on the primitive circle, and the point thus found on the right upper part of the primitive circle is joined to $m = (2\bar{1}\bar{1})$. Where it cuts the horizontal diameter is the pole P of the arc. To find the facial pole x we then set off 12° , the measured angle from $m = (2\bar{1}\bar{1})$, to the left along the primitive circle, and join the point thus obtained to P, the pole of the arc. Where this line cuts the arc is the facial pole x . These constructions are shown in Fig. 352.

Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 297^\circ \ 5' \\ r' \ 230 \ 11 \ A \\ r \ 184 \ 0 \ A \\ s \ 154 \ 55 \ A \\ m \ 117 \ 5 \end{array} \right.$	$\left. \begin{array}{l} mr' \ 66^\circ \ 54' \\ r'r' \ 46 \ 11 \ A \\ rs \ 29 \ 5 \ A \\ sm \ 37 \ 50 \end{array} \right\} rm \ 66^\circ \ 55'$	$\left\{ \begin{array}{l} m \ 344^\circ \ 43' \\ s \ 306 \ 46 \ A \\ r' \ 277 \ 42 \ A \\ r \ 231 \ 32 \\ m \ 164 \ 43 \end{array} \right.$	$\left. \begin{array}{l} ms \ 37^\circ \ 57' \\ sr' \ 29 \ 4 \ A \\ r'r' \ 46 \ 10 \\ rm \ 66 \ 49 \end{array} \right\} mr' \ 67^\circ \ 1'$

In addition to these complete zone measurements, two further good values of mr were obtained, namely, $66^\circ 45' A$ and $66^\circ 40'$, and one value of rr' , $46^\circ 35'$, from the other three similar zones on which the s and x faces do not occur, the reflections from other faces in these zones not being adequately trustworthy.

The various values of the same angle obtained from these six similar zones were collected and the following mean values extracted:

$$mx = 12^\circ 1', xs = 25^\circ 54', ms = 37^\circ 55', sr \text{ or } sr' = 28^\circ 58', mr \text{ or } mr' = 66^\circ 50', rr' = 46^\circ 18'.$$

This completes the angular measurements necessary for the perfect demonstration of the symmetry of the crystal, and for the determination of the elements.

We are now, therefore, in a position to proceed to the calculation

of these six angles, from the basal angle $mr = (2\bar{1}\bar{1}) : (100) = 38^\circ 13'$, in order to confirm the measured values thus arrived at; and after this is done we shall be able to compile our final table of angles and to calculate the elements.

Calculation of Angles and Elements.

The basal angle $mr = (2\bar{1}\bar{1}) : (100) = 38^\circ 13'$, forms one of two known sides, the other being $mm = 60^\circ$, in the right-angled triangle mmr ($2\bar{1}\bar{1}$), ($11\bar{2}$), (100), already alluded to, and which will readily be recognised in the projection, Fig. 352.

We may conveniently begin the calculations by finding the third side of this triangle, namely $rm = (100) : (11\bar{2})$. Constructing the Napierian diagram, Fig. 353, we have, by Napier's rules :

$$\cos rm = \cos 60^\circ \cos 38^\circ 13'. \quad rm = 66^\circ 52'.$$

The mean measured value of rm was $66^\circ 50'$, a satisfactory agreement.

We may next conveniently pass to the calculation of the angle $ms = (2\bar{1}\bar{1}) : (4\bar{2}1)$. The first step is to calculate the angle at m , ($11\bar{2}$), in the same triangle mmr . By Napier's rules :

$$\cos m = \tan 60^\circ \cot 66^\circ 52'. \quad m = 42^\circ 16'.$$

This angle, however, is the same as the angle at m , ($2\bar{1}\bar{1}$), in the right-angled triangle $m = (2\bar{1}\bar{1})$, $s = (4\bar{2}1)$, ($1\bar{1}0$), by reason of the symmetry, and we also know a second side in this triangle, namely $m = (2\bar{1}\bar{1}) : (1\bar{1}0)$, for it is obviously 30° , the pole ($1\bar{1}0$) being at the end of one of the Bravais-Miller axes, and although the face corresponding to the pole is not developed we can, of course, utilise our knowledge of its position in the calculations, for it depends entirely on the symmetry, which we have proved to be trigonal. Constructing the Napierian diagram, Fig. 354, and applying the rules :

$$\begin{aligned} \cos s &= \sin 42^\circ 16' \cos 30^\circ & \cos ms &= \cot 42^\circ 16' \cot s \\ s &= 54^\circ 22\frac{1}{2}'. & ms &= 37^\circ 58'. \end{aligned}$$

Or we might have calculated ms in one step from the equation $\cos 42^\circ 16' = \tan 30^\circ \cot ms$, for by transposition $\cot ms = \cos 42^\circ 16' \cot 30^\circ$. The value yielded for ms is the same, namely $37^\circ 58'$.

The mean measured value of ms was $37^\circ 55'$.

Again, by reason of the symmetry, as the six zonal arcs are in pairs symmetrical to the diameters at the ends of which they intersect, and as the three pairs of such arcs are equal (as indeed are all six arcs), the value of rm obtained from the first triangle, namely $66^\circ 52'$, is the same as the value of $mr' = (2\bar{1}\bar{1}) : (2\bar{1}2)$. Hence, as we have now found the value of part of this angle, namely, ms , we can obtain the value of another angle required to be calculated, sr' , by difference. That is,

$$sr' = (4\bar{2}1) : (2\bar{1}2) = mr' - ms = 66^\circ 52' - 37^\circ 58' = 28^\circ 54'.$$

The mean measured value of sr' (or of sr) was $28^\circ 58'$.

The remaining angle common to these six zones, rr' , is also at once afforded, for

$$rr' = (001) : (2\bar{1}2) = 180^\circ - 2mr = 180^\circ - 2(66^\circ 52') = 46^\circ 16'. \quad (\text{Measured } 46^\circ 18'.)$$

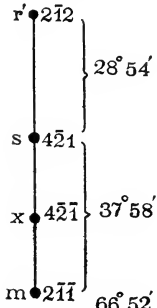
It only remains now to calculate the position of the trapezohedron face x . On

constructing the stereographic projection from the mean measured angles, it was observed that the pole x of the trapezohedron face which gave a good reflection in the zone $[m=(2\bar{1}\bar{1}), r'=(2\bar{1}2)]$ was not quite at the intersection of this zone with the other zone obtained by constructing a circular arc to pass through the ends of the Bravais-Miller diameter $(1\bar{1}0)$, $s=(4\bar{2}1)$, and through the poles $s=(14\bar{2})$ and $(41\bar{2})$. An angle of $12^\circ 1'$ from $m=(2\bar{1}\bar{1})$ places the pole x about $2\frac{1}{2}^\circ$ nearer m than the intersection of the two zonal arcs. Now the well-known trapezohedral forms shown by quartz crystals from all over the world, one pole of each of which lies in the zone msr' , are $u=\{8\bar{4}\bar{1}\}$, $y=\{10.5.2\}$, and $x=\{4\bar{2}\bar{1}\}$, the last being the nearest to $m=(2\bar{1}\bar{1})$ and the other two in consecutive order in the zone towards $s=(4\bar{2}1)$. If now we calculate the indices of the pole at the intersection of this zone with the zone $(1\bar{1}0):(41\bar{2})$, by cross-multiplication, we get:

$$\begin{array}{rcl} 4\bar{2}1 & 4\bar{1}2 & 4\bar{1}2 \\ \times \times \times & = [360] = [120]; & \times \times \times = [\bar{2}\bar{2}\bar{5}] = [225]; \\ 2\bar{1}\bar{1} & 2\bar{1}\bar{1} & 1\bar{1}0 \\ 120120 & & \\ \times \times \times & = (10.5.2). & \\ 225225 & & \end{array}$$

Evidently the pole at the intersection is $(10.5.2)$, so that x cannot have these indices, not being at the intersection, but must be the next trapezohedron in order towards $m=(2\bar{1}\bar{1})$, namely $(4\bar{2}\bar{1})$. Assuming these latter, therefore, to be the indices of the trapezohedral pole x which gave the good reflection, we can proceed to find its position in the zone $[msr']$ by means of the anharmonic ratio of the four poles bearing these letters in the zone, for all their indices are known, and the angular positions of three of them with respect to each other are also known. Fig. 355 shows the data.

To find mx and xs :



$$\begin{array}{l} \sin mx \cdot \sin r's = \frac{2\bar{1}\bar{1}}{4\bar{2}\bar{1}} \cdot \frac{2\bar{1}2}{4\bar{2}\bar{1}}; \\ \sin ms \cdot \sin r'x = \frac{2\bar{1}\bar{1}}{4\bar{2}\bar{1}} \cdot \frac{2\bar{1}2}{4\bar{2}\bar{1}}; \\ \frac{\sin mx}{\sin r'x} \cdot \frac{\sin 28^\circ 54'}{\sin 37^\circ 58'} = \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}; \\ \frac{\sin mx}{\sin r'x} = \frac{\sin 37^\circ 58'}{5 \sin 28^\circ 54'} = \tan \theta \text{ (according to the convention explained in Chapter VI., page 78).} \\ \theta = 14^\circ 17'. \end{array}$$

FIG. 355.

Transforming now to the sum and difference of two tangents, as discussed in Chapter VI., we have:

$$\begin{aligned} \tan \frac{r'x - mx}{2} &= \tan \frac{66^\circ 52'}{2} \tan (45^\circ - 14^\circ 17') \\ &= \tan 33^\circ 26' \tan 30^\circ 43'. \\ \frac{r'x - mx}{2} &= 21^\circ 25'. \end{aligned}$$

We thus find:
and we know that:

$$\begin{array}{l} \text{Hence, adding:} \quad 2r'x = 109^\circ 42', \text{ and } r'x = 54^\circ 51'; \\ \text{and subtracting:} \quad 2mx = 24^\circ 2', \text{ and } mx = 12^\circ 1'. \end{array}$$

We thus arrive at the angle $12^\circ 1'$ for mx , a value identical with the single measured angle, a result which is thus very satisfactory, and definitely decides that the indices of x are $(4\bar{2}\bar{1})$.

The last angle remaining to calculate, xs , can be got by difference from ms , and is:

$$xs = ms - mx = 37^\circ 58' - 12^\circ 1' = 25^\circ 57'. \quad (\text{Measured } 25^\circ 54').$$

It may perhaps be worth while to confirm the assumption that α cannot be (10. 5. 2), by actually calculating the position of α on the assumption of these indices instead of (421). The procedure is just the same, the only difference being introduced on the right-hand side of the first equation by the different indices, this side working out to $\frac{1}{4}$ instead of $\frac{1}{5}$, and a 4 being substituted for a 5 in consequence in the denominator of the expression for $\tan \theta$. Working out the value of $m\alpha$ on this basis we find it to be $14^\circ 35'$, an angle $2^\circ 34'$ larger than the observed angle between the faces m and α , a difference far greater than any possible error even with a badly reflecting face, while the reflection from α was distinctly good. We are thus finally sure that α is (421) and the angle $m\alpha$ $12^\circ 1'$.

This completes the calculations of the interfacial angles, and we proceed lastly to calculate the elements.

The Millerian rhombohedral axes being of equal length, there is no ratio of the axes to calculate, but we have to determine the angle α between the axes, that is, the angle which any pair of axes make with one another, for the three axial angles are equal, and therefore all are equally represented by α .

The Millerian Axial Angle α .—As the three polar edges of the primary rhombohedron {100} are chosen as the directions of the Millerian axes, the angle between the axes is equal to the plane angle between these polar rhombohedral edges. In the projection, Fig. 352, the arc joining the two poles of the primary rhombohedron (100) and (010), and passing also through the possible pole of the negative rhombohedron (110), represents one of these polar edges, the zone-axis of the zone-circle of which the arc is part being parallel to this polar edge. Similarly, arcs joining (100) to (001), and (001) to (010), represent the other two edges. The angle of intersection of any two of these three arcs will consequently be the supplement of the axial angle α .

We can readily calculate the half of this supplementary angle, $\frac{180-\alpha}{2}$, for it is the angle at (100) in the triangle (100):(111):(110), in which the angle at (110) is a right angle. For we know two other elements in this triangle, namely, the angle at (111) which is 60° , and the side (100):(111), which is the complement $51^\circ 47'$ of the basal angle $m\alpha$ $38^\circ 13'$.

Constructing the Napierian diagram, as in Fig. 356, we at once obtain :

$$\begin{aligned} \cos 51^\circ 47' &= \cot 60^\circ \cot [\text{angle at (100)}] \\ \text{or } \cot [\text{angle at (100)}] &= \cos 51^\circ 47' \tan 60^\circ. \end{aligned}$$

$$\text{Angle at (100)} = 43^\circ 1\frac{1}{2}'.$$

Thus

$$\frac{180-\alpha}{2} = 43^\circ 1\frac{1}{2}'$$

$$180-\alpha = 86^\circ 3'$$

$$\alpha = 93^\circ 57'.$$

This value of α agrees precisely, to within half a minute, with the Millerian axial angle of quartz given by von Groth.

From the same triangle we can calculate another important angle which is often given along with the elements, namely the angle of the primary rhombohedron, $rr = (100):(010)$, the angle over each polar edge. From the diagram, Fig. 356, we at once get the value of half this angle, (100):(110), for

$$\sin [(100):(110)] = \sin 60^\circ \sin 51^\circ 47'.$$

$$(100):(110) = 42^\circ 53'$$

\therefore the rhombohedron angle

$$rr = (100):(010) = 85^\circ 46'.$$

This is the currently accepted value of the rhombohedral angle of quartz.

The Bravais-Miller Axial Ratio.—In order to calculate the relative lengths of the

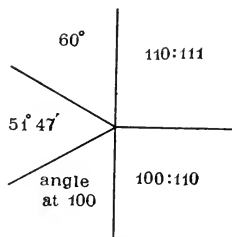


FIG. 356.

vertical axis c and of any one of the three equal horizontal axes, we only require to know the inclination of any face inclined to both axes, and preferably for simplicity of one the pole of which lies in the plane of the two axes, that is, the inclination of any face inclined to both axes and perpendicular to their plane. The pole of such a face will lie in the projection on the Bravais-Miller axial diameter representing the horizontal axis in question. The indices, of course, must be known, and the case is simplest when the face is a primary one.

In the case of the crystal of quartz under investigation there is such a primary form on a Bravais-Miller axis, namely, the form s , the face $(2\bar{1}1)$ of which, $(4\bar{2}1)$ in Millerian notation, lies on the axial radius $+A_1$ (see Fig. 317 on page 340 for Bravais-Miller stereographic projection). The indices inform us that this face cuts the A_1 axis at the unit parametral distance a , and that it meets the c axis at twice the unit length, for the intercepts are $1\bar{2}\bar{2}2$; and that it is normal to the plane of these two rectangular axes. The conditions are those of the triangle in Fig. 357. Now, the angular position of the face, although not measured directly on the Bravais-Miller axial diametral zone owing to the absence of other faces in that possible zone, can be readily calculated, for the angle between the possible prism face $(2\bar{1}10)$, the Millerian $(1\bar{1}0)$, the pole of which is at the end of the same diameter, and the face s in question, is one of the angles in the triangle the Napierian diagram for which is given in Fig. 354, and from the latter we at once get:

$$\sin [(2\bar{1}10) : s] = \tan 30^\circ \cot s = \tan 30^\circ \cot 54^\circ 22\frac{1}{2}' \\ (2\bar{1}10) : s = 24^\circ 26\frac{1}{2}'.$$

Now this angle $24^\circ 26\frac{1}{2}'$ is that between the normals to $(2\bar{1}10)$ and $(2\bar{1}\bar{1}1)$, and is equal to the top acute angle in Fig. 357. From this triangle we then have:

$$\frac{OA}{OC} = \frac{a}{2c} = \tan 24^\circ 26\frac{1}{2}', \text{ and therefore } \frac{a}{c} = 2 \tan 24^\circ 26\frac{1}{2}'.$$

Hence

$$\frac{c}{a} = \frac{\cot 24^\circ 26\frac{1}{2}'}{2} = \frac{2 \cdot 2002}{2} = 1 \cdot 1001.$$

We can also readily find $\frac{c}{a}$ from the angle between any face of the primary rhombohedron and the hexagonal prism face adjacent to it, say the front face $(10\bar{1}1)$ and the prism face $(10\bar{1}0)$. If Fig. 358 represent the upper half of the combination of the two primary (direct and inverse) rhombohedra, the face $+A_1, -A_3$, C is the one in question. OA_1 and OA_3 are the two horizontal Bravais-Miller axes which it meets, of unit length a , and OC is the vertical axis, of the unit length c . Thus:

$$\frac{c}{a} = \frac{OC}{OA_1} \text{ or } \frac{OC}{OA_3}.$$

If $A_1 A_3$ be bisected at M , then:

$$\frac{OC}{OM} = \tan [\text{complement of basal angle } (10\bar{1}1) : (10\bar{1}0)] \\ = \tan 51^\circ 47'.$$

Now in the plane triangle OMA_1 or OMA_3 the angle at O is 30° and the angle at A_1 or A_3 is 60° , the angle at M being a right angle. Hence:

$$\frac{OM}{OA_3} = \frac{\sqrt{3}}{2} \text{ or } OA_3 = \frac{2}{\sqrt{3}} OM.$$

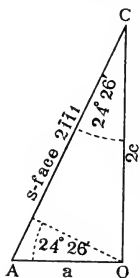


FIG. 357.

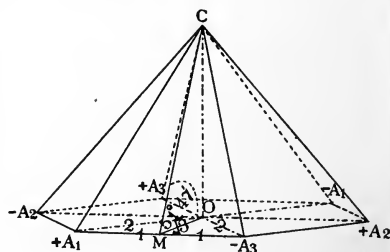


FIG. 358.

Substituting this value for OA_3 (or OA_1) we get :

$$\frac{c}{a} = \frac{\sqrt{3}OC}{2OM} = \frac{\sqrt{3}}{2} \tan 51^\circ 47' = 1.0999.$$

Taking the mean of the two values thus obtained by independent methods for the ratio, namely, 1.1001 and 1.0999, we may confidently state the ratio of the axis as :

$$a : c = 1 : 1.1000.$$

This value agrees precisely with that given by Miers for the ratio.

TABULAR PRESENTATION OF RESULTS FOR CRYSTAL OF QUARTZ, SiO_2 .

We may now concisely express the results of the investigation of the crystal of quartz in the following table.

Crystal-system : Trigonal. Class : 18, trapezohedral.

Millerian rhombohedral axial angle : $\alpha = 93^\circ 57'$. Angle over polar edges of primary rhombohedron, $85^\circ 46'$.

Bravais-Miller ratio of axes : $a : c = 1 : 1.1000$.

Forms observed : $m = \{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$; $r = \{100\} = \{10\bar{1}1\}$; $r' = \{\bar{1}22\} = \{\bar{1}011\}$; $s = \{4\bar{2}1\} = \{2\bar{1}\bar{1}1\}$; $x = \{4\bar{2}\bar{1}\} = \{6\bar{1}\bar{5}1\}$.

Table of angles : the basal angle used as the foundation of the calculations is marked with an asterisk in the following table. Angles in the same zone are linked together by a bracket.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$mm = (2\bar{1}\bar{1}) : (1\bar{1}\bar{2}) = (10\bar{1}0) : (01\bar{1}0)$	6	$59^\circ 53' - 60^\circ 8'$	$60^\circ 0'$	$60^\circ 0'$	$0'$
$\left\{ \begin{array}{l} mr = (2\bar{1}\bar{1}) : (100) = (10\bar{1}0) : (10\bar{1}1) \\ mr' = (2\bar{1}\bar{1}) : (\bar{1}22) = (10\bar{1}0) : (\bar{1}011) \end{array} \right\}$	6	$38 \ 10 - 38 \ 16$	$38 \ 13$	*	...
$rr' = (100) : (\bar{1}22) = (10\bar{1}1) : (\bar{1}011)$	3	$103 \ 28 - 103 \ 39$	$103 \ 33$	$103 \ 34$	1
$mx = (2\bar{1}\bar{1}) : (4\bar{2}\bar{1}) = (10\bar{1}0) : (6\bar{1}\bar{5}1)$	1	...	$12 \ 1$	$12 \ 1$	0
$xs = (4\bar{2}\bar{1}) : (4\bar{2}\bar{1}) = (6\bar{1}\bar{5}1) : (2\bar{1}\bar{1}1)$	1	...	$25 \ 54$	$25 \ 57$	3
$ms = (2\bar{1}\bar{1}) : (4\bar{2}\bar{1}) = (10\bar{1}0) : (2\bar{1}\bar{1}1)$	4	$37 \ 50 - 37 \ 59$	$37 \ 55$	$37 \ 58$	3
$\left\{ \begin{array}{l} sr' = (4\bar{2}\bar{1}) : (2\bar{1}\bar{2}) = (2\bar{1}\bar{1}1) : (\bar{1}\bar{1}01) \\ sr = (2\bar{1}\bar{2}) : (001) = (\bar{1}\bar{1}21) : (0\bar{1}\bar{1}1) \end{array} \right\}$	4	$28 \ 48 - 29 \ 5$	$28 \ 58$	$28 \ 54$	4
$r'r = (2\bar{1}\bar{2}) : (001) = (\bar{1}\bar{1}01) : (0\bar{1}\bar{1}1)$	4	$46 \ 10 - 46 \ 35$	$46 \ 18$	$46 \ 16$	2
$\left\{ \begin{array}{l} mr' = (2\bar{1}\bar{1}) : (2\bar{1}\bar{2}) = (10\bar{1}0) : (\bar{1}\bar{1}01) \\ mr = (2\bar{1}\bar{1}) : (001) = (10\bar{1}0) : (0\bar{1}\bar{1}1) \end{array} \right\}$	8	$66 \ 40 - 67 \ 1$	$66 \ 50$	$66 \ 52$	2

(2) Example of Class 21. Calcite, $CaCO_3$.

The crystal of calcite selected for measurement as the second example of a substance crystallising in the trigonal system was an elongated hexagonal prism, about two centimetres long and half a centimetre thick, rough at one end, where it had been attached to the rock from which it had grown, but exhibiting at the free end faces of the primary (direct or positive) rhombohedron, two other rhombohedra, and two scalenohedra. As the forms of calcite are so well known, it may as well be at once

stated what were the actual forms present, and in Fig. 359 a reproduction is given of the drawing made of the crystal to scale in the usual conventional parallel-lined manner, after the completion of the measurements and calculations, in elaboration of the freehand drawing preliminarily made. The mode of construction will be given in the next chapter for this particular prism of calcite.

The hexagonal prism, the faces of which were lettered *m* as is usual, proved to be that of the first order, $\{2\bar{1}1\} = \{10\bar{1}0\}$. The best-developed and most brilliant-faced of the rhombohedra was the primary one, $r = \{100\} = \{10\bar{1}1\}$, the three faces of which were readily recognised because they are the directions of the excellent cleavage so characteristic of calcite, and several cleavage fractures were clearly visible in the

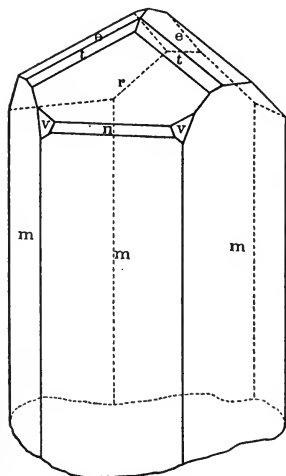


FIG. 359.—The Measured Crystal of Calcite.

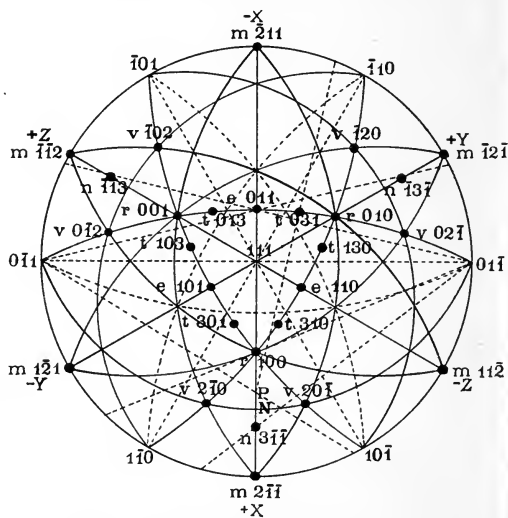


FIG. 360.—Stereographic Projection of Calcite Crystal.

lower part of the prism which were obviously parallel to these three faces. The faces of the inverse or negative obtuse rhombohedron $e = \{110\} = \{01\bar{1}2\}$ were also brilliant and afforded excellent images of the goniometer signal-slit, although they were smaller on the average than those of the primary direct or positive rhombohedron just referred to. One brilliant narrow face of the direct or positive acute rhombohedron $n = \{3\bar{1}1\} = \{40\bar{4}1\}$ was also present, as shown in the drawing, but the other two faces of this form were not developed. The common scalenohedron of calcite $v = \{20\bar{1}\} = \{21\bar{3}1\}$ was represented on the crystal by two faces at the ends of the *n*-face, the other four of the possible six being absent; and another scalenohedron, $t = \{310\} = \{21\bar{3}4\}$, was also represented by a pair of faces symmetrically arranged to the same Millerian axis as the pair of $\{20\bar{1}\}$ faces.

The disposition of all these faces will be rendered still clearer by the stereographic projection given in Fig. 360, which was drawn finally to

scale, from a rough draft constructed during the progress of the work, after the completion of the measurements and calculations, the method of procedure with which will be indicated as the measurements of the various zones are described.

The first measurements to be carried out were obviously those of the prism zone, which gave the following results :

Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 303^{\circ} \ 10' A \\ m \ 243 \ 9 \ A \\ m \ 183 \ 8 \\ m \ 123 \ 11 \\ m \ 63 \ 14 \\ m \ 3 \ 9 \ A \\ m \ 303 \ 10 \ A \end{array} \right.$	$\left\{ \begin{array}{l} mm \ 60^{\circ} \ 1' A \\ mm \ 60 \ 1 \\ mm \ 59 \ 57 \\ mm \ 59 \ 57 \\ mm \ 60 \ 5 \\ mm \ 59 \ 59 \ A \end{array} \right.$

It is useless to take the mean of these six values of the prism angle, for in any case, as they together make up a complete circle, such mean would be $60^{\circ} 0'$. But this is obviously the zone of a hexagonal prism, for the two adjacent A-values are each within $1'$ of $60^{\circ} 0'$. Hence, the primitive circle of the stereographic projection may at once be drawn in, showing a pole of this form at every 60° , arranged at the end of a Millerian and not a Bravais-Millerian axis in each case, as the position of the cleavage rhombohedron clearly indicates that the prism is the first order one $\{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$ and not the second order prism $\{10\bar{1}\} = \{11\bar{2}0\}$.

The three equal zones, the projections of which are the three diameters obtained by joining parallel pairs of prism poles, were next measured. Each contains the two parallel prism faces m , and between them a face of the primary positive rhombohedron $r = \{100\}$ and a face of the obtuse negative rhombohedron $e = \{110\}$, and one of the zones also comprises the single developed face of the acute positive rhombohedron $n = \{3\bar{1}\bar{1}\}$. The measurements were as follows :

Circle Readings.	Angles.	Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 352^{\circ} \ 15' A \\ n \ 338 \ 2 \ A \\ r \ 306 \ 52 \ A \\ e \ 236 \ 2 \ A \end{array} \right.$	$\left\{ \begin{array}{l} mn \ 14^{\circ} \ 13' A \\ nr \ 31 \ 10 \ A \\ mr \ 45 \ 23 \ A \\ re \ 70 \ 50 \ A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 228^{\circ} \ 27' A \\ e \ 164 \ 47 \ A \\ r \ 93 \ 50 \ A \\ m \ 48 \ 26 \ A \end{array} \right.$	$\left\{ \begin{array}{l} me \ 63^{\circ} \ 40' A \\ er \ 70 \ 57 \ A \\ rm \ 45 \ 24 \ A \end{array} \right.$	$\left\{ \begin{array}{l} m \ 236^{\circ} \ 20' \\ e \ 172 \ 32 \ A \\ r \ 101 \ 38 \\ m \ 56 \ 16 \end{array} \right.$	$\left\{ \begin{array}{l} me \ 63^{\circ} \ 48' \\ er \ 70 \ 54 \\ rm \ 45 \ 22 \end{array} \right.$

We may now collect the three values of mr , of re , and of em , and take the mean in each case ; for this is quite legitimate as the three angles of each such zone are widely different, while the three zones are undoubtedly exactly alike. For instance, the three values of mr only differ by $2'$. We then find these mean values to be $45^{\circ} 23'$, $70^{\circ} 54'$, and $63^{\circ} 44'$ respectively.

We had next another series of equal zones to measure, namely those represented by the circular arcs drawn symmetrically to each diameter, beginning at one end of the diameter and finishing at the other end, thus connecting a pair of parallel prism faces again, but including in this zone only a primary rhombohedron face r , and, when developed, a common scalenohedron face v . As two of the scalenohedron faces were

developed, the two such zones containing them were measured, and gave the following results :

Circle Readings.	Angles.	Circle Readings.	Angles.
$\left\{ \begin{array}{l} m \ 18^{\circ} \ 40' \text{ A} \\ r \ 309 \ 17 \text{ A} \\ v \ 226 \ 52 \text{ A} \\ m \ 198 \ 40 \text{ A} \end{array} \right.$	$\left\{ \begin{array}{l} mr \ 69^{\circ} \ 23' \text{ A} \\ rv \ 82 \ 25 \text{ A} \\ vm \ 28 \ 12 \text{ A} \end{array} \right.$	$\left\{ \begin{array}{l} m \ 230^{\circ} \ 17' \text{ A} \\ v \ 202 \ 21 \\ r \ 119 \ 49 \\ m \ 50 \ 17 \end{array} \right.$	$\left\{ \begin{array}{l} mv \ 27^{\circ} \ 56' \\ vr \ 82 \ 32 \\ rm \ 69 \ 32 \end{array} \right.$

The mean values of mv , vr , and rm in these zones are thus $28^{\circ} \ 4'$, $82^{\circ} \ 29'$ and $69^{\circ} \ 28'$ respectively. As regards the construction of the arcs representing these zones in the projection, it is only necessary both for this purpose and that of constructing the only remaining arcs on the projection, those for the next series of zones to be considered, to find the positions of the poles r and e on any one of the Millerian diameters. This is done, as indicated in Chapter VI., by setting off from the end of the diameter, the pole m , an arc along the primitive circle equal to the measured angle mr or me , and joining this point thus found on the primitive circle to the pole of the zone of which the diameter is the projection, that is, to the end lying on the primitive circle of a diameter perpendicular to the one in question, dotted in Fig. 360. The line of junction passes through the pole of the face r or e as the case may be. It is necessary to draw all three dotted diameters, perpendicular to the three Millerian ones joining prism poles, for these diameters are the Bravais-Miller axes, and their ends are the ends also of arcs representing zones passing through the facial poles v , r , t , e , a second t and a second r . Having found the poles r and e on one of the Millerian diameters by the usual method just indicated, the poles of these forms on the other two Millerian diameters can be pricked off at similar positions with the compasses, the radius can be found of an arc to pass through m and the facial pole r lying to right or left of the diameter on which m is situated, and the arc drawn. Six such equal and similar arcs can be constructed as shown in Fig. 360, and in the case of the two comprising also the scalenohedron faces v the pole of one such v -face will also lie on the arc, in a position to be presently indicated. Similarly three further arcs may also be drawn, terminating at the ends of a Bravais-Miller diameter in each case, to pass through the facial pole e which lies on the Millerian diameter at right angles to it; such an arc will also, as above mentioned, pass through two r facial poles, and two poles of the scalenohedron t , as well as one of the common scalenohedron poles v in the case of two of the three arcs so drawn. The fact that each pole of the common scalenohedron v lies both on one of the six former arcs ending at an m facial pole, and on one of the three latter arcs terminating at the ends of a Bravais-Millerian diameter, determines at once the position of the v pole, namely, at the intersection of the two zonal arcs. The position of the acute positive rhombohedron n must be found in the same manner as the poles r and e were found, namely, by cutting off an angle along the primitive circle equal to the measured angle mn , and joining this point to the pole of the diametral zone, that is, to the end of the perpendicular (dotted) diameter on the other side to that on which the arc was marked off. The poles of all the forms present have now been located on the projection except those of the scalenohedron t . We have next to proceed to the measurement of the three zones last alluded to, terminating at the ends of Bravais-Millerian diameters, and in doing so we shall locate the position of the t poles, and enable their places also to be marked on the projection. The actual measurements gave the following results :

Circle Readings.	Angles.	Circle Readings.	Angles.	Circle Readings.	Angles.
v 237° 5'	vr 29° 4'	v 271° 34'	vr 29° 4' A	r 264° 10'	re 37° 28'
r 208 1 A	rt 16 27	r 242° 30 A	rt 16 27	e 226 42	er 37 28
t 191 34	te 20 58	t 226 3	te 20 55	r 189 14	rr 74 56
e 170 36	re 37 25	e 205 8	re 37 22		
r 133 7 A	er 37 29	r 167 38	er 37 30		
	rr 74 54		rr 74 52		

Collecting similar angles and taking means we find that the mean value of vr is 29° 4', of rt 16° 27', of te 20° 57', of re 37° 27', and of rr 74° 54'.

We can now find the positions of the poles of t on the projection. Taking the t pole to the right of the vertical diameter ($2\bar{1}\bar{1}$):(211) for instance, on the arc ending on the horizontal dotted diameter, we first find the pole of the arc itself, namely, that which bears the poles ($0\bar{1}1$), $r=(001)$, $e=(011)$, $t=(031)$, $r=(010)$, $v=(02\bar{1})$, ($0\bar{1}\bar{1}$). To do so we join ($0\bar{1}1$) to $e=(011)$ and produce the straight line of junction to meet the primitive circle; from this point on the primitive circle downwards we mark off a quadrant arc, and join this point thus marked off to the starting-point ($0\bar{1}1$). Where this junction line cuts the diameter ($2\bar{1}\bar{1}$):(211) is the required pole P of the zonal arc. We have now only to mark off the angle $et=20^\circ 57'$ (or adequately closely 21°) from $m=(2\bar{1}1)$ along the primitive circle to the right, and join this point to P, the pole of the zone just found. Where this line cuts the zonal arc is the position of the pole $t=(031)$.

One further measurement was made, namely of the angle between the two common scalenohedron faces, vv , which was found to be $35^\circ 39'$. There are no other faces in the same zone, those of the second order hexagonal prism, which would be in the same zone if present, not being developed on this crystal of calcite.

This completes the angular measurements necessary for the determination of the crystal elements and the verification of the symmetry.

The indices of the forms n , v , and t , which were preliminarily assigned to these forms as the result of the knowledge of their positions gained during the measurements, and which corresponded to the well-known positions of the chief forms of calcite, will be found to be fully confirmed during the course of the calculations, to which we may now proceed. The indices of v are, moreover, at once confirmed by cross-multiplication of the indices of the two zones on which each scalenohedron face lies.

Calculation of Angles and Elements.—Basal Angle, $mr=(2\bar{1}\bar{1}): (100)=45^\circ 23'$.

It follows immediately, from the value of the basal angle mr , that its complement $(100):(111)$ is $44^\circ 37'$.

To find $re=(100):(110)$ and $(111):(110)$.

These angles can both be calculated from the triangle $r=(100)$, (111) , $e=(110)$, in which the angle at e is a right angle, the angle at (111) is 60° , and the side $(111):(100)$ has just been shown to be $44^\circ 37'$.

Constructing a Napierian diagram, as in Fig. 361, we at once derive from it according to Napier's rules:

$$\begin{aligned} \cos 60^\circ &= \cot 44^\circ 37' \tan (111:e) \\ \tan (111:e) &= \cos 60^\circ \tan 44^\circ 37' \quad | \quad \sin re = \sin 60^\circ \sin 44^\circ 37' \\ 111:e &= 26^\circ 16'. \quad re = 37^\circ 28'. \end{aligned}$$

From which: $rr=(100):(010)=2re=74^\circ 56'$.

The measured value of rr was $74^\circ 54'$, which agrees satisfactorily with this

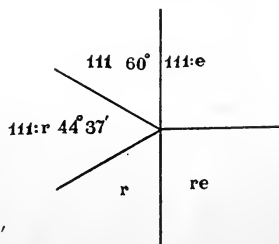


FIG. 361.

calculated value. This angle over the polar edges of the primary rhombohedron $\{100\}$ is the well-known rhombohedral angle of calcite, and the angle of the cleavage rhombohedron of the mineral.

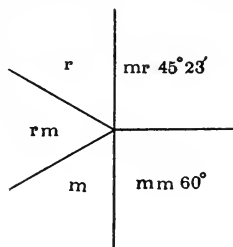


FIG. 362.

To find $rm = (100) : (11\bar{2})$.

This angle can be found from the triangle $m = (2\bar{1}\bar{1})$, $r = (100)$, $m = (11\bar{2})$, in which there is a right angle at $m = (2\bar{1}\bar{1})$, the side mm is 60° , and the side $m = (2\bar{1}\bar{1}) : r = (100)$ is the basal angle $45^\circ 23'$. From the Napierian diagram, Fig. 362, we at once get by the rules:

$$\cos rm = \cos 45^\circ 23' \cos 60^\circ. \quad rm = 69^\circ 26'.$$

This agrees well with the mean of the two measured values, $69^\circ 28'$.

We come next to the problem of finding the position of the scalenohedron face $v = (20\bar{1})$, in the zone $(10\bar{1})$, $v = (20\bar{1})$, $r = (100)$, $e = (101)$, that is:

To find the angle $rv = (100) : (20\bar{1})$.

This is achieved by means of the anharmonic ratio of four poles in a zone, and the problem is the simpler by reason of the fact that the whole angle from $(10\bar{1})$ to $e = (101)$ is 90° . The conditions are clearly shown in Fig. 363, from which we at once deduce, in accordance with the method explained in Chapter VI.:

$$\frac{10\bar{1}}{20\bar{1}} \cdot \frac{101}{100} = \frac{\sin(10\bar{1}) : v}{\sin 52^\circ 32'} \cdot \frac{\sin 37^\circ 28'}{\sin ev}$$

$$\frac{10\bar{1}}{100} \cdot \frac{1}{20\bar{1}} = \frac{\sin(10\bar{1}) : v}{\sin ev} \cdot \frac{\sin 37^\circ 28'}{\sin 52^\circ 32'}$$

$$\frac{1}{1} \cdot \frac{1}{3} = \frac{1}{3} = \frac{\sin(10\bar{1}) : v}{\sin ev} \cdot \frac{\sin 37^\circ 28'}{\sin 52^\circ 32'}$$

$$\frac{1}{3} = \frac{\sin(10\bar{1}) : v}{\cos(10\bar{1}) : v} \cdot \frac{\sin 37^\circ 28'}{\cos 37^\circ 28'}$$

$$= \tan(10\bar{1}) : v \cdot \tan 37^\circ 28'$$

$$\tan(10\bar{1}) : v = \frac{1}{3} \cot 37^\circ 28'. \quad (10\bar{1}) : v = 23^\circ 30'.$$

Then $vr = [(10\bar{1}) : r] - [(10\bar{1}) : v] = 52^\circ 32' - 23^\circ 30' = 29^\circ 2'$.

The mean observed value of this angle was $29^\circ 4'$.

To find $mv = (2\bar{1}\bar{1}) : (20\bar{1})$.

The magnitude of this angle mv may be readily calculated from the triangle $m = (2\bar{1}\bar{1})$, $r = (100)$, $v = (20\bar{1})$, in which we know the two sides $mr = 45^\circ 23'$ and $vr = 29^\circ 2'$ and the angle between them. For this latter is the same as the angle at r in the first triangle dealt with, for which Fig. 361 is the Napierian diagram. Working out first this angle r from that diagram we have:

$$\cos r = \sin 60^\circ \cos 26^\circ 16' (111 : e). \quad r = 39^\circ 3'.$$

Having now our three known elements in the oblique-angled triangle mrv , we apply formula (b) of the few ordinary essential formulæ of spherical trigonometry given in Chapter VII., page 96, from which we get:

$$\tan \theta = \tan 29^\circ 2' \cos 39^\circ 3'; \quad \cos mv = \frac{\cos 29^\circ 2' \cos (45^\circ 23' - \theta)}{\cos \theta}.$$

$$\theta = 23^\circ 19'.$$

$$mv = 28^\circ 4'.$$

This value $28^\circ 4'$ is identical with the mean of the measured values of mv .

To find $vv = (20\bar{1}) : (2\bar{1}0)$.

The half of this angle, vN , can easily be found, as it forms one of the sides adjacent to the right angle in the triangle of which the other two sides are $r = (100) : v = (20\bar{1})$,

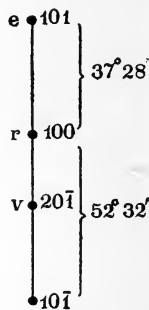


FIG. 363.

and the part Nr of $m=(2\bar{1}\bar{1}):r=(100)$ on the upper or r side of its intersection by the arc vv in question. The Napierian diagram is given in Fig. 364. The angle at r is the same as occurred in the last triangle, namely, $39^\circ 3'$, the side rv is $29^\circ 2'$, and the angle at the intersection N of mr and vv is a right angle. Hence, from the Napierian diagram we have:

$$\sin \frac{1}{2} vv = \sin 29^\circ 2' \sin 39^\circ 3'. \quad \frac{1}{2} vv = 17^\circ 48', \quad vv = 35^\circ 36'.$$

The measurement of vv gave the value $35^\circ 39'$.

It now only remains, as regards the angular magnitudes, to determine the positions of the faces of the rhombohedron $n=\{3\bar{1}\bar{1}\}$ and of the scalenohedron $t=\{310\}$.

To find the position of $n=\{3\bar{1}\bar{1}\}$ in the zone $[mnr]$.

The four poles consisting of the three just mentioned and (111) are situated on and together make up a quadrant, so we have the simpler case of the anharmonic ratio of four poles in a 90° zone. The conditions are defined in the diagram of Fig. 365. Setting forth the ratio we obtain the following:

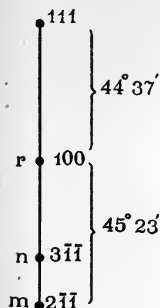


FIG. 365.

$$\begin{array}{r} 2\bar{1}\bar{1} \quad 111 \\ \times \quad \times \\ 3\bar{1}\bar{1} \quad 100 \\ \hline 2\bar{1}\bar{1} \quad 111 \\ \times \quad \times \\ 100 \quad 31\bar{1} \end{array}$$

$$\frac{\sin mn}{\sin 45^\circ 23'} \cdot \frac{\sin 44^\circ 37'}{\sin (111:n)} = \frac{2\bar{1}\bar{1}}{3\bar{1}\bar{1}} \cdot \frac{100}{111}$$

$$\frac{\sin mn}{\cos mn} \cdot \frac{\sin 44^\circ 37'}{\cos 44^\circ 37'} = \frac{1}{1} \cdot \frac{\bar{1}}{\bar{4}} = \frac{1}{4}$$

$$\tan mn = \frac{1}{4} \cot 44^\circ 37'. \quad mn = 14^\circ 13'.$$

$$\text{Then : } nr = mr - mn = 45^\circ 23' - 14^\circ 13' = 31^\circ 10'.$$

Each of these values is identical with the single measured value.

Similarly, to find the position of $t=\{310\}$ in the 90° -zone comprising the four poles $(10\bar{1})$, $r=(100)$, $t=\{310\}$, $e=(101)$.

The conditions are set out in Fig. 366, from which we at once derive the ratio:

$$\frac{\sin 52^\circ 32'}{\sin (10\bar{1}):t} \cdot \frac{\sin et}{\sin 37^\circ 28'} = \frac{10\bar{1}}{100} \cdot \frac{101}{301}$$

$$\frac{\sin et}{\cos et} \cdot \frac{\cos 37^\circ 28'}{\sin 37^\circ 28'} = \frac{1}{4} \cdot \frac{\bar{2}}{\bar{1}} = \frac{1}{2}$$

$$\tan et = \frac{1}{2} \tan 37^\circ 28'. \quad et = 20^\circ 58'.$$

$$\text{Then } tr = 37^\circ 28' - 20^\circ 58' = 16^\circ 30'.$$

The measured values of these two angles, each the mean of two measurements, were $20^\circ 57'$ and $16^\circ 27'$ respectively.

The agreement of both these values for the position of the scalenohedron $t=\{310\}$, as also of both those for the position of the rhombohedron $n=\{3\bar{1}\bar{1}\}$, with the measured values, is so satisfactory that absolute proof is afforded that the indices provisionally assigned to those two forms, namely $\{310\}$ and $\{3\bar{1}\bar{1}\}$, are correct. It will be obvious that the method of the anharmonic ratio can be used, as shown in Chapter VI., either for the calculation of the angles, assuming the indices, as has just been done, or for the determination of the indices, assuming the angles. As a matter of fact, as soon as the angles had been measured, the method was used to arrive provisionally at the proper indices, assuming the measured values to be correct. For instance, as regards the indices of the pole n , the left-hand side of the equation

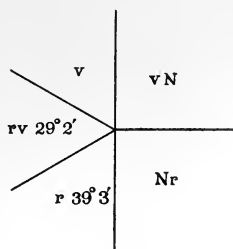


FIG. 364.

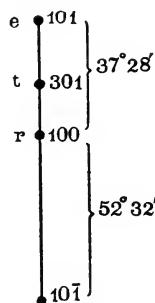


FIG. 366.

derived from Fig. 365, assuming the measured angle, $mn=14^\circ 13'$, to be correct, became: $\tan 14^\circ 13' \cdot \tan 44^\circ 37'$, the actual value of which is $\frac{1}{4}$.

For $\tan 14^\circ 13' \cdot \tan 44^\circ 37' = 0.24998 = \frac{1}{4}$, or $\frac{\tan 44^\circ 37'}{\tan 75^\circ 47'} = \frac{0.9867}{3.9471} = \frac{1}{4}$.

The right-hand side of the equation, if the indices of n are (pqq), is:

$$\begin{array}{ccc} 2\bar{1}\bar{1} & 111 \\ \times & \times \\ pq q & 100 \end{array} = \frac{2q+p}{1} \cdot \frac{\bar{1}}{q-p} = \frac{-2q-p}{q-p};$$

$$\begin{array}{ccc} 2\bar{1}\bar{1} & 111 \\ \times & \times \\ 100 & p q q \end{array}$$

Hence, equating the two sides to each other:

$$\frac{1}{4} = \frac{-2q-p}{q-p}, \text{ or } q-p = -8q-4p,$$

that is:

$$9q = -3p, \text{ or } \frac{p}{q} = \frac{9}{3} = \frac{3}{1}.$$

From which we gather that $p=3$, and $q=-1$, so that the indices of n are ($3\bar{1}\bar{1}$).

In a similar manner, the indices of the particular face of t in the zone shown in Fig. 366 were provisionally found to be (301), assuming the angle ct to have the measured value $20^\circ 57'$. The final confirmation of the measured angles, by the calculations of those angles from the basal angle, now completes the proof of the correctness, both of the indices and of the positions of the forms n and t .

The Millerian Axial Angle α .—As shown in the case of quartz, the angle of intersection of a pair of zones [rer] is the supplement of the axial angle. For each arc rer represents the angle over the polar edge between two primary rhombohedron faces, the well-known rhombohedral angle, and the angle between such a pair of edges is by the very mode of choice of the Millerian axes the angle between the axes, for the edges are the chosen directions of the axes. Now half the angle of intersection of these arcs, namely the angle at r in the triangle $r=(100)$, $o=(111)$,¹ $e=(110)$, has already been calculated and shown to be $39^\circ 3'$. Hence, the angle of intersection itself is $78^\circ 6'$. This is obviously the acute angle between the axes, and the one required, representing the obtuse angle between the two polar edges bounding the rhombohedron face, is consequently the supplement of this, or:

$$\alpha = 180^\circ - 78^\circ 6' = 101^\circ 54'.$$

Von Groth gives $101^\circ 55'$ for this angle, so that our measurements have evidently been satisfactorily accurate.

The Bravais-Miller Axial Ratio.—There is no face developed on the crystal, the pole of which would lie on a Bravais-Miller axis (dotted in Fig. 360). But we can find the ratio $\frac{a}{c}$ by the same method as was employed (second method) in the case of quartz, from the angle $r=(100):o=(111)$ in the same triangle as has just been used for obtaining the Millerian axial angle, namely, roe . This angle is the complement of the basal angle, namely $44^\circ 37'$, and from the discussion of the same case for quartz (page 372) it will be clear that the relationship between this angle and the Bravais-Miller axial ratio is:

$$\frac{c}{a} = \frac{\sqrt{3}}{2} \tan 44^\circ 37'.$$

Calculating this out, we obtain $\frac{c}{a} = 0.8545$.

Hence:

$$a:c = 1:0.8545.$$

¹ It is convenient to denote the pole of the undeveloped basal plane (111) by the letter o .

The value of the ratio given by Kuppfer is 1:0.8543, which is a very good concordance.

TABULAR PRESENTATION OF RESULTS FOR CRYSTAL OF CALCITE, CaCO_3 .

Crystal system: Trigonal. Class: 21, scalenohedral.

Millerian rhombohedral axial angle: $\alpha = 101^\circ 54'$. Angle of primary rhombohedron $\{100\}$ (over polar edges) $= 74^\circ 56'$.

Bravais-Miller ratio of axes: $a : c = 1 : 0.8545$.

Forms observed: $m = \{2\bar{1}\bar{1}\} = \{10\bar{1}0\}$, $r = \{100\} = \{10\bar{1}1\}$, $e = \{110\} = \{01\bar{1}2\}$, $n = \{3\bar{1}\bar{1}\} = \{40\bar{4}1\}$, $v = \{20\bar{1}\} = \{21\bar{3}1\}$, $t = \{310\} = \{21\bar{3}4\}$.

Table of angles: The basal angle is marked with an asterisk; the large brackets indicate the zones.

Angle.	No. of Measurements.	Limits.	Mean observed.	Calculated.	Difference.
$mm = (2\bar{1}\bar{1}) : (11\bar{2}) = (10\bar{1}0) : (01\bar{1}0)$	6	$59^\circ 57' - 60^\circ 5'$	$60^\circ 0'$	$60^\circ 0'$	0'
$mn = (2\bar{1}\bar{1}) : (3\bar{1}\bar{1}) = (10\bar{1}0) : (40\bar{4}1)$	1	...	14 13	14 13	0
$nr = (3\bar{1}\bar{1}) : (100) = (40\bar{4}1) : (10\bar{1}1)$	1	...	31 10	31 10	0
$mr = (2\bar{1}\bar{1}) : (100) = (10\bar{1}0) : (10\bar{1}1)$	3	$45^\circ 22' - 45^\circ 24'$	$45^\circ 23'$	*	...
$ro = (100) : (111) = (10\bar{1}1) : (0001)$	44 37	...
$oe = (111) : (011) = (0001) : (1012)$	26 16	...
$re = (100) : (011) = (10\bar{1}1) : (1012)$	3	$70^\circ 50' - 70^\circ 57'$	$70^\circ 54'$	$70^\circ 53'$	1
$em = (011) : (211) = (1012) : (1010)$	3	$63^\circ 40' - 63^\circ 48'$	$63^\circ 44'$	$63^\circ 44'$	0
$mv = (2\bar{1}\bar{1}) : (20\bar{1}) = (10\bar{1}0) : (21\bar{3}1)$	2	$27^\circ 56' - 28^\circ 12'$	$28^\circ 4'$	$28^\circ 4'$	0
$vr = (20\bar{1}) : (010) = (21\bar{3}1) : (1101)$	2	$82^\circ 25' - 82^\circ 32'$	$82^\circ 29'$	$82^\circ 30'$	1
$rm = (010) : (211) = (1101) : (1010)$	2	$69^\circ 23' - 69^\circ 32'$	$69^\circ 28'$	$69^\circ 26'$	2
$vr = (20\bar{1}) : (100) = (21\bar{3}1) : (10\bar{1}1)$	2	$29^\circ 4' - 29^\circ 4'$	$29^\circ 4'$	$29^\circ 2'$	2
$rt = (100) : (301) = (10\bar{1}1) : (31\bar{2}4)$	2	$16^\circ 27' - 16^\circ 27'$	$16^\circ 27'$	$16^\circ 30'$	3
$te = (301) : (101) = (31\bar{2}4) : (1102)$	2	$20^\circ 55' - 20^\circ 58'$	$20^\circ 57'$	$20^\circ 58'$	1
$re = (100) : (101) = (10\bar{1}1) : (1102)$	6	$37^\circ 22' - 37^\circ 30'$	$37^\circ 27'$	$37^\circ 28'$	1
$rr = (100) : (001) = (10\bar{1}1) : (0\bar{1}11)$	3	$74^\circ 52' - 74^\circ 56'$	$74^\circ 54'$	$74^\circ 56'$	2
$vv = (20\bar{1}) : (2\bar{1}0) = (21\bar{3}1) : (3\bar{1}21)$	1	...	$35^\circ 39'$	$35^\circ 36'$	3

In concluding these practical examples of the method of carrying out the morphological investigation of crystals, it may be remarked that the only difference between the method here shown and the actual investigation of a newly discovered, and therefore hitherto unmeasured, crystallised substance, is that instead of one crystal being so investigated at least ten of the very best procurable crystals should be measured, selected from at least four different crops, in order that as many excellent "A" values of the angles may be obtained as possible from which to extract the mean value, and so that we may be reasonably sure that all the forms developed by the substance, which may not all be present on a single crystal, or on any of the crystals of a particular crop, have been observed.

CHAPTER XXV

THE DRAWING OF CRYSTALS.

ALL the drawings of crystals which have been given in the foregoing illustrations have been constructed accurately to scale (the pencilled construction lines other than the crystal axes having been eliminated after inking in the outlines) on a conventional plan, the so-called clinometric or clinographic projection, a slightly inclined form of "parallel perspective," according to which the eye is supposed to be removed to an infinite distance, so that parallel lines, instead of converging to a vanishing point as in ordinary perspective, are actually parallel in the drawing. The general appearance of the crystal, as regards foreshortening from back to front, is at the same time very faithfully reproduced, which confers a great advantage over the isometric projection. The method is now universally employed, and is the accepted mode of drawing crystals for publication in the *Zeitschrift für Kristallographie*.

The advantage will at once be obvious of a representation of a crystal on paper which, while correctly giving the impression taken in by the eye, assures that edges which are really parallel to each other shall still appear so in the drawing; for it is of the first importance to the detection and appreciation of the symmetry of a crystal to know which faces belong to the same zone, a fact which we have seen is indicated by the parallelism of their edges of intersection. Now the angle between the directions of two edges on different parts of a crystal is often so small that an inspection of a drawing in ordinary perspective would not inform us whether the edges were really parallel on the crystal or not, whereas on the plan adopted they are either parallel or not exactly according as they are drawn.

There is one further convention besides the parallelism of truly parallel lines and the back-to-front foreshortening, namely, that the eye is supposed to be raised somewhat above the horizontal plane, in order that something may be seen of the foreshortened upper faces of the crystal from above, and also somewhat moved to the right, in order to see the right-hand foreshortened side faces. The designation "clinometric" or "clinographic" refers to this fact, and distinguishes the method from the orthometric or orthogonal projection, which represents

the crystal as seen directly from above, as in a plan, or from the front, as in an elevation, a plane of symmetry being usually the plane of projection.

In both the orthogonal and clinographic projections the light rays joining the eye and crystal coigns (solid angles, corners at which three or more edges meet) are all parallel, but in the former they are perpendicular to the plane of the paper on which the drawing is made, while in the latter they are inclined a few degrees to the paper.

The key to the whole drawing is that of the crystallographic axes, which are first drawn in accordance with the rules of these conventions, and in agreement with the system of symmetry exhibited by the crystal. When once these are drawn, the crystal can be readily built up about them, for intercepts on these projected axes are proportional to the actual intercepts on the axes; and when the outline of the crystal is completed the axial scaffolding can be dispensed with, unless, as has frequently been the case in respect of the drawings in this book, it be considered desirable to indicate the positions of the axes by lines broken-and-dotted in order to distinguish them from the continuous lines indicating the front faces and from the dotted lines indicating the back faces. The pencilled auxiliary construction lines, however, may be erased after the inking-in of the crystal outline and axes. The recent development of "process reproduction" by photo-engraving has rendered it usually most convenient for the purpose of subsequent reproduction, which requires an exceedingly clear and clean drawing, that the construction shall be done on Bristol-board, a fine-surfaced dense cardboard particularly suitable for such drawings in Indian ink. If the drawing be that of an original crystal, and the results of the investigation are to be published, it should be made twice or three times the size it is required to appear, in order that the engraver may reduce it to the required size during his photographic reproduction, and so greatly enhance the neatness.

It must be remembered that the stereographic projection of a crystal—which will always have been drawn during the course of the measurements of the crystal angles and before the accurate drawing in clinographic projection is undertaken from the hand sketches of the crystal—is of the greatest help in constructing the clinographic projection. For it at once informs us which faces belong to any one and the same zone; and as all such interfacial edges are parallel, and parallel lines remain as such in the clinographic projection, these edges must be drawn parallel to each other, wherever they appear unreplaced by other forms. Moreover, the general direction of any zone oblique to the axial planes, and its relation to other zones, will be clearly apparent on the stereographic projection, and will enable confirmation to be obtained of the accuracy of drawing of the interfacial edges of the zone on the clinographic projection. Indeed, it is possible to determine the directions of the interfacial edges of such zones by the graphical methods of Penfield,¹ directly from the stereographic projection, when these

¹ *Amer. Journ. of Science*, 1901, 11, 1 and 115; also 1902, 14, 249.

directions are not already known by being parallel to the crystallographic axes. The student must, however, in any case be able to use the perfectly general and simple method of intersections described in this chapter, and can then with great advantage confirm the accuracy of the construction by reference to the stereographic projection.

Rules for constructing the Axes of a Cubic Crystal.—For all classes of crystals the axes have first to be constructed as for a cubic crystal, and then, when the latter is other than cubic, modified accordingly. Hence the construction of the cubic axes is of the first importance to the drawing of crystals, and it embodies all the various conventions which have been referred to. It is shown in Fig. 367, the method being the original one of Naumann, in the form recommended by von Groth, and as it is usually employed for crystal drawings published in the *Zeitschrift für Kristallographie*.

We first draw a horizontal line XX' , and a vertical line YY' , intersecting truly at right angles in the centre O of the drawing. XX' is then divided into 6 parts, and

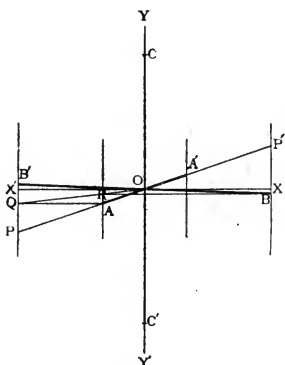


FIG. 367.
Construction of Crystal Axes.

lines parallel to YY' are drawn through the two ends and through the second and fourth divisions. From X' a length $X'P$ is set off downwards equal to one division, and P is joined to the centre O and the line produced to P' . The part AA' of this line PP' which is intercepted between the two parallel vertical lines drawn through the second and fourth divisions is the projection of the front-to-back horizontal axis of the crystal.

From A a line AQ is drawn parallel to XX' , and the point Q where it cuts the left-end vertical line is joined to the centre O , the line QO intersecting the second vertical line at R . From R another line RB is drawn parallel to XX' , meeting the right-hand vertical line at B . B is then joined to the centre O and the line BO produced to B' , equally on the other side of the centre. BB' is then the projection of the right-to-left horizontal axis of the crystal.

To determine finally the proper length of the vertical axis, OC and OC' are each made equal to OP .

As the cubic axes have to be drawn in the first instance, in the case of every crystal drawing, it saves much labour in their construction to have a permanent three-armed ruler or "templet" constructed for them once for all. By graduating each of the axial edges with a convenient scale, the larger units of which in each case are the unit lengths of the cubic axes as determined in Fig. 367, and the smaller units tenths of these, every five-tenths being emphasised by giving it an intermediate length, it is possible not only at once to mark off on the ruled axes the points corresponding to unit lengths and to twice and three times the unit lengths of the cubic axes, but also to mark off the proper lengths of the axes when the crystal is tetragonal or rhombic, the directions of the axes in these systems being equally rectangular and coincident with the cubic ones but the lengths differing. A three-limbed ruler of

this nature may be constructed on these principles out of a single piece of thick sheet brass. But the author finds one of the form shown (one quarter the real linear size) in Fig. 368 to be even more convenient. It has the advantage that both limbs of each axis are included, so that they can at once be ruled in and their lengths marked off. Moreover, it is by no means necessary to go to the expense of having it constructed of metal, although naturally it is then more durable and

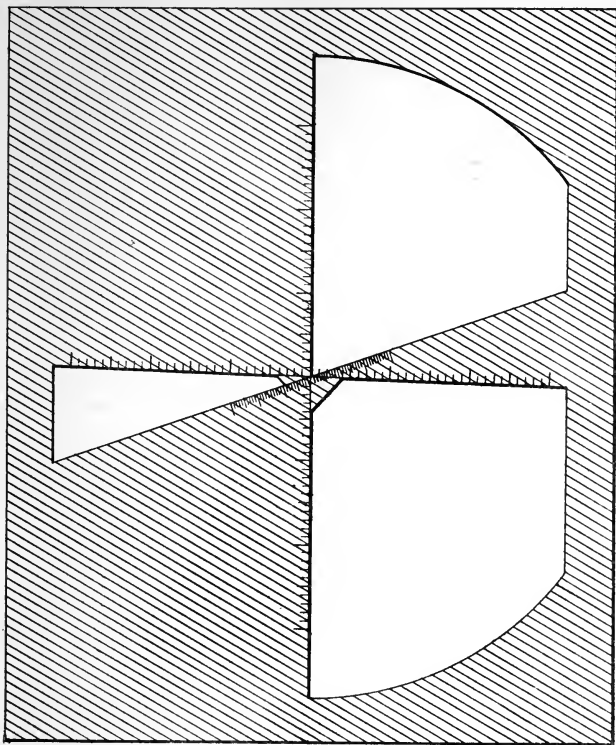


FIG. 368.—Templet for Drawing Crystal Axes.

the edges are less likely to become damaged. The ordinary full-sized sheet ($15\frac{1}{4}$ by $12\frac{1}{2}$ inches) of the stoutest Bristol-board serves admirably, and enables three unit lengths to be graduated along each semi-axis from the centre outwards, on the same scale as that which has been used for all the original drawings for the illustrations in this book.

The Drawing of Cubic Crystals.—The form of the octahedron {111} is at once given by joining the ends of the axes to each other, as shown in Fig. 369, those edges actually visible in front and at the sides being drawn in prominent but not too thick continuous lines, and those edges which are hidden from the eye if the crystal be opaque, behind the

crystal, and which even in the clearest transparent crystal would be distorted by refraction if visible, are drawn in dotted thinner lines.

The cube $\{100\}$ is obviously obtained as in Fig. 370, by drawing

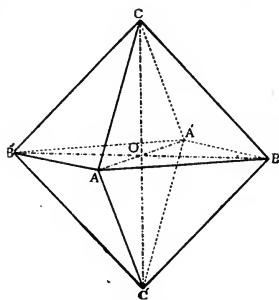


FIG. 369.
Construction of Octahedron $\{111\}$.

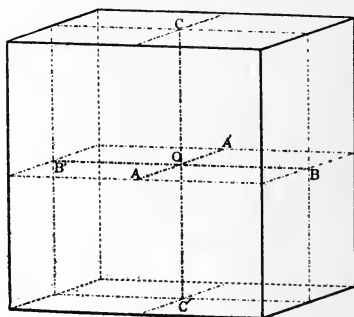


FIG. 370.—Construction of Cube $\{100\}$.

lines parallel to the axes and of the same length, in such a manner that the 6 ends of the axes are the centres of the six parallelograms representing the sides.

The edges of practically all other forms are constructed on the general principle of finding the intersection of the two planes corresponding to the pair of faces meeting in each particular edge, the two planes being simply constructed by joining points on the axes corresponding to their intercepts. The drawing of each of the principal forms of the cubic system will now be given, including the combination exhibited by the garnet crystal practically worked through in Chapter XII.; and this section will be concluded by the drawing of the somewhat more complicated combination of cubic forms, including one of a lower order of symmetry than the holohedral class 32, exhibited by the crystal of cobaltite also discussed in Chapter XII.

To obtain the direction of the edge between any two faces, we have first to convert their indices to intercepts, by the method given in Chapter V., page 69, to mark off along the 3 axes (assuming all 3 to be intercepted) the lengths corresponding to these intercepts for each of the 2 faces, and to join the 3 points in each case so as to obtain a representation of the facial plane; these two planes will then frequently meet or intercept each other at 2 points (usually an intersection and a point of contact), and the line joining these points is the direction of the edge between the 2 faces. In the not uncommon event, however, of the two planes as thus drawn only meeting at one point, one of them must be moved inwards parallel to itself until it touches the other at another point, conveniently the end of another axis, when it will intercept that other at a second point instead of merely touching at the first point referred to. We can always move the planes parallel to themselves until a convenient intersection of the two is obtained. Moreover, the direction of the edge afforded by the line of intersection is not necessarily actually the

representation of the edge itself; this latter may be drawn wherever it is required, parallel to the indicated direction.

These few simple general rules practically cover the construction of all crystals, even the most complicated.

The **rhombic dodecahedron** $\{110\}$ is a simple instance of the procedure by the method of intersections.

Each of the four front faces, meeting in an apex at A, intersects the axial plane BOC, Fig. 371, in one of the 4 thin broken-and-dotted lines DE, EF, FG, GD, which have been drawn parallel to the axes BB' and CC' through the ends of those axes. Obviously, therefore, the faces intersect each other in pairs at the corners D, E, F, G. As each corner is thus common to the planes of the two adjacent faces it must be a point on the edge between them, a second point being the apex A. The directions of the edges are therefore obtained by joining A to D, E, F, G respectively. The directions of the edges of the four similar back faces terminating at A' are in the same manner obtained by joining A' to the four corners. The other edges on the crystal are all parallel to these same directions already drawn, for every face of the rhombic dodecahedron is a regular rhombus and not merely a parallelogram. We draw, therefore, all the remaining edges from the ends of the axes BB' and CC' parallel to the proper edges terminating at A, as shown in the figure, and these lines will determine the correct lengths of the original edges terminating at A, as well as of those terminating at A' , the directions of which only were given by joining A or A' to the corners. The back faces are, of course, to be drawn in broken or dotted lines.

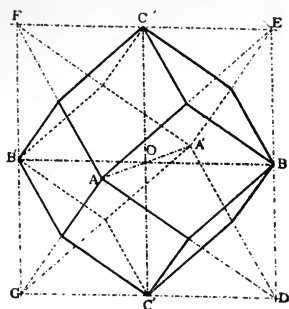


FIG. 371.—Construction of Rhombic Dodecahedron $\{110\}$.

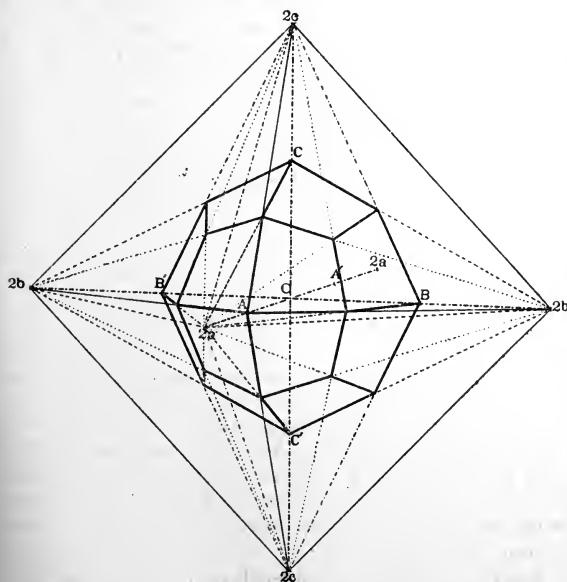


FIG. 372.—Construction of Icositetrahedron $\{211\}$.

We commence as usual by drawing in the axes by ruling from the templet; they are shown in the strong broken-and-dotted lines in Fig. 372, and two unit lengths are

The **icositetrahedron** can be readily drawn on the same principle, and it will next be shown in Fig. 372 how that represented by the indices $\{211\}$ is drawn, the commonest form of this solid, that which is exhibited by the garnet crystal dealt with in Chapter XII.

required of each, unit lengths being marked off at A and A', B and B', C and C', and the double lengths at $2a$, $2b$ and $2c$. The 3 planes corresponding to the 3 faces of each octant are then drawn in, by joining their intercepts on the proper 3 semi-axes, and their intersections give the directions of the icositetrahedron edges. Taking the upper right octant as an example, the intercepts corresponding to the 3 faces (211), (121), and (112) (for the indices see Fig. 87, page 158) are respectively $a : 2b : 2c$, $2a : b : 2c$, and $2a : 2b : c$. For the face (211) the points marking in front the unit length of a (namely A), $2b$ to the right, and $2c$ above, are joined, forming the triangle shown in thin continuous line in Fig. 372; for the face (121) the points marking $2a$ in front, unit length of b (that is, B), and $2c$ above are joined, forming the triangle shown in thin broken-and-dotted lines, and for (112) the points $2a$, $2b$, and unit length of c (that is, C) are joined, forming the triangle shown in broken lines in the figure. Each pair of these triangles intersect at one point and touch one another at a second, the latter being $2a$, $2b$, or $2c$; and the line joining these two points, drawn dotted, is the direction of the edge between the two icositetrahedron faces corresponding to the two triangles, the three edges thus found being those which meet in the centre of the octant. The parts of the dotted junction lines which represent the actual edges on the crystal are drawn in bold continuous line, as being visible edges in front. The other edges forming the intersection of the three faces of this octant with those of the adjacent ones are actually parts of the sides of the triangles representing the planes themselves, as will be obvious when we have treated the other three front octants in the same manner, in order to get the three centrally converging edges of each of these octants, for it will be clear that the parts of the triangles in question, forming the other edges of the icositetrahedron, are each common to a pair of faces of the two adjacent octants the separation of which they mark, and are thus the lines of intersection, or edges, of those faces. Hence, the whole construction for the front half of the icositetrahedron is afforded by the four sets of three triangles. The back half may, of course, be similarly obtained, but as we are dealing with a perfectly symmetrical solid, and the figure is now getting complicated with numerous lines it is perhaps simpler to make a tracing of the front half now drawn, and to reproduce it on the Bristol-board with the aid of a tracing stylus and a blacklead¹ interleaf, after rotating the tracing about the centre O (through which a pin is stuck) until the axes are again coincident, that is, for exactly 180° . As the figure is symmetrical the back faces are exactly like the front ones, but inverted, their outline being precisely that of the front half when the latter is turned round 180° in the plane of the paper. The whole figure of the icositetrahedron {211} shown in Fig. 87 (page 158) in Chapter XI., in which the indices have also been inserted, was obtained from Fig. 372 in this manner by means of a tracing, care having been taken, by use of the set-squares or parallel ruler, that such edges as are parallel were made strictly so in the inking-in.

The same tracing was made use of for the drawing of the crystal of garnet, Fig. 114 (page 174), described in Chapter XII., and the small faces of the rhombic dodecahedron, marked d in the figure (the icositetrahedral faces being labelled i) were inserted by the method of Fig. 371. For the icositetrahedron {211} directly replaces the edges of the rhombic dodecahedron, the faces of the two forms lying alternately in the same dodecahedral zone, so that the edges of intersection of the two forms are parallel to the edges of the latter form. The d -faces were given such a relative size as they possessed on the garnet itself, for we can, of course, move the edges parallel to themselves so as to represent faithfully the observed proportions, so long as the directions are those determined by the intersections.

The **triakis octahedron** may also be readily drawn on similar lines. The outline of the octahedron appears in this solid just as in the simple

¹ Pure graphite paper must be used; the carbon sheets used for duplicating typewriting must be avoided, as they give lines which are ineradicable by india-rubber.

octahedron, but on each face a low triangular pyramid is erected. In Fig. 373 the construction of the triakis octahedron $\{221\}$ is given.

The difference from the last construction is that here two of the three intercepts remain unity as for the octahedron, hence the fact that the octahedron edges persist, while the third is of double length, and for the three faces of any octant the three semi-axes become of double length in turn. The triangle representing the plane (221) (see Fig. 88, page 158, for indices) having intercepts $a : b : 2c$, is shown in continuous fine line, that for the face (212), with intercepts $a : 2b : c$, is drawn in thin broken-and-dotted line, and that for the third face (122), having intercepts $2a : b : c$, is given in ordinary broken lines, and corresponding types of lines are given for the 3 triangles of each of the other 3 front octants.

The two points where each pair of triangles representing the planes of adjacent faces meet and intersect give, when joined (by dotted lines in Fig. 373), the direction of the edge between the faces. One of these two points in each case is a point of intersection P, and the other is a point of meeting at the end of an axis. The whole figure of the front half of the crystal thus obtained is drawn in strong lines, the part of each dotted intersection line which actually corresponds to the edge between the two faces in each case, and which therefore requires to be thus drawn in strong continuous line, being quite obvious.

The illustration of the triakis octahedron given in Fig. 88 (page 158), Chapter XI., accompanying the description of the form, was obtained by preparing a tracing of the crystal outline from this Fig. 373, and subsequently transferring it to Bristol-board with the aid of the tracing stylus and the blacklead interleaf, and the back half also given in Fig. 88 was obtained by rotating the tracing paper for 180° about a pin stuck through the centre O and again tracing from the front half thus inverted, except the octahedron edges which were drawn directly, the usual precaution being taken to render all lines strictly parallel which should be so, during the inking-in, with the aid of the set-squares. The indices for the front faces are also inserted in Fig. 88, so that the two figures together enable the whole construction to be followed without undue complication of either.

Tetrakis Hexahedron.—In the case of this solid the cube forms the main skeleton, with a tetragonal pyramid erected on each face. The construction of the tetrakis hexahedron $\{210\}$ is shown in Fig. 374.

The plane representing each face is now parallel to one of the axes, it cuts another at unit length and the third at double that length. The axes, the unit lengths of

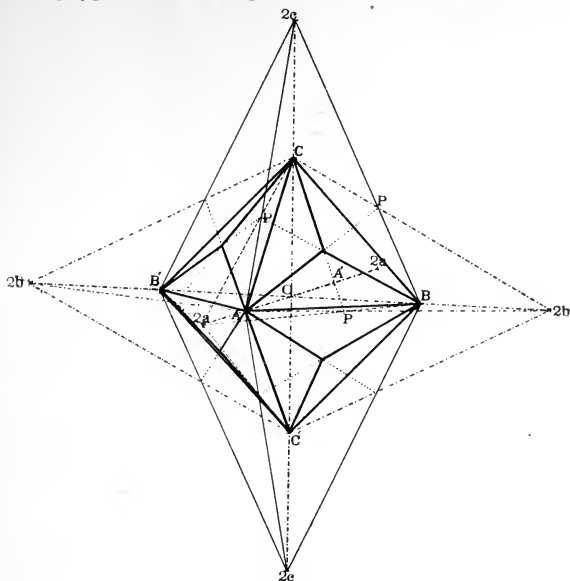


FIG. 373.—Construction of Triakis Octahedron $\{221\}$.

them, and the double unit lengths, are drawn and marked as in the two previous figures; and also the lines joining the points on the axes corresponding to the intercepts, indicating the various facial planes, are drawn in the same manner, but only such of them as are essential, by their actual intersection or meeting, to the determination of the directions of the crystal edges, in order not to unduly complicate the

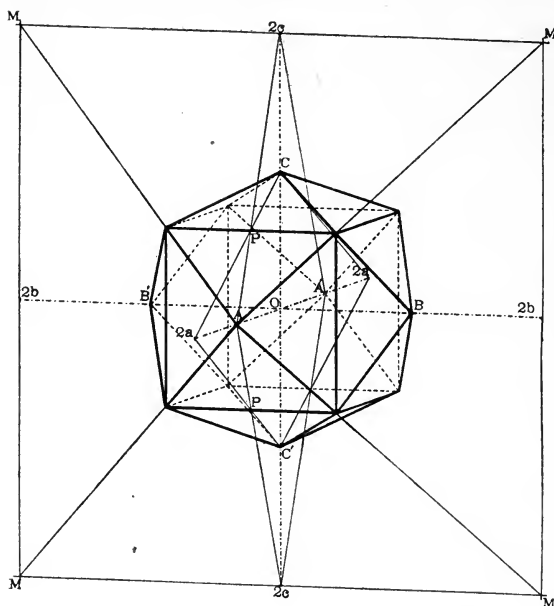


FIG. 374.—Construction of Tetrakis Hexahedron $\{210\}$.

other piece of Bristol-board, in order to obtain Fig. 89 (page 159), the illustration of the tetrakis hexahedron which accompanies the description of the solid in Chapter XI. The back half of the crystal was also traced in after rotating the tracing of the front half for 180° , and subsequently inked-in with thin broken lines. The indices are inserted in Fig. 89 for the front faces, which will enable the construction of the intersecting planes in Fig. 374 to be the more readily followed.

Hexakis Octahedron.—The construction for the $\{321\}$ representative of this 48-sided crystal form, exhibiting the fullest type of cubic symmetry, and therefore the fullest of all possible symmetry, is given in Fig. 375. Although apparently complicated owing to the large number of faces, it is really quite simple. Only such construction lines as are essential are shown, in order not unnecessarily to involve the drawing.

The indices of the faces of the front half are given in Fig. 85 (page 157) of Chapter XI, which was derived from the constructed Fig. 375 precisely as has just been described for the tetrakis hexahedron, and includes in the same way the back faces, inked-in with broken lines. It will be seen from Fig. 85 that the indices are all formed by the three figures 3, 2, and 1, and the intercepts consequently are always either 1, $\frac{2}{3}$, or 3. These lengths are, therefore, shown marked off along each of the three axes in Fig. 375, and every facial plane meets each of the axes at one of these three lengths. To construct every one of the 24 triangles representing the front facial planes completely,

figure. Thus P marks the point of intersection of two planes which determines the position of the cube edge in each case, parallel to the axis b , and M in each case marks one of the two meeting points of two of the front facial planes, the other being the end A of the axis a , so that the four lines AM are the directions of the four front pyramidal edges.

The outline of the crystal, as finally drawn in strong line, on the Bristol-board on which the construction was directly made for the purpose of this Fig. 374, was traced off on to tracing paper, and the tracing was transferred with the aid of the stylus and blacklead paper to another

by joining three of these points in each case, would produce a maze of lines, so only such of the junction lines as are really required to give the essential intersections or contacts for the determination of the directions of the interfacial edges are actually shown, although most of them were put in originally in pencil, but erased after the inking-in of the lines actually needed. For instance, the direction of the edge between the face (321), the face which gives its indices to the form, and that below it, (32 $\bar{1}$), is afforded by the intersection of the two planes having the corresponding intercepts $a : \frac{2}{3}b : 3c$ in each case, the $3c$, however, being marked off along the positive upper half

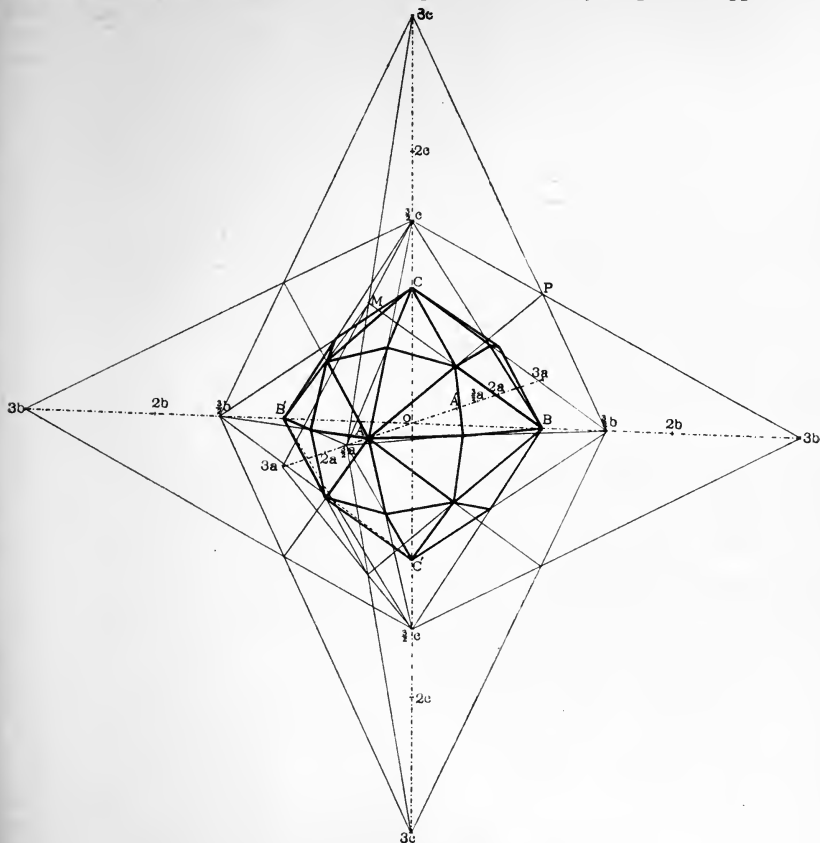


FIG. 375.—Construction of Hexakis Octahedron $\{321\}$.

of the vertical axis in the former case and along the lower negative half in the latter, the a and the $\frac{2}{3}b$ being identically marked off along the positive halves of those axes. The line joining A and $\frac{2}{3}b$ is thus common to the two triangles representing the planes, and is therefore the direction of the edge between the planes, the edge itself, subsequently inked-in strongly, being part of it. A similar process affords us all the other edges lying in the cube planes, thus, for instance, the line joining B and $\frac{2}{3}a$ is the direction of the edge between the next pair of faces to the right, (231) and (23 $\bar{1}$), and the part of it which is the actual edge itself is determined by its intersection with the edge-direction between (321) and (32 $\bar{1}$) already found, being that part of the thin junction line which lies between B and this intersection, just as the part of the

junction line between A and the intersection is the actual edge between (321) and (32 $\bar{1}$). Again, to find the edge between the same form-indicating face (321) and that above it to the left, (312), the intercepts are respectively $a : \frac{2}{3}b : 3c$ as before and $a : 3b : \frac{2}{3}c$; the triangles formed by joining these intercepts have a common meeting point at A and a point of intersection at P (the intersection of the sides $3b : \frac{2}{3}c$ and $\frac{2}{3}b : 3c$). The line AP, therefore, is the direction of the required edge. The point in the middle of the octant, at which this edge, starting at A, terminates, and to which five other edges also radiate, is found by one further construction, say by that of the edge between (231) and (132); the intercepts are $\frac{2}{3}a : b : 3c$ and $3a : b : \frac{2}{3}c$, and the two triangles formed by joining these intercepts have the point B in common and intersect at M, so that BM is the direction of the edge in question. Where AP and BM cut each other is, therefore, the radial point in the middle of the octant, and we can consequently draw in the two actual edges under discussion by thickening the parts of AP and BM starting at A and B respectively and terminating at this radial point. Moreover, we can also draw in at once the other four edges radiating to the same point, from C and from the three intermediate points between A, B, and C where the edges lying in the cube planes, already found, meet. In a similar manner the interfacial edges of the other three front octants can be drawn, the necessary triangle-sides being all given in thin lines in the figure, so that the process can be readily followed. The finished outline of the front half of the crystal thus obtained, as clearly marked by the thickened lines, was reproduced by the tracing process already fully described, in the Fig. 85 already alluded to, and the back faces inserted after the rotation of the tracing for exactly 180°.

To conclude the discussion of the drawing of cubic crystals an example of a **combination** will be taken, which includes not only the fully symmetrical forms of class 32 but also one selected from the classes of lower symmetry, a so-called hemihedral form. The example chosen is the crystal of **cobaltite** described and worked through practically in the latter half of Chapter XII. It comprises faces of the cube, octahedron, and the pentagonal dodecahedron {210} of class 30.

Its construction is shown in Fig. 376, only such structural lines being given as are essential to the determination of the directions of the various edges. Also, after having found by construction the direction of a single edge of any zone, the knowledge gained from the practical measurement of the crystal as to which faces and their interfacial edges belong to the same zone has been fully utilised, for such edges must always be parallel to each other. After the completion of the drawing of the front half of the crystal, a copy of its outline was traced for Fig. 119 (page 183) in Chapter XII. describing the work on this crystal; and the back half was subsequently inserted after the rotation of the tracing about the centre for 180°. The letters *c*, *o*, and *p* on the faces in Fig. 119 indicate which belong to the cube, octahedron, and pentagonal dodecahedron respectively. Fig. 103, page 167, representing the latter form with its indices, will be helpful.

Commencing with the front faces of the pentagonal dodecahedron (210) and (2 $\bar{1}$ 0), right and left of the front-central vertically elongated cube face which replaces the edge between them, it will be best at first to ignore the latter, and to draw in the two *p*-faces as if alone present. The direction of the edge between them is obviously that of the vertical axis, as both faces are parallel to that axis, the *c*-index of each being 0. The place where we shall draw this vertical edge may be left, however, until we have found the most convenient situation relatively to the determination of the edges between the *p*-faces and the adjacent *o*-faces. To proceed to find, for instance, the direction of the edge between (210), the right central *p*-face which gives its indices as symbol for the form, and the primary octahedron face (111) above it, we first join A,

B, and C to represent the plane (111), and then similarly draw in the plane (210), the intercepts of which are $a : 2b : \infty c$, by joining A to $2b$; vertical lines parallel to axis c from A and from $2b$ should be understood to represent the ∞ value of c , but are not actually drawn as they are not wanted. This line A : $2b$ representing the (210) plane, however, only touches the (111) triangle at A, so in order to get an intersection we must move one (either) of them parallel to itself in the direction of the other; the p -plane has been so moved nearer to the centre O until it has intercepted the b -axis at B, which is then common to the two planes instead of A, the a -axis being intercepted at a point L, half-way between A and O. From L a line parallel to the vertical axis is drawn, cutting the octahedron edge AC at P. P is thus a second point

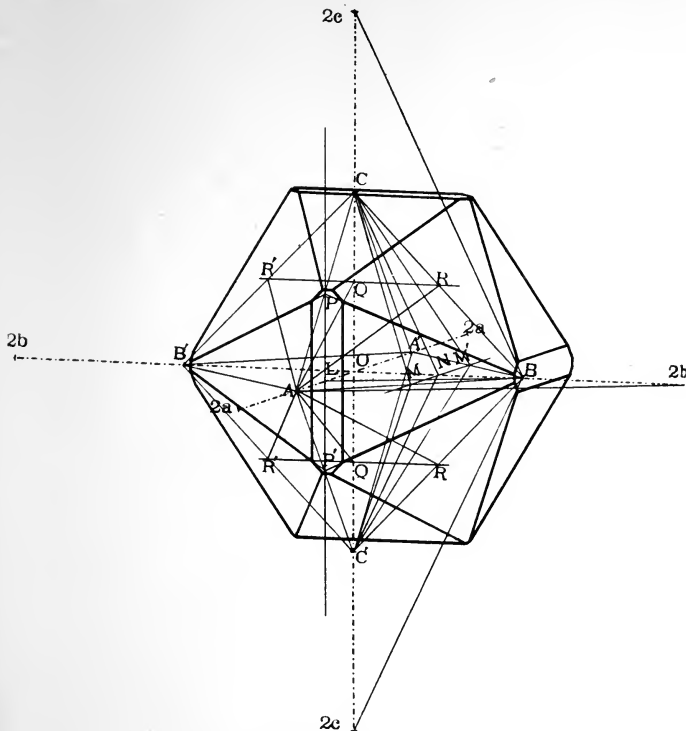


FIG. 376.—Construction of Crystal of Cobaltite.

common to the octahedron and the pentagonal dodecahedron, and if we join it to the other common point B we have the direction of the required edge. The edge between the same (210) face and the o -face below it, $(11\bar{1})$, is similarly along the direction BP' , and the edges on the left side of the centre, between $(2\bar{1}0)$ and the upper and lower left o -faces are $B'P$ and $B'P'$. The edge between the upper right o -face (111) and the p -face to the right of it (021) is determined in like manner, by constructing the plane (021) having intercepts $\infty : b : 2c$, that is, by joining B and $2c$, and then moving it parallel to itself until it cuts the c -axis at C, so as to make this a common point instead of B, when it will also cut the b -axis at N, half-way between O and B. Through N we draw a line parallel to the axis a , as the plane (021) is always parallel to that axis, and where this line cuts the octahedron edge AB, namely at M, will be the second point on the line of direction of the required edge, CM. This line CM is not the edge itself, but

only its direction, but we can move it parallel to itself until it occupies the position actually observed on the crystal, namely, so as to form a common apex at B with the two edges of the face (210). The corresponding lower edge between the right lower *o*-face and the lower *p*-face to the right of the latter is similarly parallel to C'M. Further, the direction of the two other edges bounding these right-hand *p*-faces on the extreme right of the figure, between them and the upper and lower right back *o*-face are also given by CM' and C'M', and they can be drawn in eventually so as to meet the *o*-faces at the apex in each case as shown in the drawing, and as they really appeared on the crystal itself. But before actually drawing them in it is advisable to construct the edges between the *o*-faces and the upper central and lower central *p*-faces, so as to get the complete shape of both sets of faces, and to determine where the upper and lower apices on each side come. We can, however, at once utilise our knowledge of the directions CM' and C'M' in drawing in the two edges on the extreme left of the figure, meeting at B', for these are parallel to the two edges on the extreme right and therefore their directions are equally given by CM' and C'M'. To determine the upper and lower front *p*-edges, and incidentally the upper and lower apices, we proceed as follows. The upper central *p*-face is (102), making the intercepts $2a : \infty : c$, so we join $2a$ and c , or, in order to get 2 points of intersection with the octahedron face (111) or ($\bar{1}\bar{1}1$), draw AQ parallel to the line joining $2a$ and c . On drawing through Q a line RQR' parallel to the axis b , as (102) is parallel to that axis, we find that it cuts the octahedron edges at R and R'. Joining these respectively to the second point of the intersection, the lines AR and AR' are the directions of the edges *o* : *p* required, all four of which are drawn radiating (two from each) from P or P'. These edges meet at their upper and lower ends those of the *o*-faces radiating from B, thus determining the apices from which the extreme right *p*-edges can be drawn parallel to CM' and C'M'. The top and bottom lines of the figure are then obtained by joining, parallel to the axis b , the two top and two bottom apices.

This completes the figure as regards the *o*- and *p*-faces, and the narrow cube faces can then be readily inserted, the long edges being parallel to the three axes respectively, as are also the short terminal edges, while the inclined edges between the latter and the long edges are parallel to the ordinary octahedron edges. These cube faces are given the relative size in the drawing which they actually exhibited on the crystal.

As regards the insertion of the back faces in Fig. 119, after the transfer of the front faces from the tracing of the boldly drawn outline of Fig. 376, by rotation of the tracing for 180° , it will be obvious that the centre of rotation is not O, but the actual centre of the hexagonal marginal outline. For owing to the movement of the *p*-planes parallel to themselves O no longer remains the centre, as it does in the simpler cases previously considered, in which the intersections of the various pairs of planes corresponding to adjacent faces were obtained without any such parallel movement being required.

The drawing might have been made, however, so as to retain O as centre, by moving the octahedron planes outwards parallel to themselves, instead of moving the *p*-faces inwards, in order to get their intersection. But the course adopted was taken purposely, in order to emphasise the fact that the directions of the edges are equally well afforded by either course, and that it is these **directions** which are determined, and not necessarily the actual positions or relative lengths of the edges; the latter may be drawn anywhere, and of any length, parallel to the indicated direction, so as to reproduce faithfully the general appearance and relative prominence of forms of the crystal.

Tetragonal and Rhombic Axes.—When the crystal belongs to the tetragonal or rhombic system, in both of which the axes are still rectangular, the only difference introduced in the drawing of the axes is as regards their relative lengths, the construction itself being exactly the

same as for the cubic system. In the case of a tetragonal crystal the two horizontal axes a and b are allowed to remain as if for a cubic crystal, but the vertical c -axis has to be given its proper length, corresponding to the axial ratio $c : a$. Hence, after drawing in the cubic axes from the templet, and marking off the unit lengths of a and b , we mark off OC and OC', so that these lengths above and below the centre are each equal to the value of the ratio, as given directly by the scale along the templet edge. When the crystal is of rhombic symmetry, only the right-and-left horizontal axis b is allowed to remain of the normal cubic length, and the front and back parts of the a -axis as well as the upper and lower parts of the c -axis must be given their proper lengths, corresponding to the axial ratios $a : b$ and $c : b$ (where $b = 1$), by marking off OA and OA', as well as OC and OC', with the aid of the scales along the a -edge and c -edge of the templet, equal to the values of a and c in the ratios.

If we join the ends of the axes we obtain, in the case of tetragonal axes the primary tetragonal bipyramid of the first order $\{111\}$, as shown in Fig. 377, and in the case of rhombic axes the primary rhombic bipyramid $\{111\}$, as shown in Fig. 378. The axes in the case of Fig. 377 are those which have the relative values corresponding to the axial ratio $a : c = 1 : 1.7771$ (the two horizontal axes being of unit length and the c -axis being 1.7771 times cubic unit length), being those of the typical tetragonal crystal of anatase which was worked through practically in Chapter XIV. The axes in Fig. 378 have the three different values $a : b : c = 0.5287 : 1 : 0.9539$

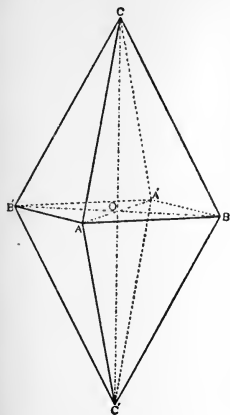


FIG. 377.

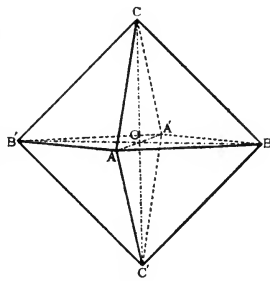


FIG. 378.

(that is, axis a is 0.5287 of cubic unit length, axis b is of unit length as in a cubic crystal, and axis c has the length 0.9539 of the cubic unit), corresponding to the axial ratio of the crystal of topaz worked through in Chapter XVI. as an example of a rhombic crystal.

As they are possibly the best examples we could choose, the method of drawing tetragonal and rhombic crystals of some little complication will next be illustrated by going through the construction, stage by stage, of these typical crystals of anatase and topaz, which will fully illustrate the mode of drawing crystals belonging to these two rectangular systems.

Commencing first with **anatase**, the construction is shown in Fig. 379. The axial lengths correspond to the ratio above given and illustrated in Fig. 377, but the axes are produced so as to afford three unit lengths of a and of b , and two unit lengths of c , the forms present requiring these lengths in order to be able to mark off their intercepts. In the cases of a and b the unit lengths are also divided into thirds, as the intercepts on these axes in the case of one form are five-thirds of unit length.

The crystal was shown in Chapter XIV. to consist primarily of the two first order bipyramids $z = \{113\}$ and $r = \{335\}$, the former occupying the two apices. Besides these prominent bipyramids, two others of the same order were also present as strips replacing the equatorial edges, namely $p = \{111\}$ and $s = \{221\}$. The faces are not

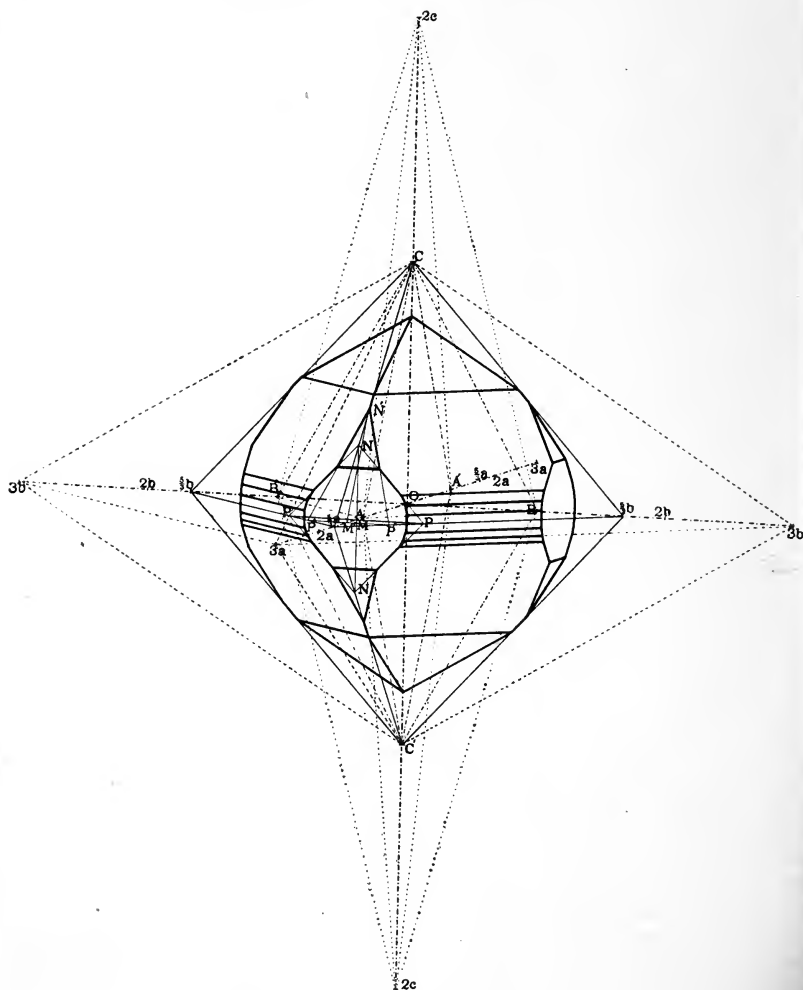


FIG. 379.—Construction of Crystal of Anatase.

lettered in Fig. 379, to save confusion with the construction letters; but the letters are given on Fig. 162 (page 212) of Chapter XIV., the illustration accompanying the description, which was reproduced by the tracing method from the original drawing for Fig. 379, the back faces being inserted (inked-in in broken lines) after the rotation of the tracing of the front half for 180° about the centre O.

The two prominent bipyramids were first drawn, $r = \{335\}$ in thin continuous lines and $z = \{113\}$ in broken lines. The intercepts corresponding to $\{335\}$ are $5a : 5b : 3c$,

or dividing by 3 all through $\frac{5}{3}a : \frac{5}{3}b : c$. The front half of the bipyramid was, therefore, obtained by joining the points on the three axes corresponding to these intercepts, which are clearly marked on them. It happens that only a very small part in the middle of each of the polar edges of this (continuous line) bipyramid actually exists on the crystal as an edge, owing to the extensive manner in which this form is modified and replaced by others. It is most convenient to take the form $z = \{113\}$ next, having the intercepts $3a : 3b : c$. The front half of a bipyramid having these intercepts is the one drawn in the broken lines in Fig. 379, but it does not intersect or touch the steeper one already drawn. All we have to do, however, is to move it inwards parallel to itself, until it cuts off as much of the apices of the bipyramid r as it appeared to do on the actual crystal itself. That is, we draw the pyramidal terminations, upper and lower, parallel to the broken lines. The edges between r and z are parallel to the equatorial edges of all first order bipyramids, that is, to the lines joining unit a and b intercepts. Tackling then the form next in prominence, the second order tetragonal prism $a = \{100\}$, in order to get the outline of its front face we draw a line parallel to axis b , so as to cut off as much of r as the front a -face was observed to do on the crystal, and intersecting the two front equatorial edges of the bipyramid r at the two points P. Where the line cuts the axis a , at M, a line is drawn parallel to the axis c , as the plane a is also parallel to this axis as well as to axis b ; this vertical line cuts the front polar edges of r at the two points N. If we join the four points P and N by the four lines PN we obtain the directions of the edges between r and a , and the middle part of each line PN is actually the representation of the edge itself. But we have a further modifying form to draw in before we can determine how much this is in each case, namely the form $e = \{101\}$, the primary tetragonal bipyramid of the second order. In order to find the shape of the e -faces we proceed in a similar manner; for the upper front face we take a line P'P' parallel to the axis b , as (101) is parallel to that axis, cutting the equatorial edges of r at P' and P' and the axis a at M'. From M' a line M'N' is drawn parallel to a line joining A and C, as the plane (101) cuts off unit intercepts from the axes a and c . Then the two lines P'N' will be the directions of the edges between r - and e -faces, and the upper part terminating at N' will be the representation of the actual edges themselves. The two points where these lines cut the directions PN of the a -faces already determined fix the position of the horizontal edge of intersection between the a and e -face, which may be drawn in parallel to the axis b ; and they also determine how much of each of the lines P'N' is the edge of the little e -face. The lower e -face is obtained precisely similarly, and the two other e -faces, as well as the a -face, to the right of the figure, are also determined in like fashion. The edges between a - and r -faces are all parallel to the polar edges of r (the continuous lines), and our knowledge of this fact alone would have enabled us to construct them; but the above is the systematic method, and the fact referred to may be used to confirm the accuracy of the drawing.

The other subsidiary faces are very easily constructed, all being merely narrow strips replacing the equatorial edges of r . Those of the tetragonal prism of the first order, $m = \{110\}$, are the central ones and relatively the broadest, and their short end edges of intersection with the adjacent a -faces are parallel to the vertical axis. The somewhat narrower faces of the first order bipyramid $s = \{221\}$ come next the m -faces, one above and one below in each case, and their end edges of intersection with the a -faces are parallel to the bipyramid constructed in thin dotted lines from the corresponding intercept-points A, B, and $2c$; and the similarly narrow faces of the primary bipyramid of the first order $p = \{111\}$ are the outermost of these strip-faces, next to the r -faces, and their short end-edges of intersection with the a -faces are parallel to the primary bipyramid constructed from the unit intercepts A, B, C in thin broken-and-dotted lines.

This completes the drawing of the front half of the crystal. The back half is also

shown in Fig. 162 of Chapter XIV., which, as already stated, was reproduced from the thickened outline of Fig. 379 now given, by the tracing method.

Proceeding now to the drawing of the **topaz** measured and discussed in Chapter XVI., the construction is given in Fig. 380, for both the back and front halves, as the two are not quite the same, owing to the presence of two important forms, $o = \{111\}$ and $x = \{123\}$, on the front half only, where each is represented by a single face. Hence the tracing method has only been employed for the transference, and repetition without construction lines, of the crystal outline including the back faces in Fig. 190 (page 235) of Chapter XVI., for the purpose of assisting the comprehension of the description of the crystal there given. All the front faces are lettered in Fig. 190, corresponding to the letters assigned

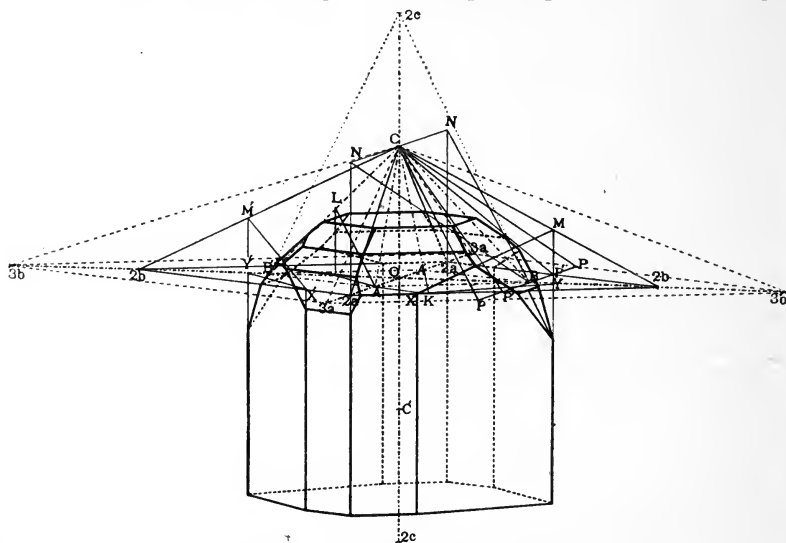


FIG. 380.—Construction of Crystal of Topaz.

to the various forms in the description, and frequent reference to it during the construction now to be described will be found very helpful, as the form letters are omitted in Fig. 380 in order not to confuse them with the construction letters.

The principal end form, which may be conveniently taken as the nucleus for the construction, is the rhombic pyramid $o' = \{112\}$, and this is drawn in Fig. 380, as it would appear if alone present, in thin continuous lines, joining the intercepts $2a$, $2b$, and C . The parts of these lines which remain after the modifying forms have been drawn will actually represent edges on the crystal, as indicated by the thickening of those parts. The other well-developed pyramid above $o' = \{112\}$ is $o'' = \{113\}$, and the directions of its edges, supposing it alone present, are drawn in thin broken lines from the intercepts $3a$, $3b$, C ; the polar edges of this form, therefore, are drawn parallel to the corresponding broken lines terminating at C , representing the polar edges of the ideal pyramid, at such positions as to make this pyramid blunt the lower one o' to the extent observed on the crystal. The horizontal edges between the two pyramids are parallel to the equatorial edges of both pyramids, and indeed to

the primary rhombus formed by joining unit a and b intercepts, and which is the base of the primary pyramid $o = \{111\}$ drawn in broken-and-dotted lines. The edges of the basal plane forming the flat upper termination of the crystal are also parallel to this same rhombus, and may at once be drawn in, at such positions as to give to the basal plane the relative size to which it was observed to be developed on the crystal. The back o' and o'' faces had also better be drawn in thick dotted lines, as the tracing method cannot be employed so well here, owing to the absence of any primary o -face behind.

It is most convenient to draw next the prism faces, assuming for the moment the presence of only the pyramids o' and o'' , as these alone are fully developed as regards the number of their faces. From $2a$ downwards the edge between the two front p -faces (110) and $(\bar{1}\bar{1}0)$ may be drawn parallel to axis c , as this would appear to be its approximate position on the crystal, the single o -face, however, to be subsequently drawn, replacing the solid angle at $2a$. We next draw one of the edges between a p -face, say the right-hand front one, and the p' -face (120) adjoining it on the extreme right, giving it the relative position seen on the crystal, that is, so as to leave the observed relative width of p -face. The same may be done for the left-front and the two back (to be dotted) pp' edges. X and X' are suitable points in front. We next determine the extreme right and left edges, those between the large p' -faces, there being no brachy-pinakoid present to blunt them. To find these edges, we draw from X and X' lines XY and $X'Y'$ cutting the b axis at Y and Y' , parallel to the lines obtained by joining $2a$ and B , the intercepts of $p' = \{120\}$. The required edges are then got by drawing lines downwards from Y and Y' parallel to the vertical axis.

The edge between p' and o' is obtained in each case by continuing these lines upwards to M and M' , where they cut the continuous lines representing the outer edges of the ideal pyramid o' ; XM and $X'M'$ are then the directions of the required edges $o'p'$.

We can next find the directions of the edges of the faces $q = (011)$ and $q' = (021)$. The edges between q and the basal plane above it, and between q and q' below it, are parallel to the axis a , the a index being 0 for all three forms. We may begin, therefore, by drawing the two basal-plane edges parallel to axis a and of such a size as represents what was observed on the crystal itself. The directions of the edges between o'' and q are obtained by finding the intersection of the two planes in the usual manner, namely (the plane o'' being already drawn in broken lines joining $3a$, $3b$, and C its intercepts), by drawing a line through B , the b -intercept of q , parallel to axis a , and joining the two points P , where this line cuts the equatorial edges of o'' , to C , the vertical intercept of both o'' and q . The two lines CP are then the directions of the required qo'' edges. Similarly CP' are the two directions of the edges between o' and q , for the two points P' are those where the line parallel to the axis a cuts the thin continuous lines representing the base of the pyramid o' . The directions of the remaining edges of q , those between it and the prism $p' = \{120\}$, are obtained by similarly drawing a line through C parallel to the axis a , and raising lines from $2a$, the intercept of p' on axis a , parallel to the vertical axis, and cutting the line through C at NN . Then NNB represents the section of the prism faces (120) and $(\bar{1}20)$ by the plane (011), assuming both to be completely developed, and the lines BN are consequently the directions of the front and back edges between p' and q , which are, therefore, drawn in parallel thereto.

To get the directions of the edges between $q' = (021)$ and $p' = (120)$ the simplest course is to draw a line from the centre of the edge between q' and q (which edge is parallel to the axis a) parallel to the dotted line joining the intercepts of q' , B and $2c$. Where this cuts the extreme right-hand edge of the prism, the edge $p'p'$, will be the apex of the isosceles-triangular face q' , and the edges required are obtained by joining this point to the ends of the edge between q and q' .

The q and q' faces on the left-hand side of the crystal are obtained similarly, but

in order not unduly to complicate the drawing some of the construction lines are omitted.

The only developed face of the primary pyramid $o = \{111\}$, a somewhat large one having the indices $(1\bar{1}1)$, and also the small single face of $x = \{123\}$ in the same left front octant, are lastly to be inserted. The edge between o and the p -face $(1\bar{1}0)$ below it, and also that made with the o' -face above it, are both parallel to the line joining unit intercepts on the a and b axes, and therefore to the edges between o' and o'' and between o'' and the basal plane; for c , o'' , o' , o and p all belong to the same zone. To find the direction of the edge between $o = (1\bar{1}1)$ and $p' = (1\bar{2}0)$, we draw the two planes and find their intersection in the usual manner. The former is already drawn in broken-and-dotted lines; if we draw the latter, its intercepts on the a and b axes are $2a$ and B' , but such a plane parallel to the vertical axis would only touch the former at the single point B' , and to bring about the intersection we move it parallel to itself until $2a$ becomes A , when the intercept on the b axis is $\frac{1}{2}b$. Raising a vertical line here, as p' is parallel to the vertical axis, it cuts the outer polar edge (broken-and-dotted) of the pyramid o at L , and LA is therefore the direction of the required edge between o and p' , and the edge itself can at once be drawn in parallel to LA .

The edge between o and o' may readily be found by producing the basal edge of the former, as ideally indicated by the broken-and-dotted line AB' , so as to meet the right-hand basal edge of the latter, joining $2a$ and $2b$, which it does at K , when CK is the direction of the required edge, which may be drawn parallel thereto. The fifth edge between o and the right-hand p -face, is then to be inserted by joining up the first and fourth edges.

As regards the small x -face, much constructive trouble is saved by remembering that the face lies in the same zone with q and o' , so that its longer edges will be parallel to the edge qo' which they replace; also that its upper inclined edge will be parallel to the edge between the two front o'' -faces, because it also lies in the same zone with those faces (see stereographic projection, Fig. 191, page 236). The fourth edge is thus automatically determined, as it merely joins the other ends of the longer edges, the relative distance apart of which will have been fixed by the direction of the upper inclined edge just referred to.

This completes the construction of the crystal of topaz, except as regards the lower cleavage plane termination, the edges of which are at once obtained by merely drawing them parallel to the lines joining the intercepts on the a and b axes proper to the two prisms, unit intercepts in the case of p , and $2a$ and B or B' in the case of p' .

Hexagonal Axes.—In the case of the axes of a hexagonal crystal we have to draw the projection of three equal horizontal axes mutually

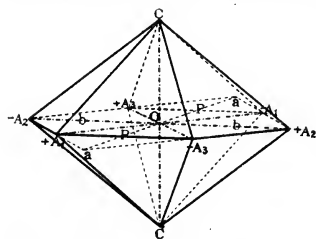


FIG. 381.

inclined at 120° , and to give to the vertical axis its proper value c indicated by the axial ratio $a:c$, in which a indicates any of the three horizontal axes taken as unity and c the proportional length of the vertical axis observed in connection with the crystal. That is, to find the length of the vertical axis we multiply OC (of the cubic axes) by c , just as in the cases of tetragonal and rhombic crystals. The readiest mode of constructing the horizontal hexagon, as shown in Fig. 381, is to construct axes as for a rhombic crystal, where the axis b is left of unit length as for a cubic crystal, and where the a -axis is made 1.732 of the cubic unit

length, a value which is such as makes the rhombus formed by joining the ends the projection of one of 120° angle. Each half of the axis a (the front and the back half) is then bisected, and through the bisecting points P lines are drawn parallel to the axis b until they intersect the sides of the rhombus, which they also do at a projected angle of 120° . The projection of a hexagon is thus produced the corners of which are the terminal points of the three horizontal axes, mutually inclined at 120° . When the vertical axis has also been given its proper length, in accordance with the axial ratio, we have a correct representation of the hexagonal axes, strictly comparable to the cubic, tetragonal and rhombic ones. By joining the ends of these hexagonal axes we at once produce the primary hexagonal bipyramid of the first order, the symbol of which is $\{10\bar{1}1\}$; for each face meets the vertical axis and two of the horizontal axes at unit intercepts and is parallel to the third horizontal axis, as will be clear from Fig. 381.

The three horizontal axes are lettered A_1 , A_2 , and A_3 in Fig. 381, and the two halves of each are marked respectively positive and negative, according to the scheme described on the first page (303) of Chapter XXI. concerning hexagonal crystals.

The most practical illustration will be to take the case of the construction of the crystal of apatite (Fig. 287, page 316), which was worked through in Chapter XXII. as the typical example of a hexagonal crystal, and this will next be proceeded with.

The length of the c -axis in Fig. 381 relative to the lengths of the horizontal axes, is that of apatite, namely 0.7345, and these axes are repeated in Fig. 382, in which the construction for apatite is given. They are produced to double their length, however, as intercepts of two unit lengths will frequently be required from the nature of the forms present. Fig. 287, page 316, was traced from the completed Fig. 382, leaving out the construction lines.

The method of procedure has now been so fully elucidated that a detailed description of this construction will not be necessary. It will suffice to say that the prism edges of $m = \{10\bar{1}0\}$, and $n = \{11\bar{2}0\}$, and of the only developed face of $p = \{21\bar{3}0\}$, are all parallel to the vertical axis; the horizontal edges of the first order prism m are parallel to the three horizontal axes in turn and are got by joining adjacent axis-ends, while those of the second order prism n are got by joining alternate axis-ends. The polar edges (those intersecting at the pyramid-apex) of the three pyramids are parallel to the directions obtained by joining the respective intercepts on axis c with the intercepts on the three horizontal axes, these directions being indicated in Fig. 382 by thin lines; the intercepts in the case of the most largely developed pyramid $o' = \{20\bar{2}1\}$ are $2c$ and unit intercepts on each of the two horizontal axes concerned, while for $o = \{10\bar{1}1\}$ the less extensively developed primary pyramid the intercept on c is also of unit length. In the case of the small upper pyramid $o'' = \{10\bar{1}2\}$ the c -intercept remains of unit length but the intercepts on the horizontal axes concerned are now of double unit length. All the horizontal edges are parallel to those between the lowest pyramid and the prism m , and these directions in the case of the top pyramid also define the contour of the terminal basal plane c . The long edges of $q = \{11\bar{2}1\}$ are parallel to the polar edges of o' , which edges they replace, and the short inclined upper edges between q and o are parallel to the other polar edge of the same o -face in each case, as each q -face lies in

the same zone with the pair of o -faces on either hand. The only face requiring much consideration is the single face of the form $s = \{21\bar{3}1\}$, and even here the fact that this face lies in the two zones with the q - and o -faces above it to the right, and with the o' - and q -faces above it to the left respectively, at once gives the directions of the two edges meeting at the upper angle. We can readily find the direction of the edge between m and s by the usual method of finding the intersection of two planes. The intercepts of s are $\frac{2}{3}A_1$, $3A_2$, $-A_3$, and $3c$. The essential lines joining these intercepts are given in the figure, and if we raise a vertical line (parallel to axis c) from A_1 (on the line $A_1 - A_3$ representing the plane m), it cuts the plane s at P , and $-A_3P$ is the direction of the required edge between m and s . The edge ns

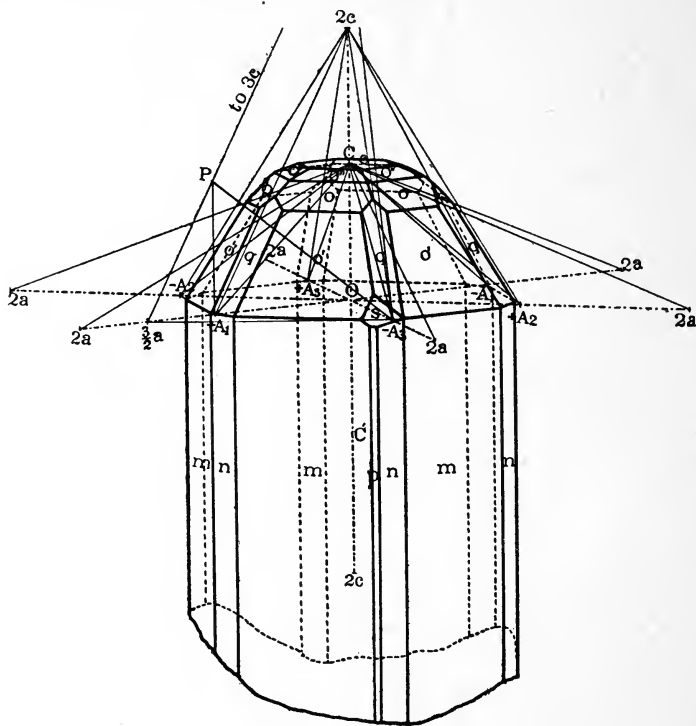


FIG. 382.

follows, the other three being known. The little edge between s and p is parallel to the line joining $\frac{2}{3}A_1$ and A_3 , as both the pyramid s and the prism p have these intercepts in common, and lie in the zone $[csp]$. (See Fig. 288 for stereographic projection.)

Trigonal Axes.—The most convenient method of drawing trigonal crystals is to treat them as if they were hemihedral hexagonal crystals and to employ the hexagonal axes. At any rate, these can and must be employed to draw the primary pyramid or rhombohedron, the edges of which are the directions of the three trigonal axes. As the Bravais-Miller ratio of the hexagonal axes, $a:c$, is usually given as well as the Millerian rhombohedral axial angle, there is no difficulty about the use of the hexagonal axes, and the symbols of the forms present are

also generally given according to both notations. But if not, it is much easier to draw first the rhombohedral axes from the hexagonal ones, rather than to start directly to draw the rhombohedral axes.

The two practical examples worked out in Chapter XXIV., quartz and calcite, will be taken to illustrate the drawing of trigonal crystals. Quartz affords an illustration of trapezohedral hemi-hexagonal symmetry, and calcite a capital instance of rhombohedral hexagonal hemihedry, and will afford us the opportunity of illustrating the construction of the rhombohedral Millerian axes.

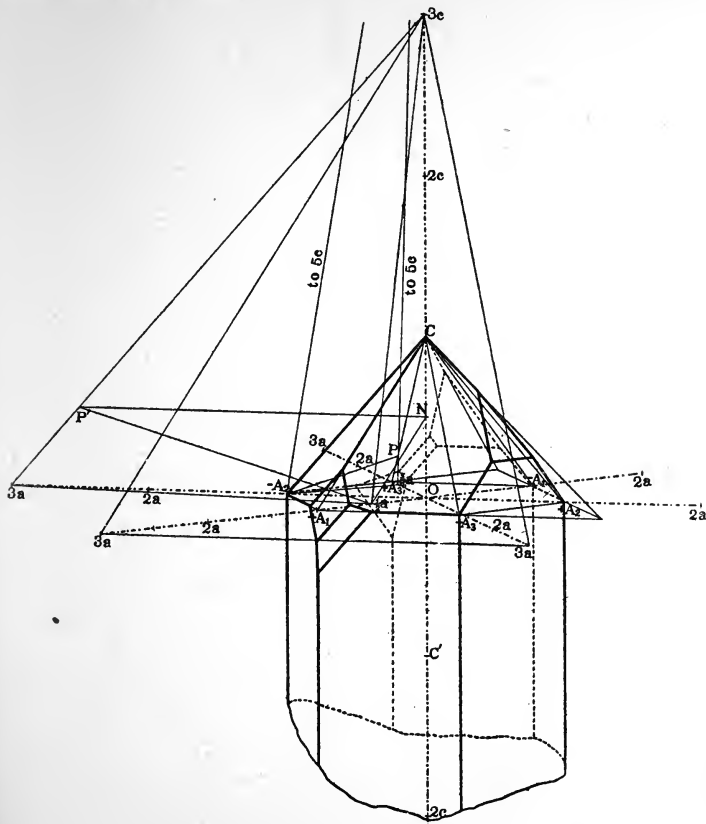


FIG. 383.

Quartz.—The construction of the left-handed crystal described in Chapter XXIV. is shown in Fig. 383, and Fig. 351 given to illustrate it in that chapter, and on which the letters indicating the various faces are marked, was reproduced by tracing the final form of the crystal, as shown by the thick lines, from Fig. 383.

The hexagonal axes were drawn as already indicated, giving the c -axis the relative length (1.1 of the cubic c) indicated by the Bravais-Miller axial ratio. The directions of the edges of the first order hexagonal prism m are obviously got by

drawing lines parallel to the vertical axis through the six ends of the three horizontal axes. Those of the two complementary rhombohedra, the primary direct one r and its inverse form r' , behave just as an ordinary hexagonal pyramid of the first order for the purposes of drawing, but the individual faces are, of course, to be given the relative amounts of development actually shown on the crystal. The directions of the six polar edges are parallel to, or actually, those obtained by joining unit C with the ends of the horizontal axes. Four of the faces, all three r -faces and one r' -face, are about normally developed, but the other two r' -faces are relatively smaller. The edge between the right-hand m -face and the front r -face, which is developed owing to the smallness of the right r' -face, is parallel to the edge rr' between the front r -face and the left r' -face, as may be seen should be the case by a reference to the stereographic projection given in Fig. 352, for all these three faces m , r , and r' belong to the same zone. Similarly, the edge between the same m -face and the right-back r -face is parallel to the edge between that right-back r -face and the smaller back-central r' -face.

The only faces requiring any considerable thought are those of s , the left trigonal pyramid of the second order $\{4\bar{2}1\} = \{2\bar{1}1\}$, and of the adjacent form x , the left positive trapezohedron $\{4\bar{2}1\} = \{6\bar{1}51\}$. It will be clear from the projection (Fig. 352), however, that the edge sx is parallel to the opposite outer edges $r's$ and mx , and that these three parallel edges are also parallel to the edge $r'r$, between the r' -face next to s and the back-left r -face beyond it. For the front m , x , s , r' , and left-back r -faces all belong to the same zone; moreover, the zone is continued by the small elongated back s -face and the middle back m -face, so that the two dotted longer s -edges are also parallel to this same direction. Next, the other two front s -edges, ms and sr , are parallel to the edge rr' between the front r -face and the right-front r' -face, for the faces concerned all belong to the same zone. The same kind of consideration applies to the two other faces of the little elongated back s -face, for these two shorter edges are parallel to the edge $r'r$ between the middle-back r' -face and the right-back r -face, already alluded to as being parallel to the extreme right mr -edge. Also the left-back (dotted) xm edge is parallel to the same direction. Thus all the s -edges and a pair of edges of each x -face are entirely determined by making use of the zonal relationships, without any intersections.

The only edges requiring intersections to be constructed are the two remaining edges of each of the pair of x -faces, and either one of the two in each case will suffice, the other being determined automatically. The Bravais-Miller indices of the front x -face are $(6\bar{1}51)$, and the intercepts may most conveniently be written $A_1, -6A_2, -\frac{5}{6}A_3, 6C$ (that is, $5A_1, -30A_2, -6A_3, 30C$ all divided by 5), and as we only need $A_1, -A_2$, and C in drawing the plane we can halve the values along these three axes in order to get the drawing into reasonable compass and thus actually use $\frac{1}{2}A_1, -3A_2, 3C$. These three intercepts are joined in Fig. 383 with thin lines to represent the plane x . To find now its intersection with the front r -face in order to determine the direction of the edge rx , we need only push the r -plane back until its intercept on A_1 is also $\frac{1}{2}A_1$, when it will cut C at $\frac{1}{3}C$ marked by the letter N ; and as r is parallel to the axis A_2 a line NP' may be drawn parallel to that axis. The points P' and $\frac{1}{2}A_1$ will thus be common to the two planes, and the line joining them will consequently be the direction of their intersection, and the edge rx should be drawn parallel thereto, from the corner of the s -face already drawn. The fourth edge bounding x , namely mx (left m), may also be at once drawn in, for the third edge mx being known to be parallel to sx can be drawn so, and the ends of the fourth are thereby fixed. The back x -face is to be completed in the same way. The indices of this face are $(\bar{1}561)$, and the most convenient mode of writing the intercepts is $-5A_1, -A_2, \frac{5}{6}A_3, 5C$. Joining $-A_2, \frac{5}{6}A_3$, and $5C$ to represent the plane x , and also remembering that the back-left r -face is represented by the lines joining $-A_2, +A_3$, and C , it will be obvious that the point $-A_2$ is common to the two planes, and that P is a point of

intersection; the line joining these two points is, therefore, the required direction of the edge rx . The fourth edge of this back x -face is determined automatically, for three edges are now known, and the ends of the fourth are thereby fixed.

Calcite.—The crystal of calcite described in Chapter XXIV., and illustrated in Fig. 359, is primarily a combination of the same first order hexagonal prism m as in the case of quartz with the direct primary rhombohedron $r = \{100\} = \{10\bar{1}1\}$ and the inverse obtuse rhombohedron $e = \{110\} = \{01\bar{1}2\}$. Before proceeding to the actual construction of the measured crystal, it will be useful to draw separately these two rhombohedra, which are not the complementary direct and inverse pair, together making up a hexagonal pyramid as in the case of quartz, but the direct and inverse representatives of quite distinct rhombohedra, which under the older method of regarding crystal structure would be considered as the opposite hemihedral halves of two different hexagonal pyramids. Moreover, the construction of the primary rhombohedron $r = \{10\bar{1}1\}$ gives us the construction of the rhombohedral Millerian trigonal axes.

The construction of this primary rhombohedron is shown in Fig. 384, and that of the obtuse inverse rhombohedron $e = \{01\bar{1}2\}$ in Fig. 385. In calcite the two rhombohedra are often formed on the same crystal, the obtuser one (Fig. 385) blunting the upper termination of the primary one.

In the case of the primary rhombohedron $r = \{10\bar{1}1\}$ the directions of the three upper and three lower polar edges, terminating respectively at the apices C and C' , are found by drawing the triangular bases of the two alternating (upper and lower) trigonal pyramids and joining their corners to the points C and C' , which are such that OC and OC' have the length corresponding to the value of C in the axial ratio $a : c$. Drawing, therefore, first the hexagonal axes and giving OC and OC' the length

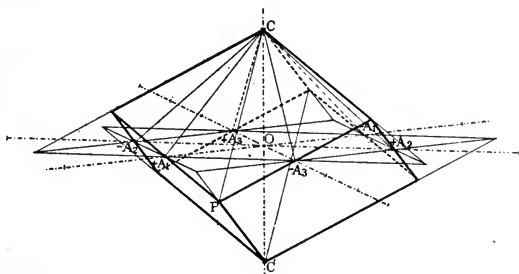


FIG. 384.—Construction of Primary Rhombohedron $r = \{100\}$.

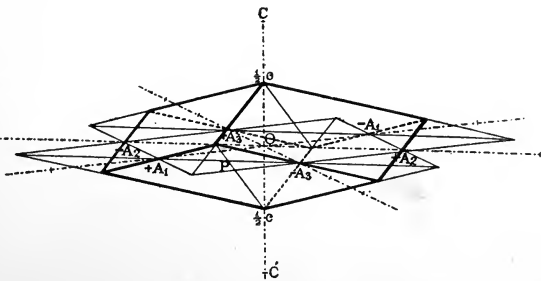


FIG. 385.—Construction of Inverse Obtuse Rhombohedron $e = \{110\}$.

0.8545 of the usual length for a cubic crystal, corresponding to the axial ratio of calcite, we proceed to find the two bases in question. The upper front r -face is a lateral expansion of $+A_1, -A_3, C$, and so we produce the line $+A_1, -A_3$ indefinitely on each side. We similarly produce $-A_1, +A_2$ and $-A_2, +A_3$ corresponding to the two other (back) upper faces of the trigonal pyramid, and from the corners of the basal triangle produced by the intersection of these three lines we draw lines to, and

meeting in, C, the apex of the upper pyramid. We then next construct the base of the lower pyramid by similarly producing the other three edges of the basal axial hexagon until they meet, and from its three corners draw lines to C', the apex of the lower alternate pyramid. We have now only to find how these two trigonal hemi-pyramids intersect. We know that opposite faces are parallel, and also that all the edges of the rhombohedron fall into three sets of four each, those of each set being parallel to one of the three polar edges. We have already determined two of the four in each set, for each lower polar edge is parallel to the opposite upper one. One point of intersection will, therefore, give us all we require, the direction of the edge starting from this point being known. The simplest intersection to find is that of the lower middle polar edge C'P with the upper front *r*-face. This can be found by drawing a central line down the face, from C to the centre of the line joining $+A_1$ and $-A_3$, and producing it till it cuts the lower middle polar edge in question at P. From P we can then at once draw lines parallel to the right and left upper polar edges, and thus complete the upper front face of the rhombohedron. From the right and left corners of this face we can then also draw lines parallel to PC' and thus complete the two front lower faces. The right and left corners of these in turn determine the two remaining long back dotted edges, after drawing which the figure of the rhombohedron is complete.

The three directions of the polar edges are those of the Millerian axes. When it is desired to employ these axes in the further drawing of any trigonal crystal, it is only necessary first to draw them parallel to the directions of these three rhombohedral edges and so as to pass through and intersect at the centre O, as is done in Fig. 307 (page 333), the rhombohedron in which has been traced from this in Fig. 384.

The obtuse inverse rhombohedron $e = \{01\bar{2}\}$ was constructed in Fig. 385 in a precisely similar manner to the primary rhombohedron. The intercepts on the two horizontal axes are now, however, double of the unit value, while the intercept on the vertical axis remains of unit length. It is most convenient to take half OC as the vertical intercept and thus to retain the unit lengths of the horizontal axes. The same basal triangles are then employed as for the primary rhombohedron, but their corners are joined to $\frac{1}{2}c$, instead of to C; and as the rhombohedron is an inverse one instead of a direct one, the basal triangle which was connected to C is now connected to $\frac{1}{2}c$ below, and that which was connected to C' is now connected to $\frac{1}{2}c$ above.

In the cases of both rhombohedra it will be observed that the ends of the horizontal hexagonal axes bisect respectively the six edges other than the polar ones.

We may proceed now to the drawing of the crystal of calcite (Fig. 386). The long edges of the prism *m*-faces may be constructed by drawing lines through the ends of the horizontal axes parallel to the vertical axis. The primary rhombohedron *r* may also be drawn just as in Fig. 384. The three upper faces of the obtuse rhombohedron *e* are then to be drawn, so as to blunt and replace the upper apex of the primary rhombohedron. The longer edges of this form, and also of the two faces developed of the scalenohedron $t = \{310\} = \{21\bar{3}4\}$ (for lettering of faces see Fig. 359, which was traced from the finished outline of Fig. 386), are parallel to the polar edges of *r*, as may be seen from a consideration of the stereographic projection in Fig. 360, for the *t* and *e* poles lie in the same zone in each case as a pair of *r* poles, and between the latter. The only part of Fig. 385 which appears in the final outline of the calcite crystal is the portion of the polar edges about the apex. Having first drawn in this apex the longer *er*-edges may at once be drawn parallel to the original *r*-edges, giving the *e*-faces the approximate relative size actually observed on the crystal, ignoring for the time being the modifying *t*-faces. The edges of the *m*-faces may next be conveniently completed, as far as they are determined by the *r*- and *e*-faces, ignoring preliminarily the faces of the third rhombohedron *n* and of the scalenohedron *v*. The front edge between the front *m*- and *r*-faces is the line joining $+A_1$ and $-A_3$; that between the left-back *m*- and *r*-faces is the line, now to be dotted, $-A_2, +A_3$; and that between

the right-back m - and r -faces, also to be dotted, is obtained by joining $-A_1$ and $+A_2$. Both these dotted edges will remain unmodified, but the front edge mr is subsequently to be replaced by a narrow face, the only one developed on the crystal, of the rhombohedron $n = \{3\bar{1}1\} = \{4041\}$, as well as by two little corner faces of the common scalenohedron of calcite, $v = \{201\} = \{21\bar{3}1\}$. The edges between the r -faces and the two m -faces on either side of the middle one in front, and between the middle back m -face and the r -faces, are parallel in each case to one of the polar edges of the ideal

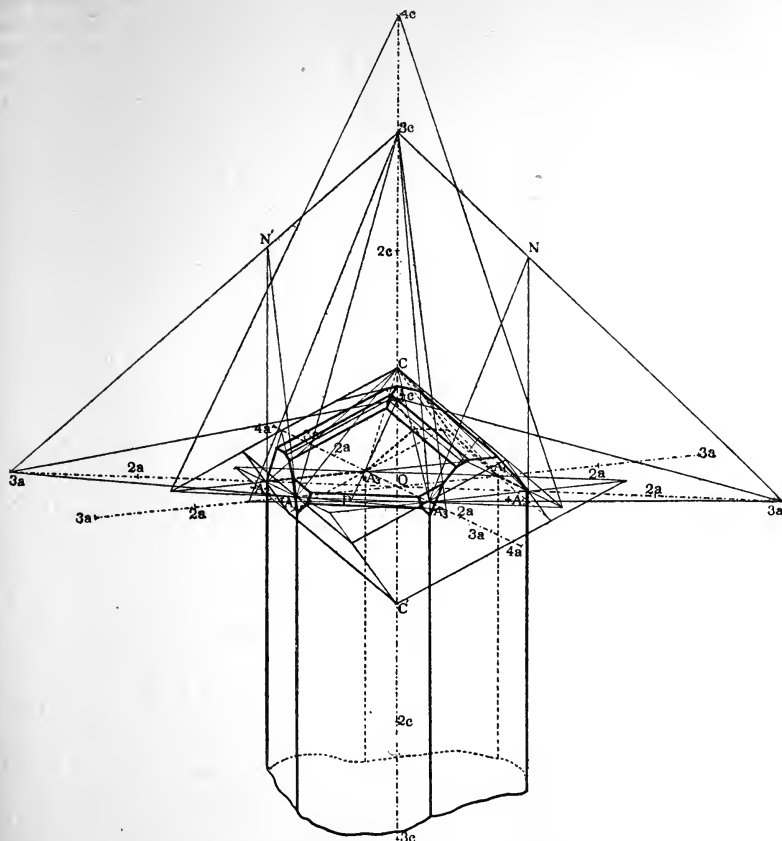


FIG. 386.—Construction of Measured Calcite Crystal.

hexagonal pyramid of which r is half, and the proper one is readily determined by a reference to the stereographic projection, or better still to that of quartz given in Fig. 352, as this latter gives the poles of all six pyramid faces (r and r'). In our drawing of calcite all six theoretical edges are drawn in lightly, for use in drawing the mr edges parallel. The edges me are parallel to those three edges of the basal hexagon which are alternate to those which form the mr edges.

It now only remains to insert the smaller modifying faces of t , n , and v . The little polar edge tt determines the two t -faces, as we know that the longer edges are parallel to the edges re . This polar edge is readily obtained by constructing the two t -planes and finding their intersection. The indices of the right-hand t -face are $(21\bar{3}4)$, and the

corresponding intercepts are $6a$, $12a$, $4a$, $3c$, and as the three latter only are needed they may most conveniently be divided by 4 and written $3a(\text{on} + A_2)$, $a(\text{on} - A_3)$, and $\frac{3}{4}c$. These intercepts are joined by thin lines in the drawing, and a similar set are drawn on the left side to represent the plane of the left-hand t -face. If the two bases of these triangles are produced inwards they meet at P , and the line joining P and $\frac{3}{4}c$ is the required line of intersection of the two t -planes, and the little edge tt is to be drawn parallel thereto.

As regards the single n -face, it replaces the front horizontal edge mr , and its two long edges are parallel thereto. This face n belongs to an acute direct rhombohedron having the indices $\{40\bar{4}1\}$, and it is well known that its edges are directly replaced by the common scalenohedron of calcite $v = \{21\bar{3}1\}$. Hence the little edges nv will be parallel to the ideal edges of n . These are drawn in thin lines, the apex being at $4c$, the intercept on the vertical axis being 4 times the unit length, while the same unit base as was used for r is still valid for n . Hence the required directions of the edges nv are parallel to the lines drawn from $4c$ to the right and left corners of the basal triangle. The edges between the v -scalenohedron faces (on the right and left front corners of the crystal) and the front r -face are parallel to the edges rt , and after having inserted them it only remains to find the directions of the edges vm . The indices of the right v -face being $(21\bar{3}1)$, its intercepts are $3a$, $6a$, $2a$, $6c$, and as the last three suffice to determine the plane we may write them as $3a(\text{axis} + A_2)$, $a(\text{axis} - A_3)$, $3c$. This plane is constructed by joining these intercepts, in thin lines in the drawing. The right m -face is represented by the line joining $-A_3$ and $+A_2$, and thus the two planes have the point $-A_3$ in common. If we raise a line from $+A_2$ parallel to the vertical axis N will be another point in common, for the line $3c \cdot N$ lies in the vertical plane. Hence a line joining $-A_3$ with N is the required intersection, and the little edge vm must be drawn parallel to it. Treating the left-hand planes v and m in the same manner, the line joining $+A_1$ with N' is the direction of this left intersection vm , and the edge between the left-front m -face and the left v -face should be drawn parallel to it.

The fourth edge of each v -face, the edge between it and the front m -face is then at once automatically determined, the other three edges having been drawn. But its direction in each case may be readily verified, for the two directions are parallel respectively to the edges of the primary pyramid $+A_2C$ and $-A_2C$.

This completes the drawing of the crystal of calcite.

Monoclinic Axes.—We now revert to the system of three axes, but while one remains perpendicular to the plane of the other two, those two latter may be arranged at any angle other than 90° in that plane. One of these two is chosen as the vertical one, and labelled c as usual; so monoclinic symmetry simply amounts to the other axis being inclined to the vertical axis at an angle which is not a right angle, while remaining at right angles to the first axis. This first axis, known as the symmetry axis, as it is perpendicular to an axial plane, is lettered b , and remains of the unit length of the cubic axes. The vertical axis c requires to be given the length corresponding to c in the axial ratio $a:b:c$, which is really $a:1:c$, as b is retained as unity according to the usual convention. The only new problem, therefore, is to find the direction of the inclined axis a , and then to give it its proper length as indicated by the axial ratio.

In Fig. 387 the cubic axes are represented as usual by AA' , BB' , and CC' . From the centre O we mark off a length OP along OC , such that

$$OP = OC \cdot \cos \beta,$$

where β is the acute angle between the inclined axes (the supplement of the obtuse axial angle usually given in descriptions of monoclinic crystals). The author's method of carrying this out in practice is to express OC in millimetres, reading to a tenth of a millimetre with the aid of a good scale, and to add the logarithm of this to that of $\cos \beta$, the sum being the logarithm of OP also expressed in millimetres and decimals of such, which can then be marked off along OC with the aid of the scale.

Next, from the centre O a length OQ is marked off from OA' in a similar manner, such that

$$OQ = OA' \cdot \sin \beta.$$

Lines are then drawn from P and Q respectively parallel to OA' and OC. Their point of intersection, A'', is a point on the inclined a -axis; A'' is joined to O, therefore, and the line produced for an equal distance to A''. A''A''' is then the direction of the inclined axis, and would represent it at its proper length also if the relative length to the unit axis b were also unity. As this is not so, however, the proper length must be given to it, by multiplying its actual length

in the drawing by the value of a in the axial ratio $a:b$. The vertical axis must also be given its proper length corresponding to the ratio $c:b$, by multiplying OC by the value of c ; or if the cubic-axial templet be used, the value of c is marked off from the scale on the templet edge, both above and below the centre. In Fig. 387 the lengths of the a - and c -axes, aa' and cc' , are those for ammonium magnesium sulphate, a crystal of which salt was described and worked through practically in Chapter XVIII. In this case the axial ratio is: $a:b:c = 0.7400:1:0.4918$, and the axial angle $\beta = 107^\circ 6'$.

The actual calculations for OP and OQ are as follows:

In the original drawing, of which Fig. 387 is a half-size reproduction,

$$OA' = 15 \text{ millimetres and } OC = 44.5 \text{ mm. Acute } \beta = 72^\circ 54'.$$

Log. OC	1.64836	Log. OA'	1.17609		
Log. $\cos \beta$	1.46841	Log. $\sin \beta$	1.98036		
Log. OP	1.11677	OP = 13.1 mm.	Log. OQ	1.15645	OQ = 14.3 mm.

The drawing of this crystal will next be considered, employing the axes just drawn, as an illustration of the construction for a monoclinic crystal.

Ammonium Magnesium Sulphate.—The construction for Fig. 225, the illustration in Chapter XVIII. representing the crystal measured, is given in Fig. 388. It is quite similar to that for a rhombic crystal, except that the basal plane c is now tilted downwards towards the observer, parallel to the inclined axis a , instead of forming a rectangular termination.

The edges cq and bq (see Fig. 225 for lettering of faces) are therefore parallel to the inclined axis a , which is their zone axis. The edges of the prism zone, pp , pp'' , $p''b$

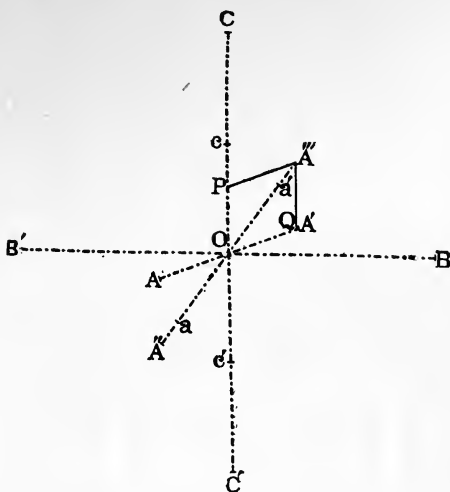


FIG. 387.—Construction of Monoclinic Axes.

plementary edge on the other side of the figure completes the construction, for the few small remaining edges are determined automatically, it being only necessary to join up the ends of those now constructed, in the few places where gaps have been left.

Triclinic Axes.—Here we have lastly the most general case, where all three axes are mutually inclined at angles other than 90° . The angle between the axes b and c is α , that between a and c is β , and that between a and b is γ . The construction is similar in principle to that for monoclinic axes, only carried out for two of the three instead of for one, the third axis remaining vertical; that is, the system of axes is arranged so that one of them is vertical, when it becomes only necessary to find the inclination of the other two to it and to each other.

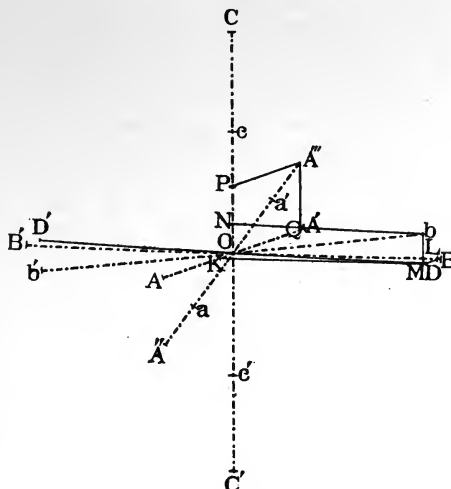


FIG. 389.—Construction of Triclinic Axes of Copper Sulphate.

We start, as always, with the cubic axes, drawn from the templet. They are AOA', BOB', COC' in Figs. 389 and 390, which represent the construction of the axes of two typical triclinic crystals, those

of copper sulphate and of methyl triphenylpyrrolone. Their half-lengths OA, OB, and OC are most conveniently expressed in millimetres and decimals of such, for the purposes of the following simple calculations, as in the case of the calculation for the inclined axis of the monoclinic system.

Along OA we take the length OK such that

$$OK = OA \cdot \cos C,$$

where C is the angle between the pinakoidal faces (100) and (010); and along OB we take the length OL, such that

$$OL = OB \cdot \sin C.$$

The lines KD and LD are then drawn parallel to the cubic axes b and a respectively, intersecting at D, and the diagonal DO of the

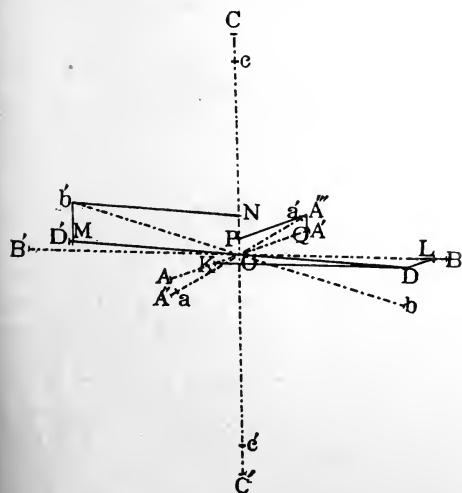


FIG. 390.—Construction of Triclinic Axes of Methyl Triphenylpyrrolone.

parallelogram is also drawn and produced to an equal length beyond O, that is to D'.

This diagonal, and the vertical axis c , form the projection of the pinakoidal plane

(100), and the diagonal OD itself is the projection of a line of unit length in that plane. We next use this diagonal as a base from which to construct the real triclinic axis b , by taking account further of the inclination α of the axes b and c (which are parallel to the edges of intersection of the pinakoid {100} in question with the other two pinakoids {010} and {001}), that is, of the inclination to the vertical direction of that edge of the pinakoid (100) which is formed by its intersection with the basal pinakoid (001). To effect this we take a length OM along either OD or OD', according as the acute angle α between the axes b and c is to the right or to the left (referring to the 2 upper quadrants), and such that

$$OM = OD \cdot \sin \alpha;$$

and along the vertical axis the length ON, such that

$$ON = OC \cdot \cos \alpha,$$

where α represents the acute angle between the axes b and c .

We could equally well let α represent the conventional angle, even though it be obtuse; but in this latter case, while OM would be unaffected, and would always be measured along OD', because $\sin(180^\circ - \alpha) = \sin \alpha$, ON would have to be measured downwards, on the lower or negative part of the vertical axis, for $\cos(180^\circ - \alpha) = -\cos \alpha$, and we are obliged to use the acute supplement in logarithmic calculation. This only produces the same geometrical result as before, however, for we simply construct Ob' (Fig. 389) instead of Ob. It is simpler on the whole always to take the acute α and to measure the resulting OM along OD or OD' in the two several cases.

We then have the sides of another parallelogram the diagonal of which is the axis b , by drawing the lines Nb and Mb parallel to OD and the vertical axis. The point of their intersection is the termination of the axis b , and this gives its real length also, as the b -axis is taken as unity even in the triclinic system.

Having thus constructed the vertical axis c and the right-and-left axis b , it only remains to construct the front-and-back axis a by the method used in the case of a monoclinic crystal, for the plane (010) alone need now be considered, its edges being parallel to the vertical axis and to the axis a in question. The crystal is so disposed that the position of this brachy-pinakoid $b = \{010\}$ in space is relatively the same as in the monoclinic and rhombic systems, as will be obvious from the fact that the two b -poles are also shown in the stereographic projection at the extremities of the horizontal diameters (Figs. 35, Chapter VI., and 252 in Chapter XX.).

We take, therefore, OP along OC and OQ along OA', such that

$$OP = OC \cdot \cos \beta,$$

and

$$OQ = OA' \cdot \sin \beta,$$

where β is the acute angle between the axes a and c . Through P and Q the lines PA''' and QA''' are to be drawn parallel to the cubic axis AA' and to the vertical axis respectively. Then the diagonal A'''O is to be drawn and produced equally on the other side of O to A''. The line A''A''' is then the projection of the axis a , of unit length, however, as if it were equal to b .

It only now remains to give to the axis a thus constructed, and to the vertical axis c , their true lengths corresponding to the axial ratio, $a : 1 : c$, for the particular crystal-line substance under consideration.

In Fig. 389 the axes drawn are those for the typical triclinic substance described and practically worked through in Chapter XX., copper sulphate, the axial ratio for which is :

$$a : b : c = 0.5715 : 1 : 0.5575.$$

The axial angles are :

$$\alpha = 82^{\circ} 16', \beta = 107^{\circ} 26', \gamma = 102^{\circ} 40'.$$

The value of α being less than 90° , the upper acute angle between the axes b and c is thus to the right (compare Fig. 26, Chapter V.), and consequently OM is taken along OD and not along OD', which latter is perhaps more commonly the case with triclinic crystals.

In order to make this commoner case equally plain, Fig. 390 is added, representing the construction of the axes of methyl triphenylpyrrolone, a substance goniometrically investigated by the author, and the crystals of which form a particularly perfect example of triclinic symmetry, all the primary forms being developed, and at quite typical angles, although the substance was not suitable for the practical work recommended in Chapter XX. on account of the difficulty and expense attending its preparation. Its axial ratio is :

$$a : b : c = 0.9059 : 1 : 0.8695,$$

and its axial angles are :

$$\alpha = 100^{\circ} 8', \beta = 93^{\circ} 57', \gamma = 109^{\circ} 34'.$$

It must be remembered that the angles conventionally quoted as the axial angles, and which, as here, are all three more commonly obtuse, are the following : α is the upper-right angle between the axes b and c , β is the upper-front angle between the axes a and c , and γ is the right-front angle between the axes a and b . This will be clear from Fig. 26, Chapter V. Sometimes the acute angles, the supplements of these, are given, but in such a case a statement ought to be made definitely specifying the angles referred to. Occasionally, as in the literature concerning copper sulphate, hopeless confusion has arisen owing to one or two out of the three axial angles having been given correctly, but the others or other having been expressed by their supplements, without any explanatory statement, and a full practical reinvestigation has alone been able to put the case straight. Hence, one cannot insist too strongly on the necessity for either following the convention, or for definitely specifying to which particular angles α , β , and γ refer ; in fact it is safest to do both.

In the case of methyl triphenylpyrrolone, the value of α being greater than 90° the supplementary upper acute angle lies to the left, and so OM is measured along OD', and not along OD as was the case with copper sulphate.

The actual calculations in the two cases were as follows :

The cubic axes in the original drawings, from which Figs. 389 and 390 are reproduced half-size, were of the lengths :

$$OA = 15 \text{ millimetres, } OB = 42 \text{ mm., and } OC = 44.5 \text{ mm.}$$

For Methyl Triphenylpyrrholone.	
C = Angle (100) : (010) = $69^{\circ} 19'$	
Acute $\alpha = 79^{\circ} 52'$	
Acute $\beta = 86^{\circ} 3'$	
Log. OA	1.17609
Log. cos C	1.54802
Log. OK	0.72411 OK = 5.3 mm.
Log. OB	1.62325
Log. sin C	1.97107
Log. OL	1.59432 OL = 39.3 mm.
OD' = 34.1 mm.	
Log. OD'	1.53275
Log. sin α	1.99317
Log. OM	1.52592 OM = 33.6 mm.
Log. OC	1.64836
Log. cos α	1.24536
Log. ON	0.89372 ON = 7.8 mm.
Log. OC	1.64836
Log. cos β	2.83813
Log. OP	0.48649 OP = 3.1 mm.
Log. OA'	1.17609
Log. sin β	1.99897
Log. OQ	1.17506 OQ = 14.9 mm.

For Copper Sulphate.	
C = (100) : (010) = $79^{\circ} 6'$	
Acute $\alpha = 82^{\circ} 16'$	
Acute $\beta = 72^{\circ} 34'$	
Log. OA	1.17609
Log. cos C	1.27668
Log. OK	0.45277 OK = 2.8 mm.
Log. OB	1.62325
Log. sin C	1.99209
Log. OL	1.61534 OL = 41.2 mm.
OD = 39.0 mm.	
Log. OD	1.59106
Log. sin α	1.99603
Log. OM	1.58709 OM = 38.6 mm.
Log. OC	1.64836
Log. cos α	1.12892
Log. ON	0.77728 ON = 6.0 mm.
Log. OC	1.64836
Log. cos β	1.47654
Log. OP	1.12490 OP = 13.3 mm.
Log. OA'	1.17609
Log. sin β	1.97958
Log. OQ	1.15567 OQ = 14.3 mm.

These values thus calculated were all marked off on the two drawings, the parallels and diagonals drawn, and then finally the proper lengths given to the axes a and c by multiplying the length of A'O and OC by the values of the a and c axial ratios. These proper lengths are aO and $a'O$ in the case of the a -axis, and cO and $c'O$ in the case of the c -axis. The b semi-axes Ob and Ob' are already of their correct unit length.

We shall next proceed to employ the triclinic axes of copper sulphate, as just drawn in Fig. 389, for the drawing of the crystal of that salt which forms the subject of Chapter XX. In so doing the process of drawing triclinic crystals in general will be well illustrated.

Copper Sulphate.—The construction for the particular crystal considered in Chapter XX., and which is typical of the greater number of the dozen crystals measured during the author's reinvestigation of copper sulphate, is shown in Fig. 391. Fig. 251 in Chapter XX. is a reproduction of the finished outline of Fig. 391, and with the faces marked with their form letters.

The prism edges are first drawn, parallel to the vertical axis. To get their positions, a section of the prism is made to represent the faces of $p = \{110\}$ and $p' = \{1\bar{1}0\}$, by joining with thin lines the ends A, B, A', B', of the a and b axes. Then the traces of the somewhat narrow α -faces are put in the section, beginning, say, with the front one, parallel to the b -axis and of the observed size (that is, at such a distance

from the centre as makes the section line of the a -face have about the right length relative to those of the p -faces on each side of it); then the back a -face is similarly inserted into the section, and the b -edges may also be introduced in like manner parallel to the a -axis, and so as to have about the right relative length. Then the pair of faces of $p''=\{1\bar{2}0\}$ are to be provided for in the section, parallel to the lines joining the intercepts $2a$ and b , and of the proper observed relative size. Lines are then drawn parallel to the vertical axis to represent the prismatic edges, at present of indeterminate length, through all the corners of the section thus constructed.

Among the end-faces the two parallel o -faces are so predominately developed, that it is simplest to assume at first that no other end-face is present, and to construct the o -faces accordingly. The outline of each of these faces will be seen in thin lines in the drawing. The zonal arrangement of the crystal, as clearly indicated by the stereographic projection, Fig. 252 of Chapter XX., will help us greatly, and we see at once that the pair of po edges (for lettering of faces see Fig. 251, page 287, Chapter XX., in which the construction lines are left out) must be parallel to the edges pt and to , for the faces p , o , and t all belong to one and the same zone. We need only find by construction, therefore, the direction of any one of these edges, and that between $p=(110)$ and $t=(021)$ will be a convenient one to choose. The p -face may be represented by the line AB , joining axes a and b , and by lines through A and B parallel to the vertical axis. The plane $t=(021)$ is represented by joining $2c$ and B and drawing a line through $2c$ parallel to axis a . This cuts the vertical line erected from A at the point P , and as the point B is also common to the two planes BP is the direction of their intersection. The edges pt , to , and op will thus be parallel to BP . The direction of the last edge may easily be found in confirmation directly as follows: Join A' , B and C to represent the plane $o=(\bar{1}11)$, and draw a line through the centre O parallel to AB to represent $p=(\bar{1}\bar{1}0)$, this cuts $A'B$ at Q , and C being common to the two planes CQ is the direction of intersection of the p - and o -planes. It will be observed that CQ and BP are parallel, as they should be from the zonal relationships referred to, proving the parallelism of the edges po and pt , and consequently also of to .

The edges ao , aq , and go are all parallel to the line joining the axes b and c at unit intercepts, that is, to BC ; for the forms a , q , o all belong to the same zone. Similarly, the edge ac between the macro-pinakoidal a -face and the very small basal-pinakoidal c -face, is parallel to the axis b from its very nature, the edge ac having been chosen as b -axis. Again, the edges bt , tq , gc and cq' are all parallel to the axis a , for t , q , and q' all belong to the pinakoidal zone $[bc]$, the possible edge between c and b , in this case modified by t , q , and q' , having been chosen in the first instance as the a -axis.

The directions of the edges $p'q'$ are next to be found. To represent $p'=(1\bar{1}0)$ we draw lines parallel to axis c through B' and A , and to represent $q'=(0\bar{1}1)$ we draw a line joining B' and C , and through C a line parallel to axis a ; this intersects the vertical from A at M , and B' being common to the two planes $B'M$ is the direction of their intersection, and the edge $p'q'$ is parallel thereto.

Similarly, we find the direction of the edge $p''t'$ by finding the intersection of the

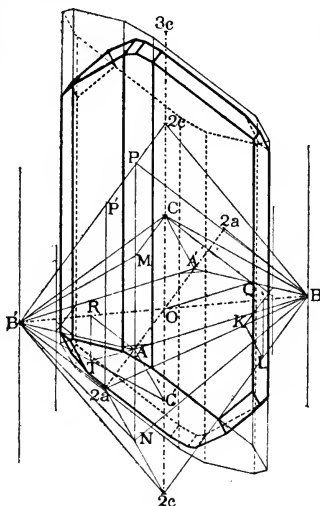


FIG. 391.—Construction of Measured Crystal of Copper Sulphate.

two planes $p''=(\bar{1}20)$ and $t'=(0\bar{2}1)$. For p'' we join $2a$ and B' , and draw a line through $2a$ parallel to axis c , and for t' we join B' to $2c$, and draw a line through $2c$ parallel to axis a ; this cuts the vertical from $2a$ at P' , and B' being again common to the two planes $B'P'$ is their intersection and the edge $p''t'$ is to be drawn parallel thereto.

The intersection of p and s' is also required, as the direction of the edges ps' . It is readily obtained by drawing a plane to represent $s'=(12\bar{1})$, that is, joining the intercepts $2a, 2c$ below, and B , and by dropping a line parallel to axis c from A , the line AB representing the prism $p=(110)$; where this vertical cuts the line $2a:2c$ at N is a point on the intersection, and B is common to the two planes. Hence BN is the intersection required, and the edges $s'p$ are parallel thereto, as are also the opposite edges $s'q'$ of the two s' faces.

It will be obvious that the edges bs' are parallel to the line KL , for K and L are two points on the intersection of the two planes $b=(010)$ and $s'=(12\bar{1})$. Or we might have taken the line $2a:2c$ itself as the direction of the intersection, for KL is parallel to it. The edge $p''s$ is parallel to the line joining B' and $2a$.

The only other direction required for the drawing of the whole crystal is that of the edges $p''o$, which is clearly AT ; for $o=(1\bar{1}\bar{1})$ is represented by joining B', C' , and A , and p'' by joining B' to $2a$, then drawing a parallel to the latter line from A , cutting axis b at R , and another line RT parallel to the vertical axis from R . T is a point of intersection and A is common, hence AT is the line of intersection of the two planes.

In drawing a crystal of this kind, where the end-faces are developed to very different extents, and each face constitutes along with its parallel fellow at the other end a separate form, the chief difficulty is to give the various faces their observed relative dimensions. In this particular case, where the o -faces so largely predominate that it is most convenient to draw them in at first as the only end-faces, one proceeds, after having settled, as has now been done, the directions of all other edges on the crystal, to replace such of the edges between o and pinakoidal or prism forms as are modified on the actual crystal, by the smaller end-faces, beginning conveniently with the upper s -face and then following on with the t, q, c, q', t' , and s' faces, after which the lower faces of these forms can be similarly proceeded with.

After having completed the drawing of a crystal, especially of a somewhat difficult triclinic one such as that of copper sulphate, the drawing should be carefully examined alongside the stereographic projection (Fig. 252, page 287, Chapter XX. in this case), in order to verify that those faces belonging to one and the same zone, whenever they touch, are bounded by edges parallel to one another and to the zone axis. Such an inspection gives confidence that the drawing has been accurately carried out.

Another drawing of a triclinic crystal, of a more general and therefore easier type, with the faces more evenly developed, is given in Fig. 34 of Chapter VI. (without construction lines, which have been erased for the sake of clearness), alongside its stereographic projection, Fig. 35, and this drawing can also readily be examined in the sense just indicated. The relations of the three pinakoids a, b , and c are here of the usual character, the goniometrical angles ab, bc , and ac being all less than 90° , whereas in the case of copper sulphate the angle bc was

just over 90° . The procedure for the drawing of the crystal is practically the same in both cases, however, the axes being first drawn as in Figs. 389 and 390, and then the various faces inserted from the knowledge either directly of their intercepts or after determinations of their intersections with each other, as in the case of the copper sulphate crystal now worked out in detail.

The drawing of all the various types of crystals has now been discussed, and from the experience of the general principles of the construction gained with the typical cases considered, it should be possible to carry out any crystal drawing without difficulty. The case of twin crystals will be specially considered in the next chapter.

The great value of the stereographic projection has again been abundantly emphasised in this chapter, and its importance as an aid to crystal drawing, so strongly urged by Penfield, been rendered very obvious. It has been shown, indeed, that in many cases the use of it renders the general method of intersections unnecessary, but it is well, all the same, that the latter should be thoroughly understood, and the applications of it in this chapter must have brought out very clearly, and fixed them indelibly on the memory, the fundamental relations of axes, intercepts, indices, and crystal elements, in crystals of the various types of symmetry.

CHAPTER XXVI

COMPOSITE CRYSTALS AND TWINNING.

THE perfectly formed single crystals required for the purpose of goniometrical measurement are, as will have been abundantly clear from the earlier chapters of this book, by no means invariably easy to obtain. The vast majority of natural mineral crystals, and the greater number of those grown artificially from solution, exhibit some form of aggregation. After having completed the accurate description of the crystallographic symmetry and its elements, and determined the interfacial angles, with the aid of single crystals of the highest attainable degree of perfection, it is both interesting and important to include in the scope of the investigation a description of such mode or modes of aggregation as are found to be characteristic of the substance. For it is a fact that usually one or two very definite types of composite crystal are found to be peculiar to a particular mineral or to a specific chemical preparation.

There are three main kinds of composite crystal, namely: (1) parallel growths, (2) twin crystals, and (3) aggregates which are neither twins nor completely parallel growths, but which are arranged so that one particular edge or face of each individual, the prism edge or the basal plane for instance, is always arranged parallel to the corresponding edge or face of each of its neighbours. Completely parallel growths and twins are much the more common occurrences, however, when the crystals have been allowed to grow free from disturbance.

Parallel Growths.

In a parallel growth of different individuals of the same substance all the corresponding edges and faces of the different individuals are parallel, supposing all of them to show the same forms. An excellent example is potash alum, the well-known double sulphate of potassium and aluminium which crystallises with 24 molecules of water of crystallisation, $K_2Al_2(SO_4)_4 \cdot 24H_2O$, or almost any of the salts isomorphous with it, all known as alums whatever the second metal may be. The cubic octahedra of the alums are often formed in columnar piles of parallel crystals, as shown in Fig. 392; the planes of contact are parallel to the cube-face (001), and the centres of successive octahedra, indicated by the dots, lie in the prolongation of the vertical axis. It will be obvious that rotation for 90° or 180° of any one or more of the octahedra effects no change in the aspect of the pile. If the same forms

are not present, the parallelism consists of a parallel arrangement with respect to the crystallographic axes. Such parallel-growth aggregates usually reveal themselves by the fact that all the parallel faces become illuminated at the same time by parallel rays of light, so that simultaneous reflections are obtained from them. This is well seen on the goniometer when the crystals are sufficiently small, and the signal-images simultaneously reflected from the individual parallel facets ought, of course, to coincide into an identical image. Such composite crystals, however, are rarely sufficiently perfect for this to happen, and usually a bundle of images is afforded, lying the closer together the greater the degree of perfection. Such parallel growths of the same substance do not call for any further structural study, for they would never be selected for goniometrical work, except to verify the fact that the substance has a tendency to produce such parallel growths.

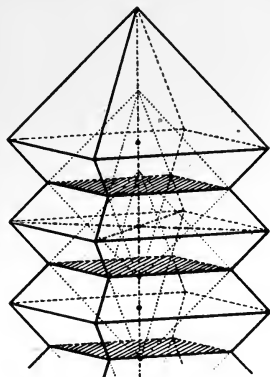


FIG. 392.—Parallel Growths of Alum.

The subject of the conditions under which parallel growths of one substance can be formed on the crystals of another has been investigated by Barker,¹ who has shown from experiments with a large number of salts that only such substances as possess similar molecular volumes and molecular distance ratios (discussed in Chapter XXXI.) exhibit the property of forming such parallel growths. It is doubtless due to the congruency of the space-lattice structure, that is, the near equality of the absolute dimensions of the structural-unit parallelepipeda of the space-lattices, in these cases, that parallel growths of the two crystal substances are capable of formation one on the other.

Twin Crystals.

Crystals are frequently found which are obviously of a composite character, that is, are composed of more than a single individual crystal, and in which the two or possibly even three parts, clearly belonging to different individuals, are united in a definite and regular manner, other than parallelism, the particular mode of union being characteristic of the substance. The twin nature is often betrayed by the presence of "re-entrant angles," forming notches, arrow-head shapes, knee-shapes, cruciform, or heart shapes; but not infrequently the two or more individuals are so intimately blended that the appearance at first sight is that of a single individual crystal, and possibly of that of a crystal of a higher degree of symmetry than the single crystals of the particular substance have been already proved to exhibit.

The "twin law" defining the manner in which composition occurs is usually stated in terms such as the following: The two individuals are first supposed to be arranged parallel to each other, with one face

¹ *Journ. Chem. Soc.*, 1906, 89, 1120; *Mineralog. Mag.*, 1907, 14, 235.

of each in mutual contact, and then in order to produce the twin one individual is supposed to be rotated for 180° upon a plane, which is usually although not necessarily the contact face, called the "twin plane," and about the normal to this plane as axis, which is termed the "twin axis." Sometimes, as just suggested, the common plane of symmetry is not the face of contact, and it is then distinguished as the plane of "composition." In general, the plane of twinning may be any actual or possible face of a form developed or possible on the crystal, except obviously a plane of symmetry, and usually it is one with low indices, and indeed very often a primary face.

But this mode of stating the nature of the regular mode of composition of twins is misleading, as it is based on or bound up with the idea of rotation of one individual with respect to the other, whereas the twin is simply the expression of the fact that the growth of the crystal has occurred simultaneously according to two distinct but intimately related orientations, owing to the molecules of the substance in the act of crystallisation taking up two alternative positions with equal facility. If stated as above, the idea that one individual has been actually joined to the other after an actual rotation for a semicircle is somehow involuntarily and erroneously conveyed.

A much preferable mode of stating it is that a twin crystal consists of two individuals united symmetrically about a plane, the "twin plane," which is not one of systematic symmetry but a possible face of the crystals of the substance, or about an axis, the "twin axis," which is a possible crystal edge. Besides being a mode of expressing the rule less liable to misinterpretation, it is also a more accurate statement; for all twins cannot be explained on the theory of rotation through a semicircle.

In the case of "juxtaposition twins" the plane of union, whether the twin plane or not, is known as the "plane of composition." "Interpenetration twins," however, have no plane of composition. Hence, the only true elements of twinning are the twin plane and the twin axis. The effect of complete interpenetration twinning without re-entrant angles is often apparently to convert a crystal belonging to a class lower than the holohedral into a holohedral crystal of the same system of symmetry. This type of twinning is, therefore, termed "supplementary."

There are well-known twins formed by the union of enantiomorphous crystals, that is, of a right-handed variety with a left-handed one, and which belong to a class of symmetry possessing neither a plane nor a centre of symmetry; these cannot be described at all as obtained by rotation, for no amount of rotation can bring a truly enantiomorphous crystal into coincidence or parallelism with its oppositely enantiomorphous fellow. Such twins are best described as "mirror-image" or "reflection twins," for one individual is symmetrical to the other about the plane of reflection, although the twin cannot be explained by rotation about this plane, so that the plane of twinning is not a rotation plane in such a case. An excellent instance is afforded by the famous twins of Brazilian quartz, shortly to be described.

Hence, it must now be clear that we have twins characterised by

possessing a twin plane, which is a possible face of the crystal and is a plane of symmetry of the twin but not of the single crystal, and which may have a twin axis normal to it; also other twins endowed with a twin-axis only, which is a possible edge of the crystal; and others again which may be ascribed to reflection across a plane, which is a possible facial plane. If the two supplementary individuals belong to a class of their system lower than the holohedral, and the twin plane or plane of reflection is a symmetry plane of the holohedral class, or if they are twinned about a twin axis which happens to be an axis of symmetry in the holohedral class, the resulting solid will exhibit the exterior symmetry of the holohedral class, and the more indistinguishably from a truly holohedral crystal the more complete the interpenetration of these supplementary twins.

It will be obvious from the foregoing that no general rule applies to the phenomenon of twinning, and that it is, therefore, incumbent to describe each variety of twin, of every specific crystallised substance, on its own merits. No investigation of a new substance is complete without the description of all the varieties of twinning which the substance is observed to exhibit. As the main purpose of this book is to give instructions for practical work, a few typical examples will now be given of well-known twins of the different varieties which have been referred to, selected from all the systems of symmetry, and these will be followed by the description of a few twins met with in the course of the author's own investigations, which will best elucidate the mode of tackling the investigation of the twins of a new substance. A description will then be given, in conclusion, of the principles underlying the drawing of twin crystals, with one or two specific examples from recent investigations.

The "notched" or grooved type of twin is very well shown by the diamond, a complementary pair of positive and negative tetrahedra, or of hexakis tetrahedra, twinned about a cube face, interpenetrating to

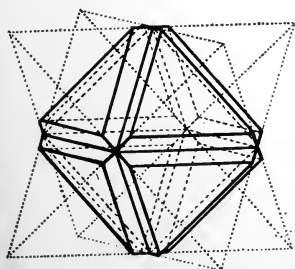


FIG. 393.—Twinned Tetrahedra of Diamond.

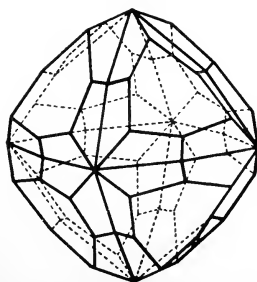


FIG. 394.—Twinned Hexakis Tetrahedra of Diamond.

produce an apparent octahedron or hexakis octahedron, but with more or less deep grooves along the octahedral edges, owing to the interpenetration not being absolutely complete.

Two such diamonds are represented in Figs. 393 and 394.

There is some doubt, however, as to whether this proves the diamond

to belong to the hexakis-tetrahedral class of the cubic system (class 31), for not only have hexakis octahedra of diamond been found, showing no trace of grooving, but also the cleavage is truly octahedral, and no proof of electric polarity has ever been brought forward, or of any kind of polarity of the trigonal axes of symmetry, all indicating that there is a possibility that the diamond may really belong to the holohedral class 32 after all. Indeed definite evidence has recently been adduced of the absence of electric polarity in a memoir by van der Veen,¹ who decides for holohedral symmetry. There is no doubt, however, as to the frequent occurrence of twin tetrahedra and hexakis tetrahedra, of the type shown in the figures, and also a few simple tetrahedra have been from time to time discovered.

The notched appearance is also typical of the well-known rhombohedral twins of calcite, twinned on the basal pinakoid (111). Precisely similar twins are also exhibited by an organic substance already mentioned in Chapter VI., methyl triphenylpyrrolone. This substance has been shown by the author to be dimorphous, having both a triclinic form and

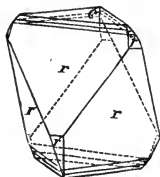


FIG. 395.—Rhombohedral Crystal of Methyl Triphenylpyrrolone.

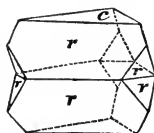


FIG. 396.—Twin of Methyl Triphenylpyrrolone.

a trigonal one. The triclinic variety is illustrated in Figs. 34 and 35, being used as a particularly typical example of a triclinic substance. The trigonal variety belongs to the calcite class, and a single crystal of rhombohedral habit is shown in Fig. 395, the chief forms being the basal pinakoid $c = \{111\}$ and the primary positive rhombohedron $r = \{100\}$, the complementary negative rhombohedron $r' = \{221\}$ being very subsidiary. Fig. 396 shows a characteristic twin, resembling one of calcite, composed of a pair of positive rhombohedra $r = \{100\}$ twinned upon the basal pinakoid $c = \{111\}$, as plane of both twinning and composition, re-entrant angles being shown at the three equatorial corners.

The arrow-head appearance is very characteristic of twins of gypsum (selenite) $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$. Fig. 397 represents a single crystal of gypsum of a very common type, and Fig. 398 an arrow-headed twin formed by the union of two such individuals. The twin plane and plane of composition is the ortho-pinakoid (100), neither of the two faces

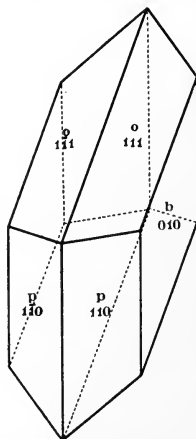


FIG. 397.—Crystal of Gypsum.

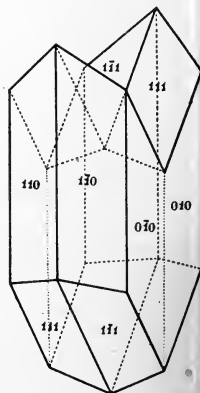


FIG. 398.—Twin of Gypsum.

¹ Kon. Akad. Amsterdam, 1907, page 182.

Another case of considerable importance is that of the two complementary pentagonal dodecahedra of pyrites FeS_2 , $\{210\}$ and $\{120\}$, twinned (reflected) about a face of the rhombic dodecahedron $\{110\}$, and interpenetrating and passing beyond one another in the manner indicated by the re-entrant angles in Fig. 404, which represents one of the beautiful twins from Minden in Prussia. A cross is formed by the long edges of the two different pentagonal dodecahedra, parallel to each cube face, the centre of the cross being at each axis-end; and in the example shown in the figure the sharp crossed edges are actually blunted by the faces of the cube, rendering the cross still more marked, and justifying the popular appellation conferred on these twins of "*Zwillinge des eisernen Kreuzes*." The star-shaped faces at the diagonal corners are octahedron faces replacing common corners of the two pentagonal dodecahedra.

When interpenetration of such complementary pentagonal dodecahedra is complete without the passage of either individual beyond the other, the solid produced is a perfect tetrakis hexahedron, indistinguishable as regards its exterior shape from the holohedral form. Hence, this is an excellent instance of **supplementary twinning**, where two individuals belonging to a class of symmetry lower than the holohedral combine to produce the holohedral form of the same system. The two diamonds shown in Figs. 393 and 394, composed of two interpenetrating tetrahedra and hexakis tetrahedra respectively, are also supplementary twins, and when the interpenetration is just complete without any sign of grooves the solid is either the octahedron or the hexakis octahedron, indistinguishable in appearance from those two holohedral solids of the cubic system. Diamonds are actually found in which this is the case, the octahedron being, however, the commoner, and although very frequently the faces are curved, many diamonds have been found which are perfectly sharp and plane-faced octahedra. In the case of pyrites there is no doubt whatever that the true class is the pyrites class 30, and not the holohedral class 32 of the cubic system. But in the case of the diamond, the question has already been said to be less clear, and it may be that while the commoner habit is that of the hexakis-tetrahedral class 31, the truly holohedral habit occasionally occurs, and that the diamond really possesses full holohedral class-32 symmetry. In this event a supplementary twin would be hard indeed to distinguish from a holohedral diamond, and the very fact of there being this interesting discussion over the diamond is of itself eloquent testimony to the close simulation of holohedral symmetry which supplementary twinning may evoke, and therefore to the need for a very sharp look-out to be kept, when investigating the crystals of a new substance, for the least sign of twinning.

There is a further case introducing the phenomenon of enantiomorphism which ought to be referred to, as the mineral has been taken in Chapter XXIV. as an example of trigonal symmetry, and forms an especially good instance of an enantiomorphous substance, namely, that of quartz. Figs. 344 and 345 (page 357) in Chapter XXIII. illustrate the

two kinds of quartz crystals, Fig. 345 the right-handed and Fig. 344 the left-handed. Fig. 405 represents a twin quartz crystal, in which two right-handed crystals are intergrown in accordance with a plan which may be described as equivalent to rotation for 180° about the vertical axis, and so completely without projection beyond each other that the crystal is apparently a single one. But it will be observed that the little faces of the right trigonal pyramid s and of the right trapezohedron x are present on two adjacent corners (60° removed) of the crystal, whereas a truly single crystal shows them only on alternate corners. Moreover, in ordinary single quartz crystals the faces of the rhombohedron $r = \{100\}$ are usually bright, and those of the inverse rhombohedron $r' = \{2\bar{2}1\}$ dull; but in this twin crystal one part of each rhombohedron face is generally bright and the rest dull, the two parts being more or

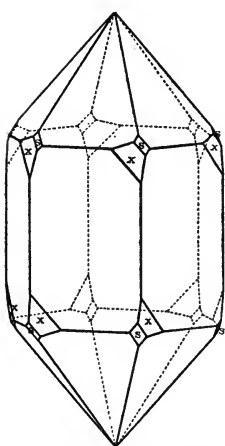


FIG. 405.—Interpenetration
Twin of Right-handed
Quartz.

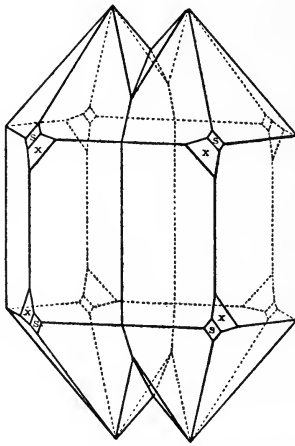


FIG. 406.—Partial Interpenetration
Twin of Right- and Left-handed
Brazilian Quartz.

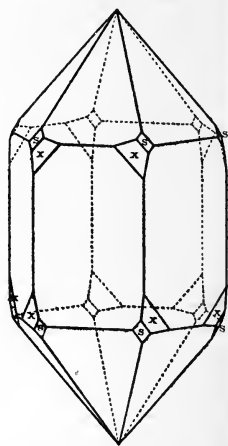


FIG. 407.—Complete Inter-
penetration Twin of
Brazilian Quartz.

less irregular and corresponding to the two different individuals. These two individuals are related to each other as if, starting with the pair parallel (both right-handed as in the case chosen for Fig. 405, or both left-handed), one of them had been rotated round the vertical axis for 180° , bringing the direct or positive rhombohedron of the one parallel to the inverse or negative rhombohedron of the other. Such twins of either right-handed or left-handed quartz are fairly common, and the amount of interpenetration varies from only very partial, showing re-entrant angles, to complete as shown in Fig. 405.

Quartz, however, exhibits yet another kind of interpenetration twin, as already mentioned, which is characteristic of crystals from Brazil. Two differently enantiomorphous individuals, that is, a right-handed and a left-handed, are here combined, and when the interpenetration is but partial, as in Fig. 406, the twin has the appearance of a mirror-image or reflection twin. But when the interpenetration is complete, the two

rhombohedra coincide and the little *s*- and *x*-faces occupy symmetrical positions, as shown in Fig. 407. The symmetry is obviously with respect to vertical planes passing through the centre of each parallel pair of prism faces, so that the twinning may be described as upon a face of the second order prism $\{10\bar{1}\}$. The rhombohedral faces of the two interpenetrating individuals, being coincident, the direct of one with the direct of the other individual, and the inverse of the one with the inverse of the other, there are no irregular patches of dull and bright natural polish exhibited, as in the case of the other variety of quartz twin shown in Fig. 405.

Repeated Twinning.—One case of repeated twinning has already been described in the triplet of rutile (Fig. 400), but the repetition is by no means confined to the formation of triplets, or even of rings of seven or eight individuals; it may occur any number of times. Carried to the extreme, this phenomenon gives rise to one of the greatest difficulties which may confront the crystallographer, for a crystal built up of innumerable twin lamellæ is not at all uncommon, and if the fact be not detected the failure leads to an erroneously high estimation of the symmetry; for the latter is lower than the apparently single crystal would appear from its exterior shape to possess. It will subsequently be shown that the optical properties exhibited by such a composite crystal are also anomalous, and often most difficult to unravel, for the individual laminae are occasionally not merely microscopic in thickness but actually ultra-microscopic. But the very fact that the optical characters are not normal is a warning to look out for possible repeated twinning.

An excellent example is afforded by the triclinic (holohedral) soda feldspar albite, $\text{NaAlSi}_3\text{O}_8$. A characteristic twin of this mineral is shown in Fig. 408, the twin plane being the brachy-pinakoid $b = (010)$.

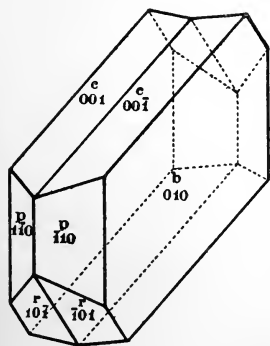


FIG. 408.

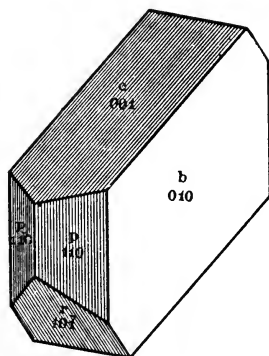


FIG. 409.

The two long upper *c*-faces show a re-entrant angle, and are the two different basal pinakoid faces (001) and (00 $\bar{1}$) of the two individuals; the two front *p*-faces are the prism faces (1 $\bar{1}$ 0) and ($\bar{1}$ 10), and the two *r*-faces are (10 $\bar{1}$) and ($\bar{1}$ 01). The twinning on this plan, however, may be so often repeated that a crystal having the shape of a single crystal as

in Fig. 409 is produced, composed of innumerable alternate lamellæ of the two kinds of individual, the only exterior sign of which is the striation indicated by the shading lines in the figure. A section-plate of such a crystal, cut transversely to the twinning, would exhibit differently coloured strips under the polarising microscope, corresponding to, and indicating clearly, the different individuals. But it can readily be imagined that a section parallel to the plane of twinning, and thick enough to include quite a number of separate individuals, would either show no indication of twinning at all or would present some anomalous phenomenon in polarised light. If the layers having the direction of one crystal individual predominate in thickness, the alternating reversed individuals being much thinner, the apparently single crystal resulting is of triclinic habit, more or less resembling a single crystal of albite, as drawn in Fig. 409. But if the layers average equal thickness, the habit becomes pseudo-monoclinic, the upper and lower edges becoming more or less parallel to the normal to $b = \{010\}$, although they are seen to be serrated when examined with a lens. Such a crystal then resembles very closely the monoclinic crystals of the corresponding potash feldspar, orthoclase, KAlSi_3O_8 . Indeed the most recent research has indicated the possibility that orthoclase itself may be really triclinic, but so extremely finely and evenly twinned in innumerable repetitions that the microscope is incapable of detecting the separate individuals. A second variety of potash feldspar, microcline, is distinctly triclinic, and shows visible multiple twinning, thus lending considerable support to this view.

A remarkably beautiful case of repeated twinning is that of right- and left-handed quartz in amethyst. The repetition of alternations of the two varieties is so frequent, the laminae so thin, that the external appearance of a single crystal is afforded, and it is only on cutting a section-plate, preferably 2-4 millimetres thick, and examining it in polarised light that the structure is revealed, although the part which shows the lamination is frequently of a pinkish purple colour, that currently described as "amethyst." This violet-coloured portion is often arranged in alternate 60° -sectors of the section. The lamination is parallel to the edges of the second order hexagonal prism and perpendicular to those of the first order prism forming the boundary of the section, and thus often appears as lines radiating from the centre and inclined at 120° or as a series of hexagons one within another, repeated many times in alternate light and dark or coloured lines from the centre to the edges of the section, or even more frequently as a combination of both, the former occupying the central portion. The more regular varieties showing lines radiating at 120° from the centre usually exhibit this phenomenon only in the central part, passing as the hexagonal edges of the section are approached into uniform regions of right- or left-handed quartz brilliantly polarising in even colours. A very beautiful section of such an amethyst, nearly 2 inches in diameter, as seen in parallel polarised light, is shown in Fig. 410. In the dark field of the lantern projection polariscope the laminae, as seen on the screen, show not only alternate bright and dark strips, but the dark lines themselves are alternately very black and less

dark. The blacker lines are black extinction bands, due to the section-plate being oblique to the plane of the lamination, which is apparently parallel to the rhombohedral planes of gliding with which quartz has been shown by J. W. Judd¹ to be endowed; so that two successive laminae overlap, and one being right-handed and the other left-handed, and producing equal and opposite retardation in the central line of the overlapping portion, a black extinction band, that of the dark field of the crossed Nicols, is here produced. This intimate repetition twinning consequently causes the crystal to exhibit an apparently non-rotatory character in the part where such lamination occurs, and to show in convergent polarised light the usual black cross and circular rings of an ordinary uniaxial crystal of an optically inactive substance. The marginal part where a region of right- or left-handed quartz is found shows both rotation and the special uniaxial figure of either right or left quartz, rings without black cross in the central part of the field, and at the junction of two such marginal regions of opposite rotations the overlapping strip shows Airy's spirals, as usual for a single overlapping of the two varieties of quartz.

An extreme case of repeated twinning is exhibited by crystals of potassium chlorate, KClO_3 . Lord Rayleigh has shown that many crystals

of this salt are composed of multiple twins of practically equal thickness. R. W. Wood² has described a crystal of chlorate of potash, a thin flake, in which there were 700 laminae each only one five-thousandth of a millimetre thick. What is most remarkable, however, is that the laminae are of constant thickness in any one and the same crystal, while the thickness varies in different crystals of the same lamellar constitution. It appears as if a crystal which starts with an alternation of a pair of plates of a specific thickness maintains this same repetition-

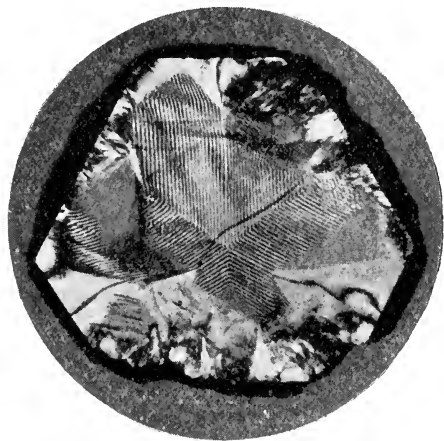


FIG. 410.—Large Section-plate of Amethyst projected in Polarised Light.

constant throughout the growth of the whole structure. The author has often been struck with the regularity in the thickness of the laminae exhibited by amethyst, in a very fine collection of amethyst sections which he possesses for projection purposes. Some of these amethysts, such as that illustrated in Fig. 410, afford simply gorgeous effects on the screen, especially as the analysing Nicol prism is rotated, and which are of a highly instructive character as regards the structure of the crystals.

Mimicry of Higher Symmetry brought about by Twinning.—We have already seen in several cases of supplementary twinning that a

¹ *Mineralog. Mag.*, 1892, 10, 123.

² *Phil. Mag.*, 1909, 18, 535.

twin pair of individuals belonging to a class of symmetry lower than the highest which the system is capable of exhibiting may simulate the holohedral class of that system. Occasionally this simulation goes higher still, to another and higher system altogether. An approach to this has just been referred to in the case of a triclinic felspar becoming apparently monoclinic. One of the best-known, however, of these cases of "mimetic twinning," and a perfectly definite one, is that of the rhombic salt so often referred to in this book, as typifying many of the most interesting properties of crystals, potassium sulphate, K_2SO_4 . It has already been shown in an earlier chapter that the prism angles between the respective two pinakoids $a = \{100\}$ and $b = \{010\}$ and the two prisms $p = \{110\}$ and $p' = \{130\}$ are very nearly 30° and 60° , and this is true for the whole series of alkali sulphates and for the isomorphous selenates. In the case of potassium sulphate the actual angles are: $ap = 29^\circ 48'$, $pp' = 30^\circ 0'$, and $p'b = 30^\circ 12'$. Thus, to begin with, the prismatic zone of a single crystal is only $12'$ from being truly hexagonal; and as the principal terminal faces are those of (1) the primary bipyramid $o = \{111\}$ corresponding to the primary prism $p = \{110\}$, and (2) the brachy-domal prism $q' = \{021\}$ corresponding to the brachy-pinakoid $b = \{010\}$ and having the same intercept on the vertical axis as the pyramid $o = \{111\}$, it will be obvious that if the faces b , p , o , and q' are developed in anything like equipoise a solid closely resembling a hexagonal prism doubly terminated by the bipyramid will be produced. The prism, moreover, may be shortened until the appearance is that of a simple hexagonal bipyramid. An intermediate case when the prism and brachy-pinakoid faces still clearly show around the equatorial base of the bipyramid, is shown in Fig. 411, which represents an actual crystal of rubidium sulphate, Rb_2SO_4 , the second member of the family series of alkali sulphates; but the habit is equally applicable to potassium sulphate, although it is not so common in the case of this first member of the series.

The remarkable thing is, however, that these salts, and most especially potassium sulphate, have a pronounced tendency to form triplets, which resemble hexagonal bipyramids even more closely than do these occasionally occurring pseudo-hexagonal single crystals. The author possesses a large number of excellent triplet-bipyramids of potassium sulphate, and one of them is faithfully represented in Fig. 412. The striking likeness to Fig. 411 is quite remarkable. But on close inspection of the two crystals the faces of that shown in Fig. 411 are observed to be truly plane and in every way goniometrically excellent, as should be the case with a perfect single crystal; whereas the crystal portrayed in Fig. 412 exhibits a very low linear ridge down each face, as indicated by the shading; it is not always or even generally symmetrically placed, but sometimes nearer the polar pyramidal edge on the one side and sometimes nearer that on the other; the face obviously consists of two parts, one of which is not truly parallel to the other, the two behaving in fact like vicinal faces. The goniometrical positions of the two portions are only a few minutes apart, but the

line of demarcation between them is quite distinct, and is almost a sure sign either of twinning or of the presence of vicinal faces.

As a matter of fact, these ridges are due to the crystal being a triplet, the three individuals interpenetrating according to a plan which

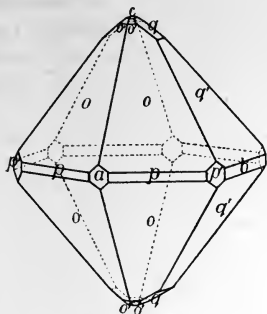


FIG. 411.—Crystal of Rubidium Sulphate.

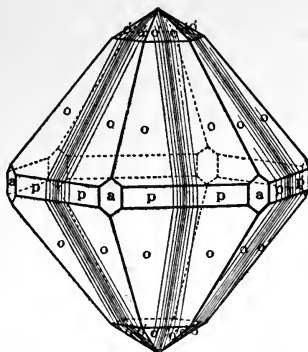


FIG. 412.—Triplet of Potassium Sulphate.

will be rendered clear by Fig. 413, which is an orthometric projection, that is, a geometric plan from directly above the crystal, projected on the equatorial plane. The circumferential edges all represent in plan the faces of the primary prism $p = \{110\}$ and of the macro-pinakoid $a = \{100\}$; and the continuous radii indicate the polar edges between the primary bipyramid faces $o = \{111\}$. The dotted radii represent the dividing

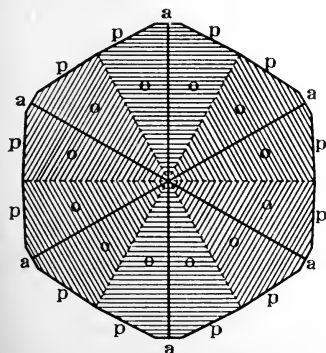


FIG. 413.—Plan of Triplet of Potassium Sulphate.

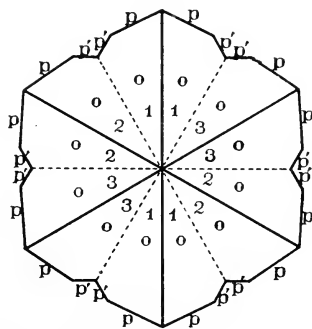


FIG. 414.—Plan of Triplet of Potassium Sulphate, showing Re-entrant Angles.

lines between the separate crystals, and the two parts of each one of the three individuals are shaded with parallel lines. The angle ap being $29^\circ 48'$, the angle pp at the corner where the little a -face truncates it is twice this, $59^\circ 36'$. There are six of these angles on the triplet, but they together only make up $357^\circ 36'$, so that if the twin plane were precisely perpendicular to $p = (110)$, and the p -faces of two individuals after twinning were exactly congruent and in the same plane, there would be a gap of $2^\circ 24'$ at the finish. But the plane of

twinning must be regarded as not precisely perpendicular to $p = (110)$ but differing therefrom by one-twelfth of the discrepancy just mentioned, namely, by $12'$, the difference of ap from exactly 30° ; for the solid is completed without a gap. That is, the twinning occurs as if the crystal were truly and not merely pseudo-hexagonal. But every pair of adjacent p -faces belonging to two different individuals are consequently not in precisely the same plane, the normal to one part making an angle of $24'$ with the normal to the other part. This accounts for the ridge, and for the fact that one half of the face reflects light at a different moment from the other half on the other side of the ridge. No gap is ever apparent if no faces of the other prismatic form $p' = \{130\}$ are present. If p' -faces are present, however, little re-entrant angles are found, as indicated in the second orthometric projection in Fig. 414. These, however, are not gaps, but true re-entrant angles. They are visible on several of the crystals of potassium sulphate in the author's collection. These faces of $p' = \{130\}$ are nearly parallel to the planes of twinning of the three individuals, as the angle pp' is $30^\circ 0'$. In Fig. 414 the three individuals are numbered, so that the parts belonging to the same individual, expressed by similar shading in Fig. 413, may be readily recognised.

It will be clear from the foregoing what an excellent instance of mimetic repeated twinning potassium sulphate forms. The presence of the ridges, and the duplicate images $24'$ apart given by the prism faces, and corresponding vicinal-face phenomena also exhibited by the two parts of each of the pyramid faces, would put an experienced crystallographer on his guard against assuming that the crystal was a simple hexagonal one. The biaxial optical properties, moreover, would render any such erroneous supposition impossible, and would establish the fact that although the habit was pseudo-hexagonal, the system of symmetry was really rhombic.

Fig. 415 gives an example of a similar triplet of the isomorphous

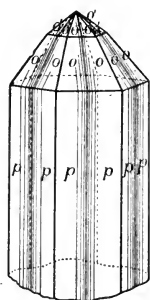


FIG. 415.—Triplet of Potassium Selenate. Complete Interpenetration.

potassium selenate, K_2SeO_4 , the composite crystal being an elongated pseudo-hexagonal prism terminated by the two pseudo-hexagonal pyramids; the construction is exceedingly like that of potassium sulphate, but the angle pp is $59^\circ 38'$ instead of $59^\circ 36'$, the difference of the twin plane from perpendicularity to $p = (110)$ is $11'$ instead of $12'$, and the angle across the little ridge in the middle of the p -faces is $22'$ instead of $24'$.

A very instructive further

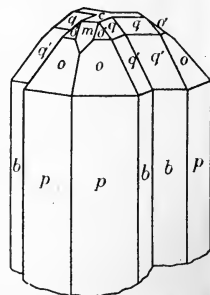


FIG. 416.—Triplet of Potassium Selenate. Incomplete Interpenetration.

triplet of potassium selenate is also shown in Fig. 416, which is very characteristic of this salt. The interpenetration is less complete, none

of the three individuals passing right through the structure so as to be continued on the other side, so that the horizontal section of the composite crystal is a three-rayed star. Re-entrant angles are formed by the brachy-pinakoid b -faces and the brachy-domal q and q' -faces.

Practical Mode of dealing with Twins.—In the investigation of a new crystalline substance careful watch must be kept for any signs of twinning among the crystals of the numerous crops prepared, and after the completion of the goniometry proper, carried out upon the best single crystals, such twins as have been found should be thoroughly examined, and measured if such a course be necessary to elucidate the plan on which the twinning is accomplished. For the kind or kinds of twin, or of still higher polysynthetic structure, which may be developed by a substance in the act of crystallisation, forms an essentially characteristic property of the substance. The chief object, of course, is to discover the exact position of the plane or axis of twinning, and to ascertain the typical mode in which the composition occurs, that is, whether by interpenetration, juxtaposition, or both. Perhaps the best mode of illustrating the *modus operandi* of this part of the investigation will be to take a concrete example from the author's own experience.

Example of Investigation of a Twin.—The case chosen, a typical and not too simple one, is that of ammonium selenate, $(\text{NH}_4)_2\text{SeO}_4$, the investigation of which was published in the year 1906.¹ This salt is dimorphous, having a rhombic form isomorphous with ammonium and potassium sulphates and the rest of the alkali sulphates and selenates, and a monoclinic form. The former is only known in mixed crystals with ammonium sulphate, potassium selenate, or best of all with rubidium selenate, which last-mentioned salt mixes in the solid crystalline state with the greatest facility with ammonium selenate on account of the closeness of their molecular volumes and molecular distance ratios (relative distances apart of the molecular centres along the crystallographic axes). The ammonium selenate crystals obtained in the usual way from a solution in water of the pure salt are invariably those of the monoclinic variety. A characteristic crystal is shown in Fig. 417, and its stereographic projection is shown in Fig. 29 on p. 73.

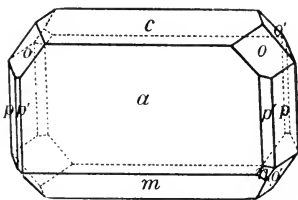


FIG. 417.—Monoclinic Crystal of Ammonium Selenate.

The habit is prismatic along the symmetry axis b . The ortho-zone parallel to this axis shows the faces of $c = \{001\}$, $a = \{100\}$, and $m = \{201\}$, so that three different angles are observed to occur, each in duplicate as two parallel faces constitute each of the three forms. These angles are :

$$ac = (100) : (001) = 64^\circ 31' ; cm = (001) : (\bar{2}01) = 68^\circ 22' ; ma = (\bar{2}01) : (\bar{1}00) = 47^\circ 7'.$$

Now in the twins this zone, in many cases indistinguishable at a cursory glance from that of a single crystal, shows only two of these angles, $64^\circ 31'$ and $68^\circ 22'$, each in duplicate, the third angle being altogether absent; its two values of $47^\circ 7'$ are replaced, one in each semicircle, by the two different values $50^\circ 58'$ and $43^\circ 16'$. It was discovered that $50^\circ 58'$ was afforded by two adjacent a -faces, and $43^\circ 16'$ by a pair of adjacent m -faces, the sequence of angles in the zone being :

¹ *Journ. Chem. Soc.*, 1906, 89, 1066.

$ca = 64^\circ 31'$
$aa = 50 \quad 58$
$ac = 64 \quad 31$
$cm = 68 \quad 22$
$mm = 43 \quad 16$
$mc = 68 \quad 22$

On some of these twins the edges aa and mm were not sharp, but were grooved along their entire length with a re-entrant angle formed by two narrow strips of a or m -faces, which gave quite trustworthy images of the signal-slit, so that there was no doubt about the nature of the faces. A section of such a prism is shown in Fig. 418, which will render the construction quite clear.

It will be obvious that we are dealing with a twin, the re-entrant angles being conclusive, and that the plane of twinning is parallel to the basal pinakoid $c = (001)$, the twin axis being normal to that face. The angles are marked in the drawing, and it will be evident from them how the two twin angles aa and mm come to be $50^\circ 58'$ and $43^\circ 16'$, as observed.

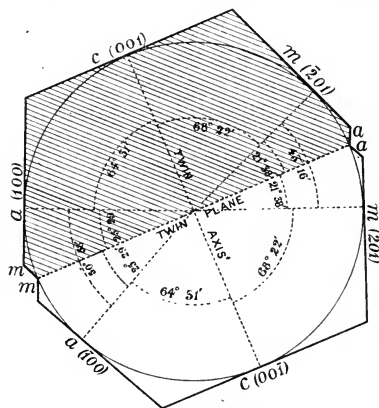


FIG. 418.—Section of Ortho-zone of Ammonium Selenate Twin.

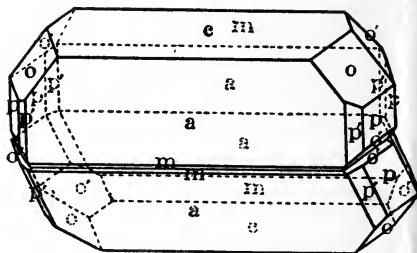


FIG. 419.—Twin of Ammonium Selenate.

Fig. 419 shows the appearance of one of these twins which had the grooves best developed, indeed that at the back was quite a wide re-entrant angle, the two a -faces forming it being quite broad; it will be apparent from the figure that the end-faces also exhibit re-entrant angles. The dotted letters in the figure refer to the back faces. The chief zone affected, involving end-faces, is $[copo'c]$. The angles of this zone in a single crystal are as follows, each being repeated in the same order in the second semicircle:—

$$\begin{cases} co = (001) : (111) = 43^\circ 53' \\ op = (111) : (110) = 33 \quad 33 \\ po' = (110) : (1\bar{1}\bar{1}) = 43 \quad 16 \\ o'c = (11\bar{1}) : (00\bar{1}) = 59 \quad 18 \end{cases}$$

There are obviously two of these zones, the one the zone symbol of which has just been given and the complementary one $[co'poc]$, corresponding to the two zonal arcs terminating at c -poles shown on the projection in Fig. 29. But in a twin each of these two zones shows at one end instead of a single p -face two p -faces inclined at an angle of $25^\circ 8'$, and at the other end also two p -faces instead of one, but in a re-entrant angle, and the angle between the two o' -faces which are now brought on either side of the re-entrant angle (instead of one o and one o' -face of the single crystal) is $61^\circ 24'$. The two angles $25^\circ 8'$ are in front, one at each end, and the two of $61^\circ 24'$ behind. Further, a little groove-like re-entrant angle is also sometimes present between the two p -faces at each end which make the angle of $25^\circ 8'$, as shown in front in Fig. 419,

due to two strips of σ' -faces being developed, one on each individual. The two angles just quoted, $25^\circ 8'$ and $61^\circ 24'$, were found by direct calculation to be exactly those demanded by a twin the plane of twinning of which is the basal pinakoid. The angle $25^\circ 8'$ is clearly shown between the two front p -faces of the two individuals at the left end in Fig. 419.

The construction of this twin is therefore perfectly clear. The best idea of it is, of course, obtained from a model, and whenever any difficulty presents itself in the geometrical imagination of the solid, the author always has recourse to the cutting of a model out of cork, when the difficulty invariably disappears. It is best as a rule to cut out a duplicate pair of single-crystal models, and then to juxtapose them in the positions actually shown by the twin crystal, which is generally too small to be manipulated as a model can be; but in a case like the present, one model will probably suffice for most purposes, cut in two halves parallel to the basal pinakoid, the twin plane.

The Drawing of Twin Crystals.—The essence of the process of drawing a twin is the construction of a second axial system, at the correct position with respect to that of the ordinary single crystal. Each case should be dealt with directly from first principles, but there are a few helpful rules which may be given for the three possible cases, which are (1) when two of the three axes remain the same and only the third has a different direction in the second individual to what it has in the first; (2) when two axes have new directions in the second individual; and (3) when all three axes have entirely different directions in the two individuals.

(1) **When only one Axis has Different Directions in the Two Individuals.**—An excellent concrete example, which will admirably illustrate the mode of procedure, is afforded by the characteristic twin of monoclinic ammonium selenate represented in Fig. 419, and the character of which has just been fully described. The twin plane has been shown to be parallel to one of the axial planes, that containing the axes a and b , parallel to the basal pinakoid $c=(001)$. Hence, it is only a question of finding the new position of the axis c of the second individual, for the axes a and b are common to the two individuals. After having found the direction of this second axis c , the second individual can be drawn in the usual way about it and the two common axes, and at such a distance from the first individual as faithfully represents what is actually observed on a typical twin.

In finding the position of the c -axis of the second individual the following considerations and Fig. 420 will be helpful. If the strong lines aa and cc represent the axes a and c lying in the symmetry plane, which is taken as parallel to the plane of the paper, then after rotation of the axial system for 180° about the normal to (001) —the twin axis, which is at right angles to the axis a and is indicated by the dotted line marked NN —the position of the axis c would be that indicated by the broken-and-dotted line marked $c'e'$, while the direction of the axis a would remain as before. The new position $c'e'$ will make with the original

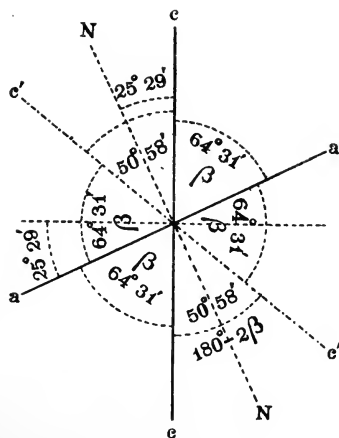


FIG. 420.

exactly on the lines so fully explained in the last chapter. The principal construction lines are shown in the upper part of the figure, and the intersection lines parallel to such crystal edges as are not simply parallel to the lines joining the ends of the various axes are BP, BQ, B'R, c'S, and c'T, and the actual edges which are parallel to these respective intersection lines are indicated clearly by dotted curved lines terminated at the edges and at the intersection lines by arrows.

The first individual, the upper half of the twin, resembles the upper part of the single crystal shown in Fig. 417. The second individual, the lower portion of the twin, is similar to the lower half of Fig. 417, but constructed about the new axis c inclined away to the back (and the same axes a and b), and with such faces as were in front now inverted at the back. The proportions of the two individuals are such as were actually shown by the twin taken as the example, and the same applies to the dimensions of the narrow faces forming the grooves along the front and the front part of each of the two ends. To render everything absolutely clear, both the indices and the form letters are stated in Fig. 421, while in Fig. 419 the form letters for both front and back faces are given, those of the latter being dotted.

(2) **When Two Axes have Different Directions in the Two Individuals.**—Suppose the axis b to be the one to remain unchanged in direction, as shown by the parallel lines BB' and $\bar{B}\bar{B}'$ in Fig. 422. Suppose further that AA' and CC' are the a and c -axes of the first individual, and that the twin plane makes the intercepts C and A' on these axes, while parallel to the b -axis. We have to draw the new axes CC' and $A'\bar{A}$ of the second individual at similar angles θ and ϕ on the other side of the twin plane $A'C$, as if by reflection across the latter; that is, we have to draw them so that the two angles marked θ are the perspective representations of really equal angles, and the two marked ϕ also similarly equal, in other words, the new c -axis is to be inclined $(180^\circ - 2\theta)$ to the old c -axis, and the new a -axis $(180^\circ - 2\phi)$ to the old a -axis. This we do in each case by a repetition of the same process as in case (1), that is, as for the construction of the inclined axis of a monoclinic crystal. Thus, for instance, at C we draw the cubic axes a and c , as shown dotted in Fig. 422, and we take along a in the backward direction the length $a \cdot \sin(180^\circ - 2\theta)$ which is the same as $a \cdot \sin 2\theta$, and along the vertical axis c we take the length $c \cdot \cos(180^\circ - 2\theta)$. We then complete the parallelogram as usual, and join those two opposite corners of which C is one, and produce this diagonal for some distance backwards; this line CC' is then the direction of the new axis c . A similar process at A' using the angle ϕ instead of θ gives $A'O'$, the direction of the new axis a , and through the point of intersection O' of the two new axes a and c thus drawn the third axis b is drawn parallel to BB' . These three new axes a , b , and c thus drawn are shown in Fig. 422 in broken-and-dotted lines to distinguish them from those of the first individual AA' , BB' , CC' . The proper length of the b -axis is afforded by drawing parallels to OO' through B and B' ,

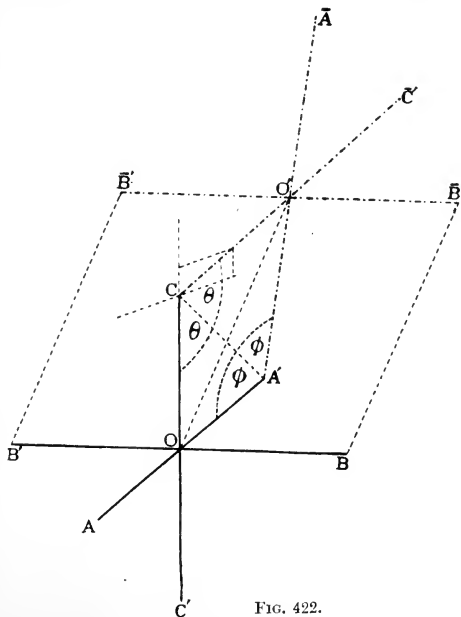


FIG. 422.

meeting the new axis b at \bar{B} and \bar{B}' , which thus mark the unit lengths on each side of the centre O' . The upper and back halves of the new a and c axes are likewise made equal to the lower and front halves.

(3) **When all Three Axes have Different Directions in the Two Individuals.**—The procedure in this, the general, case consists of three operations.

We first draw in the usual manner the axial system for one of the two individuals, as if it were an ordinary single crystal.

Secondly, we draw the representation of the twin plane (hkl), by joining its parametral points H, K, L on the axes thus drawn.

Thirdly, the axial system of the second individual is to be drawn, in such a manner that H, K, L are also the ends of the parameters of the twin plane with respect to these new axes, for the twin plane is common to the two individuals. Generally applicable rules for accomplishing this third operation are given in the following paragraphs labelled (a) to (f), and they are illustrated by a typical example the construction for which is worked out in Fig. 423.

(a) We begin by finding the position of the twin axis with respect to the first,

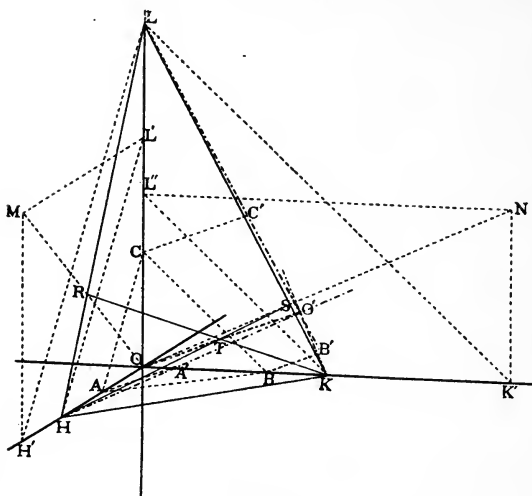


FIG. 423.

the ordinary, axial system, that is, we find the point (T in Fig. 423) at which the normal drawn from the centre O of the first axial system OA, OB, OC to the twin face (plane of twinning) meets that face.

The construction for finding this normal OT which the author finds most convenient is that recommended by Prof. von Groth. We first carry out the second operation of joining the parameters of the twin face on the three ordinary axes; in the example illustrated in Fig. 423 the points A, B , and C represent as usual the unit lengths, and the parameters of the twin

plane are H, K, L , and are respectively in the case chosen $2a, \frac{3}{2}b$, and $3c$, that is, the twin plane HKL is the face (342) , the plane ABC being (111) .

We draw parallels to AC from H and L , meeting the axes c and a respectively at L' and H' ; a similar pair of parallels to BC are next drawn from K and L , meeting axes c and b at L'' and K' . A pair of parallelograms are then completed by drawing parallels to OH and OL through L' and H' , and to OK and OL through L'' and K' ; and the two diagonals OM and ON are also drawn. These diagonals cut HL and KL , two of the continuous lines representing the twin plane, at R and S respectively; and by joining R to K and S to H we arrive at the desired point T where the normal to the twin plane from O meets that plane, for T is the point of intersection of RK and SH .

(b) The normal OT to the twin plane is produced to O' equidistant on the other side of the twin face, that is, so that $O'T = OT$. This point O' is the centre or origin of the new axial system of the second individual.

(c) The new axes themselves are at once given by joining O' to the parametral points H, K, L on the first system, as is done in broken-and-dotted lines in Fig. 423;

for the plane HKL is common to the two systems of axes, and the same points must be the ends of the two sets of parameters.

(d) The primary parametral lengths A' , B' , C' of the second system are obtained by drawing lines to meet the new axes, from the parametral points A, B, C of (111) on the first system, parallel to the normal OO' .

(e) The face determined by $A'B'C'$ will have opposite index signs to ABC, as the imaginary rotation for 180° will have brought it into the opposite octant. That is, faces appearing in the same (analogous) octant in the two individuals of the twin have opposite signs attached to their indices, as the second crystal is situated as if it had been rotated 180° with respect to the first crystal.

(f) The two individual crystals are to be drawn about their respective axial systems at such positions with respect to each other as best represent the observed facts on the actual twin crystal. The drawings of the two crystals can, of course, be transferred anywhere so long as they and their axes are moved parallel to themselves. In a perfectly interpenetrating pair of crystals the two origins will be identical, while in the case of a purely juxtaposition twin the two individuals and the centres of their axial systems will be separated at their maximum distance. Between these two extremes all degrees of interpenetration will be met with, and the two crystals and their axial centres must be so placed as to faithfully represent the observed facts.

With these general hints on the drawing of twins the subject of composite crystals may well be left. After having once constructed the two axial systems, the only real difficulty, in accordance with the particular rules among the foregoing which apply to the case under investigation—and most cases will be found to be capable of being tackled by use of one or other of these rules—no further difficulty will be presented; for the process of drawing the two individual crystals themselves will be the same in all cases, and the example, ammonium selenate, given under the first case will suffice as an illustration for all three cases. The contour of the plane of junction of the two individuals is automatically given by the points of meeting of the lines representing analogous and opposite edges on the two drawings. Finally, in investigating a twin, an open mind must above all things be maintained, and the whole structure thoroughly gone into from first principles. For the phenomena of twinning, and of composite crystallisation in general, are so varied, that something new may be met with at any moment, and the observed facts must be most carefully observed and faithfully recorded. It is hoped, however, that the rules and methods described in the foregoing pages may be of great assistance and guidance in conducting the inquiry concerning this admittedly difficult part of a crystallographic investigation.

CHAPTER XXVII

PLANES OF CLEAVAGE AND GLIDING.

AN exceedingly important property possessed by most crystals, and which is not only developed in a manner characteristic of the particular substance but is also of the greatest value as an indication of internal structure, is that of cleavage, the capability of being more or less readily split along definite plane directions, which are usually parallel to certain of the principal faces. If these faces belong to the same form, that is, are of equal value as regards the crystallographic symmetry, the facility of cleavage is equal parallel to all of them, but if there be cleavage developed parallel to the faces of more than one form, then the facility is different for each form. The degrees of facility of cleavage are usually indicated by the terms "perfect" and "imperfect," corresponding respectively to facile splitting and to cleaving with some difficulty. One of the best modes of testing for cleavage is to arrange a pen-knife blade so that its sharpened edge is pressed along the right direction of the firmly held crystal, and then to strike its blunt upper edge smartly with a light hammer. When the "knack" of properly carrying out this operation has once been learnt, the crystal may be cleanly split into two fragments without any crushing or other destructive effect, even if the crystal be one of a fairly soft substance.

A perfect cleavage is quite a special kind of fracture, and is sharply distinguished from the various forms of ordinary fractures, such as the "conchoidal" fracture of glass and ice and of the very imperfectly cleavable quartz, the "hackly" or ragged fracture of native metals, the "uneven" fracture of tourmaline, and the "splintery" fracture of fibrous hæmatite (iron oxide Fe_2O_3). For a perfect cleavage surface is as truly plane and exquisitely polished as the best natural face, and this fact is of the utmost practical importance, for it affords us the means of determining the position of the cleavage plane or planes, with respect to the crystal faces and the crystal symmetry, on the goniometer itself. An excellent image of the signal-slit is afforded by each of the two cleaved surfaces torn asunder from each other, and when a face of the natural crystal, parallel to which the cleavage has been effected, is present on either or both of the fragments into which the crystal has been cleaved, the image from the cleavage surface will be found to be situated at $180^\circ 0'$ from the image reflected from that natural face, in

the case of either fragment. Two or three minutes of aberration from this theoretical position are all that are ever observed with good crystals, the limits of error being the same as for goniometrically perfect natural faces.

The two best-known examples of highly perfect cleavage are: (1) mica, of which there are several varieties; muscovite mica is a silicate of aluminium and potassium, $\text{KH}_2\text{Al}_3(\text{SiO}_4)_3$; lepidolite is a similar silicate of aluminium and lithium, and biotite is one of aluminium, iron and magnesium; (2) gypsum (selenite), hydrated sulphate of lime $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$.

The thin cleavage sheets of transparent colourless muscovite mica are now so well known as every-day articles of commerce, being so useful for lamp-shades and similar purposes on account of their non-inflammability and poor conduction of heat, that no further description is necessary. They may readily be obtained thinner than sheets of paper, and the films are flexible; they thus exhibit cleavage in its most highly developed form, although no trace of it is visible in a well-formed crystal plate, unless the edges have been weathered or injured, as is, unfortunately, however, almost always the case with mica. A particularly

good tabular crystal of muscovite mica is shown in Fig. 424, an apparently hexagonal plate. As a matter of fact the simulation of hexagonal symmetry is as extraordinary as in the case of potassium sulphate, and here the real symmetry is not even rhombic, as in that case, but monoclinic. Moreover, the angle β between the vertical and the inclined axis is $90^\circ 0'$ within the limits of error of measurement, which are not easy to fix in the case of mica on account of the difficulty of finding crystals with perfect edge-faces to the plates; hence, mica was long thought to be rhombic. The forms shown are the vastly predominating basal pinakoid $c = \{001\}$, to which the plates are parallel and which is the direction of cleavage, the clino-pinakoid $b = \{010\}$, the prism $m = \{221\}$, and also narrower faces of the negative primary hemi-pyramid $o' = \{111\}$. The primary prism $\{110\}$, the faces of which are inclined at approximately 60° to each other and to the clino-pinakoid faces, and at 90° to the c faces, is not usually developed, although $\{110\}$ is the twin plane of the micas. If present its faces would replace the edges mo' . The downward perspective of the horizontal axis a in Fig. 424 is diagrammatically somewhat exaggerated, in order to emphasise the fact that this axis does not emerge in the centre of the mm edges. Its true direction is parallel to the cb edges.

That the symmetry is really monoclinic was shown by Hintze and Tschermak, who discovered that the ellipsoid, which expresses the values of the refractive index and the optical characters in general in different directions of the crystal, was not situated with its principal axes identical in direction with all three crystallographic axes, as would be the case if the

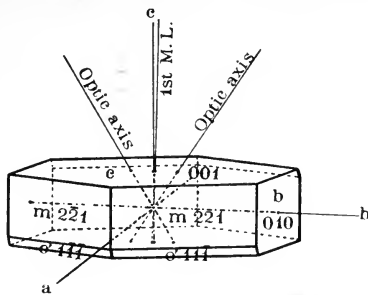


FIG. 424.—Crystal of Muscovite Mica.

crystal were rhombic, but that it was rotated slightly about the *b*-axis, which was common, and so that the axis nearly coincident with the vertical crystallographic axis was inclined backwards from that axis at an angle of $1\frac{1}{2}^{\circ}$ to 2° . This fact can readily be confirmed by studying the optic axial interference figure which is afforded when a plate of muscovite mica is examined in the polariscope, between crossed Nicols and in convergent light. The method will be fully gone into in subsequent chapters, and it will suffice to say here that the two optic axes are indicated by the vertices of dark hyperbolic "brushes," arranged like two letters V with rounded apices placed horizontally on each side of the centre of the field thus: $> <$, and which are sharply defined curves at the vertices themselves, to which two vertical lines are tangential, but which spread out as the curve in each case tails away from the centre above and below the horizontal diameter of the field; the vertex marking the optic axis in each case is surrounded by a series of spectrum-coloured interference rings (really curves termed "lemniscates") when observed in ordinary white light, and which become better defined dark rings in homogeneous (monochromatic) light, and which are looped together as figure-of-eight ∞ lemniscates after the first few rings, all passing eventually into enveloping ellipse-like curves (see Fig. 558 on Plate I. facing page 670). This beautiful figure is the one seen when the crystal plate is arranged so that the line joining the optic axes is at 45° to the crossed rectangular directions of the Nicols. When the plate is rotated 45° so as to bring the optic axial line parallel to one of the latter, the hyperbolic brushes rotate about their respective vertices (the optic axes) and become a dark cross, one narrower bar of which joins and passes through the axes while the other broader one crosses this narrower one rectangularly half-way between the two optic axes (see Fig. 557 on Plate I.). The lemniscates, however, retain their general appearance. It will be obvious that the plate necessary to afford this interference figure symmetrically placed to the centre of the field must be one either naturally developed, or artificially cut out of the crystal, perpendicular to the bisectrix of the angle (the acute one) between the optic axes. Now this bisectrix is one of the principal axes of the optical ellipsoid, and in a rhombic crystal it would be identical with one of the three crystallographic axes, for these three latter are identical in direction with all three axes of the ellipsoid.

In the interesting case of muscovite mica, it is a fortunate fact for the ease of observation of the optic axial interference figure, that the acute bisectrix is almost perpendicular to the direction of cleavage, so that a cleavage plate of the right thickness (that of a piece of card) shows the figure apparently symmetrical to the centre of the field, and recourse to the cutting of a plate is unnecessary. Obviously also a natural tabular crystal of the same thickness shows it equally well. For many years it was thought to be absolutely symmetrical to the centre, but the work of Hintze and Tschermak made it certain that the bisectrix is inclined at an angle not exceeding 2° backwards from the normal to the cleavage plate. Hence, the crystal must be monoclinic and not rhombic, and is therefore to be described as monoclinic with a strongly pseudo-hexagonal habit.

In order to render the symmetry of muscovite mica and the relations of the optic axes to it clear, the directions of the crystallographic axes a , b , and c are indicated in Fig. 424, as are also those of the two optic axes and their bisectrix, which latter is also called the first median line (and is therefore marked 1st M.L. in the figure), to distinguish it from the bisectrix of the obtuse angle (the supplement of the acute), which is called the second median line. It will be noted that the optic axes are separated in a plane at right angles to the plane of monoclinic symmetry, the latter being $b = (010)$ containing the axes a and c , which in this remarkable case are inclined at an angle β which is $90^\circ 0'$ within the error of possible determination, although it is probable that if that error could be reduced, by the discovery of crystals with perfect edge-faces, it would be found to differ by a few minutes from 90° .

It is interesting also that the symmetry plane $b = (010)$ is indicated by one of the three lines inclined at 120° , making up a star of six rays inclined at 60° , which form the well-known "percussion figure" of mica, which is produced by indenting a plate of mica with a sharply pointed punch, struck smartly by a light hammer; moreover, the particular ray of the star parallel to the symmetry plane is usually much more strongly developed than the others parallel to the m -faces $\{221\}$. So that both the percussion figure and the optic-axial interference figure at once enable the direction of the symmetry plane to be ascertained in a cleavage plate of mica, when no natural edges are present on the margin of the plate.

The separation of the optic axes in muscovite mica varies from 60° to 75° as seen in air, but the true angle within the crystal is smaller, the difference between the true and apparent angle being due to the different refractive indices of the crystal and air. The exact angle differs also for different wave-lengths of light, being greatest for red light and least for violet light. An angle in air of 60° - 75° is one that is very convenient in size for the demonstration of the rings and brushes round the optic axes of biaxial crystals, as the whole figure can be seen at once in most ordinary wide-angle polariscopes. Hence mica is a very favourite substance for the demonstration of this very beautiful optical phenomenon, and all this is due to the development of such perfect cleavage in a direction which is so fortunate for the display of the phenomenon.

In some of the other varieties of mica, biotite for instance, the separation of the optic axes is so small that they were formerly thought to be uniaxial; indeed biotite was named after Biot, who described it so long ago as 1816 as uniaxial. In all cases, however, the apparently single axis, really the bisectrix of a very small angle, is never quite perpendicular to the cleavage plate, that is, to the basal pinakoid; the axes, moreover, are separated at this minute angle in the symmetry plane and not perpendicular to it as in the case of muscovite, and the angle for red light is the least instead of the greatest. Extreme cases have been described in which the deviation from the normal to the cleavage plane has been as much as 8° in the cases of these so-called

uniaxial micas. Lastly concerning mica, the first median line moves in the symmetry plane on raising the temperature, another proof that the symmetry is monoclinic and not rhombic; for if it were the latter, the bisectrix would be identical with the crystallographic axes for all temperatures, as symmetry is independent of temperature.

Gypsum (selenite), $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$, a typical crystal of which is shown in Fig. 397 in the last chapter, is also an instance of a monoclinic mineral having a perfect cleavage, which in this case is parallel to the symmetry plane $b = (010)$. Films of great tenuity can be obtained as in the case of mica, and such selenite films are largely used for optical purposes, being very convenient for the production of interference colour-tints in plane-polarised parallel light, and for determining the sign of the double refraction of such crystals as are birefringent. These uses will be fully explained in a subsequent chapter (XLI., pages 649 to 656), together with the reason for the succession of colour-tints which selenite films display in polarised light according to their thickness, and for the distinction between tints of the first, second, third and higher orders. The symmetry plane, besides being the plane of cleavage, also contains the two optic axes, separated at an apparent angle in air of about 100° . Hence the optic axial interference rings and brushes are not seen on looking through a cleavage plate of selenite placed in the convergent light polariscope, a section-plate requiring to be cut at right angles to the symmetry plane and perpendicular to the acute bisectrix of the optic axial angle in order to produce them; this is an operation of some difficulty, as it is across the direction of the very facile cleavage, instead of being parallel to it as in the case of mica. Selenite will be taken as a special example of optic-axial-angle phenomena later on, as it is one of the most remarkable substances known to us from this point of view, showing among other things extraordinary sensitiveness to change of temperature.

Many other minerals cleave so readily that they are frequently found in the form of their cleavage solid. This in the cases of mica and selenite would simply be a plate, as the cleavage is parallel to a form of the monoclinic system which in each case consists of merely a pair of parallel faces, the clino-pinakoid $b = \{010\}$ in the case of selenite, and the basal pinakoid $c = \{001\}$ in the case of mica. But in the case of calcite, for instance, the cleavage occurs parallel to the faces of the primary rhombohedron $r = \{100\}$ (which is, in fact, taken as such because it is so obviously important with regard to the internal structure), and the mineral is very commonly found in such cleavage rhombohedra; whereas a complete natural crystal, which could be distinguished by the presence of small faces of other forms modifying its edges, such, for instance, as the crystal shown in Fig. 329 on page 346, is rarely found of the primary rhombohedral habit, the other rhombohedron of calcite $e = \{110\}$, Fig. 327, being on the other hand very common, as is also the scalenohedron $v = \{201\}$, Fig. 320, well seen in the form of calcite known as "dog-tooth" spar. The cleavage rhomb of calcite is an excellent illustration of the fact that cleavage occurs equally well along all the planes belonging to a form, in this case three.

Cleavage directions are obviously those of minimum cohesion, and the laws governing the symmetry of crystals are applicable as much to the force of cohesion exerted in different directions within the crystal structure, between the particles composing that structure, as to the exterior facial planes; for both are equally dependent on the structural arrangement of the particles, that is, on the nature of the space-lattice formed by their representative points or centres of gravity.

Another instance is the octahedral cleavage of the diamond, the cleavage occurring absolutely equally along the four planes of the regular octahedron $o = \{111\}$, the eight faces of which obviously comprise only four different planes, for opposite pairs of faces are parallel; and each cleavage surface would be found to be situated at precisely $180^\circ 0'$ to the face of the octahedron to which it is supposed to be parallel, if that face were developed in good measurable condition (and not curved, as is so often the case with the diamond) on the same fragment.

The little cubes into which galena, sulphide of lead PbS , so readily cleaves, and which may be reduced down to microscopic dimensions, afford another excellent instance of three planes of cleavage being equally developed parallel to the whole of the faces (here six, falling into three pairs of parallel faces) of a form, the cube $a = \{100\}$.

From the foregoing it will have become clear that cleavage is not a mere tendency to fracture with production of two more or less plane fracture-surfaces, one on each of the two separated fragments, and along an approximately definite direction. Cleavage is much more than this, namely, the facility for splitting along an absolutely true plane, having an orientation within the crystal definitely fixed to one or two minutes of arc, and which definite direction is identical with that of an important face of low indices, very often a primary one, to within the same minute limits of accuracy as natural faces exhibit when grown undisturbed; and the plane surfaces of fracture are endowed with the same high degree of natural polish as the best formed faces, and afford equally sharp and brilliant images of the signal-slit of the goniometer. If more than one face or pair of parallel faces constitute a form, as in the systems of symmetry higher than the triclinic, as many such cleavage directions are developed as there are planes,—each of which may correspond to two natural faces, one on each side of the crystal—constituting the form in accordance with the degree of symmetry exhibited by the crystal. Moreover, compared with the facility for splitting along the cleavage plane or planes, the crystal usually offers relatively enormous resistance to fracture along any other direction, and when it does occur the fracture is very irregular, and indeed is very frequently stepped or zigzagged, the steps or zigzags being composed of an alternation of irregular unreflective bits of crystal surface and of little true planes having the direction of the nearest cleavage plane characteristic of the crystalline substance, and which is, of course, oblique to the direction in which it was attempted to cleave the crystal. A crystal of a substance endowed with a good cleavage may be smashed up to small fragments or even to a coarse powder, and

these fragments will mostly display cleavage surfaces yielding good signal-images.

The cleavage cube of galena, the cleavage octahedron of the diamond, and the cleavage rhomb of calcite, are all obviously closed forms, so that a cleavage solid results entirely from one "cleavage," using the term in its fullest significance. But in the systems of lower symmetry, it may require three different "cleavages" (forms) to produce a closed solid, each corresponding to a form of two parallel faces and therefore of one plane. All three, however, may not be developed, and even if they are, their degrees of facility will be different. In many cases not more than two of them are developed, thus giving rise to an open prism by cleavage, terminated by natural faces. The same result is produced when the symmetry is such that a single form, parallel to which there is cleavage, consists of four faces, two pairs of parallel faces, so that here one cleavage would produce the prism. In a large number of cases, however, only one cleavage is developed, and parallel to a form comprising only a pair of parallel faces, giving rise to a plate by full exercise of the cleavage, having natural faces for its edges. Such a case is that of mica.

A concrete case of the intermediate order is afforded by our typical rhombic crystal of potassium sulphate, which possesses two directions of cleavage, a fairly perfect one in the direction of the brachy-pinakoid $b = \{010\}$ and a less perfect one parallel to the basal pinakoid $c = \{001\}$, both being single-plane forms composed of two parallel faces in accordance with rhombic symmetry.

The cleavage form is not necessarily that of the crystal. Thus fluor-spar, calcium fluoride CaF_2 , is frequently found in simple cubes, but its perfect cleavage is octahedral, and the corners of the cube may be readily split off until the crystal resembles a combination of the cube and octahedron in equipose; and the process can, of course, go further until nothing but a simple cleavage octahedron remains.

Connection between Cleavage and Structure.—In Chapter IX. it was shown that a crystal is a homogeneous structure composed of an assemblage of structural units, the chemical atoms, which may be considered as represented geometrically by analogous points within them, their centres of gravity for instance, each of which, as we know so little yet about the chemical atom, may be considered as standing for the sphere of influence dominated by the atom. These structural-unit-points were shown in the general case to make up an interpenetrant number of precisely similar space-lattices, the number corresponding to that of the atoms composing a molecule of the crystallised substance. It was also shown that it is the form of the space-lattice which determines the system of crystalline symmetry, the elements and interfacial angles, and the concordance with the law of rational indices, while the particular class of symmetry within the system is determined by the nature of the interpenetrating combined assemblage, that is, by the arrangement of the atoms forming the chemical molecule. To make this somewhat intricate but very essential point quite clear, any particular atom (or

its representative point) of the molecule may be considered as the point representing the whole molecule, so long as in every molecule of the assemblage the analogous atom is also chosen, and these analogous atoms form the space-lattice, so that there are as many similar space-lattices as there are atoms in the chemical molecule; the intermolecular arrangement of all the adjacent atoms in the assemblage which together correspond to the chemical formula of the substance, and which molecular arrangement is repeated throughout the whole assembly as often as there are points in the space-lattice, determines the details of the symmetry, that is, the particular class. A number of more complicated arrangements were also shown to exist, involving the introduction of mirror-image repetition, but the majority of arrangements are of this general character.

Thus we arrive at the important conclusion that the molecules—considered as the structural units and as each, therefore, corresponding to one point of the point system—are built up to form the crystal edifice on the plan of a space-lattice, and the architecture of this decides the crystallographic system; while the stereographic arrangement of the atoms in the molecule, that is, the architectural plan of the internal structure of the molecule itself, fixes the particular class of the 32 possible classes to which the crystal shall conform. It must thus be clearly understood that it is neither the atom nor the molecule which determines the class, for as we have seen, if we take any atom (each atom in turn if desired) of the molecule, this, together with analogous atoms taken from all the molecules, only gives us again the space-lattice and crystal system; and the same result is again attained by considering not a particular atom but any point such as the centre of gravity as representative of the whole molecule corresponding to the chemical formula. It is on the contrary the **stereographic** arrangement, the geometry of the distribution in space, of the whole of the atoms composing the molecule, or what comes to the same thing, the stereographic mode of interpenetration of the atomic space-lattices, which determines the class.

Now in a space-lattice all parallel planes are of identical structure, and are separated from each other at equal intervals. But the different sets of parallel planes, at different inclinations to each other, in the space-lattice, differ in the number of points, representing the molecules or a particular atom in each of the molecules, which they contain in a given area; that is, some planes are more thickly strewn with points than others. Moreover, those in which the points are most thickly congregated are furthest separated from the parallel planes nearest to them. Now all the evidence up to the present accumulated goes to show that the commonest faces characteristic of a crystallised substance are those parallel to planes most densely strewn with points, that is, with structural units, be they considered as whole molecules or specific atoms; also that the cleavage planes correspond to those of the space-lattice which are the most widely separated from each other. For obviously the cohesion should be a minimum between

particles at the greatest distance apart, and a maximum between those nearest together, in accordance with the universal law of inverse squares applying to forces of this description between particles. Hence, the fact that the directions of cleavage are usually those of the commonly developed crystal forms is in perfect agreement with the theory of crystal structure advanced in Chapter IX. It also now becomes perfectly clear why cleavage can only occur along certain specific plane directions identical with forms of low indices, for it can only occur along a plane or planes of the space-lattice. Further reference to this subject will be made in Chapter XXXI., when considering the 14 space-lattices, the types of cleavage characteristic of each of which will be described.

Practical Mode of investigating Cleavage.—The mode of testing a new substance for cleavage has been already incidentally referred to. It consists in attempting to split a crystal of not too small a size and which is rigidly supported or held during the operation,—say by placing it with a broad face, supposing one to be developed, perpendicular to the direction to be tested for cleavage, on a small block of hard wood resting in turn on the steady working bench or table,—by means of a pen-knife blade used as a miniature wedge, laid with its edge horizontal and firmly pressed downwards on the crystal without cutting, and along the direction it is desired to test. The upper blunt edge of the blade is then struck a smart blow with a very light hammer, when, if the direction be really one of perfect cleavage, the crystal will fall away on each side into two halves, with the minimum of actual cutting by the knife along the edge of application, the minute fissure made affording the blade-wedge the opportunity for pressing asunder the two parts of the crystal along the cleavage plane; the vastly preponderating portions of the two cleaved surfaces do not touch the knife at all, and are thus quite uninjured by the operation when skilfully performed. The two fragments should then be placed in succession on the goniometer, and the signal-images reflected from the cleavage surfaces examined. If sharp and clear single images are observed to be afforded, there can be little doubt that a real cleavage direction has been discovered; and if several repetitions of the operation on different crystals yield the same result, any lingering trace of doubt disappears. The position of the signal-image should in each case be determined with respect to that of the face parallel to which it was endeavoured to produce a cleavage, and when the cleavage direction is what it was supposed to be, it will be found that the angular difference between the readings for the cleavage surface and this natural face on the same fragment will be within a couple of minutes or so of $180^{\circ} 0'$, the deviation depending probably more on the character of the natural face than on that of the cleaved one. No true investigator will be satisfied, of course, with a single experiment and measurement; all the crystals of a substance should show the property in common, and therefore specimens from several different crops should be tested, say four or five. If all unite in giving a like result, the cleavage direction may be regarded as well established. The facility

of the cleavage should, needless to say, be noted, in order that it may be described, for the benefit of future observers as well as for reasons connected with a true appreciation of the structure, as "perfect" or "imperfect." The character of the signal-images will assist materially in deciding whether it is to be described as the one or the other, an imperfect cleavage rarely yielding such sharp single images as a perfect one. Moreover, when the direction corresponds to a face of a crystallographic form comprising more than one pair of parallel faces, the cleavage in both or all three planes of the form ought to be tested on one of the larger crystals available, in order to confirm the fact that cleavage occurs along all the planes of the form.

Glide-Planes.

Crystals which are not high in the scale of hardness often show another property dependent on the internal structure, somewhat akin to cleavage, which involves a permanent deformation of the crystal, namely, that of possessing planes of gliding of the structural particles over one another, under the influence of a moderate amount of force. These planes are most frequently, but not always, identical with the planes of cleavage. They were discovered by Reusch, who invoked them by pressing the blunt point of a conical steel punch on the crystal surface and striking it with a light hammer, just as for the production of the percussion figure of mica. The punch or cone, however, requires to be bluntly and not sharply pointed in order to produce these pressure figures, and the crystal is best laid on an elastic surface and not on a rigid one. In the case of mica, which shows the phenomenon well, this "pressure figure" consists of a six-rayed star similar to the percussion figure, but the rays are perpendicular and not parallel to the edges of the plate and bisect the angles between the rays of the percussion star. They indicate planes of gliding parallel to the faces of monoclinic pyramid forms.

Rock-salt, sodium chloride NaCl , which crystallises in cubes and belongs to class 32 of highest crystal symmetry, affords a still better example of a crystallised substance possessing planes of gliding. Again they are not parallel to the cleavage planes, which are those of the cube, $\{100\}$, but are on the contrary parallel to the six planes of the rhombic dodecahedron $\{110\}$.

By far the most interesting case, however, is that of calcite, in which the phenomenon was discovered by Reusch, and a most remarkable method of developing the sliding movement in which was subsequently described by Baumhauer, which is illustrated in Figs. 425 and 426. A cleavage rhomb of calcite is taken, such as that represented in Fig. 425, and laid so that the position of the trigonal axis, usually vertical, is as indicated in the figure, one set of polar edges, the longer ones of this particular rhomb, being brought horizontal and so that the plane *edh* is also horizontal. The crystal should be held firmly in this position in a vice with soft wooden or cork-lined jaws. A knife blade

is then pressed downwards and slightly sideways towards c , and with the edge horizontal, at about the middle point a of the upper edge of the rhomb. It is then observed that layer after layer of crystal substance parallel to the plane edh moves away in the direction of the sideward pressure, and if the force on the knife is adequate to cause it to reach the centre of the crystal, an apparently perfect reflection twin is produced, for the moved portion $a'bc'de$ (Fig. 426) is the exact counterpart of the portion of the original crystal which lies below it, as if the latter were reflected at the plane edh . This latter plane is parallel to one of the faces of the other common rhombohedron of calcite $e = \{110\}$, which replaces the polar edges of the primary one $r = \{100\}$, to which the cleavage planes are parallel, shown in Fig. 425. The glide-planes of calcite are, therefore, parallel to the faces of $e = \{110\}$, and the term secondary twinning has been given to this kind of twin formation, to distinguish it from ordinary natural twin growth. The glide-planes are thus again distinct from the cleavage planes. In the

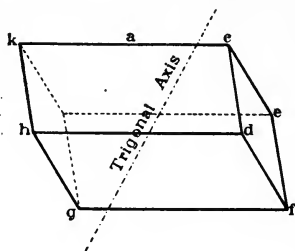


FIG. 425.—Cleavage Rhomb of Calcite arranged for Glide-Plane Experiment.

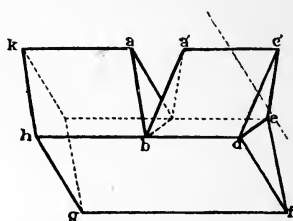


FIG. 426.—Result of Glide-Plane Experiment with Calcite Rhomb.

cases of many other crystals, however, they are identical with the cleavage planes.

It is very interesting to enquire what has happened during this apparently simple push-movement of the particles along glide-planes. If we investigate the physical properties of the newly placed part $a'bc'de$ of the calcite crystal (Fig. 426), we shall find that the position of the single optic axis (the crystal being trigonal and therefore optically uniaxial) has changed with the movement. It was formerly situated as shown in the broken-and-dotted line in Fig. 425, the optic axis being identical with the trigonal axis, but it is now rotated over in the same vertical plane $cfgk$ to the position shown by the broken-and-dotted line in Fig. 426. In other words, the corner c' is now no longer a polar solid angle but an equatorial one, of the same character as that at f . The actual amount of rotation of the optic axis from one position to the other is $52\frac{1}{2}^\circ$, while the angular movement of the face cde is 38° . In order to account for this there must not only be a transference but a rotation of the particles, probably corresponding in amount to the angular change of the optic axis. Voigt has further shown from determinations of the elasticity of calcite in different directions, that the glide-plane corresponds to one of the two mutually perpendicular

planes of minimum resistance to elastic movement of the crystal particles; but this is not sufficient to account altogether for the glide-plane, for the second plane of minimum resistance at right angles to the first is not a glide-plane. Hence there can be no doubt that there is another factor concerned in the operation, and that rotation of the particles occurs as well as the elastic movement of transference. It is highly interesting in this connection that Reusch observed that in order to get the gliding of a particular lamella to occur it was necessary to overstep a certain limit of force, and that if the force stopped short of this the lamella moved back again by its purely elastic force, no permanent deformation being suffered. Hence the weaker force can only have brought about the elastic movement along this plane of minimum resistance, but when the critical force was exceeded rotation of the particles occurred, and no return was then possible, the deformation being a permanent one.

CHAPTER XXVIII

GONIOMETRY AT HIGHER TEMPERATURES.

THE effect of change of temperature on crystals is to bring about a homogeneous deformation, not permanent in the sense used in the last chapter in connection with the movement along glide-planes, but only persisting as long as the temperature remains constant at the new level. It will be clearly apparent that change of temperature must affect the forces which hold the structural units together, so that not only the exterior shape of a crystal but also the homogeneous internal arrangement of the structural units,—be they considered as the chemical atoms or molecules or Sohnckian points representing them,—must be a function of the temperature. This homogeneous deformation is of such a nature, however, that the symmetry of the crystal, the mutual relations of the forms it displays, and the law of rational indices which connects them, are all quite independent of it. In the most general case of a triclinic crystal the dilatation induced by change of temperature varies with the direction within the crystal, and there may even be contraction along certain directions while expansion occurs along others. The net effect, however, is usually for an increase of volume to accompany rise of temperature and a diminution in bulk to follow cooling. The exact laws which have been found to govern the dilatation of crystals belonging to the various systems of symmetry will be discussed in a subsequent chapter (LIII), but it will be obvious that they will in general lead to alteration of the crystal angles, and we must therefore be provided in the laboratory with a means of measuring such changes of angular magnitude in order to complete our goniometrical study of a crystallised substance. They are always relatively very small within the limits of the temperature intervals corresponding to the lives of crystals, that is, within which they may be preserved intact as solid chemical substances, without any trace of fusion, volatilization, or chemical decomposition. The angular changes have never yet been found to exceed a very few minutes of arc.

For the measurement of crystal angles at temperatures higher than the ordinary it is necessary to employ a goniometer constructed on a somewhat larger scale than the excellent instruments for ordinary temperature work described in Chapter III. For although sometimes,

as in the case of gypsum (selenite), the changes brought about in the crystal angles by a rise of temperature of 100°C . amount to four or five minutes, such cases are exceptional and the changes rarely reach a single minute, and often indeed only amount to a few seconds. Hence the goniometer which is employed must read to seconds. The No. 2a Fuess goniometer and the similar Troughton and Simms horizontal-circle goniometer described in Chapter III. read to half minutes, a fineness which is quite adequate for measurements at the ordinary temperature, the readings being usually only required to the nearest whole minute. For the perfection of formation of the crystal faces is only very rarely indeed of higher degree than to permit those angles which are of equal value as regards the symmetry to agree among themselves to within one or two minutes, and the crystallographer considers himself fortunate when he obtains crystals exhibiting this perfection of facial planeness. Hence a larger instrument reading to seconds is not only of no more use in ordinary crystal measurements than the two excellent goniometers referred to, but is positively a drawback, as it must of necessity be larger in order to include a larger and more finely divided circle, and therefore heavier, less handy, and more tiring to manipulate, besides absorbing more time in an equal number of measurements. But for work at higher temperatures Fuess constructs an admirable larger goniometer, No. 1a, which is also an excellent instrument for the measurement of refractive indices by the method of the 60° prism to be described later in Chapter XLV. It is shown in Fig. 427 on the scale of one-sixth its size, and a section of its axial system is added in Fig. 428.

The fixed outer cylinder *a*, in which the axes of the rotatable circle and the adjustable crystal holder successively rotate, is rigidly supported by a strong tripod *b* furnished with levelling screws. Its boring is cylindrical in its central part but conical above and below, in order to support the circle axis *c* within it. The circle in this instrument is a simple uncovered circular plate *d*, adequately stout to resist deformation, and carried at the head of the axis *c*, which is capable of rotation in the cylinder *a* by means of the large milled head *e* or the ring *f* carried rigidly below it, which is added for the greater convenience of rotation by the hand. The circle may be clamped to the outer fixed cylinder and tripod by means of the screw *g*, and finely adjusted by an adjacent screw *h* at right angles to *g* in a manner which will be clear from the figure. The axis *c* is hollow within, with a boring cylindrical for its greater part, but conical at the top, and within it there rotates a second axis *i* shaped to fit the boring and capable of being rotated from its lower termination by the milled head *j*. This axis carries the crystal centring and adjusting apparatus, of like construction to that of the goniometer No. 2a but larger; it is not carried directly by *i*, but by a solid steel cylindrical innermost axis *k* having a screw thread cut on its lower portion, so that it may be raised or lowered by means of the milled head *l* of a flanged nut, in order to bring the crystal carried at the head of the adjusting apparatus to the exact level of the common optic axis of the telescope and collimator, the steel axis being keyed to prevent rotation otherwise than along with *i*. This latter axis *i* can be clamped to the circle axis by the screw *m*, and finely adjusted to the circle by means of the adjacent fine adjustment screw *n* at right angles to *m*.

The collimator and telescope are supported similarly and independently about the exterior of the fixed outer axis *a*, in such a manner that both are free to move round the circle and to be fixed and finely adjusted in any desired position with respect either

to each other or to the circle. Indeed the two optical tubes *o* and *p* are similar in all respects, so that either may be used at will as collimator or as telescope, an eyepiece *q* or a signal-slit *r* being inserted accordingly at the outer end of the tube, the objectives *s* at the inner ends being alike. In Fig. 427 the right optical tube is shown arranged as telescope, and the left as collimator, to which latter the lengthening illumination tube *t* is also attached. Each optical tube is supported on a stout column *u* arising from the end of an arm *v* or *w* radiating from and solid with a stout ring surrounding the fixed axis *a*, the arm being continued on the other side into a counterpoise *x* or *y*.

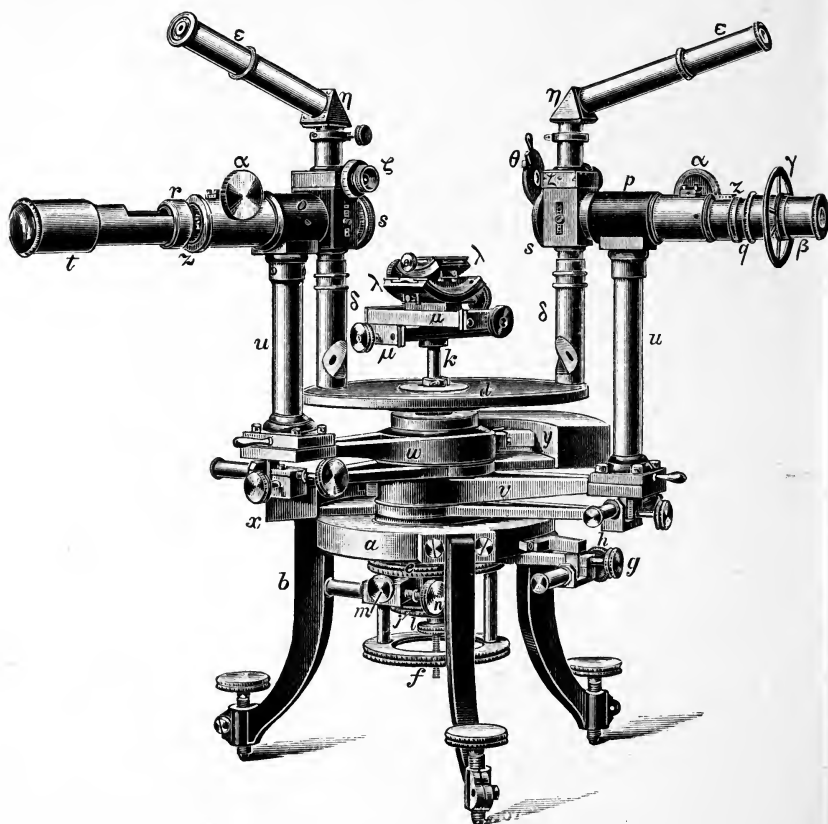


FIG. 427.—No. 1a Fuess Goniometer.

Each arm carries beneath it a fixing and a fine adjustment screw, in order that either or both of the optical tubes may be adjusted to any desired position about the circle and fixed there. The signals and eyepieces are similar to those of the No. 2a model, and are arranged to fit in automatic adjustment by means of the notched tightening collar *z*, which gears with a little V-shaped projection carried by a collar on the signal or eyepiece; a rack and pinion movement, manipulated by the milled head *α*, is provided, which enables the last refinement of focus to be attained. If it be desired to employ polarised light, as is essential in refractive index determinations, a Nicol prism *β* may be attached in front of the eyepiece, the prism being provided with a finely divided circle *γ*. A pair of such Nicol prisms are provided, corresponding to the duplicate

eyepieces, to meet the eventuality of one being required on each optical tube. Two much larger Nicol prisms, mounted on circles of considerable size, reading on silver arcs with the aid of a vernier in each case to single minutes, are also provided for attachment in front of the objectives of the two optical tubes, for use in certain optical observations to be described later, for which such a position of the polariser or analyser is preferable. A Babinet compensator (see p. 842) can also be attached in front of one of these large Nicols.

The circle is divided on its silver annulus (inlaid nearly flush with the surface of the circle plate) directly to every $10'$, and is read by two micrometer microscopes $\delta\epsilon$, one of which is carried by each of the optical tubes, near the objective, in a manner which will be clear from Fig. 427. The objective part of each microscope δ is fitted below the objective end of the telescope, and terminates below in an oblique diaphragm with central aperture, enamelled white in order to reflect diffused light on to the graduations of the circle, the silver annulus bearing them on its horizontal surface being immediately below the aperture. Above the telescope tube and immediately over the tube δ a micrometer ζ is fitted, the drum of which is divided into 60 parts; as one revolution corresponds to the movement of the parallel pair of spider-lines from the position in which one graduation of the circle is set symmetrically between them until the next graduation is similarly set, each division of the drum corresponds to ten seconds, and the tenths of a division, corresponding to one second, can be estimated. The very low power eyepiece ϵ is not arranged vertically over the objective tube of the microscope as usual, but is inclined so as to bring it more conveniently near the observer's eye, just above the eyepiece of the telescope, the rays from the objective being directed into it by means of a totally reflecting prism η .

On the author's instrument a small tubular electric glow-lamp is fixed opposite the inclined white diaphragm at the bottom of δ , and a switch for it is fitted on the mahogany basal plinth on which the instrument stands, over which fits a glass protective shade covering the whole instrument when not in use. The light can thus be switched on momentarily for the reading of the circle, and extinguished immediately afterwards for the observation of the signal-image reflected from the crystal face or refracted through the prism, according as crystal angles or refractive indices are being measured. These arrangements will be clear from Fig. 430, which represents the No. 1a goniometer in actual use by the author for higher temperature measurements.

An additional lens θ is provided, to be fitted before the objective of the optical tube used as telescope, in a manner which permits of its being instantly swung in or out of position, to convert the telescope into a low power microscope for the observation and adjustment of the crystal, as in the No. 2a model.

The crystal-adjusting and centring apparatus, lettered respectively λ and μ , is similar to that provided with No. 2a, but larger.

The heating apparatus for bringing the crystal to the required temperature is shown in partial section in Fig. 429 (in elevation above and in plan below), and in position in Fig. 430. It consists of a spherical double-walled air-bath of copper a , continued at opposite sides into horizontal cylinders a' , along the interior of each of which runs a stout copper rod b , terminating at the inner end in a shallow copper cup c concentric with the walls of the bath, and at the outer end projecting

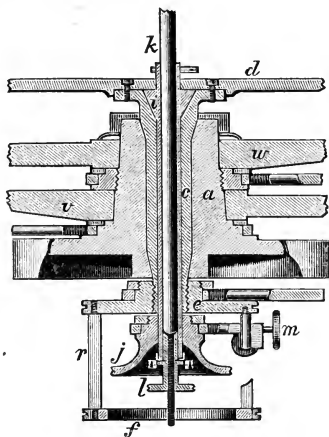


FIG. 428.—Section of Axial System of No. 1a Goniometer.

free sufficiently far to enable it to be heated by a special form of Bunsen gas lamp *d* with fan-shaped burner. All the space within the cylinder not occupied by the rod is filled up with asbestos, as is also the space between the outer and inner spheres forming the air-bath. The heat communicated to the ends of the two rods by the Bunsen flames is conducted to the air-bath, and if the burners are nicely regulated to afford approximately the same heating effect the air-bath becomes very evenly raised in temperature. The heat is warded off from the goniometer by the tin-plate screens *e*, and the cylinders are wrapped round with thick felt to prevent undue radiation. The apparatus is supported near the outer ends by two semicircular rests *f*, each forming the upper termination of an adjustable column carried by a tripod base, these two supports being adequately compact to stand conveniently on each side of the goniometer, when the apparatus is in position over the latter, without interfering with the manipulation of the instrument. Besides being adjustable itself for height the column carries an adjustable arm which bears the Bunsen burner at its end, so that the burner may be delicately adjusted for distance below the rod which it heats; the size of its horizontally elongated flame—

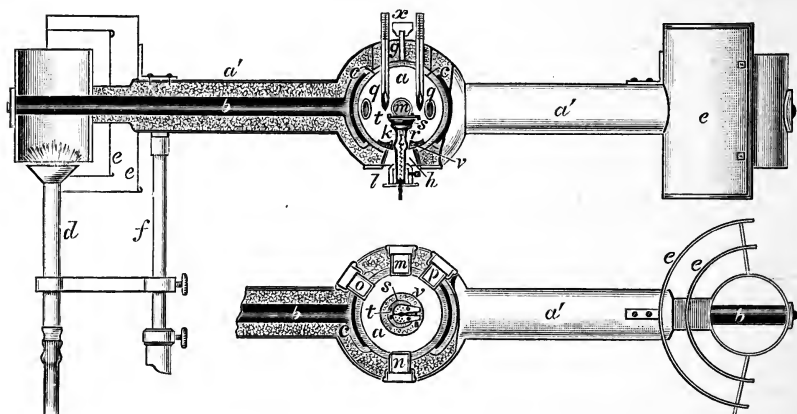


FIG. 429.—Crystal-heating Apparatus of No. 1a Goniometer.

which latter rarely requires to be more than half an inch high for the highest temperatures which crystals and the instrument itself will safely withstand (that is, not exceeding 200°C.)—is regulated by a screw pinchcock on the indiarubber gas-supply tubing just below where it is attached to the burner.

The air-bath has two circular openings vertically above and below one another, the vertical diameter of the sphere passing through the centre of each. The upper opening is of sufficient size to admit the two thermometers through the similarly constructed copper and asbestos plug *g* which closes it, but the lower aperture *h* is only just large enough to allow the special crystal holder *k* to pass through it, with sufficient play to prevent contact during rotation of the goniometer circle and the crystal holder *l* which moves with it. The air-bath requires to be placed so that its vertical diameter is as nearly as possible coincident with the goniometer axis. It is pierced with four side windows in the case of the author's instrument (one more than usual), each filled with truly plane and parallel-surfaced glass plates. Two, *m*, *n*, are placed opposite each other at the extremities of that horizontal diameter which is perpendicular to the direction of the cylinders and rods, so that when the collimator and telescope are arranged in the same straight line with one another (also at right angles to the heating apparatus between them) the Websky signal of the collimator

can be seen through the telescope and the windows; the other two windows, *o*, *p*, are arranged one on each side of the front window *m* (the one nearer to the telescope of the two just referred to), at angles of 55° and 40° to the latter respectively. The instrument as supplied by Fuess has only one of these at about 55° , but this angle has been

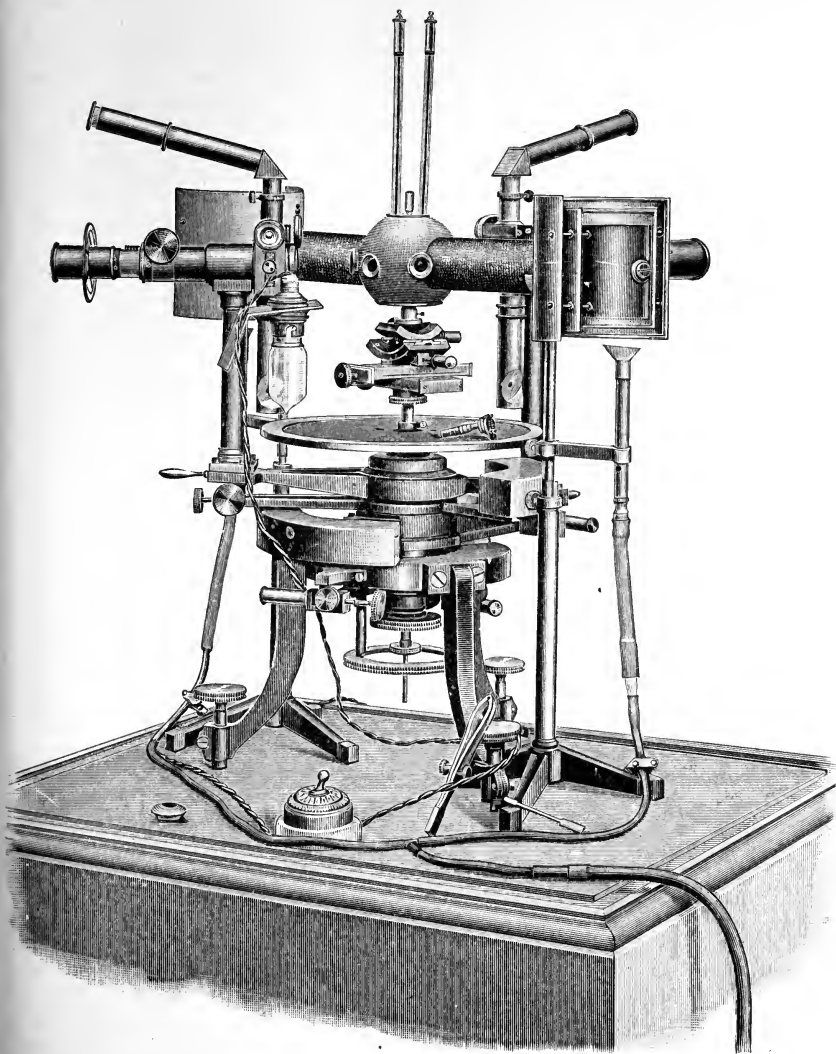


FIG. 430.—No. 1a Goniometer with Crystal-heating Apparatus in Position.

found inconveniently large by the author when determining the angle of minimum deviation with a 60° prism, except for substances of very high refraction, although it is quite convenient for crystal angle work. With two windows at 55° and 40° practically all cases are provided for. There is not room to cut both on one side, so one is cut on each side of the front central window.

The crystal holder is a special feature. There fits first into the socket at the top of the crystal-adjusting and centring apparatus the peg of a second and more widely socketted fitting *l*, into which fits the crystal holder proper *k*. The tubular stem of the latter is of metal in its lower half but of glass in its upper half, in order to prevent conduction of heat from the crystal; the latter would otherwise never quite attain the temperature of the bath as indicated by the thermometers, the bulbs *qq* of which are brought quite close to the crystal on each side. The glass part is of narrow glass tubing, which is blown out into a bulb just above the level of entry into the interior of the bath; it is again expanded at its upper termination into a shallow cup *r*, in which rests a slightly larger metallic button *s*, spherically shaped below with the same radius of curvature as the cup, so as to be capable of adjustment in the cup in any direction. The button carries above a miniature adjustable vice *t* for gripping the crystal, either directly or with the intervention of a little crystal holder which one can make for oneself out of stout platinum foil to suit the particular crystal and which can be held in the vice, one jaw of which is fixed to the button while the other is only pinned to the button and can be adjusted with reference to the fixed jaw by means of a screw, so as to grip anything inserted between. The button and all that it carries is kept pressed down on its cup by means of a spiral spring attached to the button below its centre, and which passes down through the glass tube and its metallic continuation in a stretched condition to a hook screwing in a nut which forms the lower termination of the tube and which has the same diameter as the outside of the tube, so that it forms no impediment to the fitting of the latter in the socket. The whole holder, removed from its socket, is shown lying on the circle plate in Fig. 430. After its insertion in the socket this special holder can be clamped there by a side screw manipulated by a key supplied for the purpose, and which also fits and manipulates the screw of the crystal-holding vice. This key is shown, leaning against the tripod of the right pedestal of the heating apparatus, in Fig. 430. It is obviously impossible to use any form of wax for attaching the crystal, as it would melt when the temperature was raised. Each jaw of the vice is pierced by a number of vertical holes for the reception of a little hard-metal pin, which may be fixed in the most convenient of the holes to enable the crystal to be gripped between the two upright pins, the screw manipulated by the key regulating the tightness of the grip. If, however, the shape of the crystal does not lend itself to being held in this manner during the measurements, the platinum foil holder already mentioned may be used and the pins discarded. A number of such little pincettes of strong platinum foil should be kept ready, of different sizes to suit different crystals; the two prongs of the pincette can be bent in any way which is desirable for the better gripping of the crystal, and the lower part of the holder, where the platinum strip has been bent on itself at its centre so as to form the pincette, is to be held directly between the jaws of the vice.

For convenience of inserting this special glass-stemmed crystal holder in the socket fitting, which replaces the ordinary crystal holder, when the bath is in position, a pair of spring tongs are provided, shaped at their ends so as to grip the button-head of the holder while avoiding the glass cup. They are shown in Fig. 430 leaning against the right-front foot of the goniometer tripod. The aperture at the base of the air-bath should first be closed by a little slate washer *v*, shown resting on the mahogany base, in front to the left, in Fig. 430; it is shaped below to the curvature of the bath and has a central hole of the right size to permit the glass stem but not the bulb to pass freely; it is placed loosely inside the bath before lowering the holder into position with the tongs. During rotation of the goniometer circle together with the holder and its crystal any play of the stem due to slight imperfection of centring is still permitted by the loose slate annulus, whilst the bath is at the same time effectively closed.

When the crystal has been mounted on the holder and the latter is ready for being placed in the bath for the measurements, the bath itself is first adjusted on the stirrups of its two supporting columns at the height which brings the windows opposite the optical tubes, that is, so that the horizontal axis of the bath and its cylindrical prolongations and the optical axis of the collimator and telescope (arranged in the same straight line for direct vision of the signal) intersect each other at approximately right angles, and so that the point of intersection also lies on the vertical axis of rotation of the goniometer, which latter axis then passes through the centre of the basal and upper apertures of the bath. The centre of the socket for the crystal holder should then be properly centred, so as to remain apparently immovable on rotation of the circle and all that it carries, when regarded from above through the upper opening of the sphere, the slate annulus being absent during this centring. The final adjustment of the bath is at once attained after this by making the socket occupy the centre of the lower aperture of the bath, as seen from above. The annulus is then placed in position in the bath, and the crystal holder inserted through it into the socket with the aid of the tongs, and gently clamped by the screw.

The crystal should obviously have been mounted, either between the two pins of the vice or in one of the platinum pincettes held in the vice, so that the zone of faces the mutual inclinations of which are to be measured, or the two faces which form the prism the angle and refractive minimum deviation of which are to be determined, will be approximately adjusted parallel to the goniometer axis when the holder is fixed in its socket. The goniometer observing lamp is now switched on and the images of the collimator signal reviewed in the telescope. Further adjustment will in general be required, and this is done by gently pressing down that side of the adjusting button in its glass cup which experience shows causes the signal-image under observation to come towards the centre of the field, with the end of the rod-shaped key for the vice and socket screws, this key being just conveniently shaped and long enough for the purpose. The image from the next face should then be observed, and if any adjustment is required it must be done by pressing down that side of the button which is both about 45° from the side formerly depressed and which causes this second image in question also to come to the centre; if the pressure be properly directed the first adjusted face will then only move in its own plane and its image will consequently not be put out of adjustment again. Of course adjustment by hand in this way, by pressure on the button with the key-end, will not be perfect on the first attempt, but only one or two approximations ought to be necessary in order to adjust almost perfectly the edge between the two crystal faces and their whole zone, without having recourse to the adjusting segments of the goniometer until the last moment, when a mere touch of one or both of them will perfect the adjustment. Any considerable use of the segments is very inconvenient, as it renders the glass stem of the special crystal holder oblique and puts the crystal seriously out of centring, so that it

may not remain visible through the windows on rotation ; for this crystal holder is necessarily much longer than the normal crystal holder for the centring of which the movements are calculated.

The stopper may now be inserted by means of its handle-knob α in the top of the air-bath, and the two thermometers placed in the two holes drilled for their passage through the stopper. These holes are lined with cloth or soft paper, to prevent contact of the glass stems of the thermometers with the metal. Care must be taken that the crystal has been so well centred that on rotation it does not touch the thermometer bulbs, which are arranged at its level and as close as may be to the crystal, and the stopper itself must be rotated, if necessary, so that the bulbs are not in the way of the passage of the light rays between the collimator, crystal, and telescope. Presuming now that crystal angles are to be measured (refractive indices will be dealt with in Chapter XLV.), the collimator should be fixed so that the light from it passes normally through the back window of the spherical bath, and the telescope should be directed at the side window which is inclined 55° from the front window parallel to the back one, so that the two optical tubes are inclined at about 125° , a very convenient angle for the observation of the signal-images reflected from the adjusted faces. (In the case of refractive index determinations the window at 40° on the other side is used as a rule, and this is the case actually shown in Fig. 430, as it was the one which revealed the details of the goniometer to greatest advantage.)

Before lighting up the Bunsen lamps the angle or angles to be measured at higher temperatures should first be measured at the ordinary temperature as recorded by the thermometers, both of which should have recently had their fixed points verified or corrected. The angle or angles will already be known to minutes, from the ordinary goniometrical investigation, but the measurements now require to be carried out to seconds. The last signal-image observed should be left adjusted to the vertical spider-line, in order that it may be observed from time to time during the heating operation, so as to verify that there is no sudden disarrangement owing to cracking of the crystal or other disturbing cause. It should move steadily away from the spider-line parallel to itself during the heating if all goes well.

All being now ready for the determination, the two Bunsen burners are lighted. The fan-shaped flat nozzles enable the flames to be checked down, by the pinchcocks on the caoutchouc gas-supply tubing, to narrow low strips of flame about an inch long, which should be adjusted exactly under the rods. This enables the heating to occur very slowly and deliberately. The author finds a "Stott" governor to be of considerable use in preventing sudden fluctuations of the gas supply ; it is inserted between the gas-supply tap and the burners. The supply tap is turned fully on, and the gas passes thence by a short length of indiarubber tubing to the governor, thence another short length of tubing delivers the gas to a second gas tap fitted with a long lever arm moving over a quadrant graduated in degrees, which enables the amount of gas

delivered to the burners to be regulated to a nicety. From this fine adjustment tap a further short length of caoutchouc tubing conducts to a metallic T-piece, where the gas divides and is conducted to the two burners, as shown in Fig. 430, by separate indiarubber tubes of equal narrow bore, each fitted with a screw pinchcock, as already mentioned, near the burner in order to obtain any necessary separate control of either flame, in case the thermometers are not found to indicate an equal rise of temperature on each side of the air-bath. The rise should be carefully watched, and the flames regulated so that the temperature moves deliberately up to the required height, where it eventually attains constancy. With the aid of the governor it is easy to attain a constant temperature for a particular position of the lever arm of the tap on the graduated arc, and for an interval of time which is ample to enable the determination of the angles between the faces of a complete zone to be carried out in duplicate; and also, as will subsequently be shown, for a complete duplicate set of determinations of the minimum deviation by a 60° prism to be made for a numerous series of wave-lengths of light, as well as repeated measurements of the angle of the prism. The illuminating light for crystal angles or the angle of a prism is, of course, the ordinary white light of the goniometer lamp; but for refractive index determinations of minimum deviation the spectroscopic monochromatic illuminator is employed in the manner detailed in Chapter XLV.

As soon as constancy at the required temperature is attained, the steady movement of the signal-image having correspondingly become arrested, the measurement or measurements of the crystal angles are carried out just as were those for the ordinary temperature. The measurements should be repeated, and if there be any slight difference a third set of readings should be taken in confirmation. When two series of identical readings have been obtained to within two or three seconds at most, the measurements may be accepted as representing the truth for this temperature, and as proving that the temperature indicated by the thermometers has actually been attained by the crystal.

CHAPTER XXIX

TWO AND THREE CIRCLE GONIOMETERS, AND USE THEREWITH OF THE GNOMONIC PROJECTION.

THE goniometer No. 2*a* of Fuess and the similar Troughton and Simms instrument, described in Chapter III., are the best types of ordinary single-circle goniometers. They have been shown to be fully adequate for all the purposes of crystal measurement to single minutes of arc at the ordinary temperature, while the larger single-circle goniometer No. 1*a* of Fuess, described in the last chapter, is equally the best instrument yet constructed for similar measurements to seconds of arc at either the ordinary or higher temperatures, and is usually reserved for the higher temperature work as the No. 2*a* is handier and its reading to half-minutes is all that is ever necessary for ordinary-temperature work. With their aid every zone on the crystal can be measured through the whole 360° , and the measurements may be repeated without a break if desired, in order to verify that the readings are the same and that there has consequently been no movement of the crystal on its wax setting on the crystal-holder. It is obviously necessary, however, from time to time, after the measurement of two or three or more adjacent zones, to re-set the crystal on the wax, in order to place a further number of zones conveniently for adjustment and measurement. The number of settings is not large, however, and can often be reduced to four or five at most when the general plan of the crystal is known, as is the case during the measurement of all the crystals after the first, of the ten usually measured of a new substance. When full use is made of the range of motion of the segmental adjusting movements, a crystal possessing numerous zones may often be completely measured with three settings. For the purposes of the most refined measurement, in the case of original investigation, the necessity for a few settings is as nothing compared with the accuracy, rigidity, and simplicity of such a goniometer, and particularly with the advantage of being able to complete the whole 360° of each zone, and thus to verify the absence of slipping on the part of the crystal by proving that the first and last readings, for the same face originally started with, are identical. Lastly, the single-circle goniometer pre-eminently emphasises the occurrence of crystal faces in zones, and lends itself admirably to the determination of the zonal

relationships and of the indices of the crystal faces, thus bringing home to the mind the law of rational indices in an unmistakable manner.

While these considerations render the single-circle goniometer the basis of all sound knowledge of the symmetry of crystals, there have of recent years been coming into use two- and three-circle goniometers. They have chiefly been employed by their inventors for work of an intermediate character between student work and the highest kind of research, such for instance as the rapid investigation of new specimens of known crystallised minerals, many of which crystals are only developed at one end of a prism zone, having been attached to a cavity wall at the other end, but are very rich in faces at the end which is developed. The completion of the zones other than the prism zone will here be a matter of no consequence, as only half a zone will be present, and all these half-zones will be able to be measured with a three-circle goniometer with a single setting, and the single complete zone can also be completely measured on one of the circles. Moreover, three-circle goniometers have been employed for the experimental solution of all the unknown elements of the numerous spherical triangles on a stereographic projection. Hence it is essential that a description of the best forms of these instruments should be given. It has been left to the conclusion of the goniometrical part of this book in order to emphasise the fact that the single-circle goniometer described in Chapter III. and referred to throughout is the only essential form, and that these multiple-circle goniometers may well be left until considerable experience with crystal measurement has been acquired; also, that if expense of equipment is, as usual, a consideration of moment, they may be disregarded, an excellent single-circle instrument being fully adequate for all the purposes of either teaching or research.

It is interesting that the British father of our science, Miller, so long ago as 1874 constructed a two-circle goniometer, by arranging a vertical-circle goniometer on the top of an ordinary horizontal-circle instrument, and an account of it and of the work done with it has been published by his successor in the Cambridge chair of Mineralogy, Professor Lewis.¹ In 1889 von Fedorow gave a description of a two-circle goniometer² similar to that of Miller, but instead of the separate collimator and telescope an auto-collimating telescope was used, that is, one in which the same optical tube was made to serve for both purposes, with the aid of a reflecting prism only half closing the aperture at the common focus of the objective and eyepiece. This, however, has been found to be a disadvantage, as the images from small faces are faint owing to the perpendicular incidence of the light on the crystal face. In 1893 Professor Goldschmidt³ quite independently described another similar two-circle goniometer, but in which the ordinary collimator and telescope were used, and in the same year Czapski⁴ published his account of the now well-known instrument which bears his name, and which, in the form now constructed by Fuess, will next be described.

¹ *Zeitschrift für Krystallographie*, 1883, 7, 619.

² *Proc. Min. Soc.*, St. Petersburg, 26, 458.

³ *Zeitschr. für Kryst.*, 1893, 21, 210.

⁴ *Ibid.*, 1893, 21, 574.

The crystal is adjusted on such a two-circle goniometer so that a face of importance with respect to the symmetry is parallel to the circle carrying the crystal on its axis of rotation. The position of any other face is accurately given by its two co-ordinates, one of which, the azimuth ϕ , is read off on the same circle carrying the crystal, and the other of which, the polar distance ρ , is afforded by the second circle. The zonal relationships are not directly shown by these co-ordinates, but from the latter it is possible with some considerable trouble to calculate the crystal elements and the facial indices; the formulæ employed by von Fedorow and Goldschmidt, however, are complicated and do not lend themselves to the simple methods of logarithmic calculation which have been shown in the earlier part of this book to be so useful and convenient.

Czapski Theodolite Goniometer.—This instrument, as improved by Leiss, Wulff, and Stöckl, and constructed by Fuess, is shown in Fig. 431 on the scale of one-quarter the real size.

The horizontal circle a is carried on an axial cone rotating within a fixed cone b supported by a tripod c of the usual character in the Fuess goniometers, except that one limb is longer than the other two in order to afford room for the bearing pillars d of the vertical circle e . The horizontal circle is divided on silver directly to $20'$ and reads with the aid of the usual pair of verniers to minutes of arc. It is rotated by the capstan wheel f from below, by means of the five handles. The crystal g and its adjusting segments h and centring movements i are carried as in the ordinary No. 2a goniometer, at the head of an innermost axis of steel j , bearing a screw thread on its lower portion, and lying within and keyed to another axis rotatable within the circle axis; the steel axis is thus capable of being raised or lowered without rotation, by means of the milled-headed nut near its lower end. The adjusting movements are capable of fixation to the steel axis by the screw k . The circle may be clamped to the tripod by the screw l , and finely adjusted by the other screw m at right angles, which presses against an upright projection carried by an arm underneath the circle, a spring piston pressing the projection against the adjusting screw on the other side.

The axis of the vertical circle is carried by the upright support n which terminates in the axial bearing cone o . The telescope p and collimator q are also carried about this same axis in theodolite fashion. The circle is rotated by a large conically shaped capstan head r fitted with five manipulating handles; it is read by two microscopes s , and affords, with the aid of the two corresponding verniers t , readings to minutes of arc, the circle itself being directly divided to $20'$ as in the case of the horizontal circle. This vertical circle is capable of being fixed to the bearing by the screw u , and of being finely adjusted by another screw v . The axle bearing is hollow to admit of the crystal being viewed through a lens carried near the inner end of the boring, a small diaphragm w being placed at the other and outer end. The crystal may thus be observed and set to the intersection of the two axes of the two circles. The telescope o is carried by a radial arm continued into a counterpoise w , and is provided with the usual viewing lens x at the objective end, a totally reflecting prism y between the two lenses directing the rays from the crystal into the telescope. The collimator p is carried on the elbow z of another radial arm a , the boss of which rotates round the fixed bearing outside the telescope boss, and which is also continued into a counterpoise β . The collimator can be arranged at any angle to the telescope and clamped there by means of the fixing screw γ ; the angle is read off on the circle with the aid of the index δ . The collimator carries at its outer end a little tubular fitting ϵ in which is contained a 4-volt glow-lamp, or else an alternative one carrying an adjustable

mirror, either fitting also containing a condensing lens for the illumination of the signal, the Schrauf cross-signal being employed. When the mirror is used, it is arranged to reflect the light from an ordinary electric glow-lamp placed a yard or two distant along a prolongation of the horizontal axis of the vertical circle *e*, so that for all positions of the collimator the light is reflected along the axis; a little circular opaque screen is used to ward off the central part of the light (not required by the mirror) from the crystal. An iris diaphragm may also be fitted in front of either the collimator or telescope, for the purpose of isolating the light reflected from any one part of a crystal face, the viewing lens being placed in position against the telescope objective in order to observe the crystal, so that the observer can see when he has properly screened off with the iris diaphragm the light from those parts

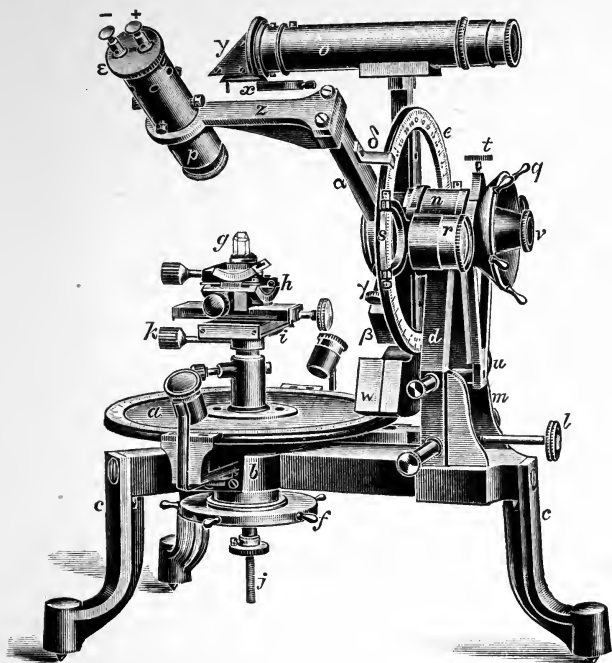


FIG. 431.—Czapski Two-circle Theodolite Goniometer.

of the face which are not desired to afford an image of the signal. On removing the viewing lens the image from the required portion of the crystal face will be seen. This renders the goniometer well suited for the study of vicinal faces, a discussion of which will be entered into in the next chapter.

This goniometer enables all the faces of the upper half of the crystal to be adjusted, and also those for 30° below the equator.

Method of Use of Theodolite Goniometer.—The crystal is observed through the microscope within the boring of the axle of the vertical circle, from *v*, and a prominent zone of faces is adjusted. The telescope *o* is then rotated down about 20° below the equator (that is, till its arm which is vertical in Fig. 431 is inclined downwards 20° below the horizontal). When the arm is horizontal the circle readings (those for the equator) are

90° and 270°. The collimator p is then arranged at the same angle upwards from the equator on the same side, so that the vertically adjusted zone of faces of the crystal will afford a series of reflected images of the signal in the field of the telescope as the crystal axis is rotated. The adjustment is then perfected so that all the images pass at the same height in the field of the telescope. This only occurs when the normals to the faces are all precisely in the horizontal plane, that is, in the plane perpendicular to the vertical axis of the instrument. The collimator is then to be raised or lowered slightly until the images pass through the centre of the field of the telescope, when the two optical tubes will make the same angle with the horizontal plane. They should then be fixed in this position and retained so during the whole series of measurements. The positions of the poles of all the faces in the zone should then be read off on the horizontal circle a . The reflections of all other faces will be found by simultaneous rotation of both circles e and a (the former with the telescope and collimator remaining in the fixed positions), and they should be finely adjusted and their positions on the two circles read off. The reading on the horizontal circle a gives the azimuth co-ordinate ϕ , and that on the vertical circle e gives the other co-ordinate, the polar distance ρ . The determination is much easier when there is a face developed perpendicular to the zone of adjusted faces; this is recognised by its image remaining fixed when the crystal and the vertical axis carrying it are rotated. The positions of all faces lying in a zone with this face and with one of the vertically adjusted faces are then easily determined by rotation of the horizontal axis and reading of the vertical circle e alone, as their azimuth ϕ (reading on horizontal circle a) is the same as that of the vertical face.

The poles of the adjusted faces are drawn on the primitive circle of the stereographic projection, and the other measurements are subsequently also graphically expressed on the projection in the usual manner.

Herbert Smith Three-circle Goniometer.—In order to render the theodolite method free from the grave disadvantage of ignoring zonal relationships and the law of rational indices, an additional circle was added by Herbert Smith, which enables the crystal to be adjusted once for all, and at the same time both the angles between the faces of any one zone and the position of this zone on the crystal, with reference to other zones and the general symmetry, to be determined by direct measurement. The first form¹ was simply obtained by adding to an ordinary No. 2a Fuess goniometer a second circle arranged vertically, which carried by means of a rectangular elbow bracket-bearing a third circle supporting the crystal at the end of its axis, with the usual adjusting and centring movements. With this adapted instrument, however, measurements can only be made through little more than a right angle in any particular zone other than the zone of reference, without readjustment of the vertical and third circles. In a second instrument,² which was constructed by Messrs. Troughton and Simms,

¹ *Mineralogical Magazine*, 1899, 12, 175.

² *Ibid.*, 1904, 14, 1.

this difficulty is largely overcome by reflecting the line of reference, —which is the line bisecting the angle between the telescope and collimator in the ordinary single-circle goniometer and the adapted one just referred to,—at right angles to its normal position by means of mirrors, so that on rotation of the vertical circle round the axis of the horizontal circle the axis of the vertical circle may be brought into coincidence with the line of reference in two positions on diametrically opposite sides of the centre, and still be free for somewhat further movement beyond the 180° of movement thus already afforded. Both these instruments will now be briefly described, for in spite of the advantage of the second form just referred to, the adapted Fuess instrument is very handy in actual practice and has been found extremely useful.

The essential principle is the same in both instruments. There are three circles arranged as follows :

A. The horizontal circle, the axis of which is vertical and fixed in space, permitting only rotation of the circle about it ; the axis is also perpendicular to the line of reference of the telescope and collimator.

B. The vertical circle, the axis of which is horizontal and at right angles to that of A, but is movable in the horizontal plane.

C. The third circle, the axis of which may be brought to occupy any desired direction in space and therefore also with respect to the axis of A, but which is always at right angles to that of B.

All three axes and the line of reference (the normal to the crystal face when adjusted so that the signal-image is symmetrical to the telescope spider-lines) intersect in the optical centre of the instrument.

Fig. 432 shows the instrument obtained by the addition of a two-circle arrangement constructed by Troughton and Simms to a No. 2a Fuess goniometer, in which it replaces the crystal-holding axis. A is the horizontal circle of the original Fuess instrument, and B and C are the two new circles. The latter are carried by a rigid horizontal plate *a*, which may either be firmly attached to the horizontal circle A, or may be rotated independently of the latter, the plate having an axial cone below it capable of rotating in the conical axial boring of A. The elongated plate carries at opposite ends respectively the upright supports *b* for the cone *c* in which the vertical circle B rotates, and a counterpoise *d* for the same and all that it carries, including the third circle C. The holes for the screws by which the column is attached to the plate are slotted, in order to provide means of accurate adjustment of the two extra circles to the optical centre of the instrument. The third circle is of aluminium in order to reduce the overhanging weight to a minimum ; and it is fitted to the vertical circle by means of a similar plate and adjustable elbow-bearing *e* in the manner which will be clear from Fig. 432. The axis of the circle C has also three adjustment screws, two to bring it into a plane parallel to the vertical circle B and one to make it intersect the axis of this circle. Both circles have clamps *f* and slow-motion screws *g* and *h*, and fixed verniers reading to minutes. The crystal-holder *l* is carried by the third circle C, and the latter is counterpoised by a weight *m* at the outer end of the bearing of the vertical circle B.

These arrangements render it easy for the reading on the circle A to be made $0^\circ 0'$ for a crystal face parallel to the vertical circle B, and the graduations of this circle are so arranged that when the axis of

circle C is vertical the reading on B is also $0^{\circ} 0'$. The circles B and C are about 9 and 11 cm. in diameter respectively, which permits of the vertical circle B being turned about the axis of the horizontal circle A for about 190° , when the telescope and collimator are inclined to one another at 70° ; the circle C may be turned about the axis of circle B through about 240° , when that axis bisects the interior angle between telescope and collimator.

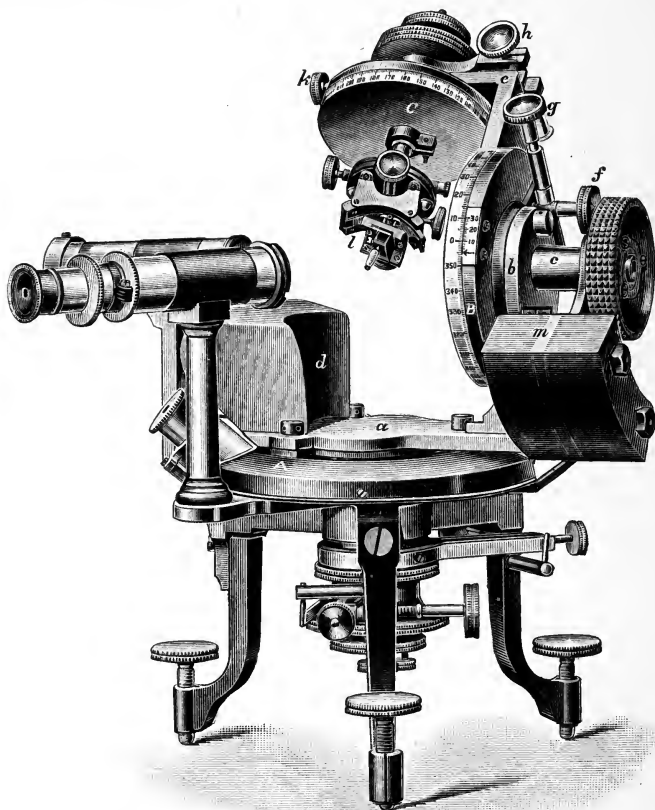


FIG. 432.—First Form of Herbert Smith Three-circle Goniometer.

In using the instrument, a zone of faces, preferably the zone of greatest symmetry if it be known, is adjusted parallel to the axis of circle C. As this circle is rotated the various faces in the zone are in turn brought parallel to circle B, and if the circle A be rotated until it reads $0^{\circ} 0'$ the image from a face parallel to B will be visible adjusted to the spider-lines. The axis of circle C should also be arranged vertically by suitable rotation of circle B. The faces of the adjusted zone may then be measured on circle C, as the circle is rotated and each face in turn comes parallel to circle B and affords an adjusted

image in the telescope. If, then, circle C be clamped when some face is thus adjusted, and circle B be rotated, any zone containing this face may be brought parallel to the axis of the horizontal circle A and measured on that circle. It thus amounts to being able to make two-circle measurements with any face of the crystal as reference face, instead of being confined to one pole-face (with any one setting), as is the case with a two-circle goniometer. Any zone may be adjusted by means of the circles B and C so as to be parallel to the horizontal circle A, and can thus be measured on the latter circle. Hence circle A gives the angles in the zone, and circle B gives the angle between this zone and the reference zone.

Fig. 433 shows the later instrument of Herbert Smith, in which a much greater angular range is obtained by the reflection of the line of reference, the telescope *a* and collimator *b* being very close together so as to be nearly parallel and pointing to one side of the centre. It is constructed by Troughton and Simms.

The tripod base *c* carries within its central cone *d* in the usual goniometrical manner the rotatable conical axis of the horizontal circle A. The scale on the bevelled edge of the circle-plate is divided to 10', and is read by a pair of micrometric microscopes *e* diametrically opposite to each other and carried by arms fixed to the tripod table; each micrometer head revolves once for every division of the scale and is divided into ten parts, each of which is subdivided into six, so that each of these smaller divisions corresponds to ten seconds of arc.

The second or vertical circle B is carried by an arm *f* attached to a thick plate *g* which is solid with the upper part of an inner stout conical axis rotating within the circle cone. The vertical circle B and all that moves with it can thus be rotated about the vertical axis of the whole instrument, quite independently of the horizontal circle A, and fixed anywhere with respect to it by means of a strong clamping screw. During this rotation the horizontal axis of the vertical circle thus moves radially with respect to the main goniometrical vertical axis (that of circle A) in the horizontal plane. The arm which carries the circle B is fixed to the cone-plate by four screws, which permit of its adjustment by other screws both towards the axis of circle A and at right angles to this direction. The circle B is read by a pair of micrometer microscopes *h* diametrically arranged on opposite sides of the divided limb, like those of circle A, and which read similarly to ten seconds. Two pairs of friction wheels are introduced between the main rotating parts, and a counterpoise *i* is provided on the other side of the cone-plate to balance the weight of the vertical circle and all that it carries.

The third circle C is carried by a twice-bent arm *j* which, together with the axle cone *k* which rotates within that of the circle B, is worked out of a solid casting, and is also rigidly attached to the latter circle in such a position that the zero reading of the circle B is 180° or 360° in the two microscopes. The scale of the circle C reads to minutes with the aid of a pair of verniers fixed to the axis in an adjustable manner by means of three pairs of adjusting screws. The counterpoise *l* for the third circle C is placed on the other side of the circle B, at the outer end of the axle.

All three circles are fitted with the usual Troughton and Simms clamping screw and screw-and-spring-piston fine adjustment movement. The crystal-holder and the adjusting segments *m* and mutually rectangular centring movements *n* are also of the usual kind given with the Troughton and Simms goniometers, as described in Chapter III.

The telescope *a* and collimator *b* are separate optical tubes (not auto-collimating), but are supported by the same pillar *o* rigidly connected with the fixed tripod table. Each optical tube is carried within an outer tube, adjustable about a vertical axis and

about a horizontal axis at right angles to the length of the tube, by means of three screws, *p*, working against strong spiral springs and passing through a flange into the fixed collar. The tubes may revolve about and be pushed along the direction of their length. A small reflecting prism is placed immediately outside the signal-slit of the collimator to permit of illumination from the side. At the inner ends of the optical tubes furthest from the observer they are provided with outer tubes *q* sliding over them like caps, each carrying an adjustable mirror of speculum metal and capable of

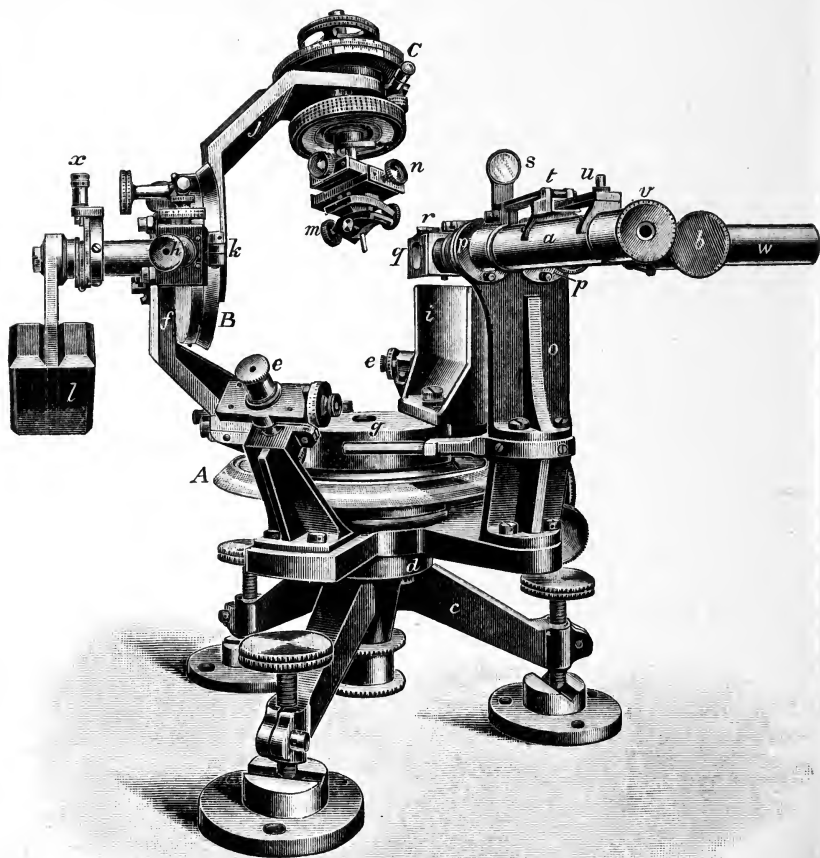


FIG. 433.—Latest Form of Herbert Smith Three-circle Goniometer.

fixation in the most desirable positions by means of split collars and tightening screws, the tubes and their mirrors being thus capable of either sliding or rotating. Each mirror is also capable of a further adjustment about a vertical hinge, by means of a screw working against a spring. The telescope mirror is placed further away than the collimator mirror, and its tube is cut away at the sides, so as to permit of the passage through it of the rays from the collimator mirror to the crystal, the reflected rays then passing back from the crystal face to the telescope mirror and thence through the telescope to the observer at the eyepiece. The speculum mirrors

may be cleansed from any tarnish which develops by means of cotton wool dipped in a mixture of two parts of absolute alcohol and one part of ammonia.

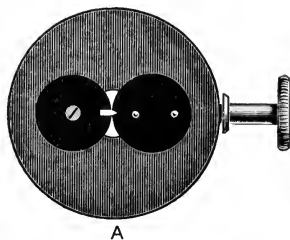
The viewing lens to be placed in front of the telescope objective is fitted on a horizontal hinge *r*, and occupies a recess on the top of the mirror tube when not in use, but can be placed in position, when it is desired to observe the crystal, by means of a lever arranged for the purpose outside the tube. Another lens *s* for placing in front of the eyepiece, to reduce the magnification when desirable, is carried on an arm *t* which is hinged on a slide movable along a bar *u* on the top of the telescope tube near the eyepiece *v*.

A special shape of collimator signal-slit is employed, which is shown at A in Fig. 434. The ordinary Websky signal is not sufficient, for the signal has to be adjusted to the horizontal as well as to the vertical spider-line of the telescope eyepiece. The illuminating light is sent down the side-tube *w*, and reflected into the slit by means of a small totally reflecting prism.

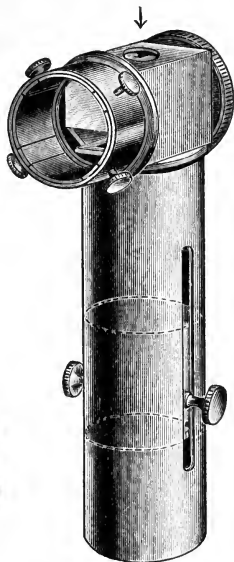
The mirrored axes of the telescope and collimator are inclined at about 22° to one another. The maximum possible rotation of the horizontal circle A is about 240° ; the axis of the vertical circle B can make with the line of reference angles of 57° towards the telescope and 3° towards the collimator. When B is near the collimator and its axis coincides with the line of reference, the greatest rotation about its axis is 188° ; when B is on the other side it may be rotated for 200° .

To prevent excessive wear of the lower half of the fixed cone of the circle B two pairs of friction wheels are provided, one pair of which are near the counterpoise and are controlled by the spring balance *z*.

A further accessory of considerable use is a camera lucida, shown at B in Fig. 434. It consists of a double Amici prism, for reflecting the object in the direction of the drawing paper supported on the base-board of the instrument, and of a vertical tube in which lenses of various focal



A



B

FIG. 434.—A, Signal-slit of Three-circle Goniometer; B, Camera Lucida of Three-circle Goniometer.

lengths, carried in an inner adjustable tube, may be inserted, to suit different scales of drawing or different distances of the eye. By this means also all parallax difficulties are eliminated. The horizontal upper shorter tube is provided with four screws by which the fitting may be attached to the eyepiece of the goniometer. It is particularly useful for the delineation of minute crystals. For the relative positions of faces on crystals with as small a cross section as 0.2 millimetre may be readily determined with the three-circle goniometer, but it is not so easy to sketch such small crystals in the ordinary way. If they are drawn by means of the ordinary camera-lucida attachment to a microscope there may be considerable uncertainty as to the identity of the faces; but when the crystal is on the goniometer any doubt is at once removed by an observation of the position of the face. It also proves very useful in the cases of faces which can only be determined by the position of the maximum light reflected from them, owing to distortion or curvature.

Adjustment of the Three-circle Goniometer.—A plate of parallel

glass is mounted on the crystal-holder, and adjusted by the eye approximately parallel to the axis of circle A. The crossed spider-lines of the eyepiece are then illuminated by the Becker illuminator (Fig. 13) or by reflection from a glass plate employed in front of the eyepiece in a similar manner, and the image of the spider-lines is also caused to be reflected from the plate of parallel glass. The tube of the telescope mirror is next adjusted until the illuminated spider-lines and their images coincide. The plate is then rotated 180° and the image from its other surface similarly reviewed. If the horizontal spider-line and its image do not coincide, the plate must be adjusted through one-half the interval and the telescope mirror through the other half. The first surface is then reverted to, and if still not in adjustment corrected until it is; a few oscillatory observations and corrections may require to be made for the two surfaces before the agreement of the lines and their images is perfect for both.

The collimator mirror is next adjusted so that the signal-image, when brought into the centre of the field, is bisected by the horizontal spider-line. On rotation about the axis of the circle A, however, the centre of the image may not continue to traverse the horizontal line, showing that the mirrored spider-line is not perpendicular to this axis; if so, the tube of the telescope and also the mirror tube must be rotated through the same angle in the opposite sense until the adjustment is effected. If the image is not upright, a similar adjustment must be made for the collimator.

To adjust the axis of B perpendicular to that of A the parallel-glass plate is rotated about the axis of C until it is approximately perpendicular to that of B, and then C is clamped. By use of the crystal-adjusting movements the plate is adjusted so that the signal-image remains stationary on rotation of the plate about the axis of B. The latter is then adjusted with respect to A by the adjusting screws, until the image is exactly bisected by the horizontal spider-line. The axis of C is next adjusted perpendicular to that of B, by making it (the axis of C) coincide with that of A, which has already been adjusted perpendicular to B. The axis of B is maintained in coincidence with the line of reference, and the glass plate is rotated about the axis of C and the images from the two sides adjusted so as to be both bisected by the horizontal spider-line; this is done by using the crystal-adjusting movements to make one image lie as far above as the other is below, and then adjusting circle C itself until both images come exactly to the spider-line.

The axis of the circle B may be readily brought parallel to the line of reference (the line to which the crystal face is brought perpendicular during crystal measurements), and will coincide with it in two positions when correctly adjusted, the circle B first lying on one and then on the other side of the optical tubes. In either position we may adjust a plane surface, such as one of the surfaces of the parallel-glass plate, perpendicular to the axis of this circle, and if the two images are bisected by the horizontal spider-line the axes of A and B are truly perpendicular.

When the axes of all three circles have been accurately adjusted as regards their direction, they are finally adjusted in absolute position, so that they pass through the same point, the optical centre, which must lie on the line of reference. This is done with the aid of a needle carried on a peg like that of the crystal-holders, in the same manner as has already been described for the adjustment of the ordinary single-circle goniometer in Chapter III. Many useful details as to this process, together with

further amplifications of the mode of effecting the directional adjustments already described, will be found in the original memoir (*loc. cit.* pp. 7 to 11).

Method of using the Three-circle Goniometer.—The zero readings of the circles A and B require to be determined before commencing measurements with the instrument. The circle A is rotated until the microscope nearest the observer reads exactly 360° . The upper part of the instrument is then rotated until the axis of B coincides with the line of reference and the circle B lies on the other side of the crystal-holder remote from the optical tubes; the parts are then clamped together. Any further necessary correction will only be to the extent of a few seconds. The circle B is then rotated until the axis of the circle C is vertical, and the microscopes for reading circle B are moved until they read as nearly as possible 360° and 180° . A small correction may again be found necessary, but only to the extent of a few seconds.

The mode of using the instrument for the actual measurement of a crystal is similar to that already sketched for the adapted Fuess instrument. The only difference is that the three circles can be utilised to measure greater arcs, owing to the fact that the line of reference is reflected into the central space, the optical tubes themselves being out of the way at the side. A zone of faces exhibiting symmetry, if any be developed on the crystal, is adjusted parallel to the axis of the circle C. To do this the axis of the circle B is brought into coincidence with the line of reference; the axis of C should at the same time be vertical, for then the images obtained by reflection from the faces of the adjusted zone will cross the field horizontally on rotation of this axis, and the adjustment is easier. The zone thus adjusted is termed the "zone of reference." Its angles are measured on the circle C as that circle is rotated (or on A if preferred by reason of its more finely divided scale, C only reading to minutes), the axis of circle B being still coincident with the line of reference. Each reading corresponds to the position in which the particular face is perpendicular to the axis of B, and therefore on rotation of this axis that face remains parallel to its own plane and the image of the signal remains immovable. The faces of this zone may be termed "origins."

The readings of circles A and B respectively for any other face on the crystal give the polar distance ρ and the azimuth ϕ , measured from the origin and the zone-plane of the zone of reference respectively. The latter would be represented by an actual face if the symmetry were of a tolerably high order, and the measurements would be much facilitated by its development.

From a face of importance with respect to the symmetry as origin, if there be such developed in the zone of reference, the crystal is to be systematically measured, and the positions of the poles of the various faces plotted out on a stereographic projection in the ordinary manner, the zone of reference being the primitive circle.

The stereographic net of Hutchinson (Fig. 39) is very useful for graphically recording the measurements as the work proceeds, the net being placed on the paper on which the drawing is to be made, and

the positions of the various poles pricked off on to the paper through the net.

With this three-circle instrument all the various spherical triangles of the stereographic projection may also be experimentally solved, if it be so desired.

The Gnomonic Projection.—In the stereographic projection the point of sight is situated on the surface of the sphere; but the gnomonic is a perspective projection in which the eye is supposed to be placed at the centre of the sphere and the poles are projected on to a tangent plane. The centre of the projection is the point of contact of the latter with the sphere, that is, the point in which a perpendicular from the centre meets the touching plane. Another mode of describing the gnomonic projection is that it is the intersection, with the plane of the paper, of the sheaf of lines drawn from a point perpendicularly to the faces of a crystal. Each face is thus represented on the projection by a point, and each zone by a line joining two such points.

As the planes of all great circles pass through the centre of the sphere, such great circles representing the zones of faces of the crystal appear as straight lines when projected on the tangent plane. The zonal relationships can, therefore, be examined by means of a linear scale, and the anharmonic ratio of four faces in a zone is afforded by the positions of the points representing them on the zonal straight line, as measured on this scale, a considerable advantage. But the gnomonic projection has the defect that all facial poles on the great circle parallel to the plane of projection are not projected at all, for their projections would lie at an infinite distance. Indeed 45° of angle on each side is all that can be conveniently included in a gnomonic projection. Hence, only a portion of the sphere and of the crystal can be represented on the same diagram, and the portion thus represented should be a characteristic one. For this reason the stereographic is a much superior projection for all ordinary purposes. For it gives us a complete plan of the distribution of the faces of a crystal, by solid dots for poles on the upper hemisphere, and by little rings for poles on the lower half, if it be desired to represent the latter distinctively on account of their not all lying on the projection identically with those of the upper hemisphere; and the primitive circle of poles in the zone perpendicular to the plane of projection is obviously the natural boundary of the figure. In the case of the gnomonic projection, however, some arbitrary boundary, such as a square or a circle, has to be given to the figure. Other difficulties which have decided the author in not employing it for general work in this book are, that the diagram suffers from distortion, owing to linear distances from the centre of projection being tangents of the corresponding angles, and that parallels of latitude on the sphere are projected as conic sections, being only circles when either the north or south pole of the sphere is the centre of projection.

On the other hand, however, a second advantage, besides suitability for use with a linear scale, is possessed by the gnomonic projection, namely, that a drawing of the crystal in orthogonal projection can readily be constructed from it. For, as the lines in a gnomonic projection are the

projections of zone planes, the edges of all faces belonging to any one zone are perpendicular to the line representing it. These perpendiculars to the gnomonic zone-lines are thus the directions of the orthogonal projections of the edges, the same plane of projection being assumed to be taken for both the gnomonic and the orthogonal projection.

The main use of the gnomonic projection comes in with the employment of a three-circle goniometer. For such a projection can well be made to represent most of the poles, if not all, investigated for one setting of the crystal on the wax of the crystal-holder. The measurements are made from poles in a certain zone, the zone of reference, as origins, and from the readings of the circle from these poles we obtain for any other pole its distance from the origin and its azimuth from the zone of reference. These co-ordinates are then used to locate the poles of the faces on the gnomonic diagram, the plane of which is taken at right angles to the edge (axis) of the zone of reference. As poles in this zone are projected to infinity, all zones passing through the same origin (a pole on the zone of reference) appear as parallel lines on the projection.

This will be rendered clearer by reference to Fig. 435, in which P is the pole of the zone of reference, and Z the direction of the origin-pole on this zone of reference. If we draw PX at right angles to PZ, and then if Q be any point (a facial pole for instance) and we draw Qk parallel to PZ, Qk will be the projection of the zone passing through Q and the origin Z. Herbert Smith,¹ to whom this and the next three figures are due, in applying the projection to work with his three-circle goniometer described earlier in this chapter, has shown that the linear co-ordinates of the gnomonic projection are related to the spherical co-ordinates of the stereographic projection by two simple equations.

If Qk in Fig. 435 be the linear distance of the point Q , which may be assumed to be a facial pole, and ρ and ϕ the spherical co-ordinates of the pole Q with respect to the origin Z , ρ being the distance from Z and ϕ the azimuth from the zone of reference, then :

$Qk = r \cot \rho \operatorname{cosec} \phi$, and $Pk = r \cot \phi$.

In order to facilitate the plotting of linear distances on the gnomonic projection and the graphical solution of angles therewith, Herbert Smith gives in the same memoir the design of a protractor, shown in Fig. 436, and also a table of values of $\cot \rho \operatorname{cosec} \phi$ for angles from 25° to 90° . The angles between poles or zones may thus be rapidly

¹ *Mineralogical Magazine*, 1903, 13, 309.

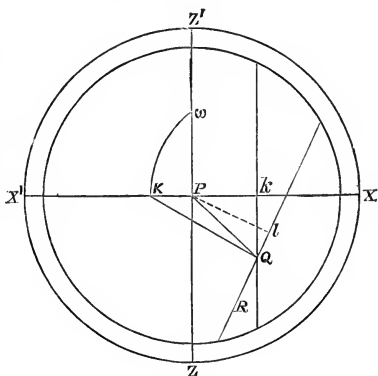


FIG. 435.

graphically determined from the projection with the aid of the protractor and table.

The protractor is a strip of celluloid on which is engraved a semi-

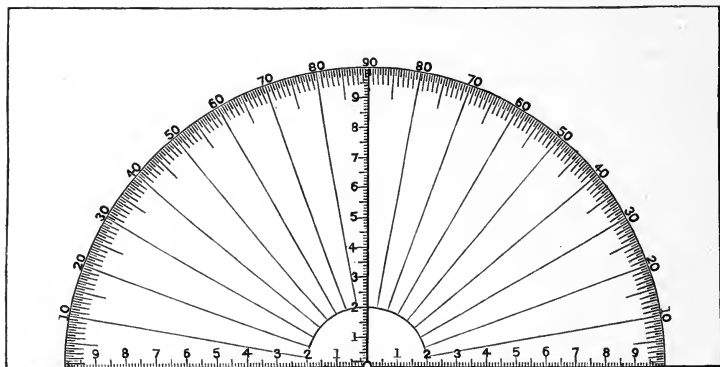


FIG. 436.—Herbert Smith's Gnomonic Protractor.

circle, the diameter of which is on the edge of the strip; both the diameter and the radius at right angles to it are divided into millimetres,

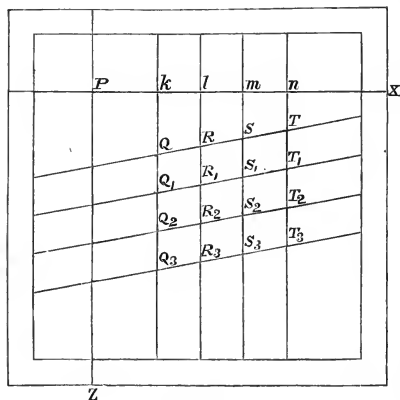


FIG. 437.

while the semicircle is divided into half degrees. By means of the two rectangular scales the poles can easily be plotted from their plane co-ordinates. It is most convenient in work with two- or three-circle goniometers to take the centre of projection itself as origin, the azimuths being then the same as the plane angles, and the linear distances proportional to the tangents of the corresponding angles.

In Fig. 437 are represented a series of poles lying in four zones passing through the same origin in the zone of reference, and therefore represented by straight lines

parallel to each other in the gnomonic projection; their azimuths may be indicated by ϕ , ϕ_1 , ϕ_2 , and ϕ_3 . It will then be an important fact that the following relationship holds, namely:—

$$\frac{km \cdot ln}{kl \cdot mn} = \frac{(\cot \phi_2 - \cot \phi) (\cot \phi_3 - \cot \phi_1)}{(\cot \phi_1 - \cot \phi) (\cot \phi_3 - \cot \phi_2)} = \frac{\sin (\phi - \phi_2) \sin (\phi_3 - \phi_1)}{\sin (\phi - \phi_1) \sin (\phi_3 - \phi_2)} = \text{a rational quantity.}$$

This rational quantity is the anharmonic ratio of four crystallographically possible zones, intersecting in the same pole.

If we suppose $\{TT_1T_2T_3\}$ to be the zone of reference n lies at infinity, and we have the simple relation left that

$$\frac{km}{kl} = \text{a rational quantity.}$$

A quite similar relation holds with respect to the poles themselves in the various zones, so that, for instance, we have:

$$\frac{QQ_2}{QQ_1} = \text{a rational quantity.}$$

Under these conditions the principal zones in the diagram are equally spaced apart, and a network of parallelograms is produced as in Fig. 437. In special cases rectangles and squares are afforded. Hence also unit linear distances on parallel lines must be the same, so that we have only to take measurements on the diagram in order to determine the simple indices corresponding to any particular face.

Two simple problems are of constant recurrence, and it may be of use to indicate how they may be solved.

(1) **To find the Angle between Two Poles represented by Points on the Gnomonic Projection.**—Suppose Q and R in Fig. 435 to be the poles. The protractor is placed so that the diametral scale lies along QR while the radial scale at right angles to the diameter passes through the centre of projection P . The zero of both scales will lie on l . Readings of the scales are then taken for the points P , Q , and R . The reading for P obviously gives the length Pl , and from the table the azimuth of the zone QR can be found. We can then further find from the table, with the aid of this information, the angular distances corresponding to the observed lengths Ql and Rl . The required angle or its supplement is then either the sum or the difference of these two angles Ql and Rl , according as Q and R lie on opposite sides or the same side of l . In the case shown in Fig. 435 the difference is taken, as both Q and R lie on the same side of l , and the angle obtained is the actual angle required. When Q and R lie on opposite sides of l the sum gives the supplement of the angle required, because the table gives the distance measured from the origin in the zone of reference.

(2) **To find the Angle between any Two Zones, represented by Straight Lines on the Gnomonic Projection.**—In order that this problem may be better understood both the stereographic and gnomonic projections are given side by side at A and B in Fig. 438. The two zones intersect in D and meet the zone of reference in p_1 and p_2 ; the zone PD is also drawn, meeting the zone of reference in d .

Assuming the angle Dd to be ϕ , the angles p_1d and p_2d to be the complements of ρ_1 and ρ_2 , and the angles p_1Dd and p_2Dd to be D_1 and D_2 respectively, we then have in the triangles p_1Dd and p_2Dd :

$$\tan D_1 = \cot \rho_1 \operatorname{cosec} \phi, \text{ and } \tan D_2 = \cot \rho_2 \operatorname{cosec} \phi.$$

The protractor is now placed on the gnomonic diagram so that the zero of the linear scales is on D while the radial scale passes over P the

centre of projection. The reading for P on this 90° -radius scale is taken, and also the readings of the two zone-lines on the circular scale; from the first we obtain the length PD , and therefore as before in case (1) from the table we find the angle ϕ , while from the other two we obtain the angles ρ_1 and ρ_2 . Treating ϕ as an azimuth and ρ as a distance we get from the table a corresponding quantity, which from the first column of the table we convert into the complement of D_1 or D_2 , as this column gives the cotangents. The sum or difference must be again taken as in problem (1), according as the angles have been measured on opposite sides or the same side of PD . The latter case affords the required angle itself, while the former gives the supplement, as in the case of problem (1).

These aids to the use of the gnomonic projection introduced by Herbert Smith, from whose memoir most of the foregoing rules for

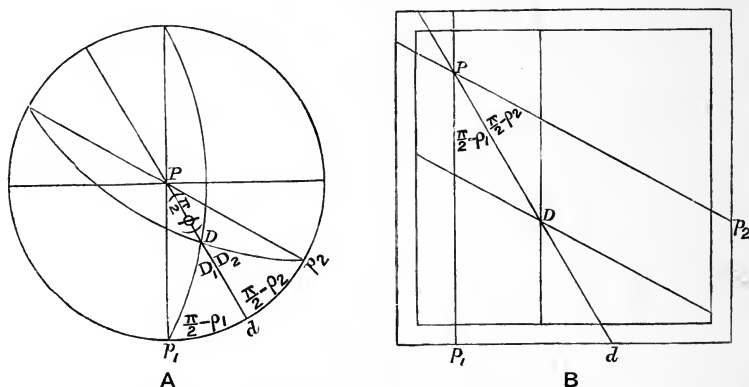


FIG. 438.—A, Stereographic Projection; B, Gnomonic Projection.

their application are taken, have greatly simplified the use of the projection, and thus indirectly rendered a still further service to three-circle goniometry. The protractor is made by Mr. Steward of the Strand, and is a very convenient accessory to possess in any case. The table of cosecant azimuths and cotangent distances is too long for reproduction here, and reference should be made to the original memoir in the *Mineralogical Magazine* (*loc. cit.*) An ordinary table of tangents and cosecants enables the protractor to be also applied to any stereographic projection.

Goldschmidt's Two-circle Contact Goniometer.—The principle of the two-circle goniometer has been applied by Goldschmidt¹ in a contact goniometer for the measurement of large crystals with dull faces, and for the demonstration of the two-circle method with large crystals and crystal models. It is shown in Fig. 439. The horizontal circle H is divided directly into single degrees, and may be read to half a degree with the aid of an indicator a . This circle H rotates with

¹ *Zeitschrift für Kristallographie*, 1896, 25, 321.

the crystal *b*, which is attached by wax on the crystal-holder *c*, the latter being rigidly fixed in the top of the crystal-adjusting movement by the screw *d*, the square head of which is manipulated by the lower end of the key *e* shown to the right. The upper end of the key is for insertion into two holes *f* and *g*, at right angles to each other, in the head of the crystal-adjusting movement *h*. The latter is a universal ball-and-socket joint, sunk in the upper part of the main central supporting column *k*. The vertical circle *V* is of hoop form, and is graduated for 100° right and left of the zero at the top of the vertical diameter. An indicator *l* is maintained in close contact with the limb by a spring, and it carries a radial rod *m* terminating in the perpendicular contact-plane *n*; the rod can be pushed further in towards the centre, or withdrawn therefrom, in order to approach or recede from the crystal face parallel to which it is desired to adjust the contact-plane; the rod is maintained throughout strictly radial by means of springs. The apparatus is mounted on a stout tripod *o*. In another form of the apparatus the contact-plane disc is replaced by a long wedge, the edge of which is adjusted parallel to the crystal face instead of the plane surface of the disc.

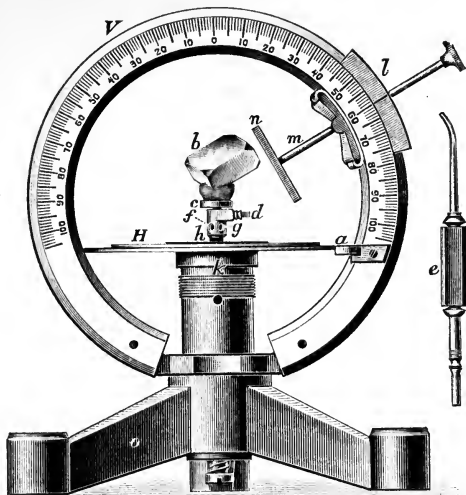


FIG. 439.—Goldschmidt's Two-circle Contact Goniometer.

In using the instrument the crystal is usually set up on the holder *c* with the prism zone, if one be prominently developed, vertical, perpendicular to the horizontal circle *H*; or, what comes to the same thing, if the crystal be tabular parallel to the basal plane, it is adjusted so that the latter face is parallel to *H*. The contact-plane *n* is then arranged with its rod vertical and the indicator at 0°, or horizontal with the indicator at 90°, and the basal plane or prism face is adjusted truly parallel to the contact-plane disc *n* by "sighting"; that is, by approaching the surface of *n* to the crystal face until it is nearly in contact with the latter, and then adjusting the crystal, by manipulation of the ball-and-socket joint with the key, until when regarded from all sides against a suitable background an equally thick or thin line of light is seen between the two surfaces. A circular background of white, black, or coloured cardboard, according to the colour or appearance of the crystal (that one being selected which offers the greatest contrast), may be attached just below the crystal, being gripped between

c and h , the stem of the crystal-holder passing through a central hole in the card. When the crystal is neither prismatic nor tabular, but more or less spherical, the ordinary method of the two-circle goniometer is followed, two faces being adjusted to the contact-plane in succession, both circles being employed. It is to be remembered, however, that the functions of the two circles are the opposite of what they are in the reflecting two-circle goniometer, for instead of the adjustment of reflections the adjustment to parallelism of the traces of the face and contact-plane is substituted. The reading v on the horizontal circle H gives the azimuth $\phi = v - v_0$, where v_0 is the reading for the selected first meridian. The reading on the vertical circle V gives the polar distance ρ immediately.

Goldschmidt's Apparatus for the Construction of Crystal Models.
—Goldschmidt¹ has further utilised the principle of the two-circle goniometer in an apparatus for the accurate construction of crystal

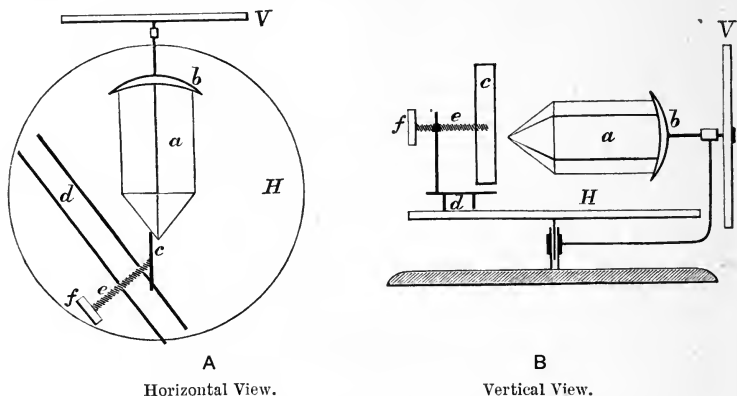


FIG. 440.—Goldschmidt's Crystal-Model-making Apparatus.

models. It is shown diagrammatically in the two views, A horizontal and B vertical, given in Fig. 440, in which V and H represent the vertical and horizontal circles respectively. The modelling material may conveniently be freshly prepared plaster of Paris (still moist), steatite, wax, or solid paraffin. It is supported as shown at a on a relatively very large crystal-holder or working piece b , at the end of the horizontal axis of the vertical circle V (the ϕ -circle in this apparatus). The horizontal circle H (ρ -circle) is massively constructed as a rotatable working table, and carries the vertical cutter c , which resembles the blade of a carpenter's plane; the latter is arranged to slide in a suitable guiding bed d in the plane of the face which it is desired to prepare on the crystal model, slice after slice being planed away until the face is sufficiently developed. The whole planing gear can be adjusted so as to approach the centre of the horizontal circle H or recede away therefrom at will, by means of a screw e with divided drum-head f and

¹ *Zeitschrift für Krystallographie*, 1899, 31, 223.

a scale to record the complete turns of the screw, so that the distance of the face from the centre of the model can be regulated as desired, in order to give the model crystal the habit contemplated.

In using the apparatus, the larger habit-conferring faces are first prepared, each of these dominating faces being produced in turn by the removal of successive shavings of plaster of Paris or paraffin, or whatever other material is being employed, the two circles being adjusted in accordance with the known values of ϕ and ρ for each particular face. When the ideal development of these primary faces is desired, the planing tool is worked in each case until the drum and scale record the same reading.

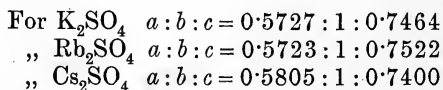
The instrument is more or less limited in its direct usefulness to the construction of the faces at one end of the crystal axis, which is arranged parallel to the axis of the vertical circle V. The other parallel faces can be prepared with a second setting subsequently, or by simply preparing them with the aid of calipers in the ordinary way. The readings of the circles are true to half a degree, which also represents the degree of accuracy of construction of the faces of the model. The central distances can be regulated to within half a millimetre. Provided adequate care be taken to set accurately the angular values of ϕ and ρ on the two circles, and the position of the planing tool to the proper central distance, the models furnished transcend in accuracy anything previously attempted. The instrument also serves as an accurate contact goniometer, for the measurement of the angles between the faces of models, or between those of real crystals of considerable size. A special crystal-holder is provided to replace the working piece *b* on these occasions.

CHAPTER XXX

CLOSENESS OF ANGLES OF ISOMORPHOUS SALTS, AND CONSEQUENT NECESSITY FOR ACCURACY—TREATMENT OF SPECIAL DIFFICULTIES IN CRYSTAL MEASUREMENT, AND PITFALLS TO BE AVOIDED.

THE object of this chapter is to emphasise how essential it is that the highest possible accuracy should be aimed at and attained in the cases of all goniometrical measurements the results of which are intended to be published as contributions to the original literature of the subject. The publication in the past of angular values derived by measurement of the crystals of substances of a low order of purity, of crystals of imperfect and more or less distorted form, and by the use of goniometers of doubtful accuracy, has introduced grave errors and indeed complete misconceptions into crystallographic literature which it will require years of patient accurate work to correct. In former times, during the infancy of chemistry and crystallography, and before the perfection of goniometrical methods, this was unavoidable; but to-day there is absolutely no excuse for the publication of immature and imperfect work, and the author desires to plead most earnestly for greater accuracy, and particularly for the avoidance of all premature publication.

The Importance of Accurate Goniometry.—A most instructive example of what is meant is afforded by the group of rhombic salts of which the typical example, potassium sulphate, K_2SO_4 , so often referred to in this book and which is illustrated in the first three figures in Chapter I., is the first member, namely, the sulphates of the alkali metals potassium, rubidium, and caesium. Previous to the author's reinvestigation of these salts, the values currently accepted for the ratios of the crystallographic axes were the early ones of Mitscherlich for potassium sulphate dating from the year 1830, Bunsen's values for rubidium sulphate published in 1861, and those given by Topsøe from measurements made by von Lang in 1867 for caesium sulphate. These values are as follows:



Now these numbers show no progression according to the atomic

weight of the alkali metal, and indeed no relationship whatsoever, beyond the fact of mere similarity in the kind of values. No definite law was obviously indicated by them. But the result of the author's investigations,¹ taking all the precautions which have been recommended in this book and employing all the resources now available for refined work, has been to show that this promiscuous order of these ratios is incorrect and totally misleading, and that their true order is as under :

$$\begin{aligned}\text{For } K_2SO_4 \quad a:b:c &= 0.5727:1:0.7418 \\ \text{,, } Rb_2SO_4 \quad a:b:c &= 0.5723:1:0.7485 \\ \text{,, } Cs_2SO_4 \quad a:b:c &= 0.5712:1:0.7531\end{aligned}$$

Remembering that the atomic weight of potassium is 38.85, that of rubidium 84.9, and that of caesium 131.9 (when hydrogen = 1), and that the atomic weight of rubidium is thus about the mean of the atomic weights of potassium and caesium, it will be obvious that the true order of the axial ratios is the same as the order of the atomic weights. This is particularly noticeable in the case of the ratio of $c:b$, where the differences between the values are far removed from the neighbourhood of the possible experimental error of these refined measurements.

The fact which it is desired to emphasise, the absolute necessity for accuracy, is even more strikingly brought out by the interfacial angles themselves. The old accepted values were of such a promiscuous character as to be worse than useless, just as in the case of the ratios, for the only conclusions possible to be based upon them were that no law governed their relationships. The important law of progression according to atomic weight, now fully established, crystallographically as well as chemically and physically, was never even suspected to apply in the case of the crystallographic constants, being entirely hidden and prevented from coming to light by erroneous measurements, or measurements with impure material. But when the true angular values were established as the outcome of most careful and prolonged work, a very different state of matters was revealed; for the magnitudes of the whole of the angles of rubidium sulphate, without a single exception, were found to lie between those of the analogous angles upon the potassium and caesium salts respectively. A progressive change is thus observed to be brought about in the angular magnitudes of the crystals of the sulphates of potassium, rubidium, and caesium, corresponding to the progressive change in the atomic weight of the contained metal, when potassium is replaced by rubidium and the latter by caesium.

This important fact will be clearly apparent from the table of angles of the crystals of the three salts, which is given on page 485, and which may also serve at the same time as a specimen of how such a table of comparative angles should be drawn up, and to afford a definite idea of the amount of the angular differences in an isomorphous series and of the limits well within which the degree of accuracy must reach. It includes also the angles of the three other isomorphous salts of these

¹ See *Crystalline Structure and Chemical Constitution* (Macmillan, 1910) for a connected account of these researches.

three strictly related metals of the same family group of the periodic classification of the elements, in which the sulphuric acid is replaced by selenic acid (sulphur and selenium forming isomorphous compounds); for the study of the three selenates of potassium, rubidium, and caesium has revealed precisely the same progressive relationships, and afforded an independent confirmation of the greatest value of the results with the sulphates. Moreover, the table also includes the analogous sulphates and selenates of ammonium and thallium, which are also in a broader sense isomorphous, and the results for these further salts are of particular interest as showing that the progressive changes, following the progressive atomic weights, are only observed when the metals belong strictly to the same family group. A second table is also appended in which the axial ratios of the whole series are set forth. The matter is put very concisely in the concluding paragraphs of the author's memoir on the thallium salts,¹ which may profitably be quoted, as they express in the clearest terms chemico-crystallographic facts which the crystal measurer should always have before him.

"The results of the investigation throw into prominent relief the main generalisation which has sprung from these researches, that in the case of an isomorphous series in the strictest sense, where the interchangeable elements belong to the same family group of the periodic classification of the elements, the whole of the properties of the crystals, morphological, optical, thermal, and physical in general, are functions of the atomic weights of these elements. For such a series the term 'eutropic,' from *εὐτροπή*, meaning a regularly progressive change (*regelmässige Änderung*), has been suggested by Linck, and is adopted by the author.

"Where we have an element like thallium, which does not belong to the same family group, the law does not apply, although the metal may be, as thallium is, capable of interchanging with the others without altering the crystal system, and without making angular and structural changes of much greater magnitude than those provoked by the interchange of family analogues.

"In the light of this research, the author defines an 'isomorphous series' as one the members of which bear some definite chemical analogy and crystallise according to the same system and in the same class of that system, and develop the same forms inclined at angles which only differ by a very few degrees, rarely exceeding 3°; and a 'eutropic series' as one in which these small angular differences, and also the structural and physical properties of the crystals, obey the law of progression according to atomic weight of the interchangeable elements which give rise to the series and which belong to the same family group.

"Thallium sulphate and selenate and ammonium sulphate (ammonium selenate is dimorphous and usually crystallises in the second, monoclinic, form) belong to the orthorhombic isomorphous series $R_2\overset{S}{Se}O_4$, while the sulphates and selenates of potassium, rubidium, and caesium belong not

¹ *Proc. Roy. Soc., A*, 1907, 79, 381.

only to this same isomorphous series but also to the still more exclusive eutropic series within it."

Comparative Table of Interfacial Angular Magnitudes of Rhombic Alkali Sulphates and Selenates.

Angle.	K ₂ SO ₄ .	Rb ₂ SO ₄ .	Cs ₂ SO ₄ .	(NH ₄) ₂ SO ₄ .	Tl ₂ SO ₄ .	K ₂ SeO ₄ .	Rb ₂ SeO ₄ .	Cs ₂ SeO ₄ .	Tl ₂ SeO ₄ .
$\{ap = (100) : (110)$	29° 48'	29° 47'	29° 44'	29° 24'	29° 3'	29° 49'	29° 43'	29° 41'	29° 2'
$\{pp' = (110) : (130)$	30 0	30 0	30 0	30 0	29 59	30 0	30 0	30 0	29 59
$\{p'b = (130) : (010)$	30 12	30 13	30 16	30 36	30 58	30 11	30 17	30 19	30 59
$\{cq''' = (001) : (012)$	20 21	20 31	20 38	20 6	20 7	20 6	20 16	20 22	19 55
$\{q'''q = (012) : (011)$	16 13	16 18	16 21	16 6	16 7	16 6	16 11	16 13	16 0
$\{cq = (001) : (011)$	36 34	36 49	36 59	36 12	36 14	36 12	36 27	36 36	35 55
$\{qq' = (011) : (021)$	19 27	19 26	19 26	19 28	19 28	19 28	19 27	19 26	19 28
$\{q'q'' = (021) : (031)$	9 47	9 45	9 42	9 51	9 50	9 51	9 49	9 47	9 54
$\{q'b = (031) : (010)$	24 12	24 0	23 53	24 29	24 28	24 29	24 17	24 11	24 43
$\{q'b = (021) : (010)$	33 59	33 45	33 35	34 20	34 18	34 20	34 6	33 58	34 37
$\{ao = (100) : (111)$	43 52	43 41	43 31	43 39	43 13	44 9	43 51	43 43	43 25
$\{oq = (111) : (011)$	46 8	46 19	46 29	46 21	46 47	45 51	46 9	46 17	46 35
$\{ao' = (100) : (112)$	58 44	58 31	58 19	58 37	58 14	59 3	58 45	58 35	58 28
$\{o'o' = (112) : (112)$	62 32	62 58	63 22	62 46	63 32	61 54	62 30	62 50	63 4
$\{bo = (010) : (111)$	65 37	65 33	65 32	65 56	66 7	65 43	65 42	65 40	66 13
$\{oo = (111) : (111)$	48 46	48 54	48 56	48 8	47 46	48 34	48 36	48 40	47 34
$\{bo' = (010) : (112)$	72 42	72 37	72 33	72 56	73 0	72 51	72 47	72 43	73 8
$\{o'o' = (112) : (112)$	34 36	34 46	34 54	34 8	34 0	34 18	34 26	34 34	33 44
$\{co' = (001) : (112)$	36 44	37 0	37 13	36 42	37 2	36 21	36 41	36 52	36 44
$\{o'o = (112) : (111)$	19 27	19 26	19 25	19 27	19 26	19 28	19 27	19 26	19 27
$\{co = (001) : (111)$	56 11	56 26	56 38	56 9	56 28	55 49	56 8	56 18	56 11
$\{oo'' = (111) : (332)$	9 45	9 42	9 40	9 46	9 42	9 49	9 46	9 44	9 45
$\{o'p = (332) : (110)$	24 4	23 52	23 42	24 5	23 50	24 22	24 6	23 58	24 4
$\{op = (111) : (110)$	33 49	33 34	33 22	33 51	33 32	34 11	33 52	33 42	33 49
$\{po = (110) : (111)$	65 8	65 2	64 54	64 31	63 52	65 17	65 1	64 55	63 56
$\{oq' = (111) : (021)$	49 12	49 22	49 30	49 24	49 47	48 57	49 13	49 20	49 36
$\{qp = (021) : (110)$	65 40	65 36	65 36	66 5	66 21	65 46	65 46	65 45	66 28
$\{po' = (110) : (112)$	72 23	72 15	72 7	71 58	71 27	72 34	72 19	72 12	71 33
$\{o'q = (112) : (011)$	34 50	35 4	35 15	34 53	35 14	34 31	34 49	34 58	34 59
$\{qp = (011) : (110)$	72 47	72 41	72 38	73 9	73 19	72 55	72 52	72 50	73 28
$\{p'o = (130) : (111)$	43 59	43 49	43 40	44 0	43 47	44 14	44 1	43 54	43 59
$\{oo' = (111) : (112)$	45 47	45 55	46 1	45 16	45 3	45 33	45 39	45 43	44 51
$\{o'q' = (112) : (021)$	46 1	46 12	46 20	46 1	46 16	45 46	45 59	46 6	46 2
$\{q'p' = (021) : (130)$	44 13	44 4	43 59	44 43	44 54	44 27	44 21	44 17	45 8
$\{p'o' = (130) : (112)$	58 48	58 35	58 25	58 50	58 33	59 6	58 51	58 42	58 48
$\{o'q' = (112) : (011)$	62 12	62 36	62 53	61 43	61 54	61 36	62 1	62 16	61 24
$\{qp' = (011) : (130)$	59 0	58 49	58 42	59 27	59 33	59 18	59 8	59 2	59 48

The point which it is now desired to emphasise is that these important generalisations, lying at the root of the goniometrical relationships

of definitely chemically related substances, are the direct outcome of exact measurement, and that the work of the future must partake of the same character if it is to have real and permanent value. The angular differences which have been substantiated between these isomorphous salts are remarkably small, the maximum met with, which occurs between thallium and potassium sulphates in the case of one angle, being only $1^{\circ} 21'$; in the great majority of cases they were very much smaller than this, the differences between a potassium and a caesium salt never reaching a whole degree, and those between a rubidium and a potassium or caesium salt only once attaining even half a degree.

Comparative Table of Crystallographic Axial Ratios.

Salt.	Axial Ratios.		
	a	b	c
K_2SO_4 . . .	0.5727	1	0.7418
Rb_2SO_4 . . .	0.5723	1	0.7485
Cs_2SO_4 . . .	0.5712	1	0.7531
Tl_2SO_4 . . .	0.5555	1	0.7328
$(NH_4)_2SO_4$. . .	0.5635	1	0.7319
K_2SeO_4 . . .	0.5731	1	0.7319
Rb_2SeO_4 . . .	0.5708	1	0.7386
Cs_2SeO_4 . . .	0.5700	1	0.7424
Tl_2SeO_4 . . .	0.5551	1	0.7243

In the similar investigation which the author has carried out of the double sulphates and selenates of the monoclinic series $R_2M\left(\overset{S}{Se}O_4\right)_2 \cdot 6H_2O$ in which R is represented by the same three alkali metals and thallium or ammonium, and which has led to the further confirmation of the law just quoted (the alkali metals exhibiting a markedly preponderating effect in determining the properties of the crystals, which consequently show precisely the same progressive angular changes following the order of the atomic weights of the alkali metals), the angular changes have been found to be somewhat larger, doubtless owing to the lower degree of symmetry (monoclinic instead of rhombic); but even here the maximum difference yet met with has only been $2^{\circ} 21'$, which occurs between potassium magnesium and caesium magnesium sulphates in the case of one angle.

It will be clear now why a high degree of accuracy is to be aimed at, and that a few minutes of error may make a difference so weighty as to render the work of measurement not only valueless, but even productive of positively mistaken conclusions. That the crystals for use in such investigations should be as pure as the refinements of modern chemical methods render possible goes without saying, and that the great care which is inculcated in this book in regard to the growth of the crystals is really necessary will be perfectly obvious. All the

precautions specified in Chapter II. should invariably be taken to avoid disturbance during the growth of the crystals, the goniometer employed should be one of the highest attainable accuracy, such as the Fuess No. 2a or the large horizontal-circle Troughton and Simms instrument, and the utmost care should be taken to exclude all angular values about which there is the slightest ambiguity, and to avoid certain pitfalls into which an inexperienced observer may very easily be precipitated, and to which attention must now be drawn.

It has been rendered abundantly plain in Chapter IV. that the image of the goniometer signal reflected by a perfect face, one which is both truly plane and brilliant, is a single, clearly defined, brightly illuminated image of the broadly terminated but centrally narrow Websky slit of the collimator. Such perfect faces are not common, and half the labour of a crystallographic investigation often consists in the many attempts which are required to grow perfect crystals before success is achieved. The author has been in the habit of measuring ten such crystals of every substance which he has investigated. But if the difficulties in the way of securing ten such perfect crystals prove too great, it is far better to rely on the measurements of even one such irreproachable specimen than to take the mean of the angular values derived from both that crystal and the other nine less perfect ones.

Distorted Signal-images.—A grossly distorted face affords naturally a correspondingly distorted image, or more frequently, an irregularly distributed bundle of distorted images. There are several pronounced types of multiple images afforded by badly distorted faces, or faces having special and peculiar properties. A very common form of image from a distorted face is a ribbon-like band of more or less overlapping images, some of which may not lie precisely in the zone, that is, may not be symmetrical to the horizontal spider-line of the telescope; the various images correspond to reflections of the signal from numerous strip-like parts of the face, the individual parts not having succeeded in quite attaining parallelism owing to disturbance of the conditions during growth. The other specific types of faces yielding multiple images fall under two main divisions, namely, the images from striated faces, and those from closely adjacent faces of the kind known as “vicinal faces.” Both these latter types of facial image merit special attention. Before proceeding to discuss them, however, it is necessary to study the effect on the signal-image of extreme narrowing of a face.

Images from Narrow Faces.—When a crystal face, while remaining truly plane, becomes reduced to a narrow strip,—that is, when development occurs almost entirely in one direction, possibly owing to the large development of two faces in the same zone with it and adjacent on each side, so that the face appears to be merely the blunted edge between those two largely developed faces and appears in the micro-telescope as an illuminated line, or under a higher power as an elongated strip with two long parallel sharp edges,—the phenomena of diffraction or the bending of light waves round sharp edges begin to appear, dependent on the interference of the light waves reflected from near the two

edges, the mass of light reflected from the intermediate parts of the surface becoming so reduced as no longer to drown these delicate diffraction effects. The narrow face, in fact, behaves just like the narrow slit in the well-known lantern experiment on diffraction. When the zone of the three faces is adjusted, the Websky signal-image from the narrow face is expanded horizontally, that is, in the direction of the measurement, rendering the latter less precise when the phenomenon becomes very pronounced. When the face is not narrower than a quarter of a millimetre little inconvenience is caused, and the image is readily placed to the vertical spider-line, the volume of light from the intermediate parts of the strip being still ample to mask all but a trace of diffraction. But when the face becomes more extremely narrow the image not only becomes inconveniently broad, but is accompanied by diffraction spectra on each side, separated from the central white image and from each other at distances which increase as the narrowness becomes more extreme, and which vary with the angle of the incident light. The weakening of the illumination of the white ordinary but now much expanded image, owing to the diminution of the area of the intermediate reflecting parts and to the production of the spectra, becomes so marked as the face approaches a mere line in width, that no trustworthy measurement can be obtained from it.

It will also be clear from the above that coloured images of the signal are always to be rejected. They are either diffraction images of the kind just alluded to, which will be the case when the face is observed to be very narrow, or they may be refracted images of the signal-slit, obtained owing to two of the faces of the zone being inclined at such an angle as to form a spectroscopic prism; they are such images as are obtained in the determination of the refractive index by means of a 60° prism, by the process described in Chapter XLV. These can at once be recognised by their brilliant spectrum character, and if the lens in front of the telescope objective be placed in position so as to enable the observer to view the crystal, the face of the prism from which the light is emerging will be seen illuminated in rainbow colours.

Striated Faces.—When a face is finely striated, it may partake both of the character of a multitude of very narrow faces arranged in close succession, and of the nature of a diffraction grating, and such striated faces are to be most suspiciously examined, and as a general rule altogether excluded from participation in the measurements. They afford a central white image the position of which may have no crystallographic significance, varying with the nature of the striation. Striation has been shown in previous chapters to be due to the rapid alternation on a single crystal of two forms of the same system, such, for instance, as the cube and the pentagonal dodecahedron in the case of pyrites; or else to repetition twinning, as in the case of albite, the two kinds of alternating lamellæ being excessively thin. The effect in either case is similar. So long as the strips belonging to the two alternating series of individuals are not below the limits referred to in the last section on narrow faces, the cumulative effect of each set of parallel strips belonging

to either of the two forms, or the two parts of the twin, is to afford a more or less trustworthy ordinary white image of the signal, the perfection of which depends as usual upon the planeness of the strips, and each set thus affords its own image. But when the narrowness of the striæ approaches that of a line the interference phenomenon of diffraction commences, and the white central image of each set, due to the directly reflected light, is widened and its edges blurred, and it is accompanied by attendant spectra on each side; as in the case of a single very narrow face, this is owing to the interference of the light emanating from near the two edges of the fine strip, just as the light from the two edges of the slit interferes in the well-known lantern diffraction experiment, the colour being due to the light of the different wave-lengths in succession being the only or predominating waves which remain unextinguished by interference at the positions where they are seen. Now, as the width of the striæ still diminishes, there must come a time when the different series of diffraction phenomena will mutually interfere, each pair of adjacent striæ acting as the two edges of a slit, and the two separate series of images will become replaced by one series, the central white image of which occupies a position intermediate between the two former separate white images, corresponding to the plane tangential to the edges of the re-entrant angles formed by the two series of strip-planes; the position of this tangential plane varies with the narrowness of the strips, and the distance from it and from each other of the attendant spectra varies with the angle of incidence of the light.

The brilliance of this intermediate white image, representing no crystallographic face, becomes greater as the striation becomes finer, and when the width of the strip-faces the alternation of which forms the striæ becomes not very much greater than the longest wave-length of light, the theory of diffraction or bending of light-waves round sharp edges is in accordance with the supposition that the tangential plane acts as if it were a real face. Hence, the white image afforded by a finely striated face is absolutely useless for crystallographic purposes in the zone parallel to the striations. In the zone at right angles, however, the images are occasionally trustworthy, but it is infinitely more satisfactory to discard a crystal altogether which exhibits striation, or at any rate not to employ the angles in the zone in which it occurs as basal angles from which to calculate the crystal elements. If either of the two strip-series belong to a zone of intermediate situation the image properly belonging to the zone may often be detected and identified, when the zone is adjusted, by the fact that it passes through the centre of the field of the telescope, the other images being obliquely echeloned on either side above and below the horizontal spider-line. The degree of accuracy with which it may be allocated to the spider-lines, however, is rarely high, on account of blurring.

Thus striated faces are to be shunned by the crystallographer for the purposes of measurement, although they are so interesting from the structural point of view. In serious investigations the only safe course is to ignore such crystals entirely during the measurements, and to seek

for crystals which are free from striation. It may happen that many crops of an artificial chemical preparation will have to be grown before one is obtained comprising crystals which are adequately free from striation.

Vicinal Faces.—The other type of multiple image to which special attention has to be called is observed when the face under examination is not the expected one, but is one of several closely-lying or "vicinal" faces replacing it. It may be that one image of the many constituting the ribbon-like series of multiple images already mentioned as afforded by an apparently distorted face vastly predominates over the others, this being characteristic of the vicinal faces of crystals belonging to the lower systems of symmetry; or it may be that three or four images, some or all of which are slightly out of the zone, appear instead of a single one, and they may either all be approximately of the same intensity, or, as more commonly happens, one image of the group may predominate largely in intensity, and cause the observer to take it for the image of a proper face of the crystal. This phenomenon of vicinal faces has been specially studied by Miers,¹ and his results published in the year 1903 are both curious and highly instructive, as affording an insight into the structural processes occurring during the growth of a crystal. The author also had frequent occasion to observe the phenomenon during his researches on the simple and double sulphates and selenates, and precautions had been taken from the beginning of the work in the year 1890 to avoid all measurements in which the phenomenon was involved.

A study of a growing crystal of potash alum (cubic) led Miers to the observation that the images from the various faces continually altered their positions, that the angle between the adjacent faces was never the theoretical angle of the regular octahedron, and that the faces usually yielded multiple images lying very close together. The latter were subsequently found to be three in number, of which one was often very much brighter than the others, due to the octahedron face being replaced by the facets of a very flat triakis octahedron, of which one face largely predominated over the others. The three images frequently changed their positions, the lines of movement being inclined at 120° to each other, but the movement did not occur continuously, but suddenly at intervals. It was very uncertain, however, whether the forms produced could be expressed by any rational indices, even by very high numbers. The angles of the vicinal faces varied at different times from the true octahedron face by $2'$ to $30'$. Similar observations were made in the cases of sodium chlorate (cubic) and the sulphates of magnesium and zinc (rhombic). In the case of zinc sulphate, the primary prism faces varied from $91^\circ 6'$ to $91^\circ 15'$, due to the production of various vicinal faces instead of the true primary prism $\{110\}$. In the case of magnesium sulphate, similar variations of $20'$ were observed. Hence with crystals of comparatively low symmetry the mean of a number of measurements may give a result very far from the truth. In the cubic system the mean may give the correct result, as the vicinal faces follow the symmetry; but in the case of a rhombic prism each face will, from the

¹ *Phil. Trans.*, A, 1903, vol. 202, p. 459.

symmetry, be replaced by one vicinal plane only, and the prism angle afforded might be considerably too large or too small.

Miers further showed that the vicinal planes follow certain specific well-defined zones only, and that to replace primary faces by vicinal planes is to replace planes of the greatest possible reticular density, referring to the space-lattice structure of the crystal, by planes of the least possible reticular density. Of all possible causes of the phenomenon, change of concentration during crystallisation appeared to be most actively influential. Hence, Miers suggests as the explanation, that the escape of the largely preponderating number of water molecules, during deposition of the relatively much fewer salt molecules from the only slightly supersaturated solution, causes the shower of solid salt particles to fall in a loosely packed condition, and if they be laid down as plane layers thus widely spaced, the plane at any moment is a vicinal face, rather than the primary face itself in which the structural units are packed in their closest order.

It may be of use to other workers to indicate briefly the precautions taken by the author to avoid the phenomenon investigated by Miers. Fortunately, even if the precautions had not been taken, the author's results would not have been affected for two reasons; firstly, because the real differences of angle between the various members of the isomorphous series, either the simple rhombic sulphates and selenates or the monoclinic double salts, although very small, as shown in the previous section of this chapter, were still much larger (in the maximum $1^{\circ} 21'$ and $2^{\circ} 21'$ respectively in the two series) than the angular differences observed between the vicinal faces and the primary planes which they replaced; and secondly, because the whole of such an isomorphous series of salts is similarly affected by the phenomenon, so that comparisons are absolutely valid.

Ever since the commencement of the work in the year 1890 great efforts, invariably with successful results, were made to eliminate vicinal faces. Of each salt a large number of crops, often exceeding a hundred, were prepared, with all the precautions described in Chapter II. against disturbance during crystallisation. The result was that some half-dozen crops of each salt were obtained, in which an adequate number of small crystals were discovered of so perfect a character as to show no sign of vicinal faces, the primary planes themselves being developed at the theoretical angles. These crystals were, as a rule, much smaller than those employed by Miers, and had been deposited under conditions of extreme quietude in a room set apart for the purpose, and from solutions screened from all rapid change of temperature.

One example may be given with advantage, as illustrative of the kind of experience which may be expected. It will be taken from the monoclinic double sulphate and selenate series, $R_2M\left(\overset{S}{\text{Se}}\text{O}_4\right)_2 \cdot 6\text{H}_2\text{O}$, of which ammonium magnesium sulphate, described in detail as an example of monoclinic symmetry in Chapter XVIII., is a member. The crystals of this series generally show striation of the faces of the basal plane

$c = \{001\}$ and the primary prism $p = \{110\}$, parallel to the symmetry plane $b = \{010\}$, due to the formation of vicinal faces instead of the true c and p faces. The directions will be clear from the shading in Fig. 441.

In the case of the c -faces, when the zone $[bqc]$ is adjusted parallel to the axis of the goniometer, two images of the Websky signal, separated by about $20'$, usually predominate very markedly in the bundle, symmetrically arranged to the symmetry plane, that is, about $10'$ on each side of the position for a true c -image. The position of the true c -plane is precisely indicated by the excellence of the reflections afforded by the faces of $q = \{011\}$, which are generally perfect on well-formed crystals, as well as by the faces of the clino-pinakoid $b = \{010\}$, which were also uniformly good, and yielded perfect signal-images. These two forms are

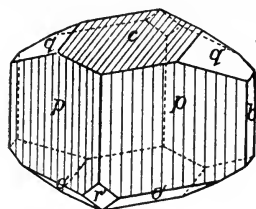


FIG. 441.—Striated Crystal of Ammonium Magnesium Sulphate.

peculiarly free from the vicinal-face phenomenon, so that not only do vicinal faces follow specific zones, as observed by Miers, but they affect only specific forms in those zones. When the zone $[cr']$ is adjusted the c -images, whether those of the true plane or of the vicinal faces, are invariably trustworthy in the case of well-formed crystals. For even if vicinal images, they are vertically over each other and do not affect the horizontal measurement. But even besides this frequent excellence of the q - and b -faces, the author eventually succeeded in preparing crystals which showed no trace of the vicinal faces at all, but only those of the true c - and p -planes.

The kind of thing that was observed with respect to the p -faces on the usual type of fairly good crystal which does show vicinal faces was as follows. The image corresponding exactly to $\{110\}$ was always present, but generally in a bundle of which the two extremes were not symmetrical (naturally from the nature of the symmetry) to it, that is, it was not the central image of the bundle. Also it was not usually the brightest, the latter being sometimes on one side and sometimes on the other. Perfect p -images from the true p -plane were, however, afforded in the cases of the few specially excellent crystals referred to above, and the values obtained with them for the angle bp were always the same, within the usual limit of $1'$ or $2'$ for even the very best of crystal faces.

The interesting fact was also observed that the true position of the c - and p -faces is always clearly indicated by a particularly trustworthy image being afforded whenever the crystal has grown with one of these planes in contact with the flat bottom of the crystallising vessel. Such contact faces are always marked with contour lines, as shown in Fig. 442 in the case of a broad c -face of one of the crystals of ammonium zinc sulphate, which had been during growth the lower one in contact with

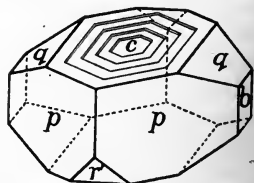


FIG. 442.—Crystal of Ammonium Zinc Sulphate, showing Contoured Contact Face.

the bottom of the dish. These contours are due to a lifting up of the crystal during growth, with eventual production of a hollow pyramid, arranged in steps, owing to the central part of the face not being in contact with more than a film of the saturated solution, whilst the margin is in contact with ample solution and has full facility for the deposition of the shower of solid salt particles. A series of shallow steps are thus produced, the horizontal strip-like surfaces of which, bounded by contour lines resembling etched figures, are parallel to the plane undergoing growth. Now the interesting point is that the whole of these contour strips, which together make up the crystal face, afford a single image of the signal, and accurately in the correct position for a true *c*- or *p*-face. For in all such cases the values for the angles *cq* or *bp* are found to be identical with the values derived from the very few exceptionally perfect crystals, free from vicinal faces, used in the actual measurements. It would thus appear that the favourable conditions for the formation of vicinal faces, that is, for the deposition of the salt particles in a loosely packed state, are not presented at the plane of contact with the crystallising vessel. Hence such contact-plane faces may be used to obtain trustworthy images in the event of absolutely perfect crystals not being eventually obtained.

Special Goniometer for the Study of Growing Crystals.—For the study of the vicinal faces of a growing crystal Miers employed a special goniometer, which is shown in Fig. 443.

This instrument is essentially a horizontal-circle goniometer with all the usual adjustments, but with the crystal-holder and the adjusting movements suspended below the circle instead of being placed above, so that the crystal may dip into the cell *a*. A cell of rectangular form containing the mother liquor is seen in position with the crystal immersed in it, and an alternative one of cylindrical shape with parallel plate-glass front is shown in Fig. 444, along with an adjustable table for it, which is often very useful. The sides of the cell through which the light rays pass are in any case formed of truly parallel plate-glass, and the cell itself is placed on an adjustable stand *b* at the height of the telescope *c* and collimator *d*. The divided circle *e* moves over a circular plate *f* which carries the verniers, and which is rigidly carried by two vertical columns *g*. The collimator is also carried by one of these columns, through an elongated aperture of which it passes; but the telescope is carried at the lower end of a stout counterpoised arm *h* suspended from a thick plate *i* below the vernier plate, which with the axial boss *j* and the counterpoise *l* which is solid with it is capable of rotation about the axis of the instrument, outside the fixed cone, and of being clamped in any desired position. The centring apparatus *m* and adjusting movements *n* are of the usual goniometrical kind, and the inner steel axis *o* from which they are suspended can be raised or lowered by means of the milled head *p* at the top of the instrument. The adjustment for height of the cell table *b* is effected by means of the screw *q* seen in front below the raised base *r* of the instrument. The temperature of the liquid is read by the thermometer *s* with conveniently bent stem shown in the figure. The trough is arranged so that one of its faces is perpendicular to the collimator, a brass guide on the little supporting table ensuring this adjustment.

The crystal-holder, or the circle, or the telescope may be independently rotated round the fixed axis of the goniometer, and any two of these parts may be coupled together in rigid connection at will, a metal disc *t* above the circle-plate and to which the clamping is effected acting as the connecting link. The goniometer is thus

rendered equally suitable for use as a spectrometer for refractive index determinations. The telescope and collimator are each fitted in collars which may be adjusted by means of three screws, and the telescope is provided with rack-and-pinion adjustment, so that when it is converted into a microscope by moving into position the usual viewing lens in front of the objective the crystal may be properly focussed when immersed in liquids of different refractive indices.

The crystal is held in a little clip of platinum foil, which in turn is held by a screw-clip holder *u* fixed in the usual manner at the base of the adjusting movements ;

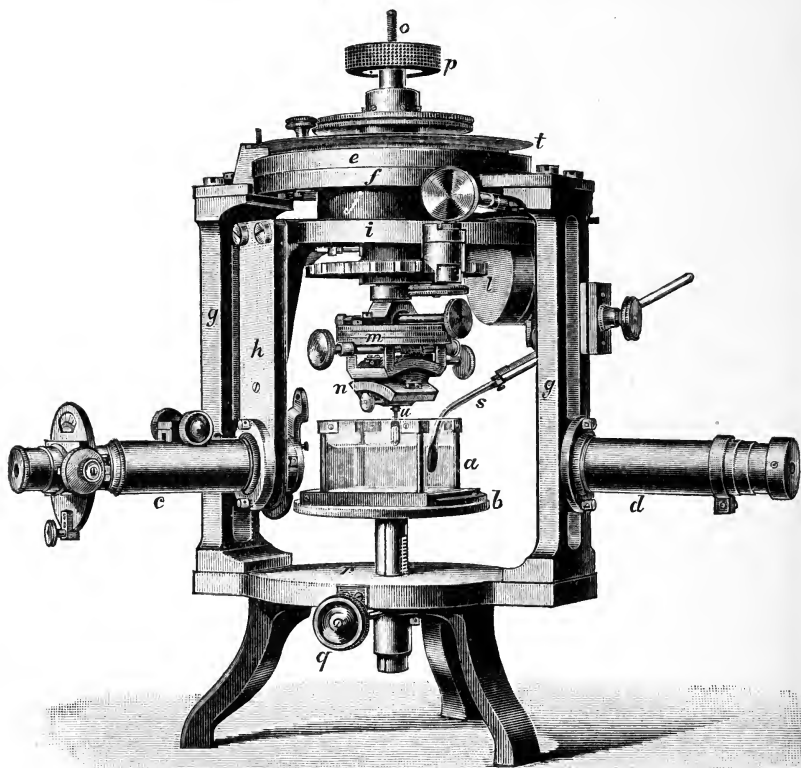


FIG. 443.—Miers Suspended Goniometer.

in the course of the growth of the crystal in the mother liquor held in the dish the platinum becomes grown over and the crystal consequently very rigidly held.

For measuring the displacement of an image in any required direction within the field of view of the telescope a special eyepiece is provided, the plan of which will be gathered from Fig. 445. A similar one was adopted by the author on his cutting and grinding goniometer (Chapter XLII.) with the kind permission of Dr. Miers. During the growth of a crystal the measurements can only be made in the adjusted zone, but variations of angle may be due to the displacement of a face out of the zone as well as in it. Suppose *O* to be the image of the collimator signal (a square one being particularly suitable for this work) as adjusted to the centre of the spider-lines ; and suppose that during the growth of the crystal this image moves out of the adjusted zone *CC* to a position *P*.

We require to measure the distance OP and the angle POC , which we may label θ . To enable this to be achieved a third movable spider-line MM is provided, which is both rotatable and capable of movement transversely to its length in the focal plane by means of a micrometer screw S , this movement being registered by a drum-head divided into 100 parts. The movable wire and the micrometer is attached to a circular frame provided with milled edges by which it can be rotated about the centre O ; the frame envelops a circular disc graduated so as to read directly to degrees, and the angle of rotation is read by means of a vernier V engraved on the inner bevelled side of an aperture cut in the frame, through which the scale on the fixed disc is viewed, which enables the readings to be made to minutes of arc. A toothed scale registers the number of complete turns of the micrometer screw. Hence, both the distance OP and the angle θ can be directly measured. The movable disc and its spider-line can be clamped to the fixed disc in any position by means of the fixing screw R shown in Fig. 445 at the lower right-hand edge of the circular frame. This arrangement enables an image to be followed, whether its movement be as usual, in the zone itself along the horizontal spider-line, or out of the zone in some other direction, or either in or out of the zone but with the image rotated more or less out of the perpendicular.

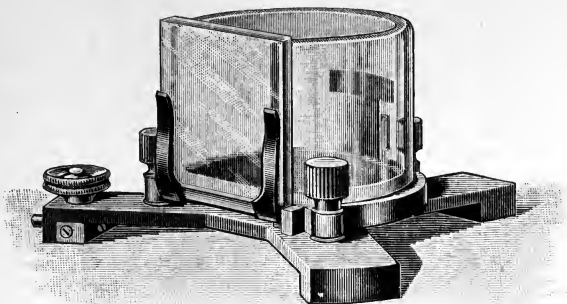


FIG. 444.—Immersion Cell for Use with Suspended Goniometer.

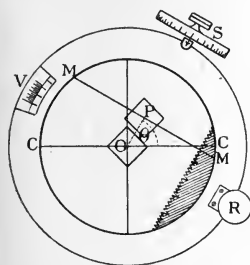


FIG. 445.—Principle of Micrometric-Goniometric Eyepiece.

to the collimator, the telescope is set perpendicular to the adjacent face of the trough by viewing the images of the crossed spider-lines reflected at its surface, by means of the light from a lamp placed beside the telescope and sent down the latter by reflection from a glass plate held obliquely before the eyepiece, the Becker fitting described on page 34, and illustrated in Fig. 13, serving admirably for this purpose.

Besides all ordinary purposes of goniometry this instrument can be used for measuring a crystal in a concentrated solution of its own substance (mother liquor) as above described; for observing the changes of form which a crystal experiences during growth; and for the measurement of deliquescent crystals, or such as are liable to alteration under ordinary atmospheric conditions, while immersed in oil or other suitable inert liquid. The action of solvents on crystals can also be studied by measuring them while immersed in acids or other solvents, and thus ascertaining the angles made by the etched-figure faces with the original crystal faces. The refractive indices of solid plates, prisms, or whole crystals can also be measured either by the method of total reflection or that

of minimum deviation (both to be described in later chapters), by immersing them in suitable liquids; and the refractive indices of a liquid can be similarly measured by means of a plate or prism immersed in it. Further, the angles of isomorphous mixed crystals may be determined as the composition changes, by measuring a crystal while in the solution of another salt of the series, or measurements may be made of the enveloping crystal grown in such a solution upon the kernel of the original salt, in order to establish the relation between the shell and the kernel.

The Measurement of Deliquescent Crystals.—As above indicated, the goniometer shown in Fig. 443 may be used for the measurement of such crystals as are affected by the moisture of the air. The fine polish of a crystal face rapidly deteriorates on removal of the crystal from the desiccator in which it had been stored, when the substance is one having the well-known affinity and attractive force which many substances possess for water, either as a result of extreme solubility or of a still more active and truly chemical affinity for the elements of water. The crystal is immersed during the measurements in an inert liquid, such as benzene, carbon tetrachloride, or other suitable liquid, the index of refraction of which must be known with great precision.

A special apparatus, however, for conducting the measurements in the usual manner with the No. 2*a* goniometer, is supplied by Fuess with that instrument. Its construction will be clear from Fig. 446.

It consists of a glass vessel *a* fitted with two truly-plane plate-glass windows *b* opposite the usual positions of the collimator and telescope; the crystal can be

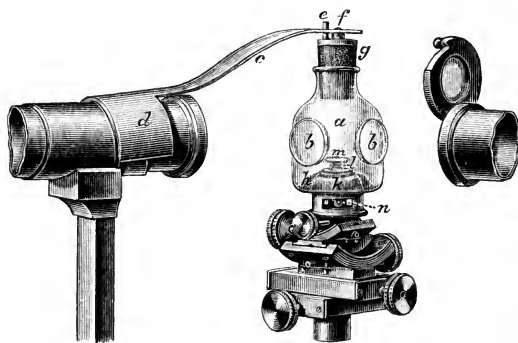


FIG. 446.—Apparatus for Use in the Measurement of Deliquescent Crystals.

enclosed in this desiccating chamber in an air-tight manner, while permitting some degree of adjustment of the crystal and providing a receptacle for a quantity of a desiccating agent such as calcium chloride or phosphoric anhydride. The ordinary crystal-holder is replaced by a special adjustable one, as the ordinary adjusting apparatus cannot be used, owing to the fact that an arm *c*, acting as a strong spring, rigidly connects the top of the crystal chamber with the collimator

tube. The arm is expanded at one end and bent round into the form of rather more than half a cylinder *d* so as to clutch the collimator, while the narrow end of the arm has a short slot and a somewhat wider hole, the former to pass over a steadying pin *e*, and the latter to rest on a central knob *f*, both carried on the top of a little metal disc cemented to the indiarubber stopper *g* which closes the neck of the crystal chamber. The base of the latter is blown in and its central part is open, the channel *h* thus left around the base serving for the reception of the desiccating material. The outside of the part blown in is ground to fit closely the upper spherical part *k* of the special crystal-holder, which consists of a hollow hemisphere mounted on a flat

base, carrying below the peg for attachment in the hole at the top of the ordinary adjusting apparatus instead of the usual crystal-holder. The upper part of the hemisphere carries a little ball-and-socket joint *l*, the ball passing upwards into a little table *m* for the usual wax crystal-holding cone which projects through the opening into the chamber, and below into a short rod *n*, by which the adjustment of the crystal can be effected; the rod is manipulated by a little handle, perforated with a hole near one end to fit the rod, and to allow the insertion of which the lower part of the fitting above the base is open, the hemisphere being continued down on to the base in short columnar fashion merely at three equidistant places.

In using the apparatus, the chamber is first furnished with a little fresh desiccating substance, calcium chloride being usually most suitable; the crystal is mounted on the wax and the whole holder then placed in position on the adjusting apparatus of the goniometer. A trace of oil is smeared on the upper part of the little hemisphere and the chamber fitted with its ground part over it, the oil thus closing the lower part completely by rendering the fit air-tight. The stopper is then inserted above, and the arm-end placed over it, the other wide semi-tubular end, which is packed inside with cloth, being fitted over the collimator end, which it just grips, being rather more than half a cylinder. When this is effected the arm will be pressing the chamber tightly down on the hemisphere, and the peg of the stopper, which fits in the slot near the arm-end, prevents any rotation of the crystal chamber during the rotation of the circle and crystal, the oil lubrication between the hemisphere and the ground base of the chamber facilitating free rotation without any drag on the chamber. The adjustment of the two faces of a zone is then carried out with the aid of the key, a few approximations being required before the adjustment is perfect. Unfortunately, only a few closely-lying zones, sometimes only one, can be adjusted with the same setting of the crystal, and every setting means opening the chamber to the air again.

The author has, consequently, not found this apparatus to be quite satisfactory in actual practice. It was subjected to a severe test during the work on the selenates of potassium, rubidium, and caesium. These three salts are progressively deliquescent, following the order of the atomic weights of the metals. Caesium selenate is so highly deliquescent that the crystals eventually dissolve in moisture attracted from the air. The desiccating chamber was used when the measurements were being carried out on moist days, but it was found better in the cases of potassium and rubidium selenates to work only on very dry days, such as when an easterly or northerly wind was blowing, and to dispense with the special apparatus. It was found possible under those conditions to complete the measurements before the deliquescence had materially affected the brilliancy of the faces, other crystals having been previously well studied while the desiccating chamber was in position, so as to familiarise one with their character and thus facilitate the subsequent measurements. Even in the case of caesium selenate this method, combined with the use of twice the usual number of crystals, so as only to carry out half the measurements on any one crystal and the remaining half on another, was found preferable to using

the desiccating chamber, for the crystals proved to be as deliquescent as calcium chloride, and phosphoric anhydride became almost fully deliquesced before the operations could be begun. Still, in the cases of other deliquescent substances the apparatus might prove more serviceable.

For extreme cases of deliquescence, such as work which the author has on hand concerning the crystalline forms of the oxides of phosphorus, P_4O_6 , P_2O_4 , and P_2O_5 , it is preferable to enclose the whole goniometer and all accessories in a desiccating chamber, such as that (Fig. 473) which will be found described in Chapter XXXII. (page 523), as used for specific gravity work on cæsium selenate. It is essentially an air-tight mahogany box, thoroughly well varnished inside and outside, and provided with plate-glass windows and two apertures closed with indiarubber gloves, in which the observers' hands are placed from outside during the manipulations. The goniometer and all the necessary accessories are enclosed overnight, along with phosphoric anhydride; one of the windows is removable for the purpose, and is immediately afterwards bolted on again. Indiarubber pads are fixed on both the side-cheeks and the window frame in order to ensure tight fitting. Next morning the air within is usually dry enough to enable successful measurements to be carried out, the only difficulty being the slight handicap incurred by the wearing of the gloves.

Testing of the Goniometer Circle.—Before concluding this chapter on goniometrical precautions and special difficulties in crystal measurement, it should be remarked that it goes almost without saying that the goniometer employed should be thoroughly tested as regards the accuracy of its circle, by repeating the measurements with the same faces of a perfect crystal in various parts of the limb. If there be any difference amounting to a minute or two, such repetitions will be necessary throughout all investigations carried out with it, the mean of the various readings being accepted as nearest the truth. This, however, has never been necessary with the author's No. 2*a* and No. 1*a* Fuess instruments, nor with the larger Troughton and Simms goniometers, the division of the circle in each case having proved to be wonderfully accurate.

CHAPTER XXXI

DENSITY, VOLUME, AND STRUCTURE—THE 14 SPACE-LATTICES— MOLECULAR DISTANCE RATIOS.

Density and Structure.—The relative density of crystals,—which is afforded by the experimental determination of their specific gravity as compared with that of water at 4° C., the temperature of its maximum density,—has acquired a far greater importance than formerly, owing to the fact that in any series of isomorphous substances of known molecular weight it enables us to calculate the relative volumes of the molecular unit cells of the structural space-lattices of the various members of the series. Also further, by combining this with the knowledge of the crystallographic axial ratios already gained from the goniometrical investigation of the substances, we are able to calculate the distances apart of the centres of gravity or other analogous points of the chemical molecules in the three (or four) directions of the crystallographic axes, these relative distances being termed the “molecular distance ratios” or “topic axial ratios.” For it was shown in Chapter IX. that the structure of a crystal is a homogeneous one, which can be represented by a system of points, each point standing for a chemical molecule or for any one of the atoms composing the molecule. The later form of Sohncke’s theory of crystal structure, as harmonised with the work of Schönflies, von Fedorow, and Barlow detailed in Chapter IX., is now generally accepted, as being in accordance with all the experimental facts concerning crystals acquired up to the present, as well as with the important advances which have been made in our geometrical knowledge of the possible types of homogeneous structures. In its essence it assumes that the crystal is built up of as many similar interpenetrating point-systems (space-lattices) as there are atoms in the chemical molecule, the relations of any one such point system with respect to any other being consequently expressed by the relative (stereometric) positions of the atoms with respect to each other in the molecule; such relative positions in most cases determine the particular class (out of the 32 possible classes of symmetry) to which the crystal conforms, while the character of any one of the point-systems (all being similar) determines which system of symmetry among the seven possible systems the crystal shall exhibit.

Suppose we consider our typical crystal of potassium sulphate once more, K_2SO_4 , a substance exceptionally rich in regard to the lessons which may be derived from it. A molecule of this crystalline salt consists of seven atoms, two of potassium, one of sulphur, and four of oxygen. The system in which the salt crystallises has been experimentally shown in Chapters IV. and VIII. to be the orthorhombic. This fact is determined by the arrangement of the space-lattice, formed by the point-system in which the points are analogous points, one taken from each chemical molecule present. Equally as well as if we took the centre of gravity of the molecule, we can take any atom within the molecule as the analogous point. We can, in fact, take all seven atoms in turn, or indeed all seven simultaneously. We can readily imagine seven exactly similar models of the space-lattice constructed, say of marbles or indiarubber balls to represent points, and if we stretch the imagination sufficiently far as to assume the structures to be held together by suitable forces and not by means of solid links, we can suppose the seven to be interpenetrable and to be made to all but coincide. We can then push these seven models one within the other till they are nearly identical, so as to bring every seven points, one similarly situated point from each model in each case, into close proximity, closer than the points of any one and the same model, and also so that the seven points thus congregated together about every point (representative of the cluster) of the common space-lattice are arranged or spaced as are the actual atoms or their spheres of influence in every molecule of the substance; thus, instead of a single point representing a molecule, seven points so represent it, and these seven are arranged among themselves stereometrically, as described on page 130 in Chapter IX. In the case of potassium sulphate the single sulphur atom will be innermost and the two potassium atoms situated about it symmetrically in such wise as affects chiefly the vertical axis of the crystal, in accordance with the experimental facts revealed by the investigation of the changes produced by replacing the atoms of one metal by those of another metal of the same alkali family group. We thus gain a very fair idea of the structure of a typical crystal of potassium sulphate, and of the nature of crystal structure in general. For it has been conclusively shown by the whole of the author's investigations of isomorphous salts, that the atoms or their spheres of influence do take up specific orientations within the molecule.

Molecular Volume. — Now, as the elementary parallelepipeda of the space-lattice will obviously correspond to the molecular cells of which the crystal substance is built up, it follows that the "molecular volume," obtained by dividing the molecular weight of the chemical compound by the specific gravity of its crystals, means in these cases of isomorphous substances, the structures of which are certainly of the same type, the relative volume of the elementary parallelepipedon of the point-system or space-lattice which represents that type; for the points are, as we have seen, strictly analogous ones within the molecules, one in each, representative of the molecules.

The quotient $\frac{\text{molecular weight}}{\text{specific gravity}}$ thus acquires from the above considerations a vastly greater importance, and when it is remembered that in the crystallographic axial ratios $a:b:c$ we have the directional relative dimensions of the elementary parallelepipedon of the crystal structure, expressed in terms of $b=1$, it will be clear that not only are the relative volumes of the parallelepipeda of different members of the series actually afforded by the corresponding values of this quotient, the so-called "molecular volume," but that if we combine the latter with the crystallographic axial ratios we obtain a relative measure of the dimensions of the molecular cell in the three (or four) directions of the crystal axes, for the several members of the series.

The meaning attached to the term molecular volume has in the past been a somewhat erroneous one, for it is certain that a comparison of the quotient $\frac{\text{mol. wt.}}{\text{spec. gr.}}$ for two substances of entirely different chemical constitution does not afford us necessarily any information of real value. For the type of crystal structure may be, and in general is, quite different. But in the case of a series of isomorphous compounds we are certain that the structure is analogous and strictly comparable, only differing slightly in dimensions as one element of the family group constituting the series is replaced by another, in accordance with the law revealed by the work on the sulphates and selenates that the structural dimensions are functions of the atomic weights of the interchangeable elements. Hence, in these cases we are on sure ground, and the comparison of the molecular volumes and of the corresponding directional dimensions of the molecular cells of the space-lattices is most instructive.

The Fourteen Types of Space-lattice.—Before passing to the discussion of the mode of combining the molecular volume with the crystallographic axial ratios, in order to arrive at these directional dimensions of the molecular cells, it is essential that the nature of the fourteen space-lattices, the elementary parallelepipeda of which represent the molecular cells, should be thoroughly understood. They may be divided into seven groups, corresponding to the seven crystal systems, which may be conveniently considered in descending order of symmetry.

First Group; Space-lattices with Cubic Parallelepipeda.—The elementary parallelepipedon of this group is either (1) a simple cube, with three equal sets of

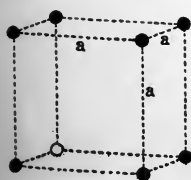


FIG. 447.—Space-lattice 1.

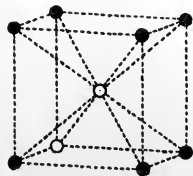


FIG. 448.—Space-lattice 2.

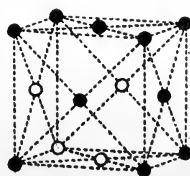


FIG. 449.—Space-lattice 3.

edges corresponding to the three equal and rectangularly intersecting crystallographic axes aaa , as represented in Fig. 447; (2) a cube with a point at its centre as shown

in Fig. 448; or (3) a cube with a point in the middle of each face, as portrayed in Fig. 449.

The densest packing of points occurs in type (1) along the cube planes, in type (2) along the planes of the rhombic dodecahedron, and in type (3) along those of the octahedron. The faces of the cube, rhombic dodecahedron and octahedron are also the planes of cleavage which alone are developed in cubic crystals, thus agreeing with the rule pointed out in Chapter XXVII, that cleavage occurs along those planes in which the points of the structure-lattice are most densely packed. The highly symmetrical nature of these three cubic structures also determines the isotropic or singly refractive optical character of cubic crystals in all directions, the optical properties being represented by a sphere. The absence of double refraction, the main optical distinction of cubic crystals, is thus the direct outcome of the highly symmetrical character of the very simple parallelepipeda of the three cubic space-lattices.

Second Group; Space-lattices with Tetragonal Parallelepipeda.—Two types of space-lattice constitute this group. The elementary parallelepipedon of the one (4)

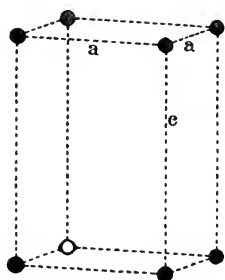


FIG. 450.—Space-lattice 4.

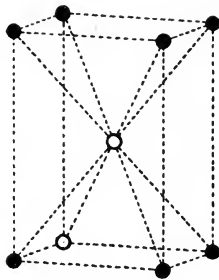


FIG. 451.—Space-lattice 5.

is a simple tetragonal prism, with two kinds of edges, a the equal and rectangular horizontal ones and c the vertical ones at right angles to the a edges, corresponding to the crystallographic axes a and c ; and that of the other (5) is a tetragonal prism with a point at its centre. They are shown in Figs. 450 and 451.

The planes of greatest reticular density may be either those of the basal plane or the sides, or the diagonal planes of the prism, or those of the tetragonal pyramid. These correspond to the possible directions of cleavage in the tetragonal system, the principal (tetragonal) axis being the direction of either minimum or maximum cohesion, just as it is the unique direction in the crystal in which optically there is no double refraction and along which the phenomenon of a single optic axis is presented, revealed by the exhibition of a black cross and circular spectrum-coloured rings, when a plate perpendicular to this direction is examined in convergent polarised light. Both the cohesion and the optical characters are represented by an ellipsoid of revolution, the principal axis of which is the tetragonal axis.

Third Group; One Space-lattice with Parallelepipeda exhibiting Trigonal-hexagonal Symmetry.—The elementary parallelepipedon of this group is a simple trigonal prism the section of which is an equilateral triangle having three 60° -angles. Every two such contiguous parallelepipeda, however, are in contact along a prism face of each, which may be considered common, in this space-lattice, and they together make up a rhombic prism of 120° angle. The 120° -prism may, therefore, equally well be considered as the elementary parallelepipedon of this space-lattice No. 6, the points being identical in the two cases.

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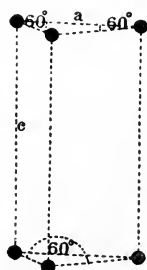


FIG. 452.—Space-lattice 6.

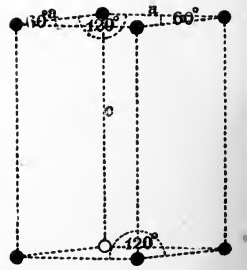


FIG. 453.—Space-lattice 6.

But if the trigonal prism be regarded as the unit parallelepipedon alternate individuals are arranged with their apices inverted. The two prisms are shown in Figs. 452 and 453; Fig. 452 is simply the right half of Fig. 453 with the front and back points also joined by dotted lines.

For we are at liberty to join the two points at the extremities of the narrow diagonal of each rhombic base if we choose. All the horizontal edges of either prism are equal and correspond to the crystallographic Bravais-Miller equal horizontal axes a of the hexagonal system of symmetry, while the relation of the vertical edges to these is that of the Bravais-Miller axis c . The plan of such a space-lattice, drawn through the points parallel to the base of the prism, is shown in both Figs. 454 and 455. In Fig. 454 the points are joined by two series of parallel lines inclined at 120° , while in Fig.

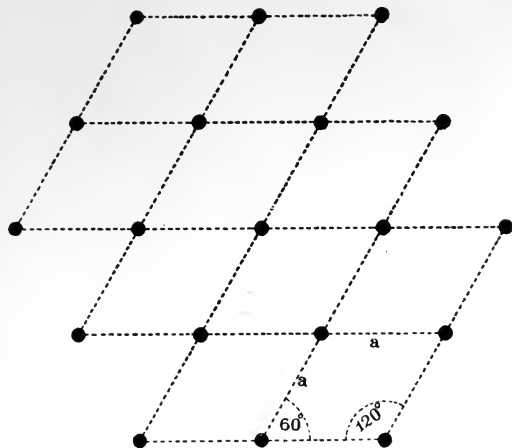


FIG. 454.

455 the same points are shown joined by these two series of lines and also by a third series which we are equally entitled to draw at 120° to the first two series, the three series having equal values as regards the symmetry. It will be obvious that the

space-lattice is hexagonal in character, but with a point in the centre of each hexagon; also that the unit parallelepipedon in Fig. 454 has a rhombic base of 120° , corresponding to the rhombic prism of that angle shown in Fig. 453, and that the parallelepipedon of Fig. 455 is the trigonal prism with a 60° -triangle for base as shown in Fig. 452. The homogeneous structure comprised of parallelepipeda each with a simple hexagonal base without a point at its centre is not a space-lattice.

If we draw a hexagonal prism, as in Fig. 456, placing a point at the centre of

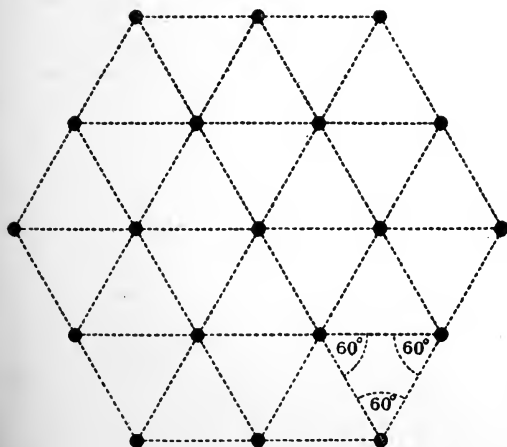


FIG. 455.

each face of the basal pinakoid, the three horizontal Bravais-Miller hexagonal axes a will be observed to divide the hexagonal prism up into three rhombic ones of 120° having the same height c , resembling that of Fig. 453. Hence one is quite justified in taking the 120° -prism as the elementary parallelepipedon of the space-lattice. But it must be remembered that cleavage parallel to the faces of the hexagonal prism divides the solid up into 60° -triangular prisms, so that we are

equally justified on this ground in taking the 60° -prism, the half of the rhombic one, as the unit parallelepipedon of the space-lattice. It will subsequently be shown that in calculating the molecular distance ratios it is immaterial which of the two

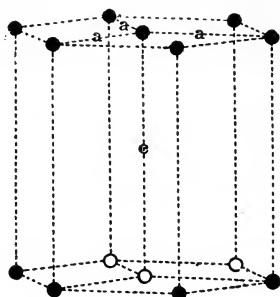


FIG. 456.

parallelepipeda is taken as unit, provided it is definitely stated which has been so chosen, and that the same choice is made throughout the whole series of compounds the molecular distance ratios of which are to be compared. For obviously the positions of the points themselves, and the lengths of the lines joining them along the crystallographic axial directions, to which the molecular distance ratios refer, are the same; the volume, however, will be double in the case of the rhombic 120° -prism to what it is in the 60° -prism, so that the absolute values of the distance ratios will be different, although their ratio will be the same, namely that of the crystallographic axial ratio $a : c$.

In this space-lattice the greatest reticular density may be along the basal plane of the prisms, or along the three planes mutually inclined at 120° intersecting in the prism axis and perpendicular to the basal plane. Thus the axial direction of the prism is a direction of optical single refraction and of minimum or maximum cohesion, and the principal axis of the ellipsoids of revolution which represent respectively the optical properties and the cohesion. Hexagonal or trigonal crystals constructed according to this space-lattice are, therefore, uniaxial like tetragonal crystals as regards their optical behaviour. Perfect cleavage may occur either parallel to the basal plane or to the faces of the hexagonal prism. Secondary cleavage may also occur, however, parallel to the faces of a hexagonal pyramid, as the plane passing through any pair of points in the basal plane and the opposite point at the other end of the same trigonal prism concerned is also one which is comparatively thickly strewn with points.

Fourth Group; one Space-lattice with Trigonal-rhombohedral Parallelepipeda.—Like the third group, only one space-lattice No. 7 is comprised in this group, the elementary parallelepipedon of which is the rhombohedron. It is represented in Fig. 457.

It possesses three planes of symmetry which intersect each other regularly, that is, at 60° , in the vertical or principal axis, the trigonal axis. The three edges of the parallelepipedon are equal, corresponding to the three equal Millerian axes a , and are equally inclined to each other; but the rhombohedral angle is different for different substances. The vertical principal axis (not a Millerian axis) is a direction of single refraction, just as is the trigonal axis of a trigonal prism belonging to space-lattice 6, and it is also the principal (or singular) axis of the ellipsoid of revolution which expresses the optical characters. Thus, like the tetragonal and hexagonal parallelepipeda, the rhombohedron is optically uniaxial. The points may be most thickly strewn either along the faces of the rhombohedron, in which case cleavage occurs along these planes, as in calcite for instance, or along planes perpendicular to the principal (vertical) axis, when the cleavage follows the direction of the basal plane; or else, thirdly, along the three symmetry planes or their bisecting planes, all of which intersect in the principal axis, and either of which confers cleavage parallel to the faces of a hexagonal prism.

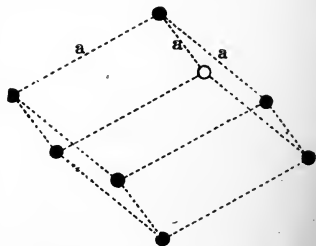


FIG. 457.—Space-lattice 7.

Fifth Group; Space-lattices with Orthorhombic Parallelepiped. — This group comprises four parallelepipeda which are symmetrical to three mutually rectangular planes. They are represented in Figs. 458, 459, 460, and 461. The first, that of space-lattice No. 8, is a rhombic prism with normal base (perpendicular to the axis of the prism); the prism faces correspond to those of the crystallographic form $\{110\}$, and the basal edges are therefore marked p in Fig. 458, while the vertical edges correspond to the axis c , to which they are parallel. The second, that of space-lattice No. 9, is the same but with a point at the centre, forming a rhombic pyramid with the four points at the base. The third, that of space-lattice No. 10, is the right-rhombic prism, a rectangular parallelepipedon with three unequal sets of edges parallel to the crystallographic axes a , b and c . The fourth, that of space-lattice No. 11, is the same as the last but with a point at

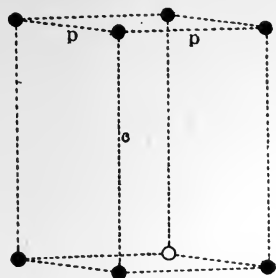


FIG. 458.—Space-lattice 8.

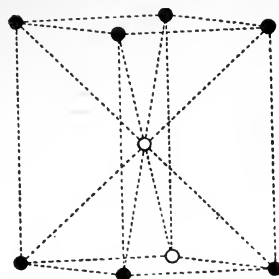


FIG. 459.—Space-lattice 9.

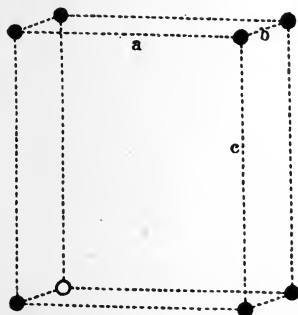


FIG. 460.—Space-lattice 10.

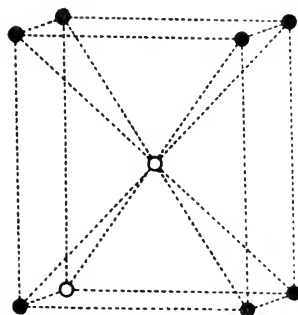


FIG. 461.—Space-lattice 11.

the centre forming a right rhombic pyramid with the four basal points. The optical ellipsoid must here be obviously one with three unequal rectangular axes, the lines of intersection of the three rectangularly intersecting symmetry planes, and there will be no longer one predominating singular axis, that of an ellipsoid of revolution and the unique direction of single refraction, but there will be a pair of directions of single refraction, those two perpendicular to which the ellipsoid of general form will have circular sections, of radius equal to the intermediate axis and of which the latter is a common diameter. The crystals are, therefore, said to be optically biaxial, and a plate perpendicular to the acute bisectrix of the angle between these two optic axes (directions of single refraction, and not to be confounded with the three axes of the optical ellipsoid, two of which bisect the acute and obtuse angles between the optic axes) actually shows their positions when examined in convergent polarised light. For these positions are marked by a series of concentric (confocal) spectrum-rings, like Newton's rings, in each case, but which are lemniscates and not true circles;

the two centres or foci where the optic axes are located are further marked by the sharply defined apices of dark hyperbolic curves ("brushes"), which join up to form a black cross when the symmetry planes coincide with the planes of the Nicol prisms of the polariscope, one arm of the cross passing through both optic axes and the other crossing this first arm rectangularly midway between the two optic axes (see Fig. 567, Plate II., facing page 674). In space-lattices Nos. 8 and 9 the symmetry planes are the basal plane and the two planes containing the diagonals of the rhombic base and the vertical axis; in space-lattices Nos. 10 and 11 they are the planes parallel to the prismatic and basal pinakoidal faces. Owing to the variable dimensions of the three sets of rectangular edges and of the pyramidal angle, the planes most densely strewn with points may be (1) one of the three planes of symmetry, (2) the planes of the rhombic prism, or (3) those of the rhombic pyramid; consequently, three corresponding varieties of cleavage can occur.

Sixth Group; Space-lattices with Monoclinic Parallelepiped.—This group comprises two types of parallelepipeda, namely (1), the rhombic prism with inclined base, Fig. 462, which builds up space-lattice No. 12, and corresponds to the monoclinic prism $\{110\}$ closed by the basal pinakoid $\{001\}$, the basal edges being consequently marked p in the figure, and the vertical prism edges c (being parallel to the axis c);

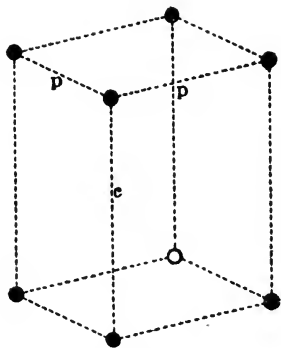


FIG. 462.—Space-lattice 12.

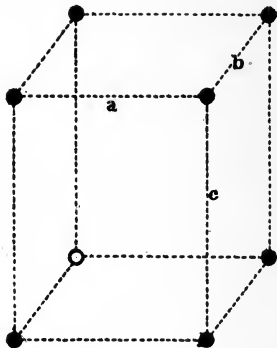


FIG. 463.—Space-lattice 13.

and (2) the parallelepipedon with two pairs of rectangular faces (oblongs) and one pair of oblique (rhomboidal) ones, which may be regarded as a rectangular rhombic prism which has been pushed over along one of its three planes and corresponds to the solid formed by the combination of the three monoclinic pinakoids, the basal pinakoid $\{001\}$, ortho-pinakoid $\{100\}$, and clino-pinakoid $\{010\}$, the latter oblique pair of faces being parallel to the symmetry plane. The edges of this second type are thus parallel to the crystallographic axes a , b , and c ; it is shown in Fig. 463, and builds up the space-lattice No. 13. Both types possess only one symmetry plane, the unique one characterising the monoclinic system, passing through the front and back points in Fig. 462 but parallel to the right and left sides of the prism of Fig. 463.

The planes of greatest density of points may be the faces of the prism in Fig. 462, or any other two planes symmetrical to the symmetry plane, or there may be only one plane of maximum density, either parallel or perpendicular to the symmetry plane. The cleavage will follow these directions. Optically, the crystals are characterised by an ellipsoid with three unequal axes, and consequently possess two optic axes of single refraction, like the fifth rhombic group, the only difference being that the ellipsoid is now only fixed in direction as regards that axis which is perpendicular to the symmetry plane, whereas in group five the three axes of the ellipsoid are all identical with the three structural and crystallographic axes.

Seventh Group; Space-lattice with Triclinic Parallelepiped.—This is the general case of a space-lattice (No. 14) the elementary parallelepipedon of which is of triclinic character, that is, the three sets of edges parallel to the crystallographic axes a , b , c are of unequal length and their three angles of inclination α , β , and γ are also unequal. It is shown in Fig. 464. These six elements for any one substance are different to those of any other substance which crystallises according to this same type of structure. The three planes of the three pairs of faces thus correspond to the three triclinic pinakoids, which combine to form the simple closed solid of the triclinic system the faces of which are parallel to the crystallographic axial planes, namely, the basal pinakoid $\{001\}$, the macro-pinakoid $\{100\}$, and the brachy-pinakoid $\{010\}$.

Any plane of points, however, may have the greatest reticular density, and the cleavage is thus subject to no definite rule, beyond the fact that only one plane can be that of perfect cleavage, perpendicular to the unique direction of minimum cohesion, there being no symmetry developed. Optically, a similar ellipsoid with three unequal axes and having two optic axes or directions of single refraction characterises the structure, as in the cases of the rhombic and monoclinic groups; but there is now no restriction whatever as to its position. Actual observation has shown, however, that when a plane of perfect cleavage is developed in a triclinic crystal, one of the axes of the optical ellipsoid is as a rule nearly perpendicular to it, and the other two axes are nearly parallel to it. The same

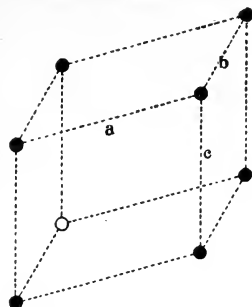


FIG. 464.—Space-lattice 14.

curious occurrence is often observed also in the monoclinic system, where, when a perfect cleavage is developed parallel to a plane perpendicular to the symmetry plane, one of the axes of the optical ellipsoid is frequently nearly perpendicular to it. Mica is an excellent example, the cleavage of which, the most perfect of that of all known crystals, is so nearly perpendicular to that axis of the optical ellipsoid which forms the bisectrix of the acute angle between the two optic axes (directions of single refraction) that the well-known interference figure of rings and brushes, above described as characteristic of biaxial crystals when a suitable plate is examined in convergent polarised light, is exhibited to perfection by a cleavage plate of mica; the two series of spectrum-coloured rings surrounding the two optic axes are apparently (although careful measurement shows that they are not absolutely) symmetrical to the centre of the field of the polariscope (see Figs. 557 and 558 on Plate I., facing page 670).

Molecular Distance Ratios or Topic Axial Ratios.—Having now described the fourteen space-lattices, the points of which represent the molecular structural units and the elementary parallelepipeda of which represent the molecular cells, we are in a position to discuss the mode of combining the crystallographic axial ratios with the molecular volume, so as to represent the relative directional dimensions of the parallelepipeda for a definitely related isomorphous series of substances, corresponding to the total change of molecular volume which occurs when one of the interchangeable chemical elements is replaced by another. The genesis of the constants is due to three distinct investigators, Prof. F. Becke of Prague, who suggested the idea,¹ Prof. W. Muthmann of Munich,² and the author,³ who simultaneously and quite unknown to each other applied it in practical investigation, the former

¹ *Anzeiger der kais. Akad. der Wiss.*, Wien, 1893, 30, 204.

² *Zeitschr. für Kryst.*, 1894, 22, 497.

³ *Journ. Chem. Soc. Trans.*, 1894, 65, 628.

in the case of the permanganates of the alkalis, and the latter to the alkali sulphates and subsequently to the corresponding selenates and to the double sulphates and selenates. These dimensional constants were termed by the author "distance ratios," and by Muthmann "topic axial ratios" from "*τόπος*," "space." The ratios really represent the relative distances apart of the points of the space-lattice,—the centres of gravity or other analogous representative points (such as a particular atom of every molecule) of contiguous chemical molecules in the directions of the crystallographic axes,—that is, the dimensions of the elementary parallelepipedon or molecular cell, and not the actual dimensions of the material molecule itself. For we do not yet know the relations of the free space and the actual matter in the unit cell or elementary parallelepipedon of the space-lattice corresponding to each molecule. Further consideration and the later development of the work in this domain only confirm the author's original view that the term "molecular distance ratios" is the most suitable appellation for these important constants.

It is now definitely proved that the structural units of the space-lattice, the chemical molecules, of the various members of any one and the same isomorphous series are built up into crystals on the same plan. The elements of the crystals, which it will be remembered are the relative lengths of the crystallographic axes a , b , and c cut off by the primary (parametral) pyramid faces and the three mutual inclinations of the axes α , β , and γ , are consequently relative measures of the sides and also the angles of the elementary parallelepipedon of the structure. The axial ratios, being usually expressed so that $b = 1$, only afford the mutual relations of the three axes to each other in that particular substance, and give no idea of the relations between the dimensions of the elementary parallelepipeda of different members of the series. But if we take the relative volumes of the parallelepipeda into consideration, by combining the molecular volume of each member of the series with its crystallographic elements, we at once obtain the relative dimensions of the elementary parallelepipeda of the various members of the series. These three new axial dimensions expressing the true relations in space are the **molecular distance ratios or topic axial ratios**, and are represented by χ , ψ , and ω .

They are expressed by simple formulæ involving only the crystallographic axial values a , b , c and the axial angles α , β , γ , the density (specific gravity at 20° compared with water at 4°) of the crystals, and the molecular weight of the chemical substance composing the crystal. If the whole space of the elementary parallelepipedon were filled with the matter of the molecule, that is, if the various atoms were packed in contact without interstices and the whole filled and took the shape of the parallelepipedon, the molecular distance ratios would indicate the relative dimensions of the chemical molecules themselves. But it is certain that the molecular cell is not thus filled with matter, hence, as above indicated, they do not represent the molecular dimensions but the distances apart of the molecular representative points along the directions of the axes. It is very important that this distinction should

be thoroughly understood, as the designation "topic axial ratios" does not fully imply it, and it is for this specific reason that the term "molecular distance ratios" is more appropriate.

In order to derive the formulæ the most general case, that of a triclinic crystal, space-lattice No. 14, will first be considered, and subsequently the formulæ will be shown to simplify, stage by stage, as the symmetry ascends.

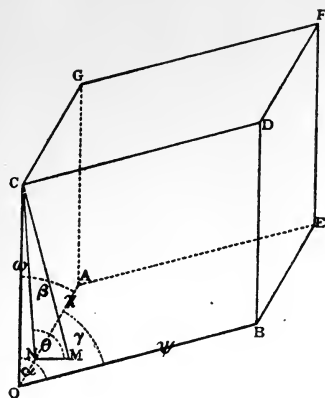


FIG. 465.

In the **triclinic** parallelepipedon shown in Fig. 465 (an enlarged Fig. 464) the axial angles are α , β , and γ , and the lengths of the sides OA, OB, and OC are χ , ψ , and ω . If we let fall a perpendicular CM from the corner C on to the face ABE, and from M draw MN perpendicular to OA, and call the angle CNM (between the planes AOB and AOC) θ , then V, the volume of the parallelepipedon, is expressed by the formula:¹

$$V = \chi\psi\omega \sin \beta \cdot \sin \gamma \cdot \sin \theta.$$

Now χ , ψ , and ω bear the same relation to each other as do the axial ratios $a : b : c$, so that:

$$a = \frac{\chi}{\psi}, \text{ and } c = \frac{\omega}{\psi}.$$

Hence:

$$\begin{aligned} \chi &= a\psi, \text{ and } \omega = c\psi, \text{ and} \\ V &= a\psi \cdot \psi \cdot c\psi \sin \beta \cdot \sin \gamma \cdot \sin \theta, \\ &= ac\psi^3 \cdot \sin \beta \cdot \sin \gamma \cdot \sin \theta, \end{aligned}$$

or

$$\psi^3 = \frac{V}{ac \sin \beta \cdot \sin \gamma \cdot \sin \theta}$$

and

$$\psi = \sqrt[3]{\frac{V}{ac \sin \beta \cdot \sin \gamma \cdot \sin \theta}}$$

Then:

$$\chi = a\psi = \sqrt[3]{\frac{a^2 V}{c \sin \beta \cdot \sin \gamma \cdot \sin \theta}}$$

and

$$\omega = c\psi = \sqrt[3]{\frac{c^2 V}{a \sin \beta \cdot \sin \gamma \cdot \sin \theta}}.$$

It may happen that the angle θ has not been calculated during the ordinary calculation of the elements $a : b : c$ and α , β , γ , and the crystal angles. In this event it may readily be derived from the following formula of the well-known type:

$$\sin \frac{\theta}{2} = \sqrt{\frac{\sin(s-\beta) \sin(s-\gamma)}{\sin \beta \sin \gamma}},$$

where

$$s = \frac{\alpha + \beta + \gamma}{2}.$$

The molecular distance ratios are finally expressed in a similar manner to the crystallographic axial ratios, namely, as $\chi : \psi : \omega$.

The above expressions for the relative values of the molecular distance ratios χ , ψ , and ω in the general case of a triclinic crystal simplify with the introduction of symmetry, the first stage being obviously that of the monoclinic system.

In the case of a **monoclinic** crystal of analogous axial-plane-pinakoidal type,

¹ Kraus and Mez, *Zeitschrift für Kryst.*, 1901, 34, 390.

constructed according to space-lattice No. 13, Fig. 463, the angles α , γ , and θ are right angles, and as $\sin 90^\circ = 1$ we have :

$$\chi = \sqrt[3]{\frac{a^2 V}{c \sin \beta}}, \quad \psi = \sqrt[3]{\frac{V}{ac \sin \beta}}, \quad \omega = \sqrt[3]{\frac{c^2 V}{a \sin \beta}}.$$

When the symmetry is **rhombic**, the structure being that of the rectangular axial-plane type of space-lattice No. 10, Fig. 460, β is also 90° , and $\sin \beta$ also consequently disappears, so that :

$$\chi = \sqrt[3]{\frac{a^2 V}{c}}, \quad \psi = \sqrt[3]{\frac{V}{ac}}, \quad \omega = \sqrt[3]{\frac{c^2 V}{a}}.$$

These formulæ were given by the author in the 1894 memoir already alluded to, and the monoclinic formulæ in a succeeding one in the year 1896.

In the case of **tetragonal** symmetry, corresponding to space-lattice No. 4, Fig. 450, $a=b=1$, so that :

$$\chi = \psi = \sqrt[3]{\frac{V}{c}}, \text{ and } \omega = \sqrt[3]{c^2 V}.$$

When, finally, the symmetry of the **cubic** system is reached, and the structure is the simple cubic one of space-lattice No. 1, Fig. 447, $a=b=c=1$, so that obviously

$$\chi = \psi = \omega = \sqrt[3]{V}.$$

The formulæ for the **trigonal system** may also be derived directly from the general triclinic ones. We have in this case the trigonal space-lattice No. 7, Fig. 457, in which the conditions are that : $a=b=c=1$ and $\alpha=\beta=\gamma$, so that the general formulæ reduce to :

$$\chi = \psi = \omega = \sqrt[3]{\frac{V}{\sin^2 \alpha \sin \theta}},$$

θ in this case being the inner angle over the polar edges of the rhombohedron, the supplement of the outer angle actually measured between two faces of the rhombohedron over their intersecting polar edge. Its relation to the crystal element α is expressed by the simple formula

$$\sin \frac{\theta}{2} = \frac{\sin \frac{\alpha}{2}}{\sin \alpha}.$$

The case of **hexagonal symmetry** is peculiar, for it has already been shown that in a hexagonal space-lattice, owing to the fact that a point also occupies the centre of each hexagon, the points are situated at the corners of a 120° -prism, which may be considered as the base of the elementary parallelepipedon ; every three such prisms together make up the hexagonal prism, as will be clear from Fig. 453 already given showing the simple prism, Fig. 456 showing the combination of three to make up the hexagonal prism, and Fig. 466 now given to show the basal plan of the three.

Assuming then that the 120° -prism is the elementary parallelepipedon, the molecular distance ratio $\chi : \omega$ is calculated in the following manner. The relation between the common height of the prisms ω and the distance χ separating the points in the horizontal plane (the sides of the rhombic base), employing Bravais-Miller axes, is that of the vertical axis c to any one of the three horizontal axes a , that is, it is equal to the crystallographic axial ratio $a : c$, or as $a=1$ it is equal to c ; hence $\frac{\omega}{\chi} = c$. All three horizontal axes being equal, we refer to them indifferently as χ . Now, if p in Fig. 466 represent the perpendicular ae from the centre

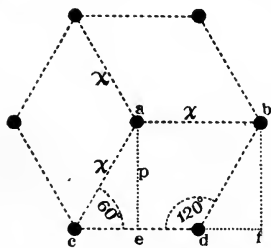


FIG. 466.

on one of the sides cd of the hexagon, $p = \chi \sin 60^\circ$, and the volume V of the 120° -prism is afforded by multiplying the height ω into the area $p\chi$ of the rectangle $abfe$, as this rectangle is equal to the base $abcd$ of the 120° -prism. Therefore,

$$\begin{aligned} V &= \omega \cdot \chi^2 \sin 60^\circ, \\ &= c\chi \cdot \chi^2 \sin 60^\circ, \\ &= c\chi^3 \sin 60^\circ. \end{aligned}$$

Hence,
$$\chi^3 = \frac{V}{c \sin 60^\circ}, \text{ and } \chi = \sqrt[3]{\frac{V}{c \sin 60^\circ}}.$$

Also, as
$$\omega = c\chi, \text{ and } \omega^3 = c^3\chi^3 = \frac{c^2V}{\sin 60^\circ},$$

we have
$$\omega = \sqrt[3]{\frac{c^2V}{\sin 60^\circ}}.$$

These simple formulæ for χ and ω thus represent the molecular distance ratio on the assumption that the 120° -prism is the elementary parallelepipedon. But it has also been shown that the half of this, the 60° -prism, is equally entitled, from the point of view of the cleavage, to be considered as the elementary parallelepipedon. If this assumption be made the ratio $\chi : \omega$ will still have the same value, that of the crystallographic axial ratio $a : c$, but the absolute numbers themselves and the formulæ expressing them will be different, on account of the volume now being only one-half of that of the 120° -prism. The points a and d must be supposed to be joined in Fig. 466, and the area of the base acd of the 60° -prism will be $p\frac{\chi}{2}$ instead of $p\chi$. The volume V is, therefore, as under :

$$\begin{aligned} V &= \omega \cdot p\frac{\chi}{2} = \omega \cdot \chi \sin 60^\circ \cdot \frac{\chi}{2} = \frac{1}{2}\omega\chi^2 \sin 60^\circ, \\ &= \frac{1}{2}c\chi \cdot \chi^2 \sin 60^\circ = \frac{c\chi^3 \sin 60^\circ}{2}, \end{aligned}$$

and
$$\chi^3 = \frac{2V}{c \sin 60^\circ},$$

from which
$$\chi = \sqrt[3]{\frac{2V}{c \sin 60^\circ}}.$$

Then
$$\omega = c\chi, \omega^3 = \frac{2Vc^3}{c \sin 60^\circ} = \frac{2c^2V}{\sin 60^\circ},$$

and
$$\omega = \sqrt[3]{\frac{2c^2V}{\sin 60^\circ}}.$$

Thus the formulæ for χ and ω , on the assumption that the 60° -prism is the elementary parallelepipedon, only differ from those for a 120° -prism parallelepipedon by having an additional figure 2 in the numerator of the expression in each case. In the above alternative cases for the treatment of hexagonal crystals it has thus been assumed that the molecular cell, corresponding to a chemical molecule of the substance, is either the 120° -prism or the 60° -prism, with normal base in each case, each point of the space-lattice representing a molecule just as in the case of the simplest of all space-lattices, the cubic No. 1; in fact, the 120° -prism may be regarded as a cubic one pushed over 30° laterally in one direction and elongated or compressed vertically according to the value of the vertical axis c .

We have now dealt with the calculation of the molecular distance ratios for the elementary parallelepipeda of the seven simplest space-lattices, one corresponding to each of the seven crystalline systems. These will be the cases most frequently required.

Two other cases of great importance and also often required are those of the rhombic and monoclinic space-lattices Nos. 8 (Fig. 458) and 12 (Fig. 462) respectively, in both of which the base is a rhombic prism and the volume of the prismatic

parallelepipedon is one-half of that of the parallelepipedon with rectangular base (space-lattices Nos. 10 and 13, Figs. 460 and 463) the edges of which are parallel to the crystallographic axes a and b . These two cases, in fact, resemble the one just dealt with, in which the 60° -prism has one-half the volume of the 120° -prism, and the formulæ can be shown to differ in the same simple manner from those for the parallelepipedon of double the volume, namely, by the addition of a 2 in the numerator in each case.

The case of the **rhombic space-lattice, No. 8**, the elementary parallelepipedon of which is shown in Fig. 458, may well be taken as an example of the mode of deriving the formulæ. It has been shown by Barker¹ to be the structure present in the barytes group of minerals and in the perchlorates and permanganates of the alkalis (the latter having been investigated by Muthmann²). The directions of perfect cleavage are parallel to the basal plane $\{001\}$ (highly perfect) and the orthorhombic prism $\{110\}$, precisely the planes of the space-lattice and those along which the reticular density is greatest. The case of these salts is thus a very clear and indeed unmistakable one.

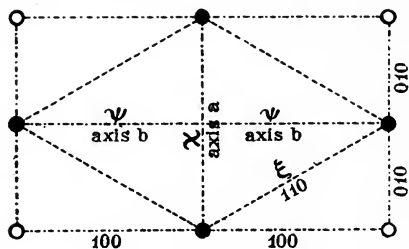


Fig. 467.

The plan of such a prism is shown in Fig. 467, which also exhibits for comparison the points of the rectangular rhombic space-lattice No. 10, these latter points being shown as rings to distinguish them. It will at once be clear that the area of the rhombic base (solid points) is one-half of that of the base of the rectangular rhombic prism the edges of which are parallel to the horizontal axes a and c ; and as the volume = area \times height, the volume is also one-half, the height being common. The axes a

and b are the diagonals of the rhombic prism, and the diagonal molecular distances χ and ψ respectively. The vertical distance between the points, corresponding to the common height of the prisms, is as usual called ω . Besides these molecular distances we have also to consider a fourth important dimension, namely, the side of the prism, which we may call molecular distance ξ . Now, the volume V of the rhombic prism is obviously

$$V = \frac{1}{2} \chi \cdot \psi \cdot \omega,$$

if we consider that of the rectangular rhombic prism as the product of the three molecular distances χ , ψ , ω . As $\omega = c\psi$ and $\chi = a\psi$, from the fundamental assumption that the molecular distance ratios bear the same relation to each other as do the crystallographic axial ratios, being in fact the structural expression of the latter, we may substitute these values for χ and ω in the expression for the volume, and thus get:

$$V = \frac{1}{2} a \psi \cdot \psi \cdot c \psi, \\ = \frac{1}{2} a c \psi^3.$$

Hence : $\psi^3 = \frac{2V}{ac}$, and $\psi = \sqrt[3]{\frac{2V}{ac}}$.

Then : $\chi = a\psi$

and : $\chi^3 = \frac{2a^3V}{ac} = \frac{2a^2V}{c}$, from which $\chi = \sqrt[3]{\frac{2a^2V}{c}}$.

Also : $\omega = c\psi$,

and : $\omega^3 = \frac{2c^3V}{ac} = \frac{2c^2V}{a}$, from which $\omega = \sqrt[3]{\frac{2c^2V}{a}}$.

¹ *Journ. Chem. Soc. Trans.*, 1906, 89, 1141.

² *Zeitschr. für Kryst.*, 1894, 22, 497.

The fourth molecular dimension, ξ the side of the rhombic base and edge (001) : (110), is at once afforded by the following simple considerations :

$$\xi^2 = \left(\frac{\chi}{2}\right)^2 + \left(\frac{\psi}{2}\right)^2 = \frac{1}{4}(\chi^2 + \psi^2).$$

Putting in the values of χ and ψ already found, and simplifying, we have :

$$\xi^2 = \frac{1}{4} \left[\left(\sqrt{\frac{3 \cdot 2a^2V}{c}} \right)^2 + \left(\sqrt{\frac{3 \cdot 2V}{ac}} \right)^2 \right] = \frac{1}{4}(1 + a^2) \left(\sqrt{\frac{3 \cdot 2V}{ac}} \right)^2$$

$$\xi = \frac{1}{2} \sqrt{1 + a^2} \cdot \sqrt{\frac{3 \cdot 2V}{ac}}.$$

Intimately connected at the same time with this case of the space-lattice having for its elementary parallelepipedon a prism with rhombic base, and with the case previously treated of the hexagonal space-lattice which has the 120° -prism for its unit molecular cell, is the very important case of the numerous **pseudo-hexagonal crystals**. Our typical crystallised substance potassium sulphate is a particularly striking example of this case, as already mentioned in Chapter XXVI., p. 430, as are also the whole of the salts of the very important isomorphous series of alkali sulphates and selenates of which it forms the first and best-known member. Fig. 411 on p. 431 shows, for instance, a crystal of rubidium sulphate closely resembling a hexagonal bipyramid modified round the basal edges by short faces of the hexagonal prism. The treatment of the pseudo-hexagonal problem will, therefore, be next worked out for this series of rhombic salts, R_2SeO_4 , and the formulæ arrived at shown to be of general application to the case.

It has been shown by the author that the prism zone of these alkali sulphates and selenates, consisting of the two parallel brachy-pinakoid faces $b = \{010\}$ and the four faces of the primary prism $p = \{110\}$, approaches a hexagonal prism so closely that the angles are within 1° of 60° . In the case of our typical crystal of potassium sulphate itself, the angles were shown in Chapter IV. to be $bp = (010) : (110) = 60^\circ 12'$, and $pp = (110) : (1\bar{1}0) = 59^\circ 36'$. The former is four times symmetrically repeated in the zone, and the latter twice, the six angles together making up 360° . The variation is but small for the different members of the series, pp varying only from $59^\circ 24'$ to $59^\circ 49'$. Now the cleavage directions common to the whole of the metallic salts of this isomorphous series are the brachy-pinakoid $b = \{010\}$ and the basal pinakoid $c = \{001\}$; both are good cleavages, but the former is the more perfect of the two in the cases of the potassium, rubidium, caesium, and thallium salts, while in the case of ammonium sulphate, doubtless owing to the stereometric difference between a metal and the radicle group NH_4 , the latter cleavage is very perfect while the former is not developed at all. The absence throughout the series of any cleavage parallel to the macro-pinakoid $a = \{100\}$ fully agrees with the assumption that the structure is not of a rectangular rhombic character, and the actual cleavages developed are quite in accordance with a pseudo-hexagonal structure, in which the basal pinakoid and the brachy-pinakoid are planes thickly strewn with points, whereas the plane of the macro-pinakoid, which would be the third plane of a rectangular rhombic structure, is not.

The conditions will be appreciated with the aid of Fig. 468, which is reproduced from the author's memoir on "topic axes, and the topic parameters of the alkali sulphates and selenates."¹ The crystal is supposed to be rotated 90° about the vertical axis, from the usual position of representation of a rhombic crystal, so as to bring it into the position corresponding to the normal hexagonal prism, usually referred to as of the first order, and to the hexagonal-space lattice as shown in Fig. 466. Indeed Fig. 468 may be taken to represent either the arrangement of points in the space-lattice, or the plan of a hexagonal prism or hexagonal pyramid. In the case of

¹ *Journ. Chem. Soc. Trans.*, 1905, 87, 1185.

the alkali sulphates and selenates the hexagonal prism (here not absolutely hexagonal but pseudo) is built up of the two faces of $b = \{010\}$ and the four faces of $p = \{110\}$ as already mentioned, while the corresponding pyramid is built up of the four primary pyramid faces $o = \{111\}$ and the pair of brachy-dome faces $q' = \{021\}$.

In Fig. 468 χ , ψ , ω represent the molecular distances on the assumption of pseudo-hexagonal structure, while χ' , ψ' , ω' refer to a rectangular rhombic structure, these latter topic axes being parallel to the crystallographic axes. The vertical crystallographic axis c is equally coincident with the vertical topic axis ω or ω' on either assumption, and is not actually shown in the figure as it is perpendicular to the plane of the paper. The direction OA of the crystallographic axis a is that of the pseudo-hexagonal topic axis ψ or of the rectangular rhombic topic axis χ' . The direction OB of the crystallographic axis b is that of the rectangular rhombic topic axis ψ' , but is no longer a topic axial direction on the assumption of a pseudo-hexagonal space-lattice; instead, we have a pair of equal pseudo-hexagonal topic axes χ , which, as the lattice is not perfectly hexagonal, are not quite equal in length to the topic axis ψ . Now, it will at once be seen on comparing Fig. 467, representing the structure with a rhombic base, with Fig. 468 that the χ of Fig. 467 is the ψ of Fig. 468, and that the pair of molecular distances χ of Fig. 468 correspond to ξ , the side of the rhombic base in Fig. 467. Also, the vertical topic axis is common. Hence, if V represent the molecular volume and a and c the crystallographic axial ratios as usual, the pseudo-hexagonal molecular distance ratios are as under:

FIG. 468.—Pseudo-hexagonal Space-lattice of Alkali Sulphates and Selenates.

$$\begin{aligned}\chi &= \frac{1}{2} \sqrt{1+a^2} \sqrt[3]{\frac{2V}{ac}}, \\ \psi &= \sqrt[3]{\frac{2a^2V}{c}}, \\ \omega &= \sqrt[3]{\frac{2c^2V}{a}}.\end{aligned}$$

This may be directly proved from first principles as follows. The area of the rhombus the sides of which are χ is obviously half the area of the rectangle $\chi'\psi'$; ψ is equal to the shorter diagonal of the rhombus parallel to the axis a , and the longer diagonal is the axis b . Now, the elementary parallelepipedon has the rhombus for its base and ω for its height (parallel to the axis c). Hence, expressing the volume in terms of the rectangular rhombic topic axes:

$$V = \frac{1}{2} \chi' \cdot \psi' \cdot \omega', \quad (1).$$

But as the sides of this rectangular parallelepipedon are proportional to the crystallographic axes:

$$\chi' : \psi' : \omega = a : 1 : c,$$

and, remembering that the two vertical topic axes ω and ω' are identical:

$$\chi' = \frac{a\omega}{c}, \quad (2); \text{ and } \psi' = \frac{\omega}{c}, \quad (3).$$

Substituting these values in (1) we obtain:

$$V = \frac{1}{2} \cdot \frac{a\omega}{c} \cdot \frac{\omega}{c} \cdot \omega = \frac{a\omega^3}{2c^2}.$$

From which we derive:

$$\omega^3 = \frac{2c^2V}{a}, \text{ and } \omega = \sqrt[3]{\frac{2c^2V}{a}}.$$

As regards ψ it has been shown to be equal to χ' ; therefore, from (2):

$$\psi = \frac{a}{c} \cdot \sqrt[3]{\frac{2c^2V}{a}} = \sqrt[3]{\frac{2a^2V}{c}}.$$

As regards χ , it will be apparent that:

$$\chi^2 = \left(\frac{\psi}{2}\right)^2 + \left(\frac{\psi'}{2}\right)^2 = 4 \left[\left(\sqrt[3]{\frac{2a^2V}{c}} \right)^2 + \left(\sqrt[3]{\frac{2V}{ac}} \right)^2 \right].$$

In the last expression the value in (3) is substituted for ψ' ; the whole expression then further simplifies to:

$$\chi^2 = \frac{1}{4}(1+a^2) \left(\sqrt[3]{\frac{2V}{ac}} \right)^2,$$

from which we finally get:

$$\chi = \frac{1}{2} \cdot \sqrt{1+a^2} \cdot \sqrt[3]{\frac{2V}{ac}}.$$

We have thus independently proved these three formulæ for χ , ψ , and ω directly from first principles.

If it should be desired to compare a pseudo-hexagonal crystal with a truly hexagonal one or a trigonal one of hexagonal habit, care must be taken in calculating the topic parameters that they are strictly comparable with the pseudo-hexagonal ones. Such has been done by Gossner¹ in the case of the double sulphate and double chromate of potassium and sodium, $K_3Na(SO_4)_2$ and $K_3Na(CrO_4)_2$, which are trigonal-hexagonal yet so similar to potassium sulphate and potassium chromate, considered as pseudo-hexagonal, that Gossner has compared their molecular distance ratios. Their Bravais-Miller axial ratios as trigonal crystals are respectively $a:c=1:1.2904$ and $a:c=1:1.2857$; but for the purposes of comparison with the pseudo-hexagonal sulphate and chromate, which are rhombic but of pseudo-hexagonal habit, the trigonal axial ratios were converted to rhombic ones, that is, the value of what would be the b axis on a rhombic assumption was found and the values of a and c referred to it as unity. The new values of a and c are, of course, such that $c/a=1.2904$ and 1.2857 for the double sulphate and double chromate respectively. In the result, Gossner found that,—while on the one hand the simple sulphate and chromate afforded molecular distance ratios extremely close in value, and on the other hand the two double salts also showed topic axial ratios very near to each other,—yet there was considerable difference between the absolute values for the two pairs of salts, indicating that while the space-lattices were similar their dimensions in space were considerably different. This is just what we should expect, for the two series are not isomorphous in the author's strict "eutropic" sense of the word (see page 484), and the structures are not such as can be so intimately blended as to produce mixed crystals, as is the case with the two members of either pair, which are thus strictly related. This example is given as an indication of the care that must be taken, in making such comparisons, that the constants and formulæ used are themselves strictly comparable; for otherwise such important conclusions can have no value whatever.

The above simple case of pseudo-hexagonal structure, in which we have a rhombic crystal of nearly 60° -prism angle, may have its monoclinic or even triclinic analogue. For deformation may not only occur as to the prism angle, out of 60° , but also the basal plane may be deformed out of the perpendicular to the prism zone, in which case the monoclinic angle β comes into consideration; also the deformation of the prism zone itself from 60° may occur asymmetrically. In both cases the volume is half that of the original monoclinic or triclinic solid the edges of which are parallel to the crystallographic axes. Each case of the kind can be worked out from first principles, from the specific conditions presented, on the lines which have been amply illustrated in this chapter.

¹ *Zeitschr. für Kryst.*, 1904, 39, 166.

One excellent example, which it may be useful briefly to consider, of a monoclinic pseudo-hexagonal crystal, is afforded by the monoclinic form of dimorphous ammonium selenate, $(\text{NH}_4)_2\text{SeO}_4$. This is the form of the salt which is commonly obtained, by crystallising a solution of selenic acid which has been saturated with ammonia gas. The rhombic form isomorphous with the other alkali selenates and sulphates is only known in mixed crystals with one or other of the latter, rubidium selenate mixing the most readily to form perfect transparent crystals, owing to its similar molecular volume and molecular distance ratios; large transparent rhombic crystals of mixed ammonium and rubidium selenates have been found to contain over 60 per cent of ammonium selenate. It is further remarkable that the monoclinic form of this latter salt retains the pseudo-hexagonal character of the prism zone, so very characteristic of the rhombic salts of the series. The angle $ap = (100) : (110)$ is $59^\circ 38'$, $ap' = (100) : (310)$ is $29^\circ 38'$, and $p'p = (310) : (110)$ is exactly $30^\circ 0'$. A typical crystal of ammonium selenate and its stereographic projection have already been illustrated in Figs. 417 (p. 433) and 29 (p. 73), and a reference to these figures will render the positions of the various faces quite clear. The primary prism $\{110\}$ of the rhombic crystals appears to correspond, however, to the tertiary prism $\{310\}$ of the monoclinic crystals, and in the latter, of course, the basal plane is tilted instead of being perpendicular to the prism zone. Fig. 469, reproduced from the author's paper on ammonium selenate,¹ represents a section through the crystal perpendicular to the prism zone $[apb]$; the plane of the paper is no longer parallel to the basal plane, as in Fig. 468, that is, the inclined axis a lies out of the plane of the paper, except where it intersects the latter at the centre, but the vertical axis is perpendicular to the plane of the paper and the symmetry axis b lies in the plane. The molecular distance ψ represents the separation of

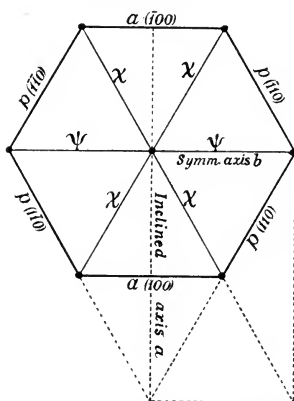


FIG. 469.—Pseudo-hexagonal Space-lattice of Ammonium Selenate.

the structural points along the symmetry axis b , and ω that along the vertical axis; χ represents the separation along two equal pseudo-hexagonal axes, inclined to the plane of the paper, and the planes of which lie at nearly 60° to each other and to the symmetry axis b . The formulæ employed by the author in calculating the molecular distance ratios were as under:

$$\chi = \frac{1}{2} \cdot \sqrt{1 + a^2} \cdot \sqrt[3]{\frac{2V}{ac \sin \beta}}, \quad \psi = \sqrt[3]{\frac{2V}{ac \sin \beta}}, \quad \omega = \sqrt[3]{\frac{2c^2V}{a \sin \beta}}.$$

It will be observed that the formulæ for χ and ω for the monoclinic crystals of ammonium selenate only differ from the corresponding formulæ for the rhombic salts, as given on page 514, by having $\sin \beta$, the monoclinic angle between the vertical and inclined axes c and a , introduced in the denominator of the fraction. An additional $\sin \beta$ also occurs in the expression for ψ , but here, as the axis a of the rhombic salts corresponds to the symmetry axis b of the monoclinic salts, the expression contains the term $\frac{2V}{ac}$ instead of $\frac{2a^2V}{c}$; the expression for ψ , in fact, is that of ψ for the rhombic space-lattice as given on page 512, with the addition of $\sin \beta$ in the denominator.

There still remain to be dealt with those five of the fourteen space-lattices in which points occur either in the centre of the parallelepipedon (Nos. 2, 5, 9, and 11) or in the centre of each of its faces (No. 3).

¹ *Journ. Chem. Soc. Trans.*, 1906, 89, 1070.

In the case of **space-lattice No. 9** (Fig. 459), in which a point occurs in the centre of a **rhombic prism** (a prism having a rhombus for base, normal to the vertical axis), if we consider, as usual, each point as representing a molecule the solid corresponding is a hexaparallelohedron, a combination of the three rhombic prisms $\{110\}$, $\{011\}$, and $\{101\}$. The volume is one-half of that of the parallelepipedon with rhombic base, that of space-lattice No. 8 (Fig. 458), and with this knowledge and that already acquired as to space-lattice No. 8 the molecular distance ratios, including those involving the central point, can be readily calculated from first principles on the lines already now so fully exemplified in previous pages.

The case of the **rectangular rhombic prism with a point at its centre, space-lattice No. 11** (Fig. 461), is similar. The presence of a point at the centre of a parallelepipedon having three pairs of unequal but mutually rectangular faces, parallel to the axial planes, renders the solid corresponding to each molecular point a heptaparallelohedron, the combination of a rhombic bipyramid with the three axial-plane pinakoids. The volume is one-half of that of the parent parallelepipedon of Fig. 460, the number of points in the same space being doubled by the introduction of the central points into the space-lattice.

With these data it is easy to work out the molecular distance ratios whenever required, from first principles.

The next is the case of the **tetragonal space-lattice No. 5** (Fig. 451), in which a point is situated in the centre of the **tetragonal prism**. This introduction of a central point doubles the number of points in the same space and halves the volume, the resulting solid corresponding to each molecular point being a hexaparallelohedron, a combination of the tetragonal prism with the bipyramid. This information is again adequate to enable the molecular distance ratios to be readily calculated.

If the original parallelepipedon is a **cube**, and a **central point is introduced** so as to convert it into **space-lattice No. 2** (Fig. 448), the solid corresponding to each molecular point is a heptaparallelohedron, a combination of the cube and octahedron, and the greatest reticular density has been shown to occur along the planes of the rhombic dodecahedron. The volume is one-half of that of the original cube, and with this knowledge the calculation of the molecular distance ratios is a very simple matter.

In the last case of a **cube with a point in the centre of each face, space-lattice No. 3** (Fig. 449), each point corresponds to a solid having the shape of the rhombic dodecahedron, and the planes of maximum density have been shown to be those of the octahedron. The volume in this case is one-fourth of that of the original cube, and this fact enables the molecular distance ratios to be calculated without difficulty.

CHAPTER XXXII

THE DETERMINATION OF THE DENSITY OF CRYSTALS.

SPECIFIC GRAVITY determinations require to be much more accurately carried out than has hitherto been the case, in order that the results may be of real value for the purpose of calculating the molecular volume and the molecular distance ratios (topic axial ratios). The crystal material employed, moreover, requires to be of the highest possible degree of purity and homogeneity. The most serious and frequent source of error is due either to the presence of cavities containing mother-liquor within the crystals, when the latter consist of pure chemical substances crystallised from solution, or to the presence of impurities and inclusions of foreign material if the crystals are those of a naturally occurring mineral.

There are only two methods of determining the density of crystals with the requisite degree of accuracy, one being the method of the pycnometer or specific gravity bottle (assuming only the most delicate and accurate form of pycnometer to be employed), within which the powdered crystals are introduced, the bottle being filled up by an inert liquid of known density; the other is the method which we owe to Retgers, of immersion of the crystal, whole and unpowdered, in a liquid of the same density, prepared by diluting a heavy liquid such as methylene iodide to the correct density with a lighter miscible liquid such as benzene.

The latter method is preferable, as the crystal is preserved intact and uninjured, whenever small and perfectly transparent crystals are available the density of which is not superior to that of methylene iodide, 3.33. The heaviest crystal, that is, the one yielding the highest result for the specific gravity, is taken as affording the nearest approach to the truth, its greater density being assumed to be due to its superior freedom from cavities. When the density of the crystal is higher than that of methylene iodide, the pycnometer method is alone available, and the highest result is again taken for a similar reason. When the care and the precautions to be now described are taken, the highest result of the pycnometer method is usually found to agree very closely with the mean result afforded by several determinations with the heaviest crystal by the immersion method, in cases where both methods

are available, several such cases having been very thoroughly tested by the author.

The Pyknometer Method.—The form of pyknometer employed and recommended by the author is shown in Fig. 470, which represents the duplicate pair actually used in all the author's investigations. Each is an ordinary specific gravity bottle of 10 c.c. capacity, with an especially accurately ground stopper centrally perforated throughout its vertical length by a capillary bore. A cap has been blown to fit over the stopper and ground to make excellent contact with the upper conically ground exterior of the neck of the bottle; it is maintained pressed down into position by a spring forming part of a little supporting brass-wire stand, which affords the means of moving the bottle to and from the desiccator and balance without contact with the hand.

Carbon tetrachloride, CCl_4 , is the liquid employed by the author,



FIG. 470.

Duplicate Pyknometers with
Caps.

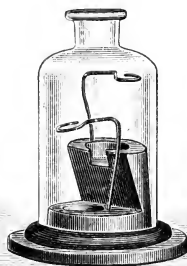


FIG. 471.

Vacuum-receiver for Exhaustion
of Pyknometers.

the chief advantages which it possesses being freedom from action on all the crystalline salts hitherto dealt with, combined with great permanence in a state of purity. The use of the cap is to prevent evaporation of the carbon tetrachloride from either the neck or the capillary, both of which are enclosed within the cap. Many forms of pyknometer, and many variations of the pyknometer method, have been tried by the author, but the method of the capped pyknometer is infinitely superior to any other. It is an advantage to employ a pair of these capped pyknometers simultaneously, affording two distinct determinations, carried out under identical conditions. Such a pair are shown in Fig. 470, standing on a glass plate which forms the base of a glass protecting cover, the whole acting also as a desiccator, a dish of vitriol being included to maintain a dry atmosphere. When it is intended to use the pyknometers, they should be thoroughly cleaned if not already so (this should obviously be done immediately after every determination, leaving them clean for the next), and allowed to stand in their wire supports under the glass cover-vase for a sufficient length

of time (a quarter of an hour is a good time) to recover surface equilibrium; they may then be weighed on the most accurate balance available, such as the most delicate Sartorius short beam balance. The weights should have been standardised, and be preferably platinised, or of platinum throughout, not merely those below one gramme which are always made of platinum.

Such a quantity of the powdered crystals should be employed as occupies about one-third or at most not more than one-half of the capacity of the bottle in each case, in order to obtain the best results, the rest of the space being filled up with carbon tetrachloride. The pulverisation of the crystals, and the filling of the bottle with the powder and carbon tetrachloride, are very important operations, the accuracy of the determinations depending largely on the care taken with them. A perfectly clean agate mortar should be employed, and the powder sifted through the very finest platinum gauze before it is accepted as adequately pulverised. Moreover, as the powdering frees the mother-liquor from such cavities as are broken into, the powder invariably requires careful drying before transference to the bottles. The temperature at which the drying can occur with safety depends, of course, on the nature of the crystalline substance. In the case of the anhydrous sulphates of the alkalis, potassium sulphate for instance, it may be well over 100°C. , and as high as 150° , a copper air-bath being employed. In the case of salts containing water of crystallisation, such as ammonium magnesium sulphate with $6\text{H}_2\text{O}$, a temperature only very slightly superior to the ordinary atmospheric can with safety be employed. After drying, it is advisable to powder the material again in the mortar and to resift through the platinum gauze.

After transferring the proper amount of the powdered crystalline substance to the two pycnometers, the interior of the necks must be well cleared of powder with a little silk cloth, the stoppers inserted, the caps adjusted, and the whole placed in the wire stand in each case. The two stands with their bottles are then allowed to rest in the desiccator for the usual interval. They are then to be each separately weighed, just as they are, stand included. The weight of the bottles in their stands when empty being known, from previous weighings (confirmed before each determination), the additional weight in each case is that of the salt introduced. The bottles are then opened again, and sufficient carbon tetrachloride is poured in to cover well the powder and about two-thirds fill the bottle in each case, not more, or some may be forced out by bubbles during evacuation. The bottles themselves, without stoppers, caps or stands, then require to be placed in a Sprengel vacuum, in order to extract the air imprisoned between the grains of the powder, which would otherwise cause a grave error. A receiver of the minimum size necessary to take the two bottles simultaneously is conveniently employed; the one used by the author is further reduced in capacity by a solid wooden stand for the two bottles, which just fits inside it, occupying nearly half its volume. It is shown in Fig. 471. As the pump is worked, and the air rises in bubbles and is rapidly

removed by the pump, the exterior of the receiver requires to be gently tapped to assist in releasing the bubbles as soon as possible after they make their appearance in the carbon tetrachloride; for otherwise they might accumulate and force the liquid into the neck or even out of the bottle. Eventually, when the vacuum gets adequately high and the drops of mercury falling down the long tube of the pump begin to click, the carbon tetrachloride commences to boil, and its vapour effectually expels the last traces of air. When the boiling has proceeded for a few minutes, with frequent tappings of the receiver, it will usually be the case that all air is finally got rid of from below the surface of the liquid. The outside air may then be readmitted into the receiver and the bottles removed. They are then to be filled up with carbon tetrachloride, the stoppers inserted but without caps, taking great care to avoid enclosing any bubbles of air, and immersed in a water-bath up to near the tops of their necks, in order that the bottles may be raised to the temperature at which the content of each bottle is to be determined, and for which the density of the salt is to be ascertained.

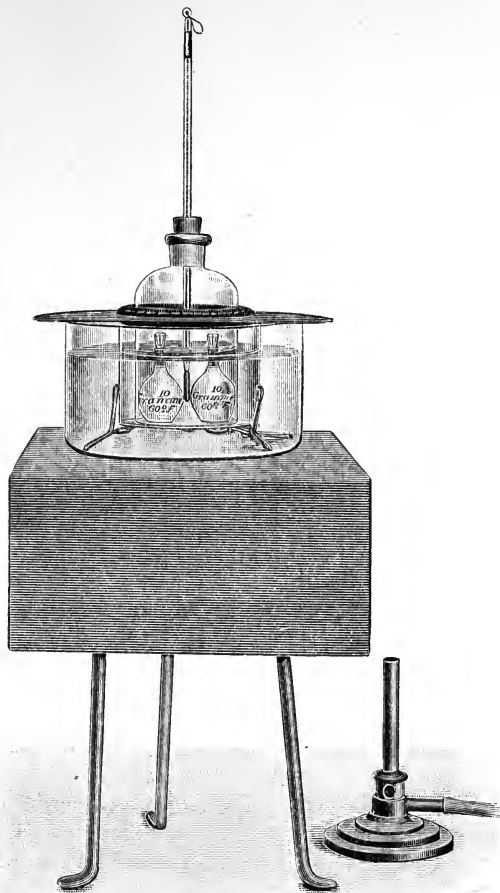


FIG. 472.—Double Water-bath in which Pyknometers are warmed to 20°.

It is convenient to determine the specific gravity at the uniform temperature of 20° C., and all the author's published pyknometer determinations have referred to this temperature. The water-bath with the pyknometers immersed in it is shown in Fig. 472. It consists of three glass vessels one inside another, an outer dish containing water, an inner one resting on a wire stand to keep it from touching the outer dish, and also containing water to the height which will enable the full

bottles to rest in it with the upper part of their necks just emerging, and an innermost small bell-jar, through a niched caoutchouc stopper in the tubular neck of which the thermometer is inserted, with the centre of its bulb at the height of the widest part of the bottles and between the two latter, the diameter of the bell-jar being ample to admit both bottles within it. The outer bath rests on a metal plate, turned down at the edges and supported on a tripod stand. By placing a very small Bunsen flame underneath the metal plate for short periods at a time the temperature of the baths may be brought very slowly up to 20° , so slowly that there is full certainty of the pyknometers taking the temperature of the inner bath as indicated by the thermometer. The latter should read directly to 0.1° C., and must be one the readings of which for 0° and 100° have been recently verified, by the usual method of immersion in ice and steam, and any necessary corrections to which have been accurately determined. Its indications are read by means of a telescope arranged horizontally a yard or more away at the same level, in order that no error of parallax may be incurred, due to eye-reading at close quarters.

As the temperature rises the carbon tetrachloride continues to ooze out of the capillary bore of the stopper of each pyknometer. When 20° is reached, the last half degree of rise having occupied ten minutes or so, the drops of carbon tetrachloride standing over the capillaries are brushed off, and the bottles are simultaneously removed from the bath, and carefully wiped with a clean handkerchief without touching them with the fingers, the liquid having already receded down the capillary owing to contraction in the colder air. The caps are then put on, the pyknometers fitted into their wire stands, and the pair placed in the desiccator and allowed to rest for the usual few minutes to take up equilibrium of surface conditions. They are then weighed in their stands, at the ordinary temperature of the room.

The wiping off of the drop of carbon tetrachloride from the top of each stopper must be done with precision and nicety, immediately before, and indeed as nearly as possible at the instant of, removal of the bottle from the bath; for otherwise, as the cooling down to the temperature of the outside air causes the liquid within the bottles to contract and recede down the capillary, some of that contained in the drop will re-enter the pyknometer. The best plan is to sweep off the drop with a finger of one hand at the instant the pyknometer is removed with the other hand, and to complete the wiping after the second pyknometer has been similarly treated, and the liquid in the capillary has receded a little way down the bore, away from any slight trace remaining on the stopper end; for by this course the object is accomplished without any risk of expansion, due to the proximity of the warm hand, causing any carbon tetrachloride to exude, any such slight expansion being then inadequate to bring the liquid again to the top of the capillary bore.

After the weighing they may be again filled up with carbon tetrachloride, and the process of warming to 20° again gone through; the water-bath will have cooled down adequately in all probability during

the weighing, or if not a little cold water can be added to it to bring it down to a good starting temperature two or three degrees below 20° . After removal of the bottles at exactly 20° as before, cleansing, capping, and fitting in their stands, they are to be allowed to stand again for the usual short interval and then once more weighed. The taking of two sets of weighings thus minimises any error due to the correct attainment of 20° or to the circumstances of the wiping of the drop, while the employment of a pair of pyknometers minimises other errors such as those due to any residual air inclusion.

Many crystalline salts are more or less deliquescent, and the property is particularly emphasised when the crystals are powdered, owing to the immense surface then presented to the moisture of the air. In order to prevent error from this cause, and to render the accuracy of the determinations in such cases as perfect as for crystals with no tendency to absorb moisture, the author performs the filling of the bottles with the powder, after it has been dried at a suitable temperature in an air-bath, in the interior of a closed air-tight chamber, a large desiccator in fact of special construction. It is shown in Fig. 473. The manipulation is carried out with the hands placed in a pair of caoutchouc gloves inserted in an air-tight manner through two adjacent sides *d* and *e* of the rectangular mahogany box which constitutes the chamber. The two other sides *b* and *c* and the top *a* are glazed, so that the manipulator can see what he is doing within. The pyknometers, agate mortar, desiccating dish of oil of vitriol or phosphoric anhydride, the substance powder, and any other required articles, are placed within the box overnight, through one of the sides *c*, generally arranged in front, the large plate-glass window of which is detachable and capable of air-tight fixation again afterwards by a series of bolts with winged nuts, the contact being made absolute between the window frame and the box side by indiarubber facings on both. The gloves are similarly clamped against the two other box sides by means of bolted frames, each having a circular aperture large enough to enable the hand to pass through, and each of which presses a wooden curtain-pole ring on which the glove wrist is stretched and which also is large enough to admit the hand, against an indiarubber pad on the box side, in which latter a corresponding circular aperture has also been cut. The whole box is rendered as air-tight as possible by several coats of

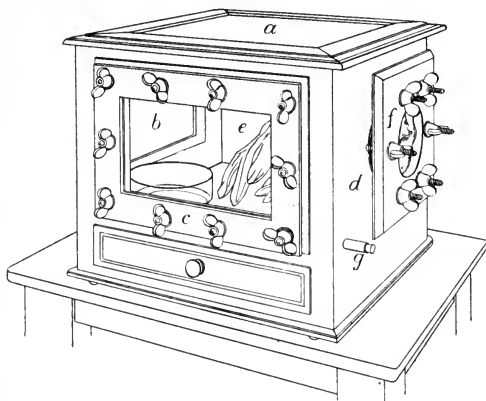


FIG. 473.—Desiccating Chamber for Use with Deliquescent Crystals.

varnish, inside and out. Having placed everything necessary in the box overnight, and hermetically closed the window again, the air within the chamber next morning is found to be adequately dry for the operation of filling the bottles with the powdered salt to be undertaken with safety. The author found the apparatus of great value in the determination of the densities of the very deliquescent selenates of the alkalis. If in any other case it should be desirable to work in an atmosphere of inactive gas such as carbon dioxide or nitrogen, provision is made for filling the chamber with it by the addition of a couple of short brass tubes, on opposite sides (diagonally) of the chamber, one of them being shown at *g*. Under ordinary circumstances these tubes are stoppered and sealed up.

Before commencing a series of density determinations the specific gravity of the liquid, carbon tetrachloride, requires to be most accurately ascertained, for the value of this constant will influence all the determinations of the crystal densities. In order to arrive at this important constant we have first to determine (*a*) how much water (by weight) the pyknometer contains at 20°, and then (*b*) to ascertain how much carbon tetrachloride is contained under the same conditions and at the same temperature of 20°. The relative density of carbon tetrachloride at 20° compared with that of water at 20°, afforded by the quotient of the two weights, b/a , has then to be corrected so as to compare with water at 4°, by multiplying by 0.99825, the density of water at 20° compared with water at 4°. The water employed for (*a*) must be pure distilled water which has just previously been freed from air by boiling, and subsequently cooled in a stoppered flask which it fills, in order to minimise any further solution of air. In all the weighings during these operations (*a*) and (*b*), as well as in the subsequent determination (*c*) of the density of the crystalline substance, correction must be made for the air displaced both by the contents of the pyknometer placed on the one pan of the balance, and by the brass weights employed to counterpoise it on the other pan; for the difference of the two displacements is often considerable enough just to affect the third place of decimals, and at any rate to affect considerably the fourth place in the value for the density. It is adequate to take the weight of one cubic centimetre of dry air as 0.0012 gramme, for the usual atmospheric conditions of temperature and pressure. The specific gravity of the brass weights is taken as 8.4, the exterior platinisation not affecting the question, and the volume of the platinum weights used for the decimal places is so inconsiderable that the whole weight on the pan can be taken as of specific gravity 8.4, an approximation which is fully near enough for the purpose, any further refinement not affecting the fourth decimal place of the density value. The density of the carbon tetrachloride and of the crystal powder, for use in calculating the correction in the first determination, before the exact corrected value has been arrived at, may be the uncorrected value in each case; it is safest to consider this as provisional, the correction being subsequently verified (or altered if it should affect the last unit in the fourth

place of decimals) when the true value has been determined. For the second and all subsequent determinations the true specific gravity as afforded by the corrected result of the first determination should, of course, be used.

It has been the author's practice to redetermine the density of the stock of pure carbon tetrachloride, after redistillation (the boiling point at 760 mm. pressure being $76\cdot7^{\circ}$ C.), before commencing each series of crystal density determinations, that is before the operations on each group of salts, the determinations for which are to be carried out on successive days during the course of a week or two.

We may now proceed to illustrate the method by an actual example, and cannot do better than take the case of the typical crystalline substance the goniometry of which was worked through in Chapters IV. and VIII., namely, potassium sulphate, K_2SO_4 . Operations (a) and (b), which are the same whatever the substance or substances to be investigated, and (c), which is similar for all substances but not identical, will each be worked through in duplicate, that is, for the two pyknometers, which we may label Nos. 1 and 2 respectively. The weights will be set out in the order in which the weighings were made. By weight of pyknometer is to be understood in all cases that of the capped bottle in its wire stand.

(a) DETERMINATION OF WATER CONTENT.

Experimental Data.

	Pyk. No. 1.	Pyk. No. 2.
Weight of pyknometer	16·4539	18·6834
Do. and water (as filled at 20°)	26·4262	28·6636
Apparent weight of water content for 20°	9·9723	9·9802

Calculations.

$$\begin{aligned}
 \text{True weight of water content of No. 1 for } 20^{\circ} &= 9\cdot9723 + [0\cdot0012 \times (10\cdot0 - 1\cdot2)] \\
 &\qquad\qquad\qquad 8\cdot8 \\
 &= 9\cdot9723 + 0\cdot0106 \\
 &= 9\cdot9829. \\
 \text{Do. do. No. 2 do.} &= 9\cdot9802 + 0\cdot0106 \\
 &= 9\cdot9908
 \end{aligned}$$

The correction for air displaced, the expression in square brackets, is the same for both determinations, the pyknometers each holding 10·0 c.c., when the volume is expressed to one place of decimals, which is enough for the purposes of the calculation of this correction. The 10 c.c., however, have to be diminished by the volume of the weights on the other pan, which is got by dividing 9·9723 and 9·9802 respectively by the specific gravity 8·4 of the weights, and which works out to 1·2 in each case. Hence 8·8, the difference of 10 c.c. and 1·2 c.c., multiplied into the weight of 1 c.c. of air, 0·0012 gramme, gives 0·0106 gramme for the weight to be added to the apparent weight in order to arrive at the true weight of the water content.

(b) DETERMINATION OF DENSITY OF CARBON TETRACHLORIDE.

Experimental Data.

	Pyk. No. 1.	Pyk. No. 2.
Weight of pyknometer	16·4539	18·6834
Do. and CCl_4 (as filled at 20°)	32·3120	34·5515
Apparent weight of CCl_4 content for 20°	15·8581	15·8681

Calculations.

$$\begin{aligned}
 \text{True weight of CCl}_4 \text{ content of No. 1 for } 20^\circ &= 15.8581 + [0.0012 \times (10.0 - 1.9)] \\
 &\quad 8.1 \\
 &= 15.8581 + 0.0097 \\
 &= 15.8678 \\
 \text{Do.} \quad \text{do.} \quad \text{No. 2} \quad \text{do.} &= 15.8681 + 0.0097 \\
 &= 15.8778
 \end{aligned}$$

The correction for air displaced is again the same for both determinations, the volume of air displaced by the CCl_4 being 10.0 c.c. and that by the weights 1.9 c.c. (15.86 divided by their sp. gr. 8.4), the difference which has to be taken into consideration being 8.1 c.c.

We now know the content of both pure water and carbon tetrachloride at 20° for each pyknometer, and the quotient of the latter by the former is the specific gravity of carbon tetrachloride at 20° compared with water at 20° . We have now to convert it, however, to the denser standard of water at 4° by multiplying by the density of water at 20° compared with its maximum density at $4^\circ=1$, namely, by 0.99825.

$$\begin{aligned}
 \text{Sp. gr. of CCl}_4 \text{ at } 20^\circ/4^\circ &= \frac{15.8678}{9.9829} \times 0.99825 = 1.5867 \text{ by pyknometer No. 1,} \\
 \text{and } &= \frac{15.8778}{9.9908} \times 0.99825 = 1.5865 \text{ by pyknometer No. 2.}
 \end{aligned}$$

A third determination of the density of carbon tetrachloride was also made with a larger pyknometer of 30 c.c. capacity, and the value 1.5867 was obtained, identical with the value from No. 1. Any slight error in any of these determinations being in all probability on the side of lowness, owing to the much greater volatility of carbon tetrachloride than water, the higher value is probably nearer the truth than the lower. Hence, as we have the confirmation by the third determination, the higher value 1.5867 is accepted as the density of carbon tetrachloride for $20^\circ/4^\circ$.

(c) DETERMINATION OF DENSITY OF POTASSIUM SULPHATE.

Experimental Data.

			Pyk. No. 1.	Pyk. No. 2.
Weight of pyknometer	.	.	16.4537	18.6834
Do.	and K_2SO_4	.	21.7144	23.1336
Do.	do.	and CCl_4 (as filled at 20°)	34.4364	36.3538
Apparent weight of K_2SO_4 in bottle	.	.	5.2607	4.4502
Do.	CCl_4	do. (when full at 20°)	12.7220	13.2202

Calculations.

$$\begin{aligned}
 \text{True weight of K}_2\text{SO}_4 \text{ in pyk. No. 1} &= 5.2607 + [0.0012 \times (2.0 - 0.6)] \\
 &\quad 1.4 \\
 &= 5.2607 + 0.0017 = 5.2624 \\
 \text{Do.} \quad \text{No. 2} &= 4.4502 + [0.0012 \times (1.7 - 0.5)] \\
 &\quad 1.2 \\
 &= 4.4502 + 0.0014 = 4.4516 \\
 \text{True weight of CCl}_4 \text{ in pyk. No. 1} &= 12.7220 + [0.0012 \times (8.0 - 1.5)] \\
 &\quad 6.5 \\
 &= 12.7220 + 0.0078 = 12.7298 \\
 \text{Do.} \quad \text{No. 2} &= 13.2202 + [0.0012 \times (8.3 - 1.6)] \\
 &\quad 6.7 \\
 &= 13.2202 + 0.0080 = 13.2282
 \end{aligned}$$

	Pyk. No. 1.	Pyk. No. 2.
Weight of full CCl_4 content for 20° from (b)	15·8678	15·8778
Weight of CCl_4 displaced by K_2SO_4 at 20°	3·1380	2·6496

This last is the difference between the weight of the content of the bottle when full of CCl_4 alone and of the CCl_4 in the pyknometer along with K_2SO_4 when full, that is, the difference between the two previous items.

Hence, at 20° , 5·2624 grammes and 4·4516 grammes of potassium sulphate occupy respectively the same bulk as 3·1380 and 2·6496 grammes of carbon tetrachloride. The quotients of the former two numbers respectively by the latter two will afford the relative density of potassium sulphate compared with that of carbon tetrachloride, both at 20° , for the two respective experiments, and as we have found that the density of carbon tetrachloride compared with the standard of water at 4° is 1·5867, the density of potassium sulphate at 20° compared with water at 4° will be as under :—

$$\text{Sp. gr. of } \text{K}_2\text{SO}_4 \text{ at } 20^\circ/4^\circ = \frac{5\cdot2624}{3\cdot1380} \times 1\cdot5867 = 2\cdot6609, \text{ by pyknometer No. 1,}$$

$$\text{and } \frac{4\cdot4516}{2\cdot6496} \times 1\cdot5867 = 2\cdot6659, \text{ by pyknometer No. 2.}$$

The corrections for air displaced (again in square brackets) are quite different for the two pyknometers, owing to the amount of salt taken in the two experiments being different; consequently, the amounts of carbon tetrachloride required to fill up the pyknometers are also different, and their corrections different. But the sum of the volumes of the salt and liquid is of course the same in both experiments, namely, the capacity of the pyknometers 10·0 c.c. (2·0 and 8·0 in the case of No. 1, and 1·7 and 8·3 in the case of No. 2); in each case the volume of the salt or liquid has to be diminished by that of the weights on the other pan, obtained by dividing them by their specific gravity 8·4, in order to arrive at the displacement of air which affects the result and which, therefore, requires to be taken into account.

As regards any difference of volume between the pyknometer itself and the weights which counterpoise it, the use of brass for the stand reduces such difference to that merely between the glass bottle, which is very light and thin, and its equivalent weight in brass weights, and this is so relatively small that it is not found to affect the fourth place of decimals, and so it can be neglected. It may be tested by actually using a duplicate bottle as a tare instead of weights corresponding to the weight of the bottle, and thus getting rid altogether of any difference of volume. But the author has not found it necessary to use this method of a tare on other than a few test occasions, as the correction for lack of it has proved to be negligible in the case of results expressed to four places of decimals.

It is absolutely useless to calculate results to more than four places of decimals, as different crops of crystals of the same pure salt often show differences of specific gravity in the third place. Indeed, the only use of the fourth place is to ensure accuracy in the third place of decimals. The actual pair of results just worked out show a difference of five units in the third place. This, however, is about the extreme difference which is ever found with good crystals, and the examples have been chosen to illustrate this fact. Four other determinations which the author carried out with different crops of potassium sulphate crystals yielded results of intermediate value between 2·661 and 2·666. A seventh determination again yielded the highest figure 2·6660.

For the reason already fully specified, that the highest value obtained represents in all probability the specimen most free from

cavity-inclusions of mother-liquor, the value 2.666 has been accepted by the author as the true specific gravity of potassium sulphate at 20°, compared with water at 4°.

The Retgers Suspension Method.—The first operation in connection with this method consists in producing such a mixture of pure methylene iodide or other heavy liquid and benzene or other suitable and miscible light liquid, as is of exactly the same density as the heaviest of the small crystals employed, which should be chosen from the best crops on account of their perfect transparency and, therefore, of freedom from visible cavities or inclusions. The author prefers methylene iodide, CH_2I_2 , to all other heavy liquids, and benzene to all light ones, and uses the mixture of the two in all cases where neither liquid reacts chemically or as a solvent upon the crystalline substance under investigation. Methylene iodide is almost colourless, only faintly yellow, when first prepared, and is so supplied by any of the first-class manufacturers, such as Merck of Darmstadt. It is issued in dark brownish-yellow bottles, and the bottle is best kept in a tin-plate case in the dark, as the liquid becomes orange-coloured after exposure to daylight. Its specific gravity when pure is 3.342 just above its melting point, 5° C., and about 3.33 at the usual temperatures of the air during the experiments. The specific gravity of benzene at 15° is 0.884, so that the mixture of the two liquids in regularly varying relative quantities affords a very wide range of density.

For crystalline salts of higher specific gravity than 3.33, such as the sulphates and selenates of rubidium, caesium, and thallium, there is unfortunately at present no liquid known which is at the same time of adequately high density and without solvent or chemical action on the salts, and recourse is of necessity had to the pycnometer method. Rohrbach's solution, for instance, of specific gravity 3.588, consists of an aqueous solution of 100 parts barium iodide and 130 parts mercury iodide, and of course the water of an aqueous solution dissolves the crystals of the salts of the alkali sulphate and selenate series.

An alternative liquid to methylene iodide which is somewhat more stable is acetylene tetrabromide, $\begin{array}{c} \text{CHBr}_2 \\ | \\ \text{CHBr}_2 \end{array}$, the specific gravity of which

is 3.001 at 6°. It is prepared by leading acetylene gas into cooled bromine. The author on the whole prefers methylene iodide, as it is much pleasanter to work with, the vapour of acetylene tetrabromide being somewhat irritating. Toluene may be used instead of benzene as the lighter liquid, and in the case of Rohrbach's solution, water is, of course, used as the lighter diluting liquid. This latter solution is to be employed when water is without solvent action while the organic liquids attack or dissolve the crystals under investigation, such as those of related organic compounds, which are often soluble in organic liquids but not in water. Rohrbach's solution may also be used for many minerals, and enables those of densities a little higher than that of methylene iodide to be dealt with.

The author's method of conducting the operations will be elucidated by Fig. 474, which shows the simple apparatus employed.

About 8 c.c. of methylene iodide, supposing this to be the heavy liquid decided on, are transferred to a miniature dropping funnel with cylindrical bulb of about 20 c.c. capacity. The ten or more perfect small crystals which have already been used for the goniometrical work and have been thereby proved to be of perfect character, and from their careful selection on account of transparency (or other perfections if the substance is opaque) will be particularly suitable for our purpose, being least likely to contain cavities, are introduced and swim on the surface, assuming them to be lighter than methylene iodide. Benzene (or toluene) is then gradually introduced from a still smaller dropping funnel supported above the larger one, until the crystals begin to regain the surface only slowly after agitation of the liquid, when all further additions of benzene are to be made only one drop at a time, followed by vigorous shaking, with the stopper in position. The agitation of the contents of the stoppered bottle can only be efficiently achieved by holding the latter freely in the hand, the neck being the best place to grip it by, between the second finger and thumb, so as to avoid warming the liquid, which is prevented from reaching the neck by the stopper (except as a thin film), the latter being held firmly in position by the first finger. As soon as the desired equilibrium of density is attained, the stopper is replaced by a centrally bored cork carrying a standardised thermometer reading to 0.1° , which should be so arranged that its bulb lies altogether immersed in the liquid. When equilibrium of temperature is also attained, the heaviest crystal should float anywhere in the liquid without rising or falling.

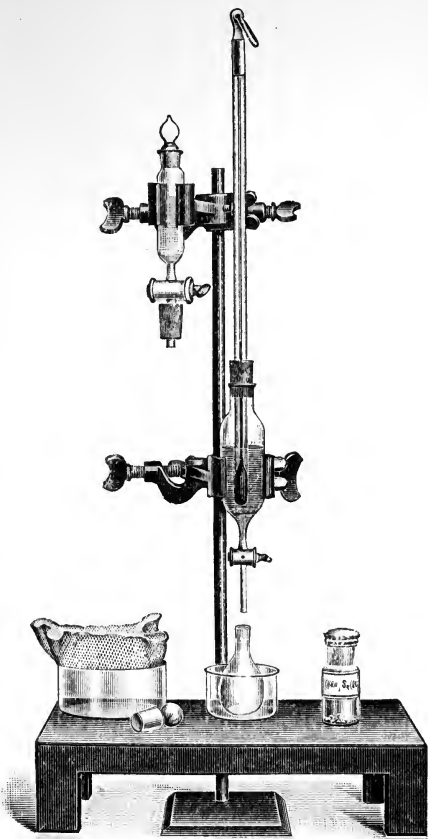


FIG. 474.—Apparatus for Retgers Suspension Method of Density Determination.

The second operation consists in determining the density of the liquid mixture by weighing a pycnometer filled with it. By working with a pair of pycnometers, the same as have been described in the previous section of this chapter, it can be so arranged that if in the first determination the tendency of the heaviest crystal should be, if anything, to rise, then in the second determination with the other pycnometer (the filling of each pycnometer corresponding to a separate determination) the tendency shall be for the crystal to sink, when it moves at all. Immediately the temperature of the liquid has been read, the latter is run off into one of the pycnometers as quickly as the stopcock allows. It is an advantage to have a small grating of platinum gauze, just smaller than the neck, loosely lying at the bottom of the cylindrical bulb of the funnel; the crystals are then retained above it during the process of running off the liquid into the pycnometer. Otherwise very small crystals may get down into the tap capillary. The perforated stopper of the pycnometer is replaced the moment the latter is full, and the drop on the top swept off; the neck is wiped dry without contact with the warm fingers, the cap fitted on, and the pycnometer inserted in its spring stand, which presses the cap tightly down on the neck. Rapidity is the essence of the process, and the pycnometer is at once weighed, before there is time for the least trace of the very volatile benzene to escape through the cap. All weighings must be corrected for air displaced, precisely as in the pycnometer determinations already described. As the usual time is not allowed for equilibrium of exterior surface condition of the pycnometer, the weight of the empty pycnometer should of course be taken under similar conditions.

It is best to work in a room the temperature of which is as near as possible to 20° , the temperature to which all the author's density results are referred. When it differs, it had better be lower than higher, to minimise loss of benzene, but preferably not lower than 15° , for all the reductions to the standard of 20° have to be made with the aid of the knowledge of the coefficient of expansion of the crystals. Retgers takes this latter to be 0.0001 for salts generally. Fortunately, the author has carried out a complete series of determinations of the thermal expansions of the sulphates of the alkalies,¹ the method of which will be described in Chapter LIII., and the results fully agree with Retgers' supposition. The actual coefficients of cubical expansion at any temperature t were found to be:—

For potassium sulphate, $0.00010475 + 0.0000001396 t$.

For rubidium sulphate, $0.00010314 + 0.0000001534 t$.

For caesium sulphate, $0.00010170 + 0.0000001620 t$.

The correction per degree is, therefore, 0.0001 multiplied by the specific gravity of the salt.

An example will now be given, in conclusion, of a pair of determinations of the density of potassium sulphate by the suspension method. The experiments were carried out in the year 1905; the

¹ *Phil. Trans.*, 1899, A, 192, 455.

previous set by the pyknometer method, of which a pair of examples are given earlier in this chapter, were made in 1893. In the meantime the weights of the pyknometers had naturally become reduced somewhat by wear, and No. 2 had lost more owing to the stopper having been ground a little shorter to be the more certain of quite clearing the cap.

The water content had also been re-determined several times, and the latest previous value is now given instead of the older 1893 value.

DENSITY OF POTASSIUM SULPHATE BY SUSPENSION METHOD.

Experimental Data.

	No. 1.	No. 2.
Weight of pyknometer	16·4522	18·6462
Do. and liquid	43·0972 at 18·4°	45·3188 at 18·1°
Apparent weight of liquid	26·6450 „	26·6726 „

Calculations.

Correcting first the apparent weight of the liquid for the air displaced, both by it and by the weights, 10 c.c. being the volume of the liquid and $26\cdot6/8\cdot4 = 3\cdot2$ being the volume of the weights, we have :

$$\text{True weight of liquid in No. 1} = 26\cdot6450 + [0\cdot0012 \times (10\cdot0 - 3\cdot2)]$$

$$= 26\cdot6450 + 0\cdot0082 = 26\cdot6532.$$

$$\text{Do. do. No. 2} = 26\cdot6726 + 0\cdot0082 = 26\cdot6808.$$

If t be the temperature of the liquid at the moment of transference to the pyknometer, in the above cases 18·4° and 18·1° respectively, then the

$$\text{Specific gravity of liquid at } t/4^\circ = \frac{\text{Weight of liquid content at } t^\circ}{\text{Weight of water content at } 4^\circ}.$$

Now the

$$\text{Weight of water content at } 4^\circ = \frac{\text{Weight of water content at } 20^\circ \text{ as directly determined.}}{\text{Spec. grav. of water at } 20^\circ/4^\circ = 0\cdot99825}.$$

Hence :

$$\text{Spec. grav. of liquid and crystal at } t/4^\circ = \frac{\text{Wt. of liquid content at } t^\circ \times 0\cdot99825}{\text{Wt. of water content at } 20^\circ}.$$

For the two different experiments this works out as under :

$$\text{For No. 1, Spec. grav. of liquid at } 18\cdot4^\circ = \frac{26\cdot6532 \times 0\cdot99825}{9\cdot9839} = 2\cdot6650 ;$$

$$\text{For No. 2 do. do. at } 18\cdot1^\circ = \frac{26\cdot6808 \times 0\cdot99825}{9\cdot9917} = 2\cdot6657.$$

These results for the two temperatures of 18·4° and 18·1° have now to be corrected to 20°, the correction being found by multiplying 0·0001 the coefficient of cubical expansion of the crystals by the specific gravity now found, 2·66, and also by the difference of the two respective temperatures from 20°. The values of the two corrections are :

$$\text{For No. 1, } 0\cdot0001 \times 2\cdot66 \times (20 - 18\cdot4) = 0\cdot0004 ;$$

$$\text{For No. 2, } 0\cdot0001 \times 2\cdot66 \times (20 - 18\cdot1) = 0\cdot0005.$$

Reducing then the specific gravities for 18·4° and 18·1° respectively by 0·0004 and

0.0005, we obtain for the specific gravity of the liquid, and therefore of the heaviest of the crystals, for $20^{\circ}/4^{\circ}$, the following numbers:

From No. 1, **2.6646**,

From No. 2, **2.6652**.

A further pair of similar determinations with fresh crystals of potassium sulphate yielded for the heaviest individual the values 2.6640 and 2.6652 for $20^{\circ}/4^{\circ}$. The mean of the two determinations for the heaviest crystal of the first batch is 2.6649, and that of the pair of estimations with the heaviest crystal of the second batch is 2.6646. Hence, accepting the former pair as having been conducted with the crystal of maximum heaviness, the final result of the determination of the specific gravity of potassium sulphate crystals by the suspension method may be taken as 2.665. Indeed, both pairs of determinations afford this same value when expressed to three places of decimals.

The highest result of the determinations by the pycnometer method has been shown to be 2.666, and as the higher figure is more likely to be correct for absolutely cavity-free crystals, when, as in this case, the results for the two methods only differ by 0.001, the author has finally accepted **2.666** as the density of potassium sulphate crystals at 20° compared with water at 4° taken as unity. It is highly satisfactory that this value is identical with that obtained by Retgers in the important work, which included the determination of the density of this salt, in which he employed for the first time the suspension method.

Calculation of the Molecular Volume and Molecular Distance Ratios.—Having thus settled the density of the crystals of the substance under investigation, we are in a position to calculate the molecular volume and the molecular distance (topic axial) ratios.

Molecular volume = $\frac{\text{molecular weight}}{\text{density}}$; hence, to obtain the molecular

volume we sum up the molecular weight from the atomic weights of the elements, multiplied in each case by the number of atoms of each present, and divide the result by the density found. In order to maintain the resulting molecular volumes for the different members of any related series, such as the isomorphous series of the alkali sulphates and selenates which has served us for so excellent an example, strictly comparable, all the densities, as we have seen, are determined for, or reduced to, the common standard temperature of $20^{\circ}/4^{\circ}$. Similarly, all molecular weights must be strictly comparable, and in his latest published values of these the author has taken the atomic weights authorised as most accurate by the International Committee on Atomic Weights in the year 1905, in which the atomic weight of hydrogen is taken as unity, brought up to date as regards any new determinations.

We will now complete our example, potassium sulphate, by calculating its molecular volume, and its molecular distance ratios, using for the latter the formulæ shown in the last chapter to apply to the series, on the most probable assumption of pseudo-hexagonal symmetry.

Molecular weight of K_2SO_4 , $H = 1$.

$$K_2 = 38.85 \times 2 = 77.70$$

$$S = 31.82$$

$$O_4 = 15.88 \times 4 = 63.52$$

$$\text{Mol. wt.} = 173.04$$

$$\text{Molecular Volume of Potassium Sulphate} = \frac{173.04}{2.666} = 64.91.$$

Molecular Distance Ratios of Potassium Sulphate.—We have thus found that the volume $V = 64.91$, and it was shown in Chapter VIII. that the crystallographic axial values were $a = 0.5727$, and $c = 0.7418$. Inserting, therefore, these values in the formulæ we get:

$$\chi = \frac{1}{2} \sqrt{1 + a^2} \cdot \sqrt[3]{\frac{2V}{ac}} = 3.8810.$$

$$\psi = \sqrt[3]{\frac{2a^2V}{c}} = 3.8574; \quad \omega = \sqrt[3]{\frac{2c^2V}{a}} = 4.9964.$$

Thus the molecular distance ratios are:

$$\chi : \psi : \omega = 3.8810 : 3.8574 : 4.9964.$$

This concluding chapter of the goniometrical and structural portion of the book may fittingly end with a summary of the structural constants of the crystals of the whole isomorphous group of which potassium sulphate, which has been taken throughout as the most illuminating example which could be found of crystal structure, is the first member. The table will form a natural appendix to the comparative tables of the crystal angles and crystallographic axial ratios of the same orthorhombic group of salts, which were given in Chapter XXX. (pages 485 and 486), and will serve as an example of how such a comparative table should be drawn up.

COMPARATIVE VALUES OF STRUCTURAL CONSTANTS OF ALKALI
SULPHATES AND SELENATES.

Salt.	Density at 20°/4°.	Molecular Weight.	Molecular Volume.	Crystallographic Axial Ratios. $a : b : c$.	Molecular Distance Ratios. $\chi : \psi : \omega$.
K_2SO_4 . .	2.666	173.04	64.91	0.5727 : 1 : 0.7418	3.8810 : 3.8574 : 4.9964
Rb_2SO_4 . .	3.615	285.14	73.34	0.5723 : 1 : 0.7485	4.0304 : 4.0039 : 5.2366
Cs_2SO_4 . .	4.246	359.14	84.58	0.5712 : 1 : 0.7531	4.2187 : 4.1849 : 5.5175
Tl_2SO_4 . .	6.765	500.50	73.98	0.5555 : 1 : 0.7328	4.0820 : 3.9644 : 5.2299
$(NH_4)_2SO_4$. .	1.772	131.20	74.04	0.5635 : 1 : 0.7319	4.0792 : 4.0051 : 5.2020
K_2SeO_4 . .	3.067	219.82	71.67	0.5731 : 1 : 0.7319	4.0291 : 4.0068 : 5.1171
Rb_2SeO_4 . .	3.902	311.92	79.94	0.5708 : 1 : 0.7386	4.1672 : 4.1315 : 5.3461
Cs_2SeO_4 . .	4.456	405.92	91.09	0.5700 : 1 : 0.7424	4.3457 : 4.3040 : 5.6058
Tl_2SeO_4 . .	6.875	547.30	79.61	0.5551 : 1 : 0.7243	4.1124 : 4.0763 : 5.3189
$(NH_4)_2SeO_4$. .	2.194	177.98	81.12	Not comparable	4.5939 : 4.2968 : 5.1506 ¹

¹ The molecular distance ratios of monoclinic ammonium selenate were calculated on the assumption of a similar pseudo-hexagonal structure to that of the nine rhombic salts. Such a structure is strikingly evident as regards the prism zone, the angles of the prism faces being 59° 38' and 30° 0', the prism zone thus resembling the well-known pseudo-hexagonal prism zone of the nine rhombic salts to a remarkable degree, considering the change of system.

The results indicated by the table are in full agreement with the important law stated on page 484, the molecular volumes and molecular distance ratios (topic axial ratios) being functions of the atomic weights of the interchangeable alkali metals potassium, rubidium, and caesium, elements belonging strictly to the same family group, in the cases of the crystals of the sulphates and selenates of those metals, which form an exclusive eutropic series within the isomorphous series. The thallium and ammonium salts, while not belonging to the inner eutropic series, show constants nevertheless which entitle the crystals of those salts to be considered as belonging to the same isomorphous series, employing the term "isomorphous" in the sense which was defined on page 484 of Chapter XXX.

The example afforded by this instructive series of salts will have emphasised in a striking manner the importance of the new conception of molecular distance ratios, and have clearly demonstrated the enhanced value which now attaches to the accurate determination of the density of crystals. In fact, no crystallographic investigation can in future be considered complete which does not include such a determination, carried out with all the precautions described in this chapter.

CHAPTER XXXIII

VON FEDOROW'S THEORY OF CUBIC AND HYPOHEXAGONAL TYPES AND OF THE CORRECT DESCRIPTIVE SETTING OF CRYSTALS. THEORY OF POPE AND BARLOW.

THE importance of choosing the correct mode of setting up a crystal for descriptive purposes has already been alluded to, and a brief account of the work of von Fedorow, who has made a special study of the question, may be here given. The facts that the greater part of von Fedorow's memoirs are in the Russian language, and that such as have been translated or abstracted in another language have suffered in the process and have not presented the complete case, besides the further fact that the ideas presented have changed somewhat in successive papers, have prevented the work of von Fedorow receiving the attention which it undoubtedly merits. Only a bare outline can of necessity be given within the limits of a single chapter, of the leading ideas which serve as the groundwork of von Fedorow's contributions to the theory of crystal structure and correct setting. But some account of von Fedorow's theory of crystal structure has already been given in Chapter IX. (page 124), and sufficient further data will be found here to be of practical use; for further details regarding both the theory and its application to practical examples, as well as for the equations for the transformation of crystal elements and indices from one setting to another, the reader is referred to the original papers and abstracts in the *Zeitschrift für Krystallographie*, and more especially to the memoir entitled "Allgemeinste Krystallisationsgesetze."¹

Parallelohedral Structure.—According to von Fedorow a crystalline substance may be regarded as built up of an immense number of small solid figures, similar and equal to each other, and of such a shape that when arranged in parallel position the whole of space is filled up, leaving no interstices whatever. Such geometrical figures are termed "parallelohedra," being bounded by pairs of parallel faces.

Parallelohedra are subdivisible into primary and secondary parallelohedra, the former possessing plane faces, and the latter being bounded either partially or wholly by curved faces. The number of varieties of the latter is obviously infinitely great; primary parallelohedra, on the other hand, are much less numerous, and moreover can be referred to five types. It is with these five types of parallelohedra that we have to deal in

¹ *Zeitschr. für Krys.*, 1903, 38, 321-490.

crystalline structures, and they are in complete harmony with the space-lattices treated of in Chapter XXXI., and with the now accepted form of the Sohnckian point-system crystal-structure theory developed in that chapter and in Chapter IX.

The five primary parallelohedra with plane faces are: (1) the cube, a triparallelohedron, as it is bounded by three pairs of parallel surfaces, and which is shown at *a* in Fig. 475; (2) the hexagonal prism, a tetraparallelohedron, being bounded by four pairs of parallel faces, as indicated at *b* in Fig. 475; (3) the rhombic dodecahedron, a hexaparallelohedron, there being six pairs of parallel faces, as represented at *c* in the figure; (4) the elongated dodecahedron, having the same six planes and pairs of parallel faces as the rhombic dodecahedron, but elongated along one axis, taken as the vertical axis, as shown at *d* in Fig. 475; and (5) the cubo-octahedron, a cube truncated by the octa-

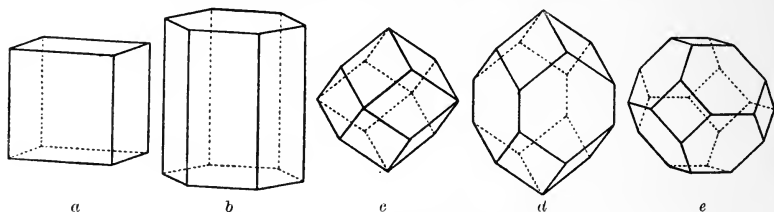


FIG. 475.—The Five Parallelohedra of von Fedorow.

hedron to the extent that each of the octahedral faces presents a regular hexagonal outline, thus forming a solid having seven pairs of parallel faces, and consequently a heptaparallelohedron, as illustrated at *e* in the figure.

The elongated dodecahedron is indistinguishable from the rhombic dodecahedron either by number or disposition of faces, and may consequently for crystallographic purposes be regarded as an unessential variety of the latter, leaving us only with four essential parallelohedra, the **cube**, **dodecahedron**, **cubo-octahedron**, and **hexagonal prism**.

A crystalline medium may be regarded as built up of a great number of any one of these four parallelohedra, and it will then be said to have one of the structures **cubic**, **dodecahedral**, **octahedral**, or **prismatic**.

Now the first, second, and third all possess the symmetry of the cubic system, whilst the symmetry of the fourth is quite distinct. We may, therefore, divide crystals into two principal types, the **cubic** and the **hypohexagonal**, the former embracing all the three forms of cubic structure, and the latter the unique prismatic structure.

The cube is capable of being transformed by homogeneous deformation into a tetragonal, orthorhombic, or monoclinic prism; also into a rhombohedron, or, the most general case, into the anorthic (triclinic) parallelepipedon. Each of these figures is a parallelohedron still, and a number of them may be packed together to completely fill space. The same holds good for the other three parallelohedra. Provided the deformation be homogeneous, a parallelohedron always remains a parallelohedron. From this it might at first sight appear that a crystal belonging

to any one of the seven systems and their 32 divisions or classes might possess any one of the four structures; but further consideration leads us to the conclusion that the choice is limited. For the symmetry of the parallelohedron must be consistent with the symmetry of the system and class to which the crystal belongs. It follows immediately from this that the prismatic structure is an impossible one for crystals of the cubic system, while crystals of the hexagonal system must necessarily possess that structure. Again, the four-fold, simple or alternating, axis of the classes of the tetragonal system is incompatible with the prismatic structure. Hence, cubic and tetragonal crystals must possess one of the first three structures, or in other words must belong to the **Cubic Type**, whereas hexagonal crystals must have the prismatic structure, and belong to the **Hypohexagonal Type**. As regards the elements of symmetry exhibited by the classes of the rhombohedral, orthorhombic, monoclinic, and anorthic (triclinic) systems, there is nothing incompatible with any of the four (appropriately homogeneously deformed) parallelohedra, and crystals belonging to these systems may belong to either the cubic or the hypohexagonal type. Rhombohedral (trigonal) crystals belonging to the cubic type, such as calcite, are termed trigonaloid to distinguish them. It will next be important to see how these four parallelohedra of von Fedorow are related to the 14 space-lattices and other point-systems which we have accepted as the basis of all crystal structure, as given in Chapters IX. and XXXI.

Relations of Parallelohedra to Space-lattices, or more generally, Point-systems.—A crystalline medium is certainly not a continuous one, composed of solid parallelohedra without any vacant space. Hence, each of von Fedorow's tiny parallelohedra does not really represent solid matter, but rather the field of influence of the structural unit placed at or around its centre. By field of influence of a unit is meant the space in which that unit exerts a paramount influence, and within which the matter of adjacent similar units never penetrates. It is obvious that by replacing a closely packed stack of cubes by points at their centres we obtain the cubic space-lattice of Bravais; and although perhaps not quite so obvious, it will be found on consideration that stacks of closely packed dodecahedra, cubo-octahedra, and hexagonal prisms, when treated similarly, give rise to the centred-face, centred-cube, and 120° -prism space-lattices respectively, Nos. 3, 2, and 6 of Chapter XXXI., as will be quite clear on referring to the descriptions of these space-lattices given in that chapter. Carrying out this operation on all the deformed parallelohedra, severally compatible with the various systems, we obtain in all 23 point-systems, corresponding to the 23 parallelohedra operated on—namely, cubic system 3; tetragonal system 3; hexagonal system 1; rhombohedral (trigonal), orthorhombic, monoclinic, and triclinic (anorthic) each 4;—which include all the 14 space-lattices as described in Chapter XXXI.

If for any particular crystal the volume of the parallelohedron be taken proportional to the molecular volume, then the distances between the centres of parallelohedra in specific directions represent the molecular distance ratios (topical axial ratios) for those directions.

The Correct Setting of Crystals.—The setting up of a crystal for descriptive purposes, that is, the choice of which direction shall be called the vertical axis and which other directions the lateral axes, can hardly be said to have been hitherto founded on any definite principles. Such rules as assigning the indices (100), (010), and (001) to the three planes of symmetry in the case of an orthorhombic crystal, or the indices (010) to the unique symmetry plane of a monoclinic crystal, are, of course, of great value as far as they go, and satisfy the purposes of classification; but they still leave considerable latitude, as to which of the three planes of symmetry shall be (100) and which (001) for instance, or which direction in the unique plane of symmetry shall be the vertical axis, and in both cases the elements (and in the former case the order of the 3 planes) depend on what plane shall be considered as the parametral one, for the choice of which the latitude is even greater. Indeed, the choice is usually more or less arbitrary from the structural point of view now under discussion. Owing to this absence of any guiding principle it is not surprising to find different authors setting up the same crystalline substance in different ways, while based on the same order (system and class) of symmetry with respect to which all agree. The habit may appear to justify the setting in each case, but the habit is often dependent on the local circumstances of growth of the crystal; for the same substance may display different habits in different crops or preparations. Moreover, it is often observed that in the description of triclinic crystals very high degrees of anorthism are given, and in the case of monoclinic crystals high inclination of the inclined axis; thus, for example, the latter angle β is given in one case which recently came under the author's notice as $28^{\circ} 27'$ or $151^{\circ} 33'$. Now, it is extremely unlikely that two of the principal directions in a crystal should deviate so greatly from 90° in the case of a crystal of cubic type, or from 60° in the case of a hypo-hexagonal crystal. It has been the object of von Fedorow to discover some true guiding principle on which the correct setting can be attained, and the following is a brief account of his methods and of the ideas on which he bases them, for a discussion of which the foregoing description of von Fedorow's views of crystal structure will have paved the way.

(1) All crystals are divisible, according to von Fedorow, into two **Types**, **Cubic** and **Hypohexagonal**, corresponding to the four structures previously enumerated. This follows deductively from his theory of crystal structure. It should be emphasised that crystals of the rhombohedral, orthorhombic, monoclinic, and anorthic systems may belong to the hypo-hexagonal type, as well as to the cubic type, that is, to either the one or the other, and their forms may, therefore, be represented by four indices.

(2) All crystals are either pseudo-tetragonal or pseudo-hexagonal in the wider sense, that is, when differences up to 20° are included. This principle was arrived at by von Fedorow from an examination of a great part of the published crystallographic data. The greater the variation from the ideal angles 90° and 60° , the fewer are the examples actually found.

It may perhaps be here suggested that as the terms pseudo-

tetragonal and pseudo-hexagonal are already employed for those cases where the deviation is one of minutes only,—as in the cases of the pseudo-hexagonal (orthorhombic) alkali sulphates and selenates fully described in previous chapters, where the differences of angle from true hexagonal symmetry (60°) are less than half a degree and often less than ten minutes only,—it would be better to employ the terms tetragonaloidal and hexagonaloidal for the wider sense (including differences up to 20°) meant by von Fedorow.

Directly derivable from these two principles are the following subsidiary ones marked (3), (4), and (5). Since crystals are either tetragonaloidal or hexagonaloidal, it results that the chief characteristic is the ratio of the principal axis to one of the horizontal subsidiary axes. Where this value is equal to unity, the crystal is said to be an isotropic complex, and the angle between the basal plane and the primary pyramid is $54^\circ 44'$ and $49^\circ 6\frac{1}{2}'$ respectively. When the ratio is greater than unity, the angle is increased and the crystal is termed **positive**. When the reverse holds, the crystal is **negative**.

(3) From the theory of crystal structure it follows that in a positive crystal (Fig. 476) the points of the network are most densely massed in a **singular plane** (the basal plane in Fig. 476), in which will therefore lie the directions of maximum cohesion; these are the directions of most intense growth, hence the crystal will be characterised by a tabular habit and a cleavage parallel to this singular plane.

(4) Conversely it follows that in negative crystals (Fig. 477) there is a **singular direction** (parallel to the vertical edges in Fig. 477) of maximum cohesion and most intense growth; we therefore meet with a prismatic habit (elongated), and cleavages, if any, are prismatic also, that is, lie in the zone the axis of which is the singular direction referred to. It will be observed that the habits of the crystals are the converse of those of their structural parallelepipeda. But fortuitous circumstances of growth frequently mask this characteristic habit by bringing about modifications.

(5) Again, from geometrical considerations, it is easy to prove that in positive crystals the forms (other than the primary pyramids and domes) which have the greatest reticular density are negative forms, that is, forms making smaller angles with the base than the primary forms. The reverse holds for negative crystals, for the forms making greater angles with the base now have the greater reticular density.

The method of obtaining the correct setting on the lines of these five principles is as follows: An accurate stereographic projection is made on one of the stereographic nets, such as that of von Fedorow himself,

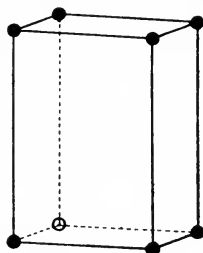


FIG. 476.—Elementary Cell of Positive Crystal.

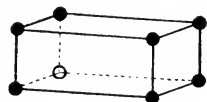


FIG. 477.—Elementary Cell of Negative Crystal.

or the still more convenient one as regards size (2·5 inches radius) of Hutchinson described in Chapter VI., and reproduced in Fig. 39 (p. 91). It must then be noted whether the angles between the principal forms approximate to 90° or to 60° , also whether the primary pyramids and domes agree better with a development corresponding to a tetragonal or to a hexagonal pyramid. Attention may also be paid to the angular distances between cleavage forms, if any exist, and the other principal forms, with a view to the discovery of positive or negative character. By inspection of this kind one is usually able to discriminate whether the crystal belongs to the cubic or to the hypohexagonal type, and in the former case, may indeed obtain some insight into the structure, as to whether it is cubic, dodecahedral, or octahedral. The most likely setting having been selected in this way, its value is then estimated in the following manner. The reticular density is calculated of all the observed parallel pairs of faces, as also that for any possible faces not actually present, which, from having simple indices, may *a priori* be predicted to have fairly high densities; if the density of one of the faces be taken as unity, the calculation of the relative densities of the remainder may be fairly rapidly made to a sufficient degree of accuracy by a graphical method based on the stereographic projection.

When all the observed forms are actually the forms of greatest possible reticular density, the form development may be regarded as ideal, and the setting as fully established. But when there are absent certain forms of reticular density greater than some of those actually present, one should proceed to estimate the relative value of the setting, compared with the ideal setting, by summing the squares of the densities (1) of the observed pairs of faces, (2) of an equal number of faces possessing as high a reticular density as possible; dividing the former by the latter we obtain a decimal fraction which is the nearer to unity the greater the approach to ideal form development, and which, if the latter be really present, actually becomes equal to unity. The same process must be carried out for any other setting considered likely to be ideal. The difference between the values of the decimal fraction, obtained for the different alternative settings of the same crystal, is usually so considerable that no difficulty attends the selection of the correct setting. The value for the correct setting will generally be greater than 0·9, whilst for incorrect settings a much lower value, often as low as 0·5, is obtained. In a crystal of the cubic type, the reticular density of a particular face will, generally speaking, vary according as the structure is cubic, dodecahedral, or octahedral, and it is by means of this property that a critical comparison of the observed forms with ideal form development for each type of structure enables one to discriminate between the three possibilities. In practice, the cubic structure is found to preponderate greatly.

Just as with an insufficiency of form development it is often impossible to determine all the elements of a crystal belonging to one of the less symmetrical systems, so it is equally true that it may also prevent the discrimination or selection of the correct setting. It may also be remarked that forms which either occur rarely, or are vicarious replacements (vicinal faces), are of little value; for they are not so much characteristic indicators

of form development as of changes in the external conditions of crystallisation. It is a good rule to take only those forms into account which are present, say, in more than half of the crystals investigated. It is interesting to note, in connection with the work of Miers on vicinal faces and his explanation of them as described in Chapter XXX., that when an old setting has been proved to be incorrect, the indices of rare forms become often still more complicated (higher index numbers) for the ideal setting, the faces then approximating more closely to those of a possible form with low indices, not actually observed, but probably replaced by the high-index vicinal faces, in the manner which has been made so clear by Miers.

Sufficient will now have been said to afford some insight into the important and highly interesting views of von Fedorow. It is much too early to express an opinion as to their complete validity, for only extensive further experimental work can decide. But it is for the very reason that in all future investigations von Fedorow's method of arriving at a proper setting of the crystal should be tested, that this chapter has been written. To the author, the wide limits of 20° , for inclusion in the cubic or the hypohexagonal type, appear very wide, and it is here that the test of further work is mostly required. In narrower cases like the sulphates and selenates of the alkalis, involving only a few minutes of deformation from truly hexagonal symmetry, the pseudo-hexagonal (and therefore certainly hypohexagonal in von Fedorow's sense) character is unmistakable, and the author has unreservedly adopted von Fedorow's method as regards the structural constants, and has given the molecular distance ratios of these orthorhombic salts on the basis of a pseudo-hexagonal structure (see table of these constants at the close of the last chapter).

It has just been communicated to the author (January 1911)—through Mr. T. V. Barker, who has spent a considerable time with Prof. von Fedorow in St. Petersburg, and to whom the author is also indebted in connection with the foregoing account of von Fedorow's theory—that Prof. von Fedorow has almost completely worked through and determined the correct setting of all the substances hitherto crystallographically investigated. He has drawn up the material into an index or table on the principles enunciated in this chapter, namely, on the basis of four structures belonging to the two types, cubic and hypohexagonal, and arranged the numerous substances belonging to each in their order, starting from the most negative and ending with the most positive. Moreover, what is more astonishing, he has succeeded in determining and identifying the majority of a large number of substances the crystals of which had been collected by Mr. Barker from various crystallographers who had measured them, and had been sent out by him to Prof. von Fedorow at St. Petersburg, without either their names or chemical formulæ appearing on the labels of the bottles, but merely a reference number, just as if the substances had been unknown new ones.

Hence, von Fedorow would appear to have discovered a purely crystallographic method of analysis which we may term "crystallochemical analysis," which is of real practical value. It will take some time to perfect and publish his index table, but in the meantime it is of great

interest to record such considerable progress in the identification of crystalline substances entirely by their crystal morphology.

Theory of Pope and Barlow.—A theory of crystal structure has recently been advanced by Pope and Barlow. The previous work of the latter investigator on the geometry of homogeneous structures has already been discussed in Chapter IX. Like von Fedorow, Pope and Barlow also regard the whole volume occupied by a crystalline structure as partitioned out into polyhedra, which they conceive to be derived by compression of a closely packed assemblage of deformable but incompressible elastic spheres, the "spheres of influence" of the chemical atoms. The most original feature of their theory is, however, that the volumes of the spheres, or rather of the polyhedra, are assumed to be proportional to the chemical valency of the atoms which the spheres represent; the volume, for instance, of the sphere of influence of a carbon atom is supposed to be four times that of a hydrogen or chlorine atom present in the same molecule. They substitute the sum of the valencies of the atoms present in the molecule—the "valency volume"—for the molecular volume in the formulæ given in Chapter XXXI. for the molecular distance ratios, thus obtaining "equivalence parameters." The theory does not concern itself with the relationships in isomorphous series (which are expressed by the molecular distance ratios), nor with the phenomena of the relative densities and molecular volumes of crystals.

In confirmation of von Fedorow, Pope and Barlow show that there are two modes of closely packing equal spheres, which give rise respectively to a cubic and to a hexagonal crystal structure, the latter endowed with a specific axial ratio; and that the crystalline chemical elements, of which the atomic spheres of influence must obviously be equal, practically all crystallise either in the cubic or the hexagonal system, and in the latter case with the specific axial ratio referred to.

The essence of their theory has been stated by Pope in a *Report on the Progress of Crystallography* published by the Chemical Society in the year 1909, and the following quotation expresses the salient propositions. They regard "the whole of the volume occupied by a crystalline structure as partitioned out into polyhedra, which lie packed together in such a manner as to fill the whole of that volume without interstices. The polyhedra can be so selected that each represents the habitat of one component atom of the material, and are termed the spheres of atomic influence of the constituent atoms. Up to this point no assumption is made other than that clearly indicated by the result of crystallographic measurements, namely, that each atom present in a crystalline structure exerts a distinct morphological effect—or, what is the same thing, appropriates a certain definite volume. The assumption is next made that the crystalline structure, which is resolvable into individual molecules and ultimately into individual atoms, exists as such by reason of equilibrium set up between opposing attractive and repulsive forces operating between the component atoms, and that this equilibrium results in the polyhedra representing the spheres of atomic influence assuming shapes which are as nearly as possible spherical. . . . The

polyhedra thus arrived at may be regarded as derived by compression of a close-packed assemblage of deformable, incompressible elastic spheres, the compression sufficing for the practical extinction of the interstitial space. When such an assemblage is released from pressure it is evident that in place of polyhedra, the shapes of which approximate as closely as possible to the spherical, closely packed spheres are presented; the distances between the sphere centres can be substantially in the same ratios as the distances between the centres of the corresponding polyhedra in the unexpanded mass, and the equilibrium condition of maximum sphericity of the polyhedra will be presented in the expanded mass of spheres by the existence of the maximum number of contacts between spheres. The whole method of treating the primary assumption thus resolves itself into finding close-packed assemblages of spheres of various sizes representing by their relative volumes the spheres of influence of the component atoms of any particular crystalline structure. . . . The conclusion is that the volumes appropriated by the polyhedra representing the spheres of atomic influence in any crystalline structure are approximately proportional to the numbers representing the valencies of the respective elements concerned. In every case hitherto studied the valency thus exhibited by an element is the lowest which its chemical behaviour assigns to it; this valency is conveniently distinguished as the fundamental valency of the element. The law thus enunciated is termed the law of valency volumes."

A considerable amount of evidence has already been presented in support of this theory. It appears to express the crystallographic relationships of the elements in the horizontal series of the periodic classification of Mendeléeff, the series of seven elements of each period regularly progressing in valency. But it does not by means of its equivalence parameters account for the further undoubted progression of the vertical groups of the periodic table, the family groups of like valency but growing atomic weight and volume. These relationships, however, have been shown in previous chapters to be clearly expressed by the molecular distance ratios, based on the accurate quantitative determinations of molecular weight, relative density, and morphological crystal elements; and the underlying cause of the progression exhibited by the members of an isomorphous series has thus been elucidated.

We may now take stock of the position reached. It is clear that the three following conclusions have been established. (1) Each atom of the molecule of the crystal substance possesses its own individual domain throughout which it exerts predominant influence; (2) the interchangeable analogous atoms in isomorphous series occupy similar relative positions in the respective structures; (3) the scheme of structure, style of architecture, type of space-lattice of the crystals of the members of isomorphous series of strictly analogous composition are identical.

If, further, with von Fedorow and Pope and Barlow we definitely apportion the whole of the space in the crystal substance among the various atoms, assigning to each atom a portion of space throughout which its influence is paramount, the acceptance of the reasoning of the latter investigators will oblige us to conclude that (4) in any one

compound the relative space occupied in the crystal structure by each atomic domain is proportional to its fundamental valency.

The volumes occupied by the atoms of different elements of similar valency in their isomorphous compounds, such as those of potassium, rubidium, and caesium, or of sulphur and selenium, or of chlorine, bromine, and iodine, cannot possibly be the same, however. For the experimental evidence is emphatic that the total volume of the elementary parallelepipedon of the space-lattice structure is considerably increased as we ascend the series. Not only will the actual volumes of the domains of predominant influence of the interchangeable atoms of different elements of the same valency present in the respective isomorphous compounds be different, but in cases where the molecular volumes are far from equality, such as those of potassium and caesium sulphates (64.91 and 84.58), there will be a wide difference in the actual magnitude of the domain of predominant influence of the same kind of atom—say the sulphur or oxygen of the alkali sulphates—when present in two different compounds. For, in passing from potassium sulphate to caesium sulphate, for instance, if we assume that the increase in the molecular volume is all traceable to the greater magnitude of the domain of the caesium atom as compared with that of the potassium atom, we must either suppose a totally different shape for the former, which enables it to appropriate outlying space among the remaining atoms of the molecule, or we must negative conclusions (2) and (3). We thus conclude that (5) in passing from one member of an isomorphous series to another most of the change of molecular volume is evenly distributed among the atoms, and is due to a proportional change of actual volume of the domains of the unchanged atoms as well as of those of the substituted atoms. That is, the spheres of influence of the whole of the atoms in the structure expand or contract, but so that their relative volumes are still proportional to the relative valencies. The total change of volume is expressed by that of the molecular volume, while the directional change is expressed by the molecular distance ratios; and the external crystallographic effect is expressed by the slight alteration of the crystal angles, in accordance with the law of progression with the atomic weight of the interchangeable elements.

The author understands that Barlow and Pope find also in this view a solution of the difficulty afforded by the exceptional behaviour of the ammonium compounds, the interchange of the five atoms of the ammonium group NH_4 for a single atom of an alkali metal having been shown to be remarkably slight, the replacement of potassium by ammonium, in fact, only calling forth almost precisely the same amount of change of molecular volume and molecular distance ratios as when rubidium is interchanged for potassium.

The extreme importance to the chemist of crystallographic research on the lines now dealt with in this first (morphological) part of the present volume will readily be perceived. Much further work remains to be done if the harmonised views above summarised are to be adequately tested and applied, and the author hopes that this presentation of them may lead more earnest students to take up this promising line of investigation.

PART II
PHYSICAL



CHAPTER XXXIV

INTRODUCTION TO THE OPTICAL STUDY OF CRYSTALS—THE NATURE OF LIGHT AND THE BEARING OF RECENT DISCOVERIES UPON IT.

THE optical part of a crystallographic investigation has in the past unfortunately been very frequently omitted, partly because of the experimental difficulties in the way of a study of the optics of so small an object as a crystal, especially if it be one of a chemically prepared substance, and partly because of the necessary knowledge of the optical properties of crystals having been more difficult to acquire than skill in the use of the goniometer. Very numerous measurements of crystals have been published without any reference whatever to their optical properties, and still more in which the reference is a very brief, cursory, and altogether inadequate one. Now the optical characters of crystals are of vital importance, for a knowledge of them is frequently essential in order that we may be able to decide definitely as to which particular system or class of symmetry is present; in many complicated cases more or less doubt is left by the goniometrical investigation, which the determination of the optical properties of the crystals is able to remove entirely. Moreover, the behaviour of light in its passage through a crystal affords a valuable indication of the nature of the internal structure, that is, of the particular point-system, not merely the space-lattice, in accordance with which the crystal edifice is erected. Further, the optical properties are often so eminently characteristic, in their details, of a specific crystalline substance, that they may be trustworthily relied upon and conveniently employed for the identification of the substance; and often this happens in cases where goniometrical investigation would be very troublesome, on account of the unsuitability of the crystals for use on the goniometer, by reason of their imperfect faces, the presence of striation, ready decomposition, or weathering, or other form of exterior deterioration. In such cases the optical investigation, being more or less independent of the nature of the exterior faces, is simply invaluable. In fine, a discussion of the optical characters, and a table of the optical constants, are essential parts of any complete description of the crystals of a substance.

The main operations comprised in the practical investigation of the

optical properties of a crystalline substance are three in number, and are as follows:—

(1) We have to determine the positions of the three principal axes of the optical ellipsoid; that is, the orientation of that imaginary ellipsoid within the crystal which we have seen in Chapter XXXI. is the expression of the optical properties (just as a similar, but not necessarily identical, ellipsoid is capable of expressing the distribution of the property of cohesion), and the orientation and character of which depend both on the nature of the space-lattice of the homogeneous molecular structure, and on the specific nature of the particular substance itself. This determination is carried out by studying the behaviour of the crystal in parallel polarised light.

(2) We have to ascertain the relative lengths of the three axes of the optical ellipsoid, and to do so we determine the refractive index of the crystals, that is, their power of bending the rays of light, along each of the three axial directions which we have ascertained as the result of (1).

(3) We independently confirm the results of (1) and (2), as well as amplify the information, by studying the optical interference phenomena displayed by the crystals in convergent polarised light, and, in the event of the crystal proving to possess two optic axes by exhibiting a biaxial interference figure of spectrum-coloured lemniscates and dark hyperbolic “brushes,” measuring the angle of separation of the so-called “optic axes” marked by the vertices of the two brushes. The optic axes are quite distinct from the axes of the optical ellipsoid, but dependent on the relative lengths of those axes, and have been preliminarily explained in Chapter XXXI. in connection with the properties of the space-lattices. When the crystal only possesses a single optic axis, the circular concentric character of the rings of the interference figure, and an unalterable rectangular dark cross radiating from the centre and which never separates into hyperbolæ, at once reveals the fact. Incidentally operation (3) affords us quite a number of phenomena in which the crystalline substance can display its individuality. Moreover, when a uniaxial interference figure is exhibited the crystal must belong to the tetragonal, hexagonal, or trigonal system; and when a biaxial figure is afforded the crystal must be of rhombic, monoclinic, or triclinic symmetry. These facts alone would enable us to decide on the true crystal system, in a case such as that of our typical crystal of potassium sulphate, which while really rhombic is of pseudo-hexagonal habit, and indeed within a few minutes of truly hexagonal morphology. The fact that potassium sulphate affords a biaxial interference figure is a decisive proof of rhombic symmetry, and would be so, even if adequately good crystals for accurate measurement and determination of the slight deviation from 60° had never been obtained.

In order to render the meaning of these three operations fully intelligible and clear, it will be advisable to review briefly the fundamental facts, first as to the nature of the agent, light, that we are employing, a subject which recent research has rendered of especial interest at the

present time, and secondly, in the next few chapters, of crystal optics. Optical crystallography often looks so appallingly difficult to a beginner, that it has been felt desirable by the author to state thus at once, in these introductory pages, that the experimental work is really of a very simple character, essentially ranging around the three operations above specified. Not only is this department of the crystallographer's work straightforward and easy, provided he thoroughly understands what he is doing and that the proper instruments are available, but it is also fascinatingly interesting. For the phenomena of polarised light are admittedly the most beautiful which experimental science has yet afforded us, and the interference phenomena exhibited by crystals, and referred to above as requiring to be studied, are among the most superb of all those which even polarised light can provide.

Consequently, the optical part of a crystallographic investigation is of surpassing interest, and if at times the phenomena prove complicated and require most careful study, the results are not only themselves rendered the more valuable thereby, but the experimental means of arriving at them are intrinsically so beautiful as to far more than compensate for all the trouble involved.

THE NATURE OF LIGHT.

It is now an established fact that light, using the term in its objective sense, consists of a vibratory motion, or oscillatory change of condition, in an everywhere pervading medium, the ether, the vibrations or oscillations being of such a definitely periodic character, **transversely** to the line of propagation, as to produce waves or undulations the impact of which on the peculiarly sensitive layer of the retina of the eye gives rise, on transmission by the optic nerve to the brain, to the sensation which we term, subjectively, light. The motion has been advisedly stated to be "in" the medium rather than "of" the medium, for there have until recently been two forms of the undulatory theory, quite irrespective of the old emission theory favoured by Newton, which was definitely proved to be incorrect by Foucault, who found experimentally that the velocity of light in water is less than it is in air, whereas according to the theory based upon the emission of "light corpuscles" it should travel more rapidly in water than in air.

According to one, now abandoned, form of the undulatory theory, the waves of light are actual transverse vibrations of the ethereal medium itself, while according to the other and more recent and generally accepted form the vibrations are due to a rapid periodic change in the electrical and magnetic condition of the ether. Both forms of the wave theory postulate a transverse vibration or oscillation in the medium; they only differ as regards the manner of its production. The later form is distinguished as the electromagnetic theory of light, and is now receiving further confirmation every day. The mechanical form of the wave theory demands the propagation of longitudinal waves as well as transverse ones, and the existence of such longitudinal waves in rays of light has not been proved. The electromagnetic theory of light does not lead to the supposition of any but transverse vibrations, and from many other considerations, derived from the developments of the remarkable discoveries of the cathode and X-rays and of radium and radio-active substances, an account of which will be given later in this chapter, it is much more likely to represent the truth.

The Medium, Ether.—All the forms of energy with which we are acquainted require media for their propagation. Thus sound requires a material substance, gaseous, liquid, or solid, for the propagation of its longitudinal vibrations, which are

incomparably grosser than those of light. It is, in fact, this all but infinitely greater length of sound waves which renders them apparently so much more capable of bending round an interposing object than the waves of light, although we now know, from the exquisite phenomena of diffraction spectra, that light waves are equally able, considering their minute size, to bend round an edge of adequate sharpness. Sound, however, cannot be transmitted through a vacuum, while light travels not only unimpeded through empty space with enormous velocity, 185,400 miles per second,—compared with which the 1100 feet per second of sound in air is an almost negligible quantity,—but with even greater velocity through space than through air or any other gaseous medium. The presence of matter does, indeed, exert more or less restraint on the vibrations of light, its velocity being more considerably reduced by its passage through transparent liquids and solids than through gases; yet even in the cases of highest resistance the velocity of light is immensely greater than that of sound. Ample proof of this reduction in velocity is afforded by the phenomena of refraction, or the bending of the rays of light by transparent material media, with which we shall have so much to do in studying the optical properties of crystals. This bending of the rays of light when they enter a transparent material substance from a vacuum, or when they pass from a lighter to a denser transparent medium, for instance, from oil into a crystal, is one of the most characteristic properties of light vibrations, and is intimately connected with, and indeed a function of, the reduction in velocity.

The conception of an ether is not at first easy to grasp. It is difficult to conceive anything other than material substances, that is to say, the chemical elements, their compounds, and mechanical mixtures of either or both. It also requires the scientific use of the imagination to picture an everywhere existing medium which interpenetrates all material substances, besides filling interstellar space. Yet recent advances have rendered the conception far less difficult than it formerly was. We have seen that there are certainly intermolecular and probably interatomic spaces in the crystal edifice, and for that reason we do not speak of the structural ratios referred to in Chapter XXXI, the dimensions of the unit cell of the space-lattice, as the relative dimensions of the molecules, but as molecular distance ratios; for although they do represent the dimensions of the molecular cell or elementary parallelepipedon of the space-lattice of the crystal structure, that is, of the bricks of which the edifice is built up, yet we do not know how much of that space is occupied by matter and how much is free space. Moreover, we now know further, from the researches of Sir William Crookes, Sir J. J. Thomson, and a rapidly growing band of other workers in the field which they have so wonderfully opened up, that the chemical atom itself is not entirely composed of matter, but of a considerable number of electronic corpuscles separated by free space; so that it is now quite easy to comprehend that what appears to be impenetrably solid matter is in reality anything but impervious and solid in the old sense of the word. It is far more difficult to believe in the possibility of action at a distance without any intervening medium, in the cases of light, magnetism, and gravitation, than to concede the existence of a medium, for which, moreover, the evidence every day becomes stronger. There can be little doubt that the influence of the sun on this earth, as on the other planets of the solar system, in constantly maintaining and reinforcing its energy, is due to the conveying power of the ethereal medium.

Of the properties of the ether we are not yet justified in stating anything. The promulgators of the earlier form of the undulatory theory assumed the ether to possess both elasticity and density, but the possibility of longitudinal vibrations in a medium so endowed, and their non-discovery experimentally, has rendered the supposition more than doubtful. The difficulty has been met by imagining the ether incompressible, which would so enhance the velocity of the longitudinal wave as to render it infinite. Moreover, it has been pointed out that all that the ether requires is torsional rigidity, or resistance to change of shape, the characteristic property of a

jelly, to enable it to transmit transverse vibrations. Again, the celebrated theory of vortices in the ether, associated with the name of Lord Kelvin, was also introduced in order to explain how the ethereal fluid could be adequately stiffened to act as an elastic vibratory fluid, capable either of mechanical or electric stress.

The difficulties surrounding this view of the ether are very numerous, however, and recent advances in electricity, especially the experimental discovery of electromagnetic waves by Hertz, and the wonderful development of their practical applications by Marconi, Lodge, Duddell and others into wireless telegraphy and telephony, have constrained us to turn to the electromagnetic form of the undulatory theory, for a truer explanation of the beautiful form of energy which we term light, and as far more promising to elucidate finally the mystery of the ether.

The Electromagnetic Theory of Light.—A rapidly alternating condition of electric polarisation of the ether forms the fundamental assumption of this theory. The ether surrounding a body which suffers rapid alternate electrical charging and discharging, or along which a rapidly alternating electric current is passed, will naturally be polarised in two opposite senses with corresponding rapidity; and provided this change of state be regularly periodic, and occurs within the limits of rapidity corresponding to the frequency of oscillation of extreme red and violet light, the vibrations of the ether which are produced will be those which are perceived by our eyes as light. If the rapidity be less than that required for red light, the vibrations will be perceived as those of radiant heat, and if the rapidity be more extreme than corresponds to violet light, the radiations will be those which are so extremely active in producing chemical activity, and which we distinguish by the term ultra violet rays.

The attractions and repulsions exerted respectively between oppositely and similarly electrified bodies, between the opposite and similar poles of magnets, and between oppositely and similarly arranged electric circuits, can only be satisfactorily accounted for by assuming the presence of a medium which conveys the energy across the intervening space. It is equally inconceivable that the generation of an electric current in one circuit, by the starting or stoppage of a current in a neighbouring circuit, can be merely action at a distance, without the intervention of a carrying medium. It is probable that when a conducting body is electrified, or an electric current is generated in a circuit, energy is stored in the ether surrounding the conductor. This energy is used up in altering the state of the medium, which may then be said to be "polarised"; and when discharge occurs the medium is restored to its original condition. This is the essence of the theory of electricity and magnetism which is associated with the name of Clerk-Maxwell, and his view that electrical phenomena are the manifestation of the energy thus stored in the ether about a conductor is one which has as completely replaced the old fluid theory of electricity, as the undulatory theory of light has replaced the emission theory. Its bearing on the wave theory of light is of the first importance, and an adequately comprehensive idea of that theory has only been rendered possible by the remarkable practical developments already alluded to, of the use of electromagnetic ethereal waves, which have resulted in telegraphy through space.

The polarisation of the ether occurs in one of two ways, one corresponding to a positive charge of electricity and another to a negative charge. We know practically nothing at present as to the nature of this polarisation, the term being merely used to indicate a changed condition. Preston has suggested that positive and negative polarisation may be related like right and left-handed screws or rotations, and we now know that such rotations or orbital motions do occur on the part of the electrically-charged corpuscles of which the elementary atoms are composed. But it is anyhow easy to see that if the ether surrounding a conductor can be polarised in alternate succession in the two opposite ways with adequate rapidity, a periodic disturbance or vibration will be produced, capable of generating waves of energy and propagating them throughout the ether of the surrounding space; just such waves, in fact, as we

are led to consider the waves of light must be. The only conditions of this vibration of which we are yet certain are, that it is strictly periodic, and that it occurs transversely to the direction of propagation. In all probability the only other condition necessary, in order that the vibration shall give rise to those waves which affect our nerves of sight, is that it shall be sufficiently rapid, that is, that the alternation of the two opposite electric charges shall occur within the already specified limits of frequency, corresponding to red and violet light respectively.

Experimental Production of Electromagnetic Waves.—An investigation of the spark discharge of an electric condenser, such as a Leyden jar, has revealed the fact, surmised on theoretical grounds by Lord Kelvin so long ago as 1853, that the discharge is not continuous, but oscillatory. The first discharge, as it were, over-reaches itself, and is followed by a recoil, the process being then repeated to and fro until equilibrium is attained. Feddersen proved the fact experimentally by means of a revolving mirror, which was caused to reflect an image of the spark into a telescope. Under suitable conditions of resistance in the circuit the image was observed to be split up into a series of bands, corresponding to separate rapidly succeeding discharges. From calculations of the time of oscillation of the discharge from a Leyden jar of ordinary size, it has been shown that such oscillatory discharges produce electromagnetic waves, varying from fifty to a hundred metres long. As the size of the jar is reduced, the wave-length correspondingly diminishes; and if we imagine the process continued until we arrive at a hypothetical condenser of the diminutive size of a chemical atom, we may expect to have waves produced corresponding in length to the extremely minute ones of light. From this it is not unreasonably deduced, that the oscillating discharge of the polarised ether about the atoms or molecules (due primarily, as we shall see later, to the electrified corpuscles of which they are composed) of incandescent substances, is the actual cause of the generation of the waves of light.

The experiments of Hertz, in detecting and investigating the longer electromagnetic waves, are of incalculable importance not only to electricity but also to light, as they go far to prove the truth of the electromagnetic form of the undulatory theory of light. This form of the theory is now, in fact, the only theory of light which has stood the test of recent work; it was originally put forward from the optical point of view by MacCullagh, and subsequently from the electrical point of view, as already mentioned, quite independently by Clerk-Maxwell. Larmor has since contributed very considerably towards finally establishing the electromagnetic theory, for he has interpreted MacCullagh's results in the light of recent work, and has shown how, starting from the optical point of view, we inevitably get back to the electrical, and has thus completely bridged over the gap between MacCullagh and Clerk-Maxwell.

When a spark discharge is caused to pass between two conducting knobs, attached in each case by a wire to an electrified brass plate, one being positively and the other negatively electrified by being attached to the terminals of an induction coil, and when another circuit, consisting of a single hoop of wire supplied with knobs at the two nearly approaching ends, is brought near, it is observed that, when the length of the wire in the hoop has been properly adjusted, an induced alternating current is set up, and eventually sparks pass between the knobs at the ends of the hoop. Moreover, experiments can actually be carried out on the reflection, refraction, and interference of the electromagnetic waves produced between the two circuits; a sheet of zinc acts as an excellent reflecting screen, a parabolic sheet best of all; and a prism of pitch acts towards these electromagnetic waves as a glass prism acts towards light waves. Indeed Hertz actually measured the refractive index of pitch for the waves produced by his apparatus. More convenient and elaborate forms of electric wave-producing and receiving apparatus have subsequently been devised by Hertz himself, Chant, Righi and others, among which may be mentioned the magnetic detector of Rutherford and the coherer of Branley, the latter of which has proved so invaluable in the hands of Marconi and others in wireless telegraphy. Several forms of apparatus have quite

recently been brought to great perfection for generating electromagnetic waves of vastly increased power, so that these waves are now at the service of mankind, messages by their means speeding thousands of miles across the oceans through the ether of space,—and of course with still greater facility within a smaller radius, such as that of the British Isles and their surrounding seas, or between a ship on the high seas and a distant coast, or between ship and ship whether in or out of sight of each other, —and being detected with certainty by the delicate feeling and receiving apparatus at their intended destination. The fact that they are now so thoroughly at command, and that they are essentially of the same nature as those of light, is the point which it is here desired to emphasise.

Length, Distance of Propagation, and Velocity of Electromagnetic Waves.—The cylindrical oscillator employed by Chant produced waves eighty centimetres long, and with the spherical oscillator having ten-centimetre spheres he obtained electromagnetic waves 38·5 centimetres long. Righi, however, obtained waves of only 7·5 centimetres wave-length, and Lebedew has actually obtained waves shorter than a centimetre, by using cylinders only four millimetres in length. The wave-length is determined by observations of the positions of stationary nodes, where direct and reflected waves interfere, a detector being moved along between the oscillator and a reflecting surface, until the needle of the galvanometer in the detector circuit ceases to indicate movement. The wave-length is twice the distance between two nodes. From such determinations and a knowledge of the frequency of the oscillations, the velocity of these electromagnetic waves can be calculated, and it appears to be identical with the velocity of light, 185,400 miles per second.

Polarisation of Electromagnetic Waves.—It can be proved with an oscillator and detector of any of these later forms of apparatus, mounted in parabolic reflectors, that the electromagnetic waves are plane-polarised, and that the vibrations are transverse to the line of propagation. The parabolic zinc screen reflects the waves, generated by the oscillator in the focal line of the screen, as a parallel beam towards the receiving screen, which in turn reflects them to its focal line where the detecting apparatus is situated, precisely in the same manner as waves of light would be reflected from a pair of parabolic mirrors. The electric force is naturally parallel to the axis of the oscillator, and the magnetic lines of force being circles concentric to it, the directions of the two forces lie perpendicular to each other in the wave-front. That the oscillator acts as a polariser in producing definitely orientated vibrations will, therefore, be clearly evident, and the detector acts similarly as an analyser, for it only responds to the electromagnetic waves when the axis of its parabolic reflector is parallel to that of the oscillator. If either be rotated until its focal line is perpendicular to that of the other, sympathetic sparking ceases, exactly as, when using a pair of Nicol prisms to demonstrate the properties of polarised light, the dark field is produced when the polarising and analysing Nicols are crossed.

Moreover, one further highly interesting and significant experiment may be performed with electromagnetic waves, which not only emphasises the undoubted occurrence of polarisation, but most strikingly brings the behaviour of these waves into line with those of light, and completes the argument for their similarity. If a screen of parallel wires be placed so as to intercept the polarised electromagnetic waves proceeding from the vibrator to the synchronising receiver, it is observed that when the wires are parallel to the oscillator and to the focal lines of the parabolic reflectors the wire screen proves opaque to the waves, because it reflects them; but when the wires are perpendicular to this direction the screen composed of them proves transparent to the waves. The former direction for opacity is parallel to the electric force, and the latter direction corresponding to transparency is parallel to the magnetic force. When, however, the oscillator and detector with their respective screens are arranged perpendicularly to each other, so that the detector is not influenced by the oscillator, and the equivalent of the dark field of the Nicols is produced, the intro-

duction of the wire screen produces no effect when the wires are parallel to either the one or the other; but when the wires are inclined at 45° to the axes of both oscillator and detector, the waves are transmitted and detected. For the waves are resolved into two components, vibrating respectively parallel and perpendicular to the wires; that component which vibrates in the former direction is naturally reflected, while the other passes to the receiver, is there again resolved into two components parallel and perpendicular respectively to the axis of the parabolic screen, and the former of these will be detected by the receiver. This result is precisely analogous to the optical one obtained when a plate of a doubly refracting crystal, cut parallel to one of the planes of optical symmetry, is introduced between crossed Nicols, first so that the two axes of the optical ellipsoid contained in that plane are respectively parallel to the two Nicols, when the dark field remains unaffected, and subsequently rotated 45° from that position, when light is transmitted, coloured when the plate is adequately thin.

Final Conclusion as to the Nature of Light.—These striking experiments, together with the fact that the velocity of electromagnetic waves is the same as that of light waves, namely, 185,400 miles or 298,500 kilometres per second,—as was predicted by Maxwell would prove to be the case,—and that electromagnetic waves of less than a centimetre in length have actually been experimentally produced, all concur in indicating that the waves of light only differ from these electromagnetic waves by their very much shorter wave-length. It is, therefore, practically proved that the oscillating discharge of the polarised ether about the atoms and molecules of glowing substances is the origin of the waves of light which they give forth. In the next section it will be shown that it is the moving electrified corpuscles, of which the chemical atoms are composed, that are the prime authors and originators of these electromagnetic waves of light; for the radiations from the simplest of atoms, such as those of hydrogen, are not of a single wave-length, but of several different wave-lengths corresponding to the lines in the spectrum of the element. The individual monochromatic rays, however, we shall see are due to the corpuscles or cycles of corpuscles building up the atom, so that the *tout ensemble* of all the corpuscular oscillations gives us the complex radiations of the atom.

THE BEARING OF RECENT DISCOVERIES ON THE FURTHER ELUCIDATION OF LIGHT RADIATION.

Beyond the unanimously accepted facts just stated very much more has lately been revealed as the result of the remarkable discoveries and investigations of cathode rays, Röntgen rays, and the so-called “radio-activity” of the two metals of highest atomic weight, uranium and thorium, and of the extraordinary new element next to them in order of atomic weight, radium. The terrestrial isolation of the light gaseous element helium, and the discovery of other inert gases in the atmosphere, neon, argon, krypton and xenon, ought also to be included. For all these discoveries have a profound bearing on the nature of light, on the constitution of the chemical atoms, and on the character of the ether itself.

We have in reality to go back to the discovery of the quantitative laws of electrolysis by Faraday, and his prescient views in explaining them, for the first inkling of the possibility of the existence of bodies smaller than the chemical atoms. He could only satisfactorily explain the definite relation between the electric current and the chemical work which it is able to perform by assuming the electrolyte to contain positively and negatively charged particles, the former of which go to the negative pole while the latter are delivered at the positive pole. The giving up of these electric charges at the two poles he conceived to be the cause of the production of the electric current.

After a long interval,—during which the investigations of MacCullagh, Clerk-Maxwell, Lord Kelvin, Fitzgerald, and Johnstone Stoney had been consolidating the groundwork

of the electromagnetic theory of light as given in the foregoing pages and as we now accept it, and the last-mentioned investigator had invented the term "electron" to signify the unit electric charge,—the beautiful work of Sir William Crookes on the passage of the electric current through high *vacua* was inaugurated. One of the many results of this most fruitful investigation was to establish the fact, which was later more fully confirmed by Sir J. J. Thomson, that the rays passing from the cathode or negative terminal—generally a platinum wire fused through one end of the vacuum tube or bulb and terminating in a plate of aluminium at right angles to the wire—to the anode or positive terminal at the other end of the tube or elongated globe now currently known as a Crookes tube, when the vacuum was a very high one (the pressure of the residual gas being less than a thousandth of a millimetre of mercury), consisted of material particles smaller than atoms and carrying along with them the negative charge of electricity. Johnstone Stoney's term "electron" is now currently applied

to them, but it must be clearly understood that when so termed the electrically charged particle is meant. The material particle itself is preferably called a corpuscle. These corpuscles are produced by the disruption of the chemical atoms of the residual

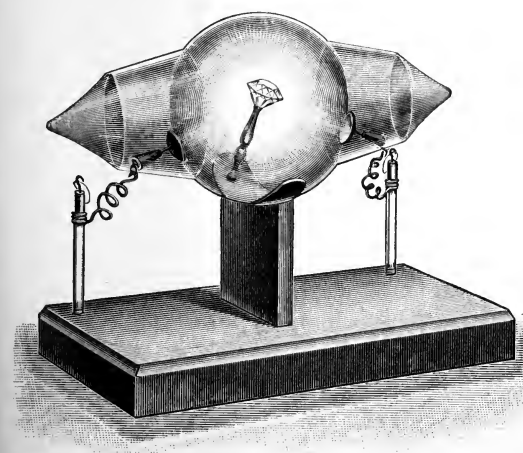


FIG. 479.—Phosphorescence of a Diamond under Impact of Cathode Rays.

gas by the electric current, and are the same whatever gas had been used to fill the tube before its evacuation, although the pressure at which the disruption occurs is slightly different for different gases. They proceed in straight lines, so that if a screen, of almost any material, be placed in their path—the cross of aluminium, for instance, interposed in the pear-shaped Crookes tube shown in Fig. 478—its shadow will be thrown on the opposite positive end of the tube or bulb, the broader end of the tube shown in the illustration. When, however, they are allowed to reach the glass walls of the

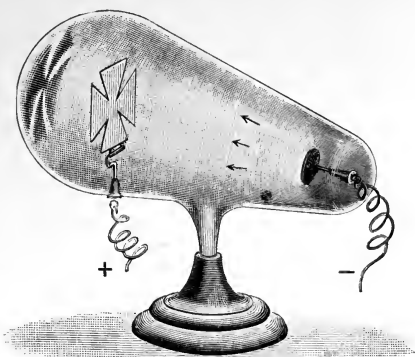


FIG. 478.—Crookes Tube, showing Shadow of Aluminium Cross interposed in Cathode Rays.

they produce brilliant phosphorescence there by their impact, lead glass giving a blue glow, soda glass a bright emerald green, and uranium glass a most brilliant green. Many crystals, such as diamonds and rubies, phosphoresce most brilliantly with characteristic colours when these cathode rays are allowed to fall on them. Fig. 479 shows a diamond belonging to Sir William Crookes phosphorescing with a bright green light in the centre of an exhausted bulb, the cathode rays being

directed on it from below. South African diamonds give a bright blue phosphorescence; other diamonds many and very various tints but often blue or greenish-blue; rubies and sapphires phosphoresce with a red colour, which in the case of the ruby is extremely brilliant.

The energy produced by the impact of the corpuscles is so great that if the cathode plate be made concave so as to converge the rays to a focus, even platinum when placed exactly at that focus is readily melted. Moreover, the rays are deflected when a magnet is held outside the tube, proving that they are composed of material particles, and are not an ethereal wave-motion. Both Perrin and Sir J. J. Thomson have independently shown that negative electricity is actually transported by them.

It was subsequently discovered by Röntgen that the impact of these negatively charged corpuscles smaller than atoms on the walls of the Crookes tube, besides producing phosphorescence and liberating heat, gives rise to a new form of vibration in the ether, the now famous Röntgen or X-rays. These are akin to light waves, merely differing in having a very much shorter wave-length and in not being a sustained wave motion, but only a series of momentary impulses, one due to the impact of each particular particle. They owe their now extensive use in surgery to their short wave-length, which enables them to pass through the molecules of all but

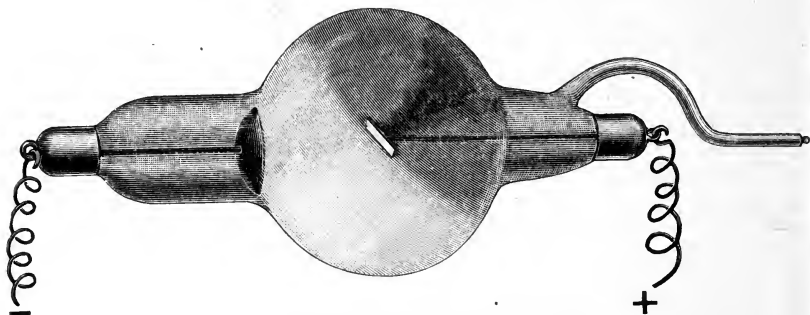


FIG. 480.—Crookes Tube for Production of X-rays.

very dense substances. Thus glass, wood, the light metal aluminium, and incidentally flesh, are more or less pervious to them, while heavy metals like lead and, as it fortunately happens, bone, are practically impervious. They affect a photographic plate, so that when the hand, for instance, is held between the excited Crookes tube and a sensitised plate in the dark, and the plate is subsequently developed, a "skiagram" or shadow photograph of the bones of the hand is obtained. For the shadows of the bones are strong, leaving nearly clear glass in the negative, the X-rays being cut off by them, while those of the fleshy parts of the fingers and hand are feeble, just sufficient to indicate their outline. A gold ring on the finger gives also a strong shadow, while the crystals on it are more or less transparent. For ocular observance of the skiagrams of the parts of the human body, the photographic plate is replaced by a glass plate coated with potassium or barium platincyanide, salts which fluoresce strongly where the rays impinge upon them. The main use of the rays in surgery is to locate any foreign object, a needle or bullet for instance, and also to indicate the presence of malformations, fractures, or abnormal growths of the bones themselves. The form of Crookes tube which is most suitable and efficacious for the production of X-rays is shown in Fig. 480. It is blown out of soda glass, as this is transparent to the rays, whereas lead glass is opaque to them. The cathode plate is of aluminium, and is concave, and the positive anode is a plate of platinum inclined at 45° to the path of the cathode rays and situated near the focus of the cathode plate, but not sufficiently exactly at the focus to cause the platinum to melt. When the tube is exhausted to a certain point the brush discharge

first observed on actuating the induction coil (to which the terminals are connected by wires) becomes stratified, and the space near the concave cathode becomes dark; this dark region rapidly extends as the exhaustion approaches the desired amount, until eventually it reaches the walls of the tube, which immediately phosphoresce. Faint blue streamers, the cathode rays, may then be perceived leaving the concave aluminium cathode and converging towards the inclined platinum plate, thus moving against the direction of the current, which is from the platinum to the aluminium electrode. The inclined plate deflects the radiations causing the phosphorescence on to the walls of the tube on one side, and thence outside where they may be used for the purpose of producing skiagrams. This use of an inclined anode is only for convenience in getting a concentration of the Röntgen rays in one direction; any body arresting the cathode rays gives off these X-rays, and when there is no broad anode the glass walls themselves serve the purpose. In the case of the use of the inclined anode plate the Röntgen rays appear to travel in straight lines from the point where the cathode rays strike the anode. The Röntgen rays, in virtue of their consisting of innumerable single impulses instead of continuous waves, are incapable of refraction and of interference, in this manner being distinguished from light waves, the originating vibrations of which are, as we have seen, continued in an oscillatory manner.

The Röntgen rays thus appear to present to us the extreme form of ethereal wave motion on the side of minuteness, while the electromagnetic waves employed in wireless telegraphy afford us the other extreme of long waves, and between the smallest of these latter vibrations yet experimentally produced and light waves come the rays of radiant heat, the wave-lengths of which form a continuation of the longer waves of light belonging to the red end of the spectrum. In the same manner the extreme blue end of the visible spectrum passes into the ultra violet chemically active rays, on the side of shortness of wave-length, after which with a considerable intervening gap we come to the extremely short wave-lengths of the Röntgen rays. It is highly interesting that the Röntgen rays are capable of producing a similar chemical action on a photographic plate to violet light waves, showing that the gap is not very great. The beautiful continuity of these forms of energy is one of the most striking results of these new discoveries, and it is absolutely necessary that we should follow them if we are to justly appreciate the nature and relative bearing of that form of energy, light, with which the crystallographer has so much to do, and which reveals for him more of the internal structure of the objects of his study than probably any other agency.

As illustrating once more how one discovery leads up to another, just as the refined studies of Sir William Crookes on the electric discharge in rarefied gases led up to the discovery of the beneficent (and dangerously otherwise if not used with due precautions) X-rays by Röntgen, so experiments by Becquerel in the year 1896 as to the nature of the fluorescence which is exhibited by the substances such as potassium and barium platino-cyanides, used in preparing the fluorescent screens for Röntgen-ray experiments, led to the discovery of the phenomenon now so well known as "radio-activity." For he made the unexpected observation that the salts of the heaviest of all known metals, uranium (atomic weight 236·7, that of hydrogen=1), which are pre-eminently fluorescent, do not require to be preliminarily excited by sunlight or Röntgen waves, but that, even when freshly prepared synthetically in the dark, they have the power of emitting rays capable of passing through opaque screens and then of acting on a photographic plate. These rays were found to consist of a wave-motion of the ether, very similar to the X-rays, and like them and violet light waves, capable of chemical action on the sensitive layer of the plate. Now it has long been known that diamonds, certain forms of fluor-spar, blende (natural crystallised sulphide of zinc), and the artificially prepared sulphides of barium and calcium, glow brightly in the dark after exposure to bright sunlight, the general rule for such phosphorescence being that the phosphorescent rays are of greater wave-length than the exciting rays. But here is a substance, uranium, which of its own intrinsic nature emits rays similar to X-rays. In the year

1898 the further discovery was made independently by Schmidt and by Madame Curie—the wife of Prof. Pierre Curie of Paris, to whose work concerning the symmetry of homogeneous structures reference was made in Chapter IX.—that this singular property was also possessed by another of the chemical elements, thorium, the next highest in order of atomic weight (230.8). But a still more surprising fact resulted from a systematic search for radio-active minerals which was then instituted by Madame Curie, namely, that the mineral from which uranium is derived, pitchblende (an impure uranoso-uranic oxide U_3O_8), was very much more radio-active than the pure oxides of uranium which are extracted from it for the preparation of fluorescent glass. The pitchblende from the Johanngeorgenstadt mines in Austria, the richest in uranium (70 per cent), proved to be four times as strongly radio-active as green uranium oxide U_3O_8 , and three times as powerful as the black oxide U_2O_5 . Moreover, the strength in radio-activity came largely from the rejected residues left after the extraction of the uranium, and after a very laborious investigation of them Madame Curie was finally rewarded by the discovery of the most wonderful of all known chemical elements, radium, which possesses the property of radio-activity *par excellence*, a million times more powerfully than uranium, and to which pitchblende very largely owes its display of the property. It was associated with the barium contained in the mineral.

Moreover, the bismuth also contained in pitchblende was found to be associated with another radio-active element, which in honour of Poland, her native country, Madame Curie named polonium. It is present in such small quantities that Marckwald, who has since discovered another mode of obtaining it from pitchblende, had to use two tons of pitchblende in order to obtain the fraction of a grain of polonium. By the end of the year 1898 these two new elements were obtained sufficiently pure to enable their spectra to be determined. In 1899 a third new element, actinium, was extracted from this important mineral, pitchblende, by Debierne, which also proved to possess radio-active properties. The atomic weights of all these five elements, uranium, thorium, radium, polonium, and actinium, are very high, the three which are known with certainty being the highest of all elements. That of radium has recently, 1908, been determined by Madame Curie to be 226 (225 when $H=1$, 226.4 if $O=16$), with a preparation of extraordinary purity, and the number has been independently confirmed in this country by Sir Edward Thorpe. We are still uncertain about the actual figures for polonium and actinium.

Metallic radium itself has actually been isolated (1910) by Madame Curie and A. Debierne by the electrolysis of the chloride, and found to be a brilliantly silver-white metal which melts at $700^\circ C.$, altering at once in contact with air with formation of a black nitride.

Radium is constantly emitting four different things: (1) α -particles, which have been shown by Rutherford, who has contributed greatly to the investigation of the radio-active elements, to consist of the positively electrically charged atoms of the gaseous element helium (atomic weight 4), which itself had only a short time before been discovered by Sir William Ramsay to exist terrestrially, occluded in the mineral cleveite, although its spectrum has been known for years to be characteristic of the solar corona. (2) β -particles, which are negatively electrically charged corpuscles smaller than atoms, similar to the cathode rays of the Crookes tube. (3) γ -rays, which are of the same nature as the Röntgen X-rays. (4) An emanation of an atomic nature, which Sir William Ramsay has proved (1910) to be another new gaseous element of the argon series, to which he has given the name "niton," the second higher in atomic weight above xenon, its atomic weight being 222, 4 below that of radium. It appears to be intimately connected with the production of helium, and that radium spontaneously decomposes into helium and niton.

Sir William Crookes has devised a simple arrangement which he terms a "spintariscope" or "spark-viewer," for observing the luminous effect on a fluorescent screen, of the α -particles, the charged helium atoms, emitted by radium.

It is shown in Fig. 480*a*, as constructed by Messrs R. & J. Beck. It consists of a short tube carrying a lens at one end, very much like the focussing glass used to focus the picture on the ground glass of a camera, but closed also at the other end by a screen coated with zinc blende, the phosphorescent sulphide of zinc, and carrying a minute speck of a salt of radium at the end of a small wire holder just in front of the centre of the screen without touching. As the charged helium atoms emitted by the radium impinge on the blende screen, they render it fluorescent at the spot, and the lens resolves the spot of light into a multitude of small sparks or flashes, momentary scintillations of bright light, like a crowd of shooting stars.

The explanation of these phenomena presented by radium is that the element is one so near the verge of instability, that some of its atoms are constantly disintegrating, and breaking up into their constituent corpuscles, the β -particles, which may be termed "electronic-corpuscles" as they emerge negatively charged with electricity; also into groups of corpuscles associated together as atoms of helium, which emerge carrying the positive electric charge, and each of which is probably composed of four times as many corpuscles as there are in the atom of hydrogen, the number in which is still a matter of discussion, the latest determination, by J. A. Growther, indicating only three; and also into the more complicated association of corpuscles, the atoms of the inert gaseous substance niton. The disintegration is thus essentially one into electronic-corpuscles, but some of these recombine as they emerge, to form helium and niton.

It would appear that this disintegration of atoms into corpuscles, which occurs spontaneously with radium, is brought about in the cases of atoms of the common gases, such as hydrogen and nitrogen, by the passage of an electric current through them in the very rarefied state in which they exist in a Crookes tube. In the case of radium, the disintegration is accompanied by the liberation of heat, so much indeed as to maintain the radium salt permanently two degrees Fahrenheit higher in temperature than the surrounding air. The energy previously locked up in the intra-atomic vibration of the corpuscles is thus liberated on the breaking up of the corpuscular arrangement constituting the atom, whatever be the nature of the latter, whether a series of definitely arranged cycles or vortex rings or any other arrangement having regular orbital movements.

The excessively minute and light β -corpuscles are matter in its ultimate simplicity as far as we yet know it, and they move with almost incredible velocity for material particles, something like 100,000 miles per second, and are capable of readily passing through cloth, paper, wood, and thin sheets of metal, especially the light aluminium; indeed the porosity to the passage of the corpuscles is in general inversely as the density of the material interposed. But they are stopped by a dense metal sheet as thick as a penny. They are emitted not only by radium, but also by thorium and uranium, and possibly by actinium, but not by polonium. They are accompanied by the γ -rays in each case, which latter are much more penetrative, passing readily through an inch of solid lead or a foot of iron, and their velocity would appear to resemble that of light waves, being nearly twice that of the β -corpuscles, which in turn is much greater than that with which the atomic α -particles of positively charged helium are emitted by radium (about 15,000 to 20,000 miles per second).

As the best source of radium is the uranium minerals, it has been suggested that radium, of atomic weight 226, is a stage in the disintegration of uranium, of atomic weight 237, just as niton and helium are further stages produced by the breaking up of radium atoms. As uranium itself emits both α -particles and β -corpuscles, it may be that radium is a product of their reassociation or combination under favourable

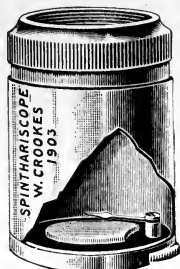


FIG. 480*a*.—The Spinthariscope.

conditions. From the experimental work of Boltwood, and also according to Joly, who has studied the question from the geological point of view, there is also an intermediate substance, ionium, formed between uranium and radium.

The facts which have thus been briefly recorded enable us to carry our knowledge of the nature of matter another stage further than the atom, which for the purposes of structural crystallography we have shown in the preceding part of this book to be the unit of crystal structure, the stereometric mode of aggregation of which in a regular point-system determines the crystal class, while the molecule determines, by its space-lattice mode of aggregation, the system; just in fact as the atom is the elementary unit of chemical combination, and the molecule the unit of the chemical compound. It was further stated, in discussing the probability of the occurrence of interspaces between the molecules and between the atoms, that the atoms themselves were probably composite, and that relatively large interspaces unoccupied by matter as we know it also occurred between the composing ultimate particles within and constituting the atom. We now perceive that the β -corpuscles are these ultimate particles, and that they are of paramount importance from the point of view of the nature of light vibrations, as will presently be more clearly explained. The interspaces within the atoms, however, are not readily penetrated, as the corpuscles are undoubtedly in rapid and probably organised motion within the confines which we regard as the atom or its sphere of influence.

The chemical atom, the conception of which by Dalton has led directly to a century of most wonderful progress in chemistry, does not lose in importance one *iota*, as the unit of chemical constitution and of crystal structure, by this discovery of component corpuscles. On the contrary, our new knowledge of its composite nature affords us an inkling of the meaning of certain properties of atoms which have long puzzled us, and in particular, of the meaning of the periodicity observed in the physical and electro-chemical properties of the atoms of the various elements, which is seen to occur when they are arranged in the order of their atomic weights. Why these properties are repeated in octaves, from basic (electro-positive) though neutral to acid (electro-negative), giving rise by their repetition to the family groups of similar elements which we have seen form the isomorphous series of crystalline compounds, is now no longer an enigma. This fundamental law of periodicity of the chemical elements, discovered by our countryman Newlands and termed by him the law of octaves, and worked out in numerous details by the master mind of Mendeléeff,¹ is indeed clearly revealed in crystals, in the beautiful law of progression of the crystal angles and structural and physical constants, according to the atomic weight of the interchangeable elements of the same family group which give rise to the isomorphous series. Indeed, the term "isomorphous" is now found to be only applicable to such series in the limited sense, involving this progression in regard to the crystal angles as to every other property of the crystals, which has been clearly stated in Chapter XXXI.²

It is obvious that such arrangements of corpuscles can readily be imagined which shall account for this atomic periodicity, and although none of the detailed schemes yet presented may give us a correct picture, there can be no doubt that their principle is on right lines. Sir J. J. Thomson, for instance, has imagined the formation of the various atoms by the regular addition of cycles of these corpuscles arranged concentrically, such cycles being found to fall naturally into octaves of similar type of progression. In some such way, though possibly with other details, the corpuscles

¹ See Chapter III. of the author's monograph, *Crystalline Structure and Chemical Constitution* (Macmillan and Co., 1910), for further details, including the periodic table of the elements. To that table the atomic weight of niton now requires to be added, 221 if $H=1$ or 222.4 if $O=16$, in Group VIII. Series 11, instead of the radium emanation, then thought to be 175 or 176 and to belong to Series 9.

² A full account of the author's researches is given in the above-mentioned monograph.

within the atom are kept in regular orbital motion within its bounds, like a miniature solar system. The number of corpuscles composing the atom of the lightest of the elements, hydrogen, was at first supposed by Sir J. J. Thomson to be as many as 800, but this number has recently been shown to be far too large, and the two latest determinations, by J. A. Growther, have agreed in indicating three as the correct number. In any case, however, the heaviest atom yet known, uranium, contains 238 times as many. There can be no doubt as to the material character of these electronic-corpuscles, meaning by the term negatively electrified corpuscles. Why the escaping corpuscles have in most cases only been observed to carry a negative electric charge, and why their associated groups, the helium atoms of the α -rays, are the chief particles yet found positively charged, are mysteries which are reserved for further investigation to solve. It is clear, however, that electric charges are intimately associated with the corpuscles, and afford the explanation of their ability to produce waves of light identical in all but wave-length with experimentally produced electromagnetic waves.

Quite recently (1910) Sir J. J. Thomson has succeeded in producing several kinds of positively electrified rays, some of which are composed of positively electrified electronic-corpuscles, while others are positively electrified atoms of the gaseous chemical elements. These experiments appear likely to lead to much more intimate knowledge of the nature of positive electricity.

A phenomenon which has much puzzled us in the past, but which is now yielding somewhat of its secret as the result of the discovery of the intra-atomic corpuscles, is that of the doubling and tripling of the bright lines in the spectra of the elements, under the influence of a powerful electro magnet, the "Zeeman effect," so called in honour of its discoverer. If the light radiations which correspond to the bright lines forming the spectrum of an element in the gaseous state are due to the vibrations of the atoms themselves as such, both the phenomenon of a spectrum of bright lines and that of their duplication under the agency of a magnetic field, are well-nigh incomprehensible. But if the vibrations are due to the orbital motions of the component corpuscles, in rings or groups or singly, not only do we see why a number of lines should constitute an elementary spectrum rather than a radiation of a single wave-length, but also a most beautiful explanation is at once forthcoming for the duplication of the lines when the gas is under the influence of a magnetic field. It appears to be clear from Zeeman's researches, that a particular orbital motion of a corpuscle or group (cycle) of corpuscles gives rise to a specific light vibration, producing waves of a definite length, which correspond to a single bright line in the spectrum; and that the magnetic force in the Zeeman experiment affects the motion in opposite parts of the orbit in an opposite manner, in one case accelerating and in the other retarding the motion, thus producing two other bright lines owing to the slight change of frequency of the vibrations in the two directions. These two new lines, one close on each side of the position of the original single line, and corresponding to enhanced and diminished frequencies respectively, do not necessarily entirely replace the ordinary single line when the magnet is excited. For, as Zeeman has also shown, the original line may persist, although less brilliantly; for the vibrations produced by the parts of the orbit parallel to the magnetic lines of force remain unaffected by the magnetic field. Many important contributions have been made to our knowledge of this beautiful Zeeman effect, notably by Lorenz, Larmor, Preston, Fitzgerald, and Michelson, the latter of whom has applied exquisite interference methods to its elucidation. They unite in showing that the vibrations productive of the light waves in the ether, which give rise both to the new and to the old lines—that is, both when the vapour of the chemical element under examination is under normal conditions and when it is under the influence of the magnetic field—are the vibrations of the charged corpuscles, or the oscillatory alternation of their electric charges or of the magnetic effect which may accompany their movement; the period of any particular light wave corresponds doubtless to the frequency

of movement of the electrified corpuscle or cycle of corpuscles, or to the frequency of oscillatory electrical alternation accompanying the movement, which produces it. We may assume with Michelson that an ellipse is the most general form of the path of such a vibrating electrically charged corpuscle, and, as this can be resolved into a plane and a circular vibration, that we obtain a plane polarised and a circularly polarised ray. Again, the plane vibration may be considered as due to two oscillations in a circle, one clockwise and another anti-clockwise, and these two are oppositely affected by the magnetic field, one being accelerated and the other retarded. As the position of a spectrum line depends on the period of the light vibration which produced it, the position of the line will be altered in two opposite senses by the magnetic field when the current is passing through the electromagnet, and doubling of the line will result. The original line persists, as already mentioned, on account of vibrations which occur parallel to the lines of force remaining unaltered. This original central line is plane polarised at right angles to the plane of polarisation of the two outer lines, which are due to vibrations in a plane perpendicular to the lines of force. This fact can be readily verified by means of a Nicol prism, either the middle line or the two outer ones extinguishing when the Nicol is at 0° and *vice versa* when it is rotated to 90° .

Summary of Conclusions as to the Nature of Light, as elucidated by Recent Discoveries.—The explanation of the Zeeman phenomenon would thus appear to have afforded the key to the relation of these new discoveries to light. At the end of the first part of this chapter we had arrived at the conclusion that the vibrations producing light waves were similar in kind, although much more minute as regards wave-length, to those producing electromagnetic waves, in the same medium, the ether, and that the vibrations were due to rapid oscillation of the electrical condition of the atoms. We know that the atoms of each chemical element produce a specific spectrum of bright lines when the element is in the gaseous state, and it would now appear that each line is due to some particular orbital motion in the nature of an electric oscillation, on the part of an electrified intra-atomic corpuscle or group of corpuscles. The inference is obvious, that the whole spectrum of the gas is due to the electric oscillatory impulses of the corpuscles composing the atoms, and therefore that the corpuscles are concerned in and are really the authors of the vibrations of light which we attribute *en bloc* to the atoms. The real nature of the impulses we do not yet know, whether they are purely electrical or also connected with the physical movement of the electrified corpuscles, or, as would appear most probable, to both combined. But we are on absolutely sure ground in concluding that the corpuscles of which the atoms are composed are the ultimate cause of the production of light waves.

It becomes clear to us now why the vibrations of an atom do not produce light of a single wave-length, and consequently a single line in the spectrum, but rather a number of such lines. For, being a complex itself, its parts, the charged electronic-corpuscles, or their groupings within the atom, are the authors of the separate radiations of monochromatic light, the whole being the spectrum of the atom. There is thus opened out to spectroscopy a magnificent further field for investigation, which may result in supplying us with accurate knowledge of the inner structure of the atom, the mode in which the constituent electronic-corpuscles are arranged, and the nature of their orbital motions.

It is for these important reasons that it has been felt necessary, in the interests of a clear understanding of the present exceedingly interesting position, when a new world, a new state of matter, is opening before the scientific worker, to review the salient facts, which are now thoroughly established, and which cannot fail to have a profound bearing on the future of optical crystallography.

CHAPTER XXXV

LIGHT WAVES AND THEIR PROPAGATION.

Characteristics of Light Waves.—The wave-motion, or wave of rapidly alternating electrical condition, with which we have to deal in the case of light, presents the following characteristics :

(1) It is energy and not matter which is propagated.
(2) Each element of ether simply oscillates, either bodily or most probably as regards its electrical condition, about its position of rest or its state of equilibrium, at right angles to the line of propagation.

(3) Through the ether of space the velocity of propagation is the same for waves of all lengths, and in all directions, namely, 185,400 miles per second.

(4) Through the ether about a material transparent substance, which is homogeneous, and either non-crystalline such as glass or crystallises in any of the forms belonging to the cubic system of symmetry, the velocity of propagation is the same in all directions, but only for waves of the same length.

(5) Through the ether about a homogeneous material transparent substance which crystallises in accordance with any of the systems of symmetry other than the cubic, the velocity not only differs for different wave-lengths, but varies with the direction according to definite laws, hereafter to be discussed, dependent on the type of symmetry.

(6) In the ether about a transparent substance which is not homogeneous, the velocity varies irregularly in different directions.

Terminology concerning Light Waves.—A graphical representation of such wave-motion is given in Fig. 481, with the important proviso that it must not be taken as implying that motion of ether particles actually occurs in the precise manner illustrated, in accordance with the older form of the undulatory theory ; but rather that such a diagram is intended to convey to the mind the idea that some change of a transverse directional character periodically occurs in the ether, probably as a rapidly alternating and oscillating change of electrical polarisation. A diagram is essential in order to explain clearly current terms regarding a wave and its parts.

In Fig. 481 a straight line of ether elements at rest is represented

by the line ABCDE of small dotted rings, and the same series in wave motion by the wavy line of solid black dots AFCGE. The wavy line AHCKE of continuous rings represents a similar wave-motion of the same particles but in an opposite phase of vibration, corresponding to total interference, extinction, and will be referred to later in this chapter. The direction of motion of the particles is in all cases transverse to the direction of propagation ABCDE of the wave, and is indicated by the dotted vertical lines. The arrows indicate the direction of motion of the particles represented by the solid black dots, along these vertical lines. Let us suppose the motion to originate at A. Each particle moves like a pendulum, with diminishing velocity, to one of the limits of its swing (indicated by the lower end L of the vertical dotted line), in a direction (that of the arrow on the vertical dotted line) transverse to the original line of particles, and after a momentary cessation of motion returns with accelerating velocity to its position of former equilibrium. It is then carried by its momentum beyond this, with again diminishing velocity, to the other limit of its swing (indicated by the upper end M of the vertical dotted line); from here it moves back

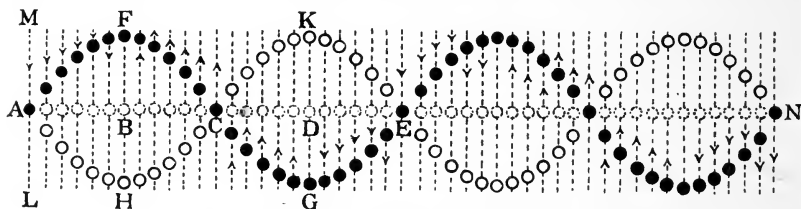


FIG. 481.—Diagrammatic Representation of Light Waves.

again with increasing velocity to its starting-point, and thence passes on its way to make another pendulic swing. The extent of the movement LM from one limit to the other, is termed the **amplitude** of the vibration. It is proportional to the initial velocity with which the element is projected out of its position or condition of equilibrium, and consequently depends on the intensity of the energy which gives rise to the oscillation. The position of the element at any particular instant is called the **phase** of the vibration. Lastly, the time occupied in completing the vibration, that is, from the position occupied at any instant until the return of the same phase in the succeeding vibration, is termed the **period**, and its reciprocal the **frequency**, of the vibration. While this particle has been making its complete excursion, however, the motion has been communicated to the neighbouring particles, one after another, each of which has taken up the impulse to execute a similar excursion, like the swing of a pendulum, a little later than its predecessor. The result is that by the time the first particle A has actually completed its excursion, the motion has been passed along the line as far as a particle E, which will be set in motion similarly, that is, with like phase, to A, and the motion of succeeding particles will correspond to that of the succession of particles following A. Similarly, still further at like distances along

the line, there will be particles such as N vibrating with the same phase. The phase of motion thus repeats itself at regular intervals in a periodic manner. The amplitude of the vibration will naturally diminish as the energy becomes used up, and the wave-motion will eventually die out. In the case of the longer electromagnetic waves, the amplitude dies out very rapidly, hence the necessity for intense energy at the origin.

The distance between any two particles having the same phase is termed the **wave-length**. Thus AE represents diagrammatically a complete wave, as does also EN.

The relation between the wave-length λ , the velocity of propagation v , and the periodic time of the oscillations T , is expressed by the simple formula :

$$\lambda = vT.$$

The periodic time T is too short to be determined directly in the case of the exceedingly rapid oscillations of light. But as the velocity and the wave-length can be determined with accuracy, the periodic time can be calculated from the equation :

$$T = \lambda/v.$$

The whole wave disturbance can be represented mathematically by the equation :

$$y = a \sin \frac{2\pi}{\lambda} (vt - x),$$

in which y represents the displacement of an ether element from its position of rest in the time t , a the amplitude of the oscillations, λ the wave-length, v the velocity of propagation, and x the distance of the element from the origin.

Rectilinear Propagation of Light Waves.—The fact that light proceeds in straight lines is familiar to every one, for we invariably find an object in the direction of the rays of light which proceed from it, provided the intermediate medium is of a homogeneous character, such as the atmosphere, for instance, under ordinary conditions. Owing to the extreme minuteness of the waves of light, they appear incapable of bending round opaque objects, and consequently throw shadows of such obstacles as are placed in their way, on any background capable of reflecting the light to our eyes. This, however, is only because the amplitude of a light oscillation is so extremely small compared with the object intercepting the light ; and the phenomenon of diffraction, which we make such important use of for the purpose of producing the spectrum by means of a diffraction grating, is entirely due to the fact that when an edge is sharp enough, or a succession of such sharp edges are presented, as in the rulings of the diffraction grating or the very fine striations not infrequently found on crystals and discussed in Chapter XXX., the light waves can bend round it to the precise extent which theoretical considerations based on the wave-length demand. As regards the production of ordinary shadows, however, the sharpness of such a shadow depends on the size of the source of light. When this

latter is a point, as in the case of the electric arc, the edge of the shadow will be quite sharp. But when the source of light is relatively large, such as the light from a studio window having a northerly aspect, the deep shadow or "umbra" will be bounded by a less defined "half-shadow" or "penumbra."

Intensity of Light.—The brightness or intensity of a ray of light is a function of the amplitude of the oscillation. The average energy of the vibration is proportional to the square of the amplitude. The illumination on unit surface arranged normally to the direction of the rays and at a fixed distance from the source of light is obviously proportional to the intensity of the light given forth by the source. The sum of all such units of surface is a sphere of equal illumination, the source occupying the centre and being supposed to radiate equally in all directions. If the radius of the sphere be altered, or the position (as regards distance from the source of light) of the unit portion actually used, the intensities of illumination are inversely proportional to the squares of the radii of the spheres, or to the squares of the distances of the unit screens from the source of light. This important "Law of Inverse Squares" thus determines that an equal area at twice the distance from the source of light receives only one-fourth as much illumination.

Range of Wave-length of Light Waves.—Just as the ear is only capable of perceiving as sounds vibrations in the air or other material medium which lie within certain lower and upper limits of frequency, so the eye is only capable of perceiving those oscillations in the ether which lie within two limits of frequency, corresponding to violet and red light respectively.

The rapidity with which the oscillations succeed each other in the case of violet light is very nearly twice as great as what it is when red light is produced. The range of the eye is thus about a single octave, whereas every musician knows that the ear is capable of appreciating about seven octaves. But waves of greater or less frequency than those of which the eye takes cognisance are recognised in other ways. Those which are more rapid than the waves of violet light, and which constitute the "ultra-violet" light, are distinguished by their chemical activity, or actinism as it is called. They are of great importance in photography on account of the energy with which they decompose the sensitive salt contained in the film of a photographic plate or sensitised paper. They probably owe this distinctive property to the fact of their shortness of wave-length, which enables them to penetrate into the molecular structure of the chemical substances which they decompose. Still shorter are the waves composing the X-rays, which are also photographically active. The waves somewhat longer and slower than those of the red rays announce themselves to our senses as those of heat; but the limit of these is reached at a wave-length of about a four-thousandth of a centimetre. We have seen, however, that we now have the means of experimenting with ether waves of lengths varying from less than a centimetre up to very great distances. As

regards the waves which are visible and those which affect us as heat, there is a certain amount of overlapping; for the luminous rays also exert some heating effect, and both the non-luminous actinic and the thermal waves are converted under certain conditions, some of which were referred to in the last chapter, into luminous waves, giving rise to the phenomena of phosphorescence and calorescence. Also, the visible waves of light exert more or less actinic power, scarcely perceptible at the red end of the spectrum, but increasing rapidly as the violet end is approached.

Colour and its Dependence on Frequency of Oscillation.—Just as the number of vibrations per unit of time determines the pitch of a musical note, so the phenomenon of colour in light is due to variations in the frequency of the oscillations which give rise to luminous waves. The velocity being the same for the same medium, and the time of the oscillation varying as the wave-length, colour is also dependent under these restrictions on the wave-length. As the velocity v differs in different media, however, while the colour remains the same when light of a definite wave-length is employed at starting, and as also the frequency remains the same,—that of the vibrating source,—in order that the equation $\lambda = vT$ may still hold (T being the reciprocal of the frequency), the wave-length must have altered proportionately to the velocity, on passing from one medium to another. Hence, speaking generally, it is the frequency of oscillation, and not the wave-length, which determines colour. It is customary, however, to associate the different colours with certain wave-lengths, but when this is done it should be understood that the original wave-length of the initial ray is meant, and strictly speaking, its wave-length *in vacuo*.

The velocity of light of all colours *in vacuo*, that is, in ethereal space, is the same, namely, about 185,400 miles or 298,500 kilometres *per second*. As the wave-length is the distance through which the undulatory movement has progressed during the period of a complete oscillation, AE or EN in Fig. 481, the wave-lengths of light of the various colours must be related similarly to the corresponding periods of oscillation. The duration of oscillation is longest for red light, less and less for yellow, green, and blue respectively, and shortest for violet light, the order being that of the colours in the prismatic spectrum. The wave-length of red light *in vacuo* is about 0·000,760 millimetre, and that of violet light about 0·000,400 millimetre. Consequently, in one second a ray of red light performs about 390 millions of millions (billions) of oscillations, and a violet ray about 750 billions.

The following table gives the wave-lengths of a few of the most useful spectrum lines, six of which,—the three hydrogen lines, red α , greenish-blue β , and violet γ , the red lithium line, the yellow sodium line, and the green thallium line,—are those of the monochromatic rays which the author employs in all optical crystallographic measurements, such as in the determinations of refractive index and of optic axial angle. The wave-lengths of the double potassium red line α , just outside the oxygen line A of the solar spectrum, and of the two solar H and K

lines due to calcium are also given as marking the approximate limits of the visible spectrum. The wave-length of the red cadmium line is also included, as this is probably the most perfectly monochromatic of all the lines of the spectrum, no resolution into a double line having been observed with dispersions transcendently high, sufficient (with the aid of the Michelson *echelon* spectroscope) to resolve most other lines; this radiation is, therefore, particularly suitable for the production of interference bands for measurement purposes. The wave-lengths of the two dark red and two violet lines of rubidium, and also of the two bright blue lines of caesium, are also included in the table, the two first because of their usefulness in marking the red beginning of the visible spectrum, even beyond the potassium line, and the two last as being very useful wave-lengths in the blue. The wave-length of the green mercury line is also added, as the mercury vapour-lamp is now much employed in monochromatic optical work. These wave-lengths represent the latest and most accurate measurements.

TABLE OF SOME PRACTICALLY USEFUL WAVE-LENGTHS.

Colour.	Spectrum Line.	Wave-Length in Millimetres.	Observer.
Dark red .	Rubidium	$\left\{ \begin{array}{l} \delta \ 0\cdot0007950 \\ \gamma \ 0\cdot0007811 \end{array} \right\}$	Kayser and Runge, 1890
" "	Potassium α	$\left\{ \begin{array}{l} 0\cdot0007699 \\ 0\cdot0007666 \end{array} \right\}$	" " "
" "	Solar A (oxygen)	0\cdot0007594	Rowland, 1893
Deep red .	Lithium α	0\cdot0006708	Kayser and Runge, 1890
Bright red .	Hydrogen α (solar C)	0\cdot0006562	Rowland, 1893
" "	Cadmium β	0\cdot0006438	Michelson, 1902
Yellow . .	Sodium (solar D)	$\left\{ \begin{array}{l} D_1 \ 0\cdot0005896 \\ D_2 \ 0\cdot0005890 \end{array} \right\}$	Rowland, 1893
Green . .	Mercury	0\cdot0005461	Kayser and Runge, 1890
" . .	Thallium	0\cdot0005351	" " "
" . .	Cadmium	0\cdot0005086	Michelson, 1902
Greenish blue	Hydrogen β (solar F)	0\cdot0004861	Rowland, 1893
Blue . .	Cadmium	0\cdot0004800	Michelson, 1902
Bright blue .	Cæsium	$\left\{ \begin{array}{l} \beta \ 0\cdot0004593 \\ \alpha \ 0\cdot0004555 \end{array} \right\}$	Kayser and Runge, 1890
Bluish violet	Mercury	0\cdot0004358	" " "
" "	Hydrogen γ (near solar G)	0\cdot0004341	Ames, 1890
" "	Solar G (calcium)	0\cdot0004308	Rowland, 1893
Violet . .	Rubidium	$\left\{ \begin{array}{l} \beta \ 0\cdot0004216 \\ \alpha \ 0\cdot0004212 \end{array} \right\}$	Kayser and Runge, 1890
" . .	Solar H (calcium)	0\cdot0003969	Rowland, 1893
" . .	Solar K (calcium)	0\cdot0003934	" "

An instrument of a spectroscopic character for the production of pure monochromatic light corresponding to any one of these wave-lengths or to any other desired wave-length, will be described in Chapter XLIII. Such an instrument is one of the first essentials for optical work with crystals.

Azimuth of Light Oscillations; Polarisation.—So far an oscillation has only been considered as occurring at right angles to the line of

propagation of the wave. But it is evident that the number of perpendicular directions around the line of propagation is infinite, and unless the motion of the ether element be a compound one, having a component tending to change the particular azimuth in which the oscillation starts, it is obvious that the pendulic swing will continue to recur in the same direction, and the wave produced will be composed of vibrations occurring in the same plane. A ray of light produced by such simple oscillations confined to one plane is said to be "plane polarised." It is evident that such a ray will exhibit peculiar characters in the plane containing the direction of oscillation and the line of propagation, and likewise in the rectangularly symmetrical plane containing also the latter direction and the line at right angles to this and to the direction of oscillation. Much discussion has occurred in the past as to whether the plane of polarisation is identical with the one or with the other of these two planes. The electromagnetic theory requires that something shall occur in both planes, so that both views are partly correct; the electric force concerned in the oscillations is found to be perpendicular to the plane of polarisation, and the magnetic force to be parallel to it.

Crystals are particularly connected with the polarisation of light, for by far the best means of producing plane polarisation is by use of them, their organised structure determining the polarisation of the light waves along certain of the most important structural planes. Special constraining circumstances are obviously necessary for the production of vibrations in a single plane, and these circumstances are most fortunately provided by the crystalline structure corresponding to systems of symmetry other than the cubic. Ordinary light consists of vibrations of every possible azimuth around the direction of propagation. It is supposed that the azimuth is constantly being altered by a very minute angle after every oscillation, and that a complete cycle of such changes of azimuth brings the path completely round the circle to the same again more frequently during the course of a second than the eye can discriminate, so that an ordinary ray gives the impression of being the same in all directions around the line of propagation.

Besides the rectilinear oscillations of plane-polarised light, in which the electromagnetic forces concerned occur in two fixed azimuths at right angles to each other and to the direction of propagation, others may be produced by the agency of certain crystals, in which the ether elements move in circles or ellipses.

The essential characteristic of polarised light, whether it be plane polarised, circularly polarised, or elliptically polarised, is that the directional character or orbit of the electromagnetic disturbance remains permanently the same. In the case of ordinary light, every kind of orbit and every possible variation in the directional character may possibly obtain.

Superposition and Interference of Light Waves.—Rays of ordinary light from different sources are capable of crossing each other to any extent without suffering destruction, each emerging as if the other did not exist. The ether thus appears capable of transmitting an infinite number of

different waves at the same time, and the actual condition of an ether element at any particular point must be due to the superposition of all the various disturbances which affect it, namely, the geometric resultant of the displacements which would occur if each disturbance acted singly. If we confine our attention to two rays of plane-polarised light, which have the same direction of propagation and of which the oscillations occur in parallel azimuths, but which originate from two different points in a row of ether elements, such for instance as the row ABCDE in Fig. 481, it will be clear that, in general, the two waves will differ in phase at any point which may be chosen for a study of their interference. But when the two points of origin are removed from each other by a multiple of a complete wave-length, the two wave systems will be always in the same phase, and will simply reinforce each other in brightness, the resulting wave having an amplitude equal to the sum of the amplitudes of the two component waves. When, however, the two origins are separated by half a wave-length, as in the case of the wave of solid black particles AFCGE and that of the ring-particles AHCKE in Fig. 481, or an odd multiple of this, the ether particles will be affected by the two disturbances in opposite senses, and, if the waves differ in intensity, the resultant wave-movement will have the phase of the stronger of the two, and an amplitude equal to the difference of the amplitudes of the two individual rays. In the special case shown in Fig. 481 where two such waves are of equal intensity, they will destroy each other completely, the light being extinguished.

We shall make very considerable use of this phenomenon of "interference" in our optical study of crystals, in distinguishing between isotropic, uniaxial, and biaxial crystals, also in connection with the measurement of the angle between the optic axes of biaxial crystals, and with the determination of the sign of double refraction by means of a quartz wedge. The principle will also be employed in the dilatometer for the determination of the thermal expansion of crystals, the interferometer, which is used in connection with the dilatometer, being the most delicate measuring apparatus yet invented, the unit of the scale of interference bands being half a wave-length of red hydrogen (or cadmium) light, one three-thousandth of a millimetre or one seventy-seven thousandth of an inch; as the interval between two such interference bands (dark straight bands parallel to each other on a brilliantly illuminated red ground, as seen through the interferometer telescope) can be divided by the eyepiece micrometer into a hundred parts, we are actually able to measure with this instrument to the extraordinary degree of fineness of the eight-millionth part of an inch or the three-hundred thousandth of a millimetre. The thousandth of a millimetre is the finest measurement possible by the most refined mechanical means, including the use of the most delicate micrometer. Moreover, so absolutely infallible is the recurrence of a dark band at every half wave-length, where the nodes of light-extinction occur, and so exactly alike are the wave-lengths themselves when the light

employed is strictly monochromatic (such as that corresponding to the red spectrum line of cadmium), that no mistake is possible. The two rays which are thus caused to interfere are reflected from two absolutely plane glass surfaces, which are inclined at a very minute angle to each other but to the eye are apparently parallel; one of them is fixed in space while the other moves with the expanding crystal, or with whatever other movable object is under investigation and the movement of which is to be measured. The amount of motion is obtained at once by multiplying the number of bands and any final fraction of a band which have moved past the centre of reference (a silver ring in the centre of one of the two glass plates, and which is focussed in the centre of the field of the observing telescope) by the half-wave-length of the light employed, usually red cadmium or red hydrogen light. The instrument itself is described in Chapter LIII., and a further application of it to the determination of the elastic bending of plates and bars of crystals in Chapter LIV.

If two waves originate from points removed by other distances than the wave-length or an odd multiple of half a wave-length, the resultant wave will possess both different phase and different amplitude. Hence two such waves as we have been considering can, according to the separation of the points of origin, vary in their resultant from the sum of the two intensities down to zero, the plane of the vibrations remaining the same.

When the planes of polarisation are different, while the direction of propagation remains the same, the resultant wave will be either circularly or elliptically polarised. The resultant is a circularly polarised wave when the amplitudes are the same, the planes of polarisation at right angles, and the difference of phase one quarter of a wave-length. The case is analogous to that of a pendulum, which is started vibrating in one plane, and to which, when at one extremity of its path, one quarter of a complete swing from the original position of rest, an impulse of equal strength is imparted at right angles to the first impulse, which results in a circular motion being described by the bob of the pendulum. The period of oscillation of such circularly polarised light is the same as that of the two component oscillations. We shall employ a "quarter-wave plate" of mica in our practical optical work with crystals, for producing a circularly polarised beam of light, which affords characteristic phenomena with crystals of different symmetry.

If the amplitude of the oscillations of the two component rays be different, or if the difference of phase be other than one quarter of the wave-length, or again, if the inclination of the two planes of polarisation be other than 90° , the resultant movement of the ether elements takes the form of an ellipse, and the resultant ray is elliptically polarised.

Propagation of Light; Wave Surface.—The consideration of single waves and simple cases of superposition just concluded lead naturally to the discussion of the whole series of waves emanating in

all directions from the oscillating source. The nature of the whole disturbance is dependent primarily on the constitution of the medium in which it is propagated, for it is this which determines whether the waves are transmitted in different directions with the same or with different velocity. The ether of space, that is *in vacuo*, where ponderable matter as we yet know it—molecules, atoms, and electronic-corpuscles,—is excluded, is a medium of the first kind, transmitting the light waves in all directions with equal velocity. Likewise the ether surrounding the molecules of transparent gases and liquids, and also the ether which penetrates between the molecules of such transparent solids as are constituted exactly alike in all directions, behaves as a medium of the first kind. Such an optically homogeneous substance is glass, and such also are all crystalline transparent solids which crystallise according to the cubic system of symmetry, the essence of which is perfect symmetry in all three rectangular directions of space, the physical properties being, as we have seen (page 502), represented by a sphere. The term "isotropic" is used to distinguish such homogeneous media. Through them light of any one particular colour is transmitted with equal velocity in every direction. The ether penetrating these isotropic substances is, indeed, only distinguished from that of vacuous space by the fact that light of different colours does not traverse them with equal velocity, the specific velocity for any particular colour being a characteristic of the particular substance. It is obvious why gases and liquids should be isotropic, for their molecules being in a state of motion, free and rapid in the case of gases but of a rolling character in liquids, it is an average effect which is produced on light, there being no congelation into an organised and rigid solid structure in which the molecules have definite homogeneously distributed positions, as in the case of a crystal. Glass would appear to owe its isotropic character to the great viscosity of the substance in the molten state at temperatures not far removed from the melting point, which causes it to retain on solidification the infinitely heterogeneous positions of the molecules which obtained in the liquid state just before solidification, the great viscosity at this point preventing the organised crystallographic arrangement of the molecules of the silicate or silicates present. That this is a correct view is proved by the formation of crystallites in some glasses by slow devitrification. The reason for the isotropy of cubic crystals is one of quite an opposite nature, namely, the most perfect development of organised crystallographic arrangement, that of the cubic system of symmetry with three equal and rectangular crystallographic axes, with its inevitable accompaniment, the display of the maximum possible number of planes and axes of symmetry.

The continuous *locus* of points just about to be disturbed in an isotropic medium, under the influence of a light oscillation of definite frequency, will evidently be a spherical surface. For instance, at the moment of the completion of a vibration of the oscillating source, the surface connecting all the ethereal elements to which the complete wave has travelled will be a sphere of radius λ , the wave-length. At the

end of a second vibration the disturbance will have reached the surface of a sphere of radius 2λ , and so on. Such a sphere of like phase is termed a wave-surface or wave-front, and the term may be extended to any surface joining ether elements in a like phase of vibration, of which those spheres having radii which are multiples of the wave-length are only special cases.

Every point on a wave-surface may be regarded as a new centre of disturbance, and the secondary waves travelling out from it may be considered as carrying on the wave-motion. The envelope of these secondary waves will be identical with the wave-front of the main disturbance, so that the aggregate effect of the secondary waves is the primary wave. The construction of radiating wave-motion as a succession of wave envelopes is the well-known principle of Huyghens. It will be rendered clearer by Fig. 482, which embodies the so-called Huyghens construction. If, instead of considering the motion as really originating

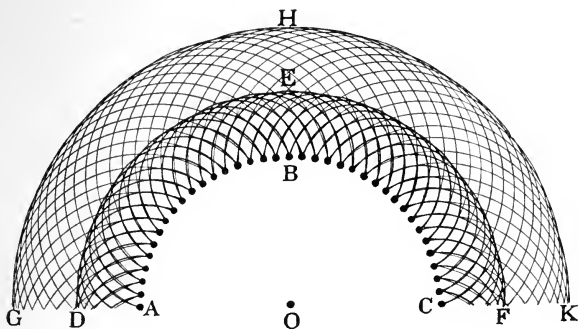


FIG. 482.

at O, we consider a circular line of particles ABC on the spherical wave-front at any moment, each of these particles may be considered as an origin, and if we draw a circle from each, having a radius corresponding to the distance the movement has travelled from the particle in a given further time t , we shall arrive at the circular section DEF by drawing the enveloping curve of all such circles. Similarly, if the time considered be $2t$, and we draw a further series of circles from the points on the circle ABC with radius corresponding to $2t$ we arrive at the circular section of the outer envelope GHK. Obviously these spherical envelopes DEF and GHK are identical with the primary spherical wave-surfaces for the same moments of time.

In the case of the simple homogeneous media now under consideration, for which it has been shown that the envelope or wave-front is a spherical surface, the individual rays will obviously be the normals to the wave-front; that is to say, the wave moves out from the oscillating centre of disturbance along the radii of its spherical front, and at right angles to the tangent to the wave-front at the point where the particular radius considered meets the wave-front. Thus, for isotropic media in

general, the direction of propagation of the wave is always at right angles to the wave-front.

The more complicated cases of the passage of light through anisotropic media, that is, through crystals belonging to the systems of symmetry other than the cubic system, will be discussed in later chapters. It will be obvious that the wave-front in these cases cannot be spherical, and it will be shown that, in general, it is an ellipsoid. Also the wave-front is not, in general, perpendicular to the direction of propagation, although it is so when the direction is one of the three principal axes of the ellipsoid.

We have thus seen that the consideration of the nature of light, and of the mode of propagation of its electromagnetic waves, leads us inevitably and naturally to the investigation of the optical properties of crystals. The two subjects of optics and crystallography are as intimately, and indeed indissolubly, bound up together as is the latter subject with chemistry and mineralogy. It has been considered both right, therefore, and advisable, inasmuch as recent investigation has so enormously widened our conception of the nature of light and of the true character of the chemical atom, that the real study of the optical properties of crystals, to which we shall now proceed, should have been preceded by these two introductory chapters, which have thus naturally paved the way for our practical studies in optical crystallography.

CHAPTER XXXVI

THE FUNDAMENTAL OPTICAL CONSTANT, INDEX OF REFRACTION; AND ITS RELATIONS TO THE ANGLES OF TOTAL REFLECTION AND OF POLARISATION.

BEFORE we can with advantage pass to the consideration of the optical ellipsoid expressive of the optical properties of crystals, and its bearing on practical crystallography, it is imperative that we should be quite clear as to the elementary facts concerning the passage of light through transparent media, and the nature of the all-important constant, refractive index. For on this knowledge depends our means of practically determining that constant. Before considering crystal-media, however, we will assume that our transparent medium is non-crystalline, glass for instance; or at any rate if crystalline that it possesses the symmetry and internal structure of the cubic system, so that it is singly refractive, or "isotropic," the optical properties being the same in all directions within the substance. We must first investigate what happens when a beam of light traverses a transparent plate or a prism of such a substance, determine the conditions for partial and total reflection at either of its surfaces, and deduce a general means of determining the index of refraction.

When a ray of light strikes a parallel-sided plate of such an isotropic transparent substance, a portion of it is reflected symmetrically to the surface of the plate. This may be otherwise expressed by saying that it is reflected at an equal angle with the normal (the perpendicular to a surface) to the plate. Thus in Fig. 483, if IO be the ray incident on the surface AB of the plate $ABCD$, and NON' be the normal to the plate, then the ray is reflected along OR , which follows a direction such that the angle RON is equal to the angle ION , IO and RO being thus symmetrical to the surface AB .

The incident light is vibrating in every azimuth perpendicular to

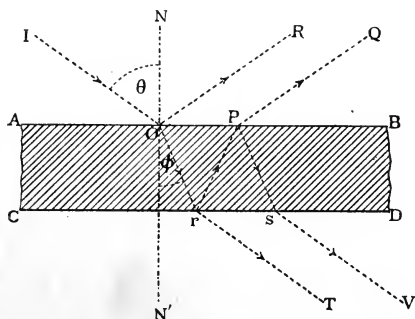


FIG. 483.

its direction of propagation, if it be ordinary light. But after reflection at the surface of the plate more or less of it, depending on the angle of incidence, is constrained to vibrate in the plane of the surface, that is, it becomes plane-polarised. The plane of polarisation is said by convention to be perpendicular to that in which the vibrations occur, that is, to be parallel to the plane of incidence, the plane ION, parallel to the paper, in Fig. 483.

For some particular angle of incidence dependent on the substance, $56^{\circ} 35'$ in the case of ordinary crown glass for instance, the whole of the reflected beam, if composed of monochromatic light (sodium light in the case of glass just mentioned), is thus plane-polarised. This angle is known as the "polarising angle."

A great deal of controversy has occurred in the past as to whether the plane of polarisation should be considered as that in which the plane vibrations occur, or the plane perpendicular thereto. It has already been shown that there are effects, both electric and magnetic, in both planes; and the convention mutually agreed upon by the great majority of physicists, that the plane of polarisation is to be considered as perpendicular to the plane in which the vibrations occur, has been so decided on the ground that the direction of vibration follows the electric effect more than the magnetic, and the electric disturbance occurs principally at right angles to the plane of polarisation.

It is thus very important to remember, that when a polarising Nicol prism (our best means of producing polarised light, to be described in Chapter XXXIX.) is arranged with its axis horizontal and its plane of polarisation, the 0° - 180° diameter, the longer diagonal of the end-face of the rhomb, also horizontal, and with the 90° - 270° diameter, the shorter diagonal, vertical, the transmitted vibrations are occurring vertically; and when the prism is rotated so that the zero or 180° graduation is at the top of the vertical diameter, the vibrations are horizontal. That is, the diameter which carries the zero mark has to be horizontal in order that the prism may let through vertical vibrations, and vertical if it be desired to transmit horizontal vibrations. The vibration plane of a "Nicol" (as the Nicol prism is currently termed) is always parallel to the shorter diagonal. Hence vibrations parallel to the refracting edge of a 60° -prism of a birefracting crystal are transmitted when the Nicol is at 90° , and those perpendicular to the edge and bisecting the angle of the prism are transmitted when the Nicol is at 0° , assuming that the prism is arranged on a horizontal-circle goniometer-refractometer with its edge vertical as usual.

The remainder of the light incident on the plate in Fig. 483, and not reflected, penetrates within the glass or whatever transparent material the plate is composed of, and is polarised in a proportion corresponding to that of the reflected light, but in the perpendicular plane, at right angles to the plane of incidence; that is, the vibrations occur parallel to the plane of incidence.

A plate of the mineral tourmaline is well known to act as a very perfect polariser of light, that is, to constrain the vibrations of such

light as passes through the tourmaline to confine themselves to a single plane. In the case of tourmaline the direction of this plane is parallel to the trigonal axis of the crystal, tourmaline crystallising in the trigonal system, class 20, the ditrigonal-pyramidal polar class, as shown on page 350 of Chapter XXIII.; the crystal plate is generally cut parallel to the trigonal axis for polarising purposes. Now tourmaline is a mineral which possesses in a high degree the property of "pleochroism," that is, it exhibits different absorptive effects on light vibrating along different directions within the crystal. Moreover, as in the cases of all birefringent crystals, the rays of ordinary light are in general constrained when they enter the crystal to vibrate along two directions only, and in this case, being trigonal in symmetry, the two component rays are an ordinary and an extraordinary one in the sense which will be fully explained in Chapter XXXVIII. Now, the difference of absorption of these two rays during their passage through the tourmaline crystal structure is so considerable, that the ordinary ray, when the crystal plate is cut parallel to the axis as usual, is practically entirely absorbed before it can emerge, and thus the extraordinary ray, the vibrations of which are parallel to the trigonal axis, is alone transmitted as a perfectly polarised beam of light.

Consequently a plate of tourmaline affords a delicate test for the polarisation of light reflected or refracted from glass, and in the case of the reflected rays, if the angle of incidence be near $56^{\circ} 35'$, they are practically all extinguished by the plate of tourmaline cut parallel to the trigonal axis, when that axis, generally parallel to the longer edge of the plate, is parallel to the plane of incidence; but the refracted rays readily pass through a tourmaline thus held.

It will be obvious also that when two such elongated plates of tourmaline parallel to the axis are held parallel to each other in a beam of light, the latter passes through as a beam of plane-polarised light, and to the eye of an observer on the other side the tourmalines appear transparent, with the slight or deeper colour corresponding to two thicknesses of the mineral; but if the two plates be crossed at right angles, the overlapping square will appear to the same eye quite black and opaque, the plane-polarised light getting through the first being altogether stopped by the second. When such a pair of plates of tourmaline,—now very rare, especially if they are of considerable size, with a minimum of colour, and that pale brown rather than green,—are mounted in a manner which permits of their independent rotation, so that they may be arranged crossed to each other or parallel at will, or at any angle of mutual inclination, they form an excellent "polariscope," an instrument consisting of a "polariser," or producer of a beam of plane-polarised light, and an identically similar "analyser," both rotatable. A section-plate cut perpendicularly to the optic axis of a uniaxial crystal, or to the acute bisectrix of the optic axial angle of a biaxial crystal, when placed between the polarising and analysing tourmalines, will exhibit the circular uniaxial rings and rectangular cross of the one, or the biaxial rings and brushes of the other very clearly, when the

arrangement is held close to the eye. The tourmalines are usually supported in independently rotatable mounts at the ends of a pair of spring tongs, between which the crystal section can be gripped with sufficient tightness to hold it firmly, as shown in Fig. 484.

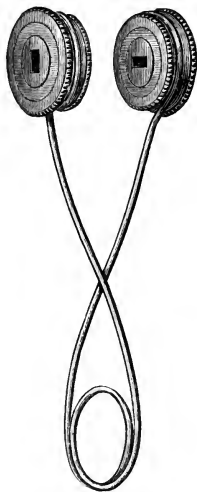


FIG. 484.—Pair of Tourmalines in Pincette.

Two large plates of colourless tourmaline are quite as good as Nicol prisms, and from their increasing rarity are becoming almost as valuable as large Nicols, the Iceland spar for which is now exhausted, only smaller pieces coming into the market, which, however, are adequate for all but projection purposes. The construction of the Nicol prism of Iceland spar will be discussed in Chapter XXXIX., after the doubly refracting peculiarities of calcite have been described in Chapter XXXVIII.

Index of Refraction.—The amount of bending or “refraction” of the rays on entering the transparent material depends on what is known as the “refractive index” of the substance. If in our example illustrated in Fig. 483 NON' be the normal to the plate of glass or other transparent isotropic substance, IO the direction of the incident ray, ION being the angle of incidence, then OR is the reflected portion of the ray, and Or the refracted portion. At r , where it meets the second surface, it is partly transmitted as the ray rT , parallel to the direction of the incident ray IO but no longer in line with it; and partly it is again reflected back to the first surface, which it meets at P ; here part escapes as the ray PQ parallel to the reflected ray OR , and part is once more reflected to the second surface at s , a portion of this ray then escaping along sV parallel to rT and part being once more reflected, a ricochet of this kind going on with partial escape of some of the light each time either surface is reached after internal reflection, the escaping ray being obviously weaker and weaker after each successive internal reflection.

Now $N'Or$ is the angle of refraction ϕ , and if the angle of incidence ION be represented by θ , then the ratio of the velocity of light in air to that within the crystal is equal to the ratio of the sines of the two angles θ and ϕ , and this ratio it is which is conventionally known as the “refractive index”; it is generally represented by the letter μ when the substance is singly refracting, that is:

$$\mu = \frac{\sin \theta}{\sin \phi}.$$

The refractive index is a constant quantity for light of the same wavelength and for the same two media, but varies with change of either medium. It is a number greater than unity when the medium the refractive power of which is measured by it, compared with air as the lighter medium, is denser than air; for ϕ will always then be a smaller angle than θ . These facts are combined in the statements of two laws

which were established so long ago as the year 1620 by Willabrod Snell, and afterwards restated by Descartes, who thus called more attention to them. They are as follows:—

Whatever be the angle of incidence, the ratio which the sine of this angle of incidence bears to the sine of the angle of refraction is constant for the same two media, but varies with different media.

The incident and the refracted ray lie in the same plane, which is perpendicular to the surface of separation of the two media.

Snell's laws apply not only to solids, but also to liquids, and as we shall afterwards require to use liquids of specific refractive powers for various operations in connection with the determination of the optical constants of crystals, it is convenient to include here a list of the refractive indices of a few such liquids which the author has found most useful, together with the indices of refraction of a number of isotropic solid substances, including crystals belonging to the cubic system. All the refractive indices in the two tables refer to sodium D-light and to the temperature of 20° C. The name of the observer is also given in each case.

REFRACTIVE INDICES OF SOME ISOTROPIC SOLIDS.

Fluor spar	1·4339, Fizeau.
Potassium aluminium alum	1·4564, Soret.
Ammonium aluminium alum	1·4594, Soret.
Obsidian	1·4964, Mühlheims.
Sodium chlorate	1·5150, Dussaud.
Crown glass, light Steinheil or Merz	1·5153, v.d. Willigen.
„ heavy Merz or Rosette	1·5339, Mascart.
Rock salt	1·5442, Langley.
Flint glass, light Hilger	1·5798, Langley.
„ heavy Rosette	1·6193, Mascart.
„ very heavy Merz	1·7515, v.d. Willigen.
Garnet, colourless grossularite	1·7438, Wülfing.
„ red almandine	1·8078, Wülfing.
„ green uwarowite	1·8384, Wülfing.
Spinel, chromite	2·0960, Thoulet.
Phosphorus	2·1442, Gladstone and Dale.
Blende	2·3692, Becquerel.
Diamond	2·4669, Schrauf.

REFRACTIVE INDICES OF CERTAIN USEFUL LIQUIDS.

Water	1·3334, Tutton.
Ether	1·3566, Gladstone and Dale.
Ethyl alcohol	1·3616, Ketteler.
Chloroform	1·4462, Lorenz.
Carbon tetrachloride	1·4607, Haagen.
Turpentine	1·4725, v.d. Willigen.
Glycerine	1·4729, Landolt.
Toluene	1·4955, Gladstone.
Benzene	1·5027, Gladstone.
Cedar oil	1·5200, Tutton.
Monochlorbenzene	1·5248, Veley.

REFRACTIVE INDICES OF CERTAIN USEFUL LIQUIDS (*continued*)

Methyl salicylate	1.5363, Wernicke.
Oil of cloves	1.5400, Tutton.
Anis oil	1.5540, Baden-Powell.
Oil of cassia	1.5862, Wiedemann.
Oil of cinnamon	1.6190, Tutton.
Carbon bisulphide	1.6276, Ketteler.
α -Monobromonaphthalene	1.6657, Tutton.
Methylene iodide	1.7421, Gladstone.

Total Reflection and Relation of its Critical Angle to Refractive Index.—If we try to follow what happens when the angle of incidence becomes greater and greater, we shall find that more and more light is reflected, and such lesser quantity as is refracted becomes relatively more and more powerfully refracted, as it approaches horizontal incidence (parallel to the plate surface), just as the converse is true that when the incidence is normal there is no refraction at all, and very little relatively for a small angle of incidence. When the limiting case is reached, and the angle of incidence is actually 90° , the ray thus gliding along the surface has no tendency to enter the denser medium at all. Incident rays, however, falling on the surface at angles just less than 90° , will be refracted partially, although mostly reflected, and the angle of refraction will be the maximum possible. If, now, we approach from the other side and investigate the case of a ray travelling *within* the denser medium towards the limiting surface, such as our plate of glass or

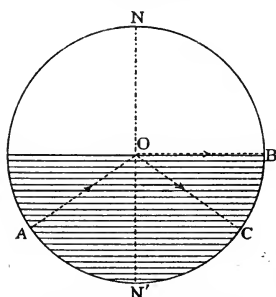


FIG. 485.—Total Reflection at the Critical Angle.

other isotropic substance or a transparent liquid, we find that if this ray meet the second surface of the plate, or the surface of the liquid, at the maximum angle just referred to, it will partially emerge; but if the angle be not only just greater than that maximum angle for emergence, but also exceeds that which would produce the gliding along the surface, corresponding to 90° of incidence, it will not emerge at all, as the ray rT is shown as doing in Fig. 483, but will be totally internally reflected at the surface, back within the denser medium again, at an angle equal to that of this interior incidence. When the interior ray AO meets the surface exactly at the "critical angle," it glides along the surface OB , as shown in Fig. 485.

Expressing the facts mathematically, if θ be the angle of incidence as before, then $\theta = 90^\circ$ for the critical angle of refraction, which latter we term ϕ as before, maintaining the same conventional signs throughout. This critical angle of refraction ϕ is now the interior angle of incidence. Then :—

$$\mu = \frac{\sin \theta}{\sin \phi} = \frac{\sin 90^\circ}{\sin \phi} = \frac{1}{\sin \phi},$$

or,
$$\sin \phi = \frac{1}{\mu}.$$

Thus the sine of the critical angle is equal to the reciprocal of the refractive index. As the latter differs for light of different wave-lengths, the critical angle also varies with the wave-length. This is the law of total reflection, of which great use will subsequently be made in crystal optics. It means that if a ray AO (Fig. 485) be incident within the transparent solid or liquid at an angle the sine of which is equal to the reciprocal of the index of refraction, the ray never gets out, but is reflected along the surface OB, and the least increase of this angle causes the ray to be totally reflected at the limiting surface of the two media, at the same angle OC, back again within the substance.

Determination of the Refractive Index by the Total Reflection Method.—The important law just stated is the foundation of the total-reflection method of determining the refractive index by means of the total-reflectometer. The instrument and the actual mode of carrying out the determination will be described in Chapter XLVI. For the limit at which total reflection begins is so sharp that by a suitable method of observation it can be caused to take the form of a straight or regularly curved line of demarcation between an illuminated part of the field of view of a telescope and a dark portion, so sharp, indeed, that the limiting line can be adjusted to a cross-wire in the eye-piece of the telescope of the total-reflectometer. The latter being merely a specially modified goniometer-refractometer, the critical angle is thus measured directly by the divided circle. The refractive index μ is afforded by the expression :

$$\mu = \frac{1}{\sin \phi},$$

where ϕ is the critical angle. That is, we have merely to calculate the reciprocal of the sine of the observed critical angle.

The critical angle for water to air for sodium light is $48^\circ 35'$, corresponding to the refractive index for 20° , 1.3334. That from glass to air, also for sodium light, is $41^\circ 18'$ for light crown glass of refractive index 1.5153, and $39^\circ 16'$ for light flint glass of refractive index 1.5798.

Application of Total Reflection to the Right-angled

Prism.—An important further application of the principle of total reflection with which we shall have to deal in experimental crystal optics, is in the right-angled total-reflection prism. It is a prism formed by three faces, the plan of which is an isosceles triangle, the two equal faces being perpendicular to each other, so that the other two angles of the prism are 45° , as illustrated in Fig. 486. All three surfaces are polished, and truly

plane if required for accurate work of the kind we shall have to describe later on. The main use of the prism is for reflecting a beam of light at right angles to its original path. One of the rectangular faces $abcd$ is arranged normally to the direction of the beam, and the

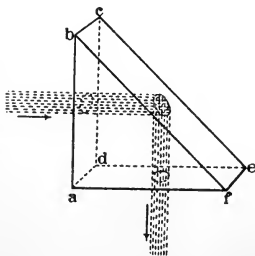


FIG. 486.—Principle of the Right-angled Totally-reflecting Prism.

latter then passes in the direction of the arrow through this face undiverted by any refraction straight to the hypotenuse-face *bcef*, striking the latter, say, about its centre; it does so obviously at 45° , which is larger than the critical angle for glass. For this has been shown to be $41^\circ 18'$ for light crown glass and $39^\circ 16'$ for light flint

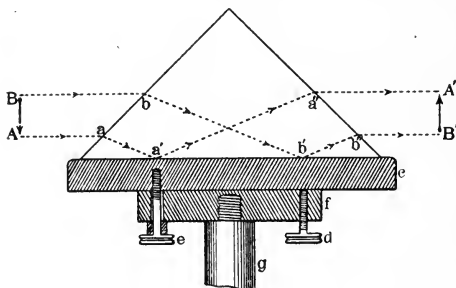


FIG. 487.—Principle of the Erecting Prism.

When used in another manner which is shown in Fig. 487 the right-angled reflecting prism is known as an “**Erecting Prism**,” as it inverts the rays passed through it, so that if a beam of light passing to the screen in a projection experiment, and producing there as usual an inverted image of the object projected by the lens, is passed through such an erecting prism, the image is reinverted by it and thus rendered upright on the screen. In the ordinary projection of lantern slides the slide is placed inverted in its carrier in the lantern front, in order that its enlarged image may be upright on the screen. But when an experiment is being projected, the apparatus being arranged between the lantern condenser and the projecting lens, it is rarely convenient to place it upside down, and here the use of an erecting prism is invaluable. The prism rests on its base, which is either horizontal or tilted slightly parallel to the beam of light when the latter is not quite horizontal; the means of tilting are provided by the three leveling or adjusting screws for the supporting table *c* on which it rests, one of these screws being shown at *d*, while three other fixing screws, one shown at *e*, serve to rigidly fix the table, if necessary, after adjusting. The adjusting screws *d* screw through the under fixed table *f*, but do not enter *c*; while the fixing screws *e* pass easily without screwing through *f* but screw into *c*. The supporting column *g* passes down into a suitable pedestal, in the column of which it is adjustable for height and is also of course rotatable. Hence all necessary adjustments are very simply provided for. A simpler mode of adjusting the prism is shown in Fig. 488, a screen with central aperture being also provided in order to exclude all rays but those of the desired beam.

A beam of light *AB* (Fig. 487) passing from the lantern to the screen enters the prism at *ab*, where it is refracted. Considering first the ray *A*, after refraction at *a* it is totally reflected at *a'* from the polished base of the prism (all three faces being polished) and leaves the prism at *a''*, where a second refraction brings it parallel to

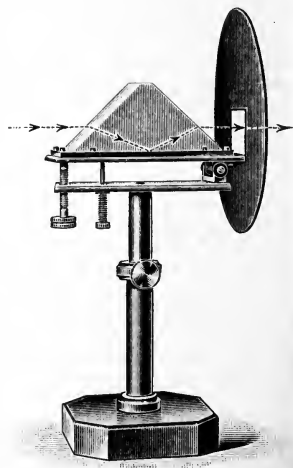


FIG. 488.—Erecting Prism on Adjustable Stand as used for Projection.

its original direction but on a higher level. On the other hand the ray B suffers the two refractions b and b' and the reflection at b' , and emerges after the second refraction at b'' parallel to its original direction but at a lower level. If the arrow at AB represent the inverted image of the object placed between the lantern condenser and the projecting lens, then after traversing the erecting prism it appears on the screen erect as A'B'.

Sometimes this erecting prism is given a larger angle than 90° , a common one being 105° . It will be observed that the upper or apex portion of the prism is not used; hence, to save glass, this upper part is often cut away, as shown in Fig. 488.

Dove's Prism is a right-angled erecting prism constructed of Iceland spar, calcite (calcium carbonate, CaCO_3), which acts as an ordinary erecting prism, as above described, for the "ordinary" ray, while the "extraordinary" ray, the second of the two rays into which a beam of light is divided on entering a doubly refractive crystal such as calcite (as will be fully explained in Chapter XXXVIII.), is not totally reflected but largely passes out of the base and is absorbed by lamp-black paint on the supporting table or other mount. It thus gives an erected (reversed) image in polarised light, the plane of polarisation of which is that of the "ordinary" ray of calcite. One often hears the common glass erecting prism referred to as a "Dove's prism"; strictly speaking the name only applies to one of calcite. The spar is cut so that the optic axis is parallel to the receiving face of the prism, and perpendicular both to the exit face and to the apex-edge between these two faces.

Determination of Refractive Index by the 60° -Prism Method.—

By far the most convenient and accurate method of measuring the refractive index is by observation of the refracted images, in various colours of light, of the collimator slit of a refractometer-goniometer through a 60° -prism, the method, in fact, of producing a line-spectrum, the lines of which, of known specific wave-lengths, give us the means of determining the refractive index for definite radiations.

As the source of light, we may employ sunlight when available, when we have the Fraunhofer dark lines as known particular wave-length radiations. A heliostat, however, is necessary with sunlight, to keep the image of the sun constantly on the slit, and there is besides the liability to interruptions by clouds to be considered. Hence, we may much more conveniently employ the bright lines of metallic vapours, such as those of sodium, lithium, and thallium, the respective yellow, red, and green lines of which are particularly convenient for the purpose. Or again, we may employ a hydrogen Geissler vacuum tube, and raise the residual hydrogen gas, at the exceedingly low pressure left in the tube, to incandescence by means of the high tension intermittent electric discharge from the secondary circuit of a Ruhmkorff induction coil, and thus produce the spectrum of three bright lines of hydrogen, the red C, the greenish-blue F, and the violet line near G. Or, best of all, we may employ the author's spectroscopic monochromatic illuminator, described in Chapter XLIII., which supplies monochromatic light to any instrument whatsoever, and to the goniometer-refractometer especially well on which the 60° -prism is to be measured as regards both its refracting angle and the angle of deviation of the rays transmitted by it. The monochromatic light furnished by this instrument is of such purity that only a 250th part of the spectrum is included, the position of which, moreover, and the actual wave-length, it directly affords, and which may be any desired wave-length whatsoever.

If in Fig. 489 ABC be the section of a 60° -prism by a plane perpendicular to the faces, the refracting edge being at C, a ray of light refracted through it obviously cannot emerge parallel to the original direction however the prism may be arranged, for this only occurs with a parallel-surfaced plate. But the amount of deviation is not constant for all positions, but has a minimum when the path of the refracted ray

through the prism is parallel to the side AB of the prism opposite to the refracting angle, and to the basal plane on which the prism stands, parallel to the paper in Fig. 489; the incident and emergent rays then make equal angles with the two refracting faces of the prism. This symmetrical position of "minimum deviation" can be very readily found in practice, when the prism is mounted on the crystal-adjusting apparatus of the goniometer-spectrometer, and thus the method is particularly convenient and accurate.

The collimator and telescope are first arranged in the same straight line, and the direct reading of the Websky slit-signal as observed through the telescope is taken on the circle; the telescope is then rotated to one side, the collimator and divided circle remaining clamped together and fixed to the tripod base, until by suitable rotation of the prism, the crystal axis being left loose, the refracted image of the signal-slit is seen directly by the eye, broadened into a spectrum by the different refraction or "dispersion" of light of different wave-lengths. It will then be found that when the arrangement is as shown in Fig. 489, rotation of the prism in either direction causes the spectrum-image of the slit to move in that one and the same direction which, if the telescope followed it in order to measure the angular movement, would increase the angle of deviation of the light rays, from the original position of the telescope for the direct reading on the circle, when it was in line with the collimator. The prism is then rotated

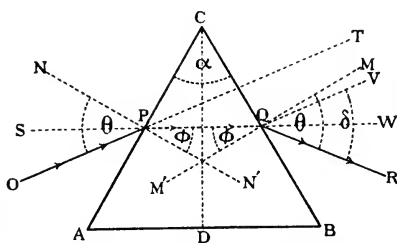


FIG. 489.—60° Refracting Prism arranged for Minimum Deviation.

back to its position corresponding to minimum deviation, and the telescope arranged so that the spectrum-image for this position is adjusted as nearly as a broad spectrum can be to the cross-wires. The least turn of the prism either way causes the image to move away from the vertical cross-wire, and on that side of it in either case which would increase the angle. After thus preliminarily adjusting the spectrum-image in white light with the ordinary goniometer lamp, the final adjustment must be made in monochromatic light, preferably sodium light, which is a radiation in a conveniently central part of the visible spectrum. The angular difference between this adjusted position and the direct reading is the desired angle of minimum deviation. Similar measurements are then made for all the radiations for which it is desired to determine the refractive index, the adjustment of the setting to minimum deviation being verified and made perfect for each colour, as tested by verifying that on movement of the prism in either direction by rotation of the crystal-holding and adjusting axis supporting it, no further approach of the image towards the direct reading occurs, and if any further slight adjustment be required, as is usual for the two ends of the spectrum, it is made in each case.

By using the author's spectroscopic monochromatic illuminator described in Chapter XLIII., determinations of minimum deviation for six or more wave-lengths of light can be very rapidly carried out. The author usually employs those corresponding to red Li-light, red C-hydrogen light, yellow sodium light, green thallium light, greenish-blue F-hydrogen light, and violet hydrogen light near G. The only other measurement required is that of the refracting angle of the prism, which is measured in the usual goniometrical manner. (See Chapter XLV. for practical details.)

The calculation of the refractive index is exceedingly simple from the data thus acquired, namely, from the angle δ of minimum deviation for a number of different wave-lengths of light, and the internal angle of the prism α , which is somewhere in the neighbourhood of 60°, and is, of course, determined to a single minute of arc.

Now if OP in Fig. 489 be the incident ray, PQ the direction of the refracted ray

through the prism at minimum deviation, and therefore parallel to the third side of the prism AB opposite the refracting angle ACB, α ; and if also QR be the emergent ray, NPN' and MQM' the directions of the normals to the prism faces employed, and CD the plan of the bisecting plane of the prism; then, the angle of incidence OPN is equal to the angle RQM made by the emergent ray with the normal to the second surface. Let this angle of equal incidence and emergence be termed θ , as usual for an angle of incidence in the nomenclature of this book. Then the angle of refraction QPN' is equal to the angle PQM'. Let these be called ϕ , in accordance with our usual labelling of an angle of refraction.

Now we have seen that the index of refraction μ is the quotient of the sine of the angle of incidence by the sine of the angle of refraction, that is:

$$\mu = \frac{\sin \theta}{\sin \phi}.$$

But by the principle of similar triangles, PQ being normal to CD and NPN' to AC, the angle of refraction ϕ is equal to $\frac{\alpha}{2}$. Also the angle of incidence θ is made up of the two triangles NPS and SPO. But NPS is equal to ϕ , that is, to $\frac{\alpha}{2}$, and SPO is equal to half the angle δ of minimum deviation, that is, to $\frac{\delta}{2}$. For the angle of minimum deviation δ is the angle between the direction of the incident ray OPT,—which is that of the direct reading of the slit image,—and the direction of the emergent ray QR. If we draw QV parallel to OPT, the angle VQR is the angle of minimum deviation δ in question. Now, as the angle WQR, part of this, is equal to SPO, and the angle WQV, the other part, is also obviously equal to SPO by construction, the whole angle δ of minimum deviation RQV must be double of the angle SPO, that is, $\text{SPO} = \frac{\delta}{2}$.

Hence, the whole angle θ of incidence NPO is the sum of half α and half δ , that is:

$$\theta = \frac{\alpha}{2} + \frac{\delta}{2} = \frac{\alpha + \delta}{2}.$$

Consequently, the refractive index μ , which by definition $= \frac{\sin \theta}{\sin \phi}$, is represented by the expression:

$$\mu = \frac{\sin \frac{\alpha + \delta}{2}}{\sin \frac{\alpha}{2}}.$$

Thus the refractive index is equal to the sine of half the sum of the angles of the prism and of minimum deviation, divided by the sine of half the prism-angle.

Relation between Refractive Index and the Polarising Angle.

As we now thoroughly understand the meaning of refractive index and its mode of practical determination, we are in a position to enquire more closely as to its relation to the most favourable angle for the complete polarisation of a ray reflected from the plane surface of a transparent solid.

An extremely simple law was discovered by Brewster to apply, namely, that the polarising angle is that angle the tangent of which is equal to the index of refraction.

If in Fig. 490 XOX' represent the surface of the glass plate and YOY' the normal to the plate, and if AO be the incident beam and θ the angle of incidence AOY , then OB is the direction of the reflected ray, making also the angle θ with OY . Now Brewster found experimentally that to produce the maximum polarisation the angle AOY must be such that the direction of the refracted ray OC within the glass (COY' being the angle of refraction ϕ) must be at right angles to the reflected ray OB .

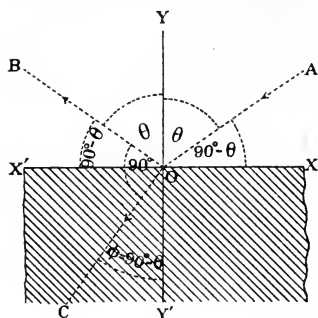


Fig. 490.

Hence the index of refraction μ becomes :—

$$\mu = \frac{\sin \theta}{\sin \phi} = \frac{\sin \theta}{\sin (90^\circ - \theta)} = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

That is, the index of refraction is equal to the tangent of the angle of incidence when maximum polarisation occurs.

Hence, if we look out in a book of mathematical tables the angle the tangent of which corresponds to the refractive index, which is known, that angle is the most favourable angle for polarisation by reflection.

The polarising angle may be also stated as that angle of incidence for which the reflected polarised ray is perpendicular to the refracted ray.

This law is absolutely true only for light of a specific wave-length, light of each wave-length having its own polarising angle.

In the following table are given the polarising angles, using sodium light, for crown and flint glass, and for natural glass, obsidian, which is much used in making the plates for glass polarisers; also for water, the still surface of which is effective as a polariser at an unusually low angle, corresponding to the low refractive index; and for diamond, the angle of polarisation of which is very large on account of its extremely high refractive index.

POLARISING ANGLES FOR NA-LIGHT.

	Refractive Index.	Polarising Angle.
Water	1.3334	53° 8'
Obsidian, natural glass	1.4964	56 15
Light crown glass	1.5153	56 35
Light flint glass	1.5798	57 40
Diamond	2.4669	67 56

CHAPTER XXXVII

THE OPTICAL ELLIPSOID OF CRYSTALS AND DETERMINATION OF ITS CHARACTER.

THE determination of the optical characters and constants of crystals becomes much simplified when the fundamental fact referred to at the conclusion of Chapter XXXV. is remembered, namely, that the optical properties are represented in general by an ellipsoid, which may express either the relative velocity of the light vibrations transmitted in all directions through the crystal, or more simply the corresponding values of the refractive index. For the practical operations concerning the optical properties really resolve themselves into the location of, and measurement of the relative directional dimensions of, this ellipsoid.

In the cases of crystals belonging to the triclinic, monoclinic, and rhombic systems the ellipsoid is one of the most general form, with three unequal rectangular axes, as represented in light and shade in Fig. 491, the axes emerging at a , b , and c , a being the maximum axis, c the minimum, and b the intermediate axis. The three principal section planes are all ellipses. In a rhombic crystal the three axes are coincident in direction with the crystallographic axes, by reason of the symmetry. In a monoclinic crystal only one is coincident with a crystallographic axis, the symmetry axis b , the other two rectangular axes lying in the symmetry plane, but anywhere in that plane; and in a triclinic crystal the orientation of the ellipsoid has no necessary relation to the crystallographic axes whatever. In the cases of crystals belonging to the tetragonal, hexagonal, and trigonal systems the symmetry necessitates that two of the three rectangular axes of the ellipsoid are equal, converting it into one of rotation, the axis of rotation being identical in direction with the singular axis of tetragonal, hexagonal, or trigonal symmetry. In the cases, finally, of crystals belonging to the cubic system the perfection of symmetry causes all three axes of the

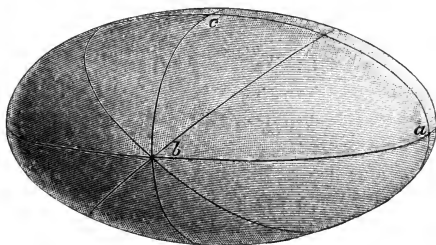


FIG. 491.—The Ellipsoid of General Form.

ellipsoid to be equal, so that the ellipsoid becomes a sphere. Obviously the ellipsoid of revolution and the sphere are but special cases of the general ellipsoid, when first two and then all three of the unequal rectangular axes become equal.

Crystals possessing the symmetry of any of the five classes of the cubic system, therefore, exhibit similar optical properties in all directions within the crystal, just as if the crystal were so much glass, although this is a consequence of the perfect symmetry of the homogeneous structure, whereas in the case of glass the effect is due to the total lack of homogeneity in the mode of arrangement of the molecules, which renders the effect equal in all directions simply because it is the average effect in a solid which is merely a congealment of a highly viscous liquid, the molecules of which are restrained by their own friction from properly arranging themselves. In a cubic crystal, therefore, the refractive index is the same for all directions, so that there is no double refraction; also there is no "optic axis" or direction of unique single refraction (shortly to be fully explained, and quite distinct from any of the three axes of the optical ellipsoid) developed, for there is only single refraction in every direction. Consequently no coloured rings and cross (uniaxial as explained below), or pair of lemniscate ring-systems and dark optic axial hyperbolic brushes (biaxial), are visible when plates of the crystal are examined in convergent polarised light. Cubic crystals are, therefore, said to be "isotropic."

Crystals endowed with the symmetry of any of the 19 classes of the hexagonal, tetragonal, and trigonal systems exhibit unique optical properties along the direction of the axis of revolution of the ellipsoid, that is, along the direction of their hexagonal, tetragonal or trigonal main axis, the vertical crystallographic axis (except when the Millerian rhombohedral axes are used in the trigonal system, when the vertical axis of trigonal symmetry is not a crystallographic axis), while the optical properties are different in all other directions. The difference is at a maximum in all directions at right angles to the singular axis. The crystals are thus endowed with the property of double refraction. For the refractive index along the direction of the singular axis is either the greatest or the least exhibited by the crystal, the two possible types being termed respectively positive and negative.

The refractive index along all the directions at right angles to the singular axis, that is, along all diameters of the circular section of the ellipsoid, is obviously the converse, being either the least or the greatest refraction which the crystal can show among all the infinite number of possible values corresponding to the various directions within the crystal.

The refractive indices for any direction in the circular section, and for the principal axis at right angles to that section, are thus the two extreme values for the crystal, and the difference is the measure of the maximum double refraction. A ray of light transmitted along the principal axis, and the vibrations of which occur therefore in the circular section at right angles thereto, is said to be an

“ordinary” ray, and the refractive index corresponding thereto is distinguished by the symbol ω . The other refractive index is labelled ϵ , and corresponds to an “extraordinary” ray of extreme properties travelling in the circular section, and the vibrations of which occur parallel to the principal axis. Thus ω is the greater in negative crystals, while for positive crystals ϵ is the larger refractive index. The double refraction is consequently $\omega \sim \epsilon$; or it may be expressed always as $\epsilon - \omega$, the sign of the result then indicating that of the double refraction.

Also the crystal exhibits the phenomenon of an optic axis along the direction of the singular axis, the position of which is rendered visible, when a section-plate perpendicular to the axis is examined in the polariscope under crossed Nicols in convergent polarised light. For a rectangular black cross is observed of which the axis forms the centre, and around that centre are seen a series of circular concentric rings; the latter are spectrum-coloured when white light is employed but black when monochromatic light is used, and resemble Newton's rings. They are similarly derived by interference, but by the retardation of one set of rays behind the other instead of by difference in the lengths of the paths of the two sets. The crystals of the hexagonal, tetragonal, and trigonal systems are therefore said to be “uniaxial.” The reproduction of a photograph, taken by the author, of the interference figure afforded by such a plate of a typical uniaxial substance, calcite, is shown in Fig. 492, the Nicols of the polariscope being crossed for the production of the dark field.

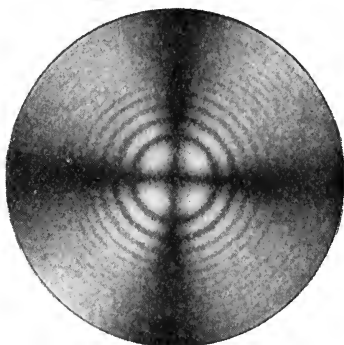


FIG. 492.—Uniaxial Interference Figure of Calcite.

Crystals exhibiting the symmetry of the 8 classes of the rhombic, monoclinic, and triclinic systems have their optical properties represented by the general ellipsoid the three rectangular axes of which are of unequal lengths, the relative measure of which is afforded by the values of the refractive index along the three respective directions. The measure is a direct one or a reciprocal one, according as we define the ellipsoid, the direct measure affording the “indicatrix” of Fletcher, and the other its polar reciprocal, the ellipsoid of Fresnel, both of which will be fully discussed and defined in Chapter XL. The three values of the index corresponding to these three axial directions are generally labelled α , β , and γ .

It has been pointed out that in the case of the rhombic system the three axial directions of the ellipsoid are identical with the three crystallographic axes, by virtue of the symmetry. The relations of the lengths, however, is not necessarily similar.

In the case of the monoclinic system the direction of only one of the

axes is coincident with a crystallographic axis, the symmetry axis b , the nature of the symmetry conferring the possibility of rotation to any extent about this axis. The ellipsoid may thus be so situated that its two other principal axes, while still at right angles to each other, lie anywhere in the symmetry plane, the exact position being definite for the same substance, but only at the same temperature and for the same wave-length of light. Indeed, it is usual for rotation actually to occur with change of wave-length, that is, as the crystal is illuminated with light of the different colours of the spectrum in succession, and also when the temperature is raised or lowered. Gypsum is a beautiful case of the change by temperature; the two rectangular maximum and minimum axes of the ellipsoid, while remaining inclined mutually at 90° in the symmetry plane, rotate together for 6° in that plane, about the symmetry axis, when the temperature is raised by 100° C.

In the case of a triclinic crystal there is no rule or restriction whatever as to the orientation of the optical ellipsoid, which as far as the symmetry is concerned (centro-symmetry only) may have any possible situation with respect to the crystallographic axes.

A natural result of the general ellipsoid having three unequal axes and of the symmetry only controlling their directions and not their amounts, is that while one of them must be the maximum or longest axis and another the minimum or shortest, the third axis may be of any intermediate length, that is, it may approximate nearer to equality with either the maximum or the minimum axis. There are thus two possible kinds of these crystals, just as there are of uniaxial crystals, and they are similarly termed positive and negative. That variety which has the intermediate axis β (supposing the ellipsoid to be the expression of the refractive index) nearer in length to the minimum axis a is termed positive, and that in which the intermediate axis β approximates most to the maximum axis γ is termed the negative variety. Hence, taking the three refractive indices as expressive of the axial dimensions of the ellipsoid, the determinations of these refractive indices for the three rectangular directions at once fixes the sign or type of the crystal; for when β works out to be nearer to a than to γ the crystal is positive, but when β is nearer to γ the crystal is negative.

If we consider next the principal section of the ellipsoid, that elliptic section the major and minor axes of which are the maximum and minimum axes of the whole ellipsoid, it will be obvious that there must be two diameters of this ellipse, somewhere between these longest and shortest diameters and symmetrically placed on each side of them, which are of equal length to the intermediate axis perpendicular to the section under consideration. Hence there are two sections of the ellipsoid which are circular, each containing the intermediate axis and one of the two diameters in question. The directions perpendicular to these two circular sections will clearly be comparable to the singular axis, which is also the optic axis, of a uniaxial crystal; for the velocity of light vibration will be equal in all directions perpendicularly around it, while

either greater or less along it. Hence in convergent polarised light the phenomenon of an optic axis, surrounded by a series of spectrum rings, but differing from a uniaxial interference figure in being traversed by only a single diametrical black bar instead of a rectangular cross, is observed through a plate cut perpendicularly to either of the two directions. Such an interference figure, afforded by gypsum, monoclinic hydrated sulphate of lime, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$, is shown in Fig. 493. If a plate be cut at right angles to that axis of the ellipsoid which is the bisectrix of the angle between the two optic axial directions, both optic axes will be seen at once in the field of view of the polariscope, provided the angle of the lens system of the latter be wide enough to include them. The single bar again stretches across the field diametrically, passing through both axes, assuming that the plate is so arranged that this line joining the axes is parallel to either of the crossed Nicols; but it forms now one arm of a black cross, the other arm of which intersects it rectangularly midway between the two axes, and is usually much broader and less deeply black. The inner rings soon give place to lemniscates like the figure 8 as the axes are receded from, and these in turn pass into outer ellipse-like lemniscates near the margin of the field. On rotating the section-plate, the black cross opens out into a pair of hyperbolic "brushes," the vertices of which are narrow and fairly well defined; each vertex passes through one of the optic axes, the position of which it thus precisely indicates, while the wings tail off into ever widening but diffusing streams on each side. For all practical purposes these brushes afford an excellent means of determining the apparent acute angle of the optic axes in air, and it will be shown later (Chapter XLVII.) that by making the observation in a highly refractive liquid instead of in air, and making also a similar observation of the supplementary obtuse angle in the same liquid, by means of a second section-plate cut perpendicularly to the first plate and to that other axis of the ellipsoid which is the bisectrix of the obtuse angle in question between the optic axes, the true angle of inclination of the two optic axes within the crystal can be readily calculated with the aid of a very simple formula. The crystals of the rhombic, monoclinic, and triclinic systems are consequently termed "**biaxial**." The interference figure afforded by a typical biaxial substance, aragonite, the rhombic form of carbonate of lime, CaCO_3 , in the dark field of the polariscope, with the Nicols crossed and parallel to the principal axial directions of the section-plate, which is cut perpendicularly to the bisectrix of the optic axial angle, is shown in Fig. 494; while Fig. 495 represents the appearance when the section-plate (or the pair of Nicols simultaneously) is rotated 45° from this position, this arrangement being that

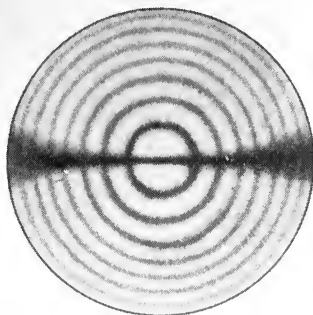


FIG. 493.—Interference Figure about one of the Optic Axes of Gypsum.

which is practically employed in the measurement of the optic axial angle. Both figures are reproductions of the author's direct photographs.

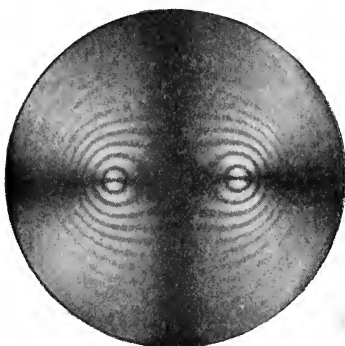


FIG. 494.

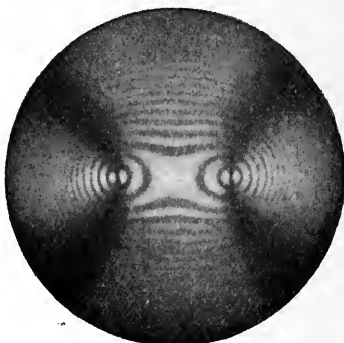


FIG. 495.

Biaxial Interference Figures of Aragonite.

The practical determination of the optical constants of crystals thus resolves itself into the following three main operations, which were briefly indicated in the introductory pages of Chapter XXXIV., but which may now be more definitely expressed.

(1) The determination of the orientation of the optical ellipsoid, if this is not already fixed by the symmetry. As this latter is the case, however, for all but monoclinic and triclinic crystals, it is only when the symmetry of either of these two systems is developed that an experimental determination is necessary.

(2) The orientation of the ellipsoid being known, to determine the relative lengths of its three principal rectangular axes. This is achieved in practice by determining the three refractive indices, α , β , and γ , corresponding to these three directions of the rectangular principal axes of the optical ellipsoid, when the crystal is a biaxial one; or if the crystal be a uniaxial one, by determining the two refractive indices ϵ and ω , corresponding respectively to the principal axis of the ellipsoid of revolution and to directions perpendicular thereto. The difference between the two extreme indices, $\gamma - \alpha$, in the case of a biaxial crystal, or the difference of the two indices ϵ and ω in the event of the crystal being uniaxial, affords the measure of the amount of double refraction. In the latter case ϵ is greater than ω when the ellipsoid (indicatrix) represented by the refractive indices is vertically elongated like an egg, which is the case with positive uniaxial crystals, such as quartz, so that the double refraction is then $\epsilon - \omega$; but ω is greater than ϵ for negative crystals such as calcite, the ellipsoid then being vertically compressed like an orange, so that the double refraction is $\omega - \epsilon$.

(3) In the case of a biaxial crystal the angle of separation of the two optic axes also requires to be determined, first of all the apparent angle in air, and secondly the true angle within the crystal. For the optic axial angle is a very characteristic property of any biaxially

crystalline substance, and the nature of the dispersion of the axes of the ellipsoid for different colours of light, in the cases of monoclinic and triclinic substances, is often of such a specific character as to enable the substance to be identified by means of it.

In all cases in which the position of the optical ellipsoid is not fixed by the symmetry, the orientation requires to be determined for light of a graduated series of different wave-lengths, at suitable intervals throughout the whole spectrum; for the position differs, as has just been indicated, for different colours of light. Determinations in ordinary white light are inadequate, although a crystal is always first explored in white light, before proceeding to use monochromatic light, and any preliminary adjustments are also carried out in white light. Both the refractive index and the optic axial angle must imperatively be determined in monochromatic light of the same graduated series of convenient wave-lengths, so that it is essential to possess a ready means of procuring monochromatic light, and of supplying it directly to the observing instruments. Such an apparatus for producing light of any desired wave-length will be described in Chapter XLIII.; it is one which has proved of inestimable service to the author in all his crystallographic researches.

For the purpose of carrying out the three operations above specified, which are the fundamental essentials of every crystallographic optical investigation, certain parallel-faced section-plates and 60° -prisms are required to be cut or ground (or both) out of the crystal, each having two truly plane polished surfaces, accurately orientated with respect to the optical ellipsoid. It is the difficulty presented by the problem of preparing these plates and prisms that has so seriously hindered in the past the investigation of the optical properties of crystals, but an instrument for carrying out this all-important part of the work, the cutting-and-grinding goniometer, has been devised by the author, which accomplishes the task with the minimum of trouble and the maximum of accuracy and perfection, rendering, indeed, what was formerly the most difficult part of a crystallographic research at the same time comparatively easy and very interesting. It will be described in Chapter XLII.

CHAPTER XXXVIII

THE TRANSMISSION OF LIGHT THROUGH UNIAXIAL CRYSTALS.

It has been shown in the last chapter that the sign of the double refraction of a uniaxial crystal is negative when $\omega > \epsilon$, and positive if $\epsilon > \omega$. Hence, the velocity of vibration being inversely as the refractive index, the extraordinary ray moves with the greatest velocity in negative crystals, while the ordinary ray is the quicker in positive crystals.

Quartz, SiO_2 , and calcite, CaCO_3 , are two typical examples of positive and negative uniaxial crystals respectively, and their pairs of refractive indices for sodium light, and the amounts of their double refraction, are the following :

For quartz (positive) $\omega = 1.5443$, $\epsilon = 1.5534$; $\epsilon - \omega = 0.0091$.

„ calcite (negative) $\omega = 1.6583$, $\epsilon = 1.4864$; $\omega - \epsilon = 0.1719$.

Both these colourless transparent minerals are of great use in crystallographic optics.

When a beam of parallel rays is allowed to pass through a rhomb of calcite normally to the two faces, as in Fig. 496, or in any other

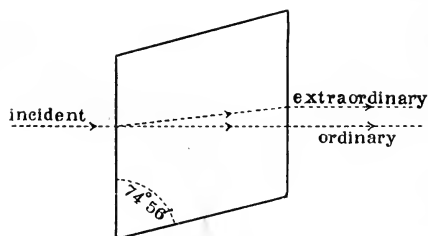


FIG. 496.—Passage of Light through Rhomb of Calcite.

direction than the principal axis, it separates into two rays on entering the crystal. One, termed **the ordinary ray**, behaves as if the crystal were cubic, or a piece of glass, and leaves the crystal parallel to its original direction, indeed in the same straight line when the incidence is strictly normal. The other, conventionally known as **the extra-**

ordinary ray, is refracted to one side in any case, and emerges laterally displaced, but of course parallel to the original path as the two faces of the rhomb through which it passes are strictly parallel, being either natural or cleavage faces, each inclined $74^\circ 56'$ to the other pair of faces. The extraordinary ray does not obey the law of sines, except when the plane of incidence is perpendicular to the principal axis, and therefore a circular section of the optical ellipsoid of revolution.

When the rhomb is rotated about the direction of the incident ray, the two parallel faces through which it passes being thus turned in their own plane, the extraordinary ray rotates round the ordinary one. This may be well shown by focussing on the screen, by means of a projection lens, the image of a round aperture in a metal plate placed in front of the lantern condenser. The arrangement shown later in Fig. 518 (page 612), in connection with another experiment, consisting of four apertures in a rotatable diaphragm, is a very suitable one, enabling such an aperture to be chosen as is of most convenient size; the lens on its adjustable stand is also shown in the figure. On interposing the calcite rhomb in the path of the rays, either between the aperture and the lens or after leaving the lens, preferably at their crossing-point where the beam is narrowest, two images are seen on the screen, and if the rhomb is mounted for rotation (like the double-image prism seen to the extreme right nearest the screen in Fig. 518) one image, that produced by the ordinary ray, will be observed to remain stationary if the incidence be normal, while the other image due to the extraordinary ray rotates round the fixed one, on rotating the rhomb. When such a rotating rhomb on adjustable stand is used, the whole apparatus is precisely as shown in Fig. 518 without the large Nicol prism.

If, instead of a natural rhomb, a 60° -prism of calcite be employed, and a rectilinear slit instead of a round aperture in front of the lantern, two spectra are afforded on the screen, both rays being now refracted to one side as with a spectroscopic prism of glass, but one ray, the ordinary, at a much more considerable angle of deviation than the other, the extraordinary, corresponding to its much higher refractive index.

The experiment is shown in Fig. 497. The electric lantern is to the left, with a water cell 2 inches thick in front of the $4\frac{1}{2}$ -inch condenser, the cell being readily attachable on a short tubular fitting also $4\frac{1}{2}$ inches in diameter, to the lantern front. The ordinary projecting lens and the carrier bar on which it is adjustable by rack and pinion (shown in Fig. 520, page 615) are entirely removed, the bar simply unscrewing from a boss in the lantern front; an additional support for it is the little stirrup column standing up from the water-cell fitting.

The slit is carried in front of the water cell, and in the experiment as performed by the author a disc 6 inches in diameter is employed having a fixed permanent slit 2 inches long and an eighth of an inch wide cut in it, and two vertical bevelled jaws in front sliding in horizontal dovetails so as to be able to vary the width of the opening; there are also two similar horizontal jaws behind, adjustable in vertical dovetails, which enable the length of the slit to be varied. The disc is mounted on a standard which is adjustable for height, and the base of which is also provided with a lateral adjustment in a dovetailed bed, which rests in turn on a laterally elongated mahogany base, one of a series of sliders in a mahogany grooved and rabbeted bed, forming the optical bench of the large projection polariscope. At 4, 5, or 6 inches distance comes a similarly mounted projection lens of $2\frac{1}{2}$ inches diameter, and either 4, 5, or 6 inches focus. Next is a large Nicol prism to be described in the next chapter, and then the calcite prism, mounted on a little circular table carried on a similar adjustable standard to those bearing the slits and the lens.

The amount of separation of the two spectra depends on the direction in which the prism is cut, and of the two refractive indices, corresponding

to the positions of the two images when the prism is adjusted for minimum deviation, one always has the value 1.6583, namely, that belonging to the ordinary ray; while the other has a value which may vary from 1.4864 to 1.6583, the latter occurring when the separation of the two rays has become zero, that is, when no double refraction occurs, which only happens when the prism is so cut that the ray travels along the optic axis.

This latter case of a 60° -prism of calcite cut to afford only one

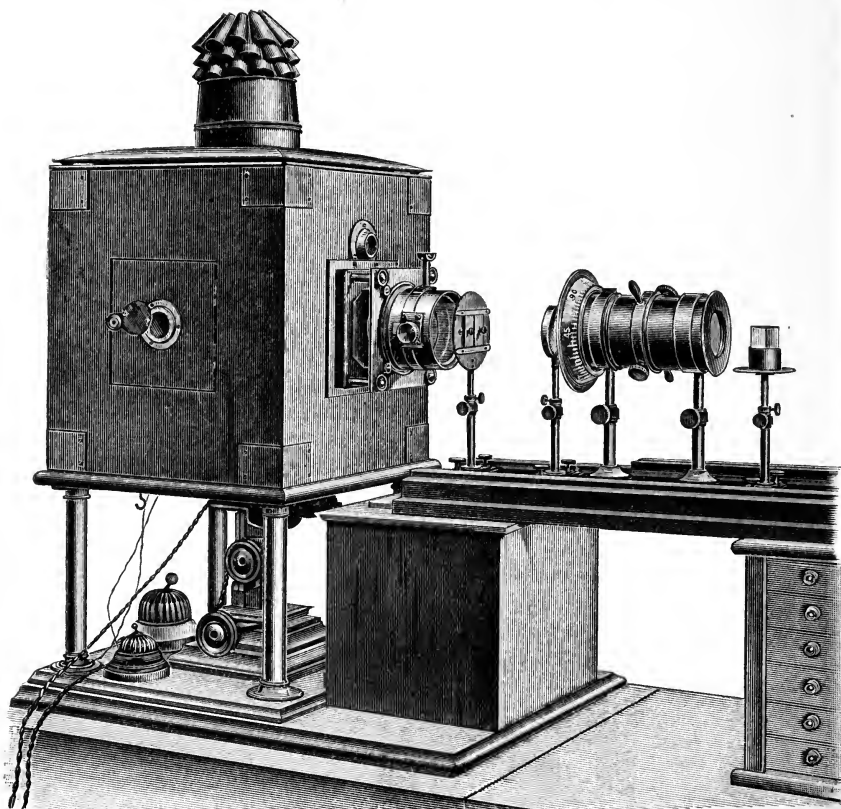


FIG. 497.—Experiment demonstrating Rectangular Polarisation of the two Spectra afforded by Calcite Prism.

spectrum, when arranged for minimum deviation, as if it were a prism of glass, is one much employed in optics for spectroscopic purposes. The refracting edge is made perpendicular to the trigonal axis (the optic axis), and the basal face of the prism parallel both to this edge and to the axis as shown in Fig. 498. When arranged at minimum deviation for any one colour—the incident and refracted light rays of that wave-length being then equally inclined to the base—the direction of transmission within the crystal-prism is parallel to the base, along the

optic-trigonal axis, and the vibrations thus occur entirely in the circular section of the ellipsoid perpendicular thereto, with equal velocity in all azimuths, corresponding to $\omega = 1.6583$.

The condition for extreme separation of the two images is afforded when the prism is cut in either of two ways, both of which ensure that one ray vibrates parallel to the axis, while the other vibrates perpendicularly to it. The first mode of cutting is shown in Fig. 499, and the only condition is that the refracting edge shall be parallel to the axis, the prism being orientated anyhow around that axis. Whatever be the orientation, a beam of monochromatic light entering at minimum deviation is transmitted through the crystal perpendicularly to the axis, and is separated by the crystal into two distinct rays, composed of two sets of vibrations, both occurring at right angles to its path, one set vibrating in the circular section perpendicular to the axis, and the other set vibrating parallel to the axis. Obviously, therefore,

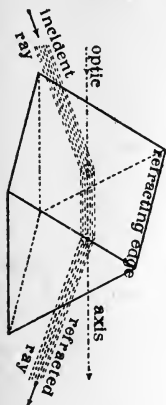


FIG. 498.

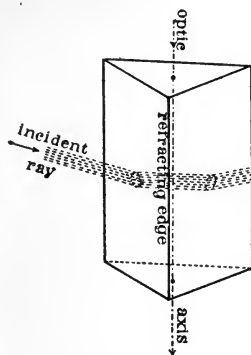


FIG. 499.

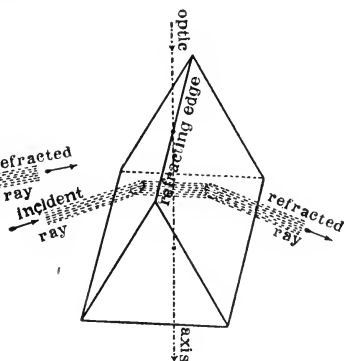


FIG. 500.

there will be two images of the slit afforded, corresponding respectively to $\omega = 1.6583$ (as always for a ray vibrating in the circular section), and to $\epsilon = 1.4864$ (that formed by rays vibrating parallel to the axis).

The second mode of cutting the prism is so that the refracting edge is perpendicular to the principal axis, and also so that the axis bisects the refracting angle, as shown in Fig. 500. In this case the monochromatic beam traverses the crystal-prism at minimum deviation in the circular section perpendicular to the axis, and the two rays into which it divides vibrate parallel and perpendicular to the axis respectively, the former corresponding to $\epsilon = 1.4864$ and the latter to $\omega = 1.6583$.

The images of the slit in monochromatic light, or the spectra in white light, are thus separated at the maximum possible distance for calcite by these two latter prisms, so that, as we shall see when we come to the practical determination of refractive index in Chapter XLV., such prisms afford the two indices ω and ϵ directly. Moreover, the images are formed by rectangularly plane-polarised light, one image extinguishing when a Nicol

prism introduced in the path of the rays is arranged at its 0° position, and the other disappearing when the Nicol is set at 90° , the former being that produced by light vibrating parallel to the refracting edge of the crystal-prism. In the case of the prism represented in Fig. 498, the unique image never extinguishes at all, for any position of the Nicol, the prism behaving in this respect also as if the crystal were singly refractive.

The Nicol is shown in position in Fig. 497 for demonstrating these facts, the prism actually used being of the type shown in Fig. 499, its refracting edge being parallel to the optic axis.

These phenomena with calcite are typical of uniaxial crystals in general, and the three above described modes of cutting 60° -prisms are equally applicable to any uniaxial substance. The values of the indices, the corresponding amounts of separation of the images, and the dispersion of each spectrum in the cases of the two latter prisms (Figs. 499 and 500), as well as the order of the values corresponding to positive or negative character, will vary for different substances, but the principles are the same.

Thus, in uniaxial crystals the ordinary ray always vibrates perpendicularly to the axis (along a direction parallel to a diameter of the circular section) and with the same velocity; it is consequently transmitted equally in all directions, and has a spherical wave-surface. Its refractive index ω is the maximum index in the case of calcite (1.6583) and all other negative uniaxial crystals, but the minimum index in the case of quartz (1.5443) and all other positive uniaxial crystals. Its velocity is the converse, namely, the minimum for a negative crystal and the maximum for a positive one.

On the other hand, the extraordinary ray is unequally transmitted in different directions, other than those lying in the circular section, and its wave-surface is thus an ellipsoid of revolution. In negative crystals such as calcite, the velocity of transmission becomes greater, and the refractive index conversely less, as the direction of the principal axis is left and that of the circular section approached, until for directions lying in the latter the maximum velocity and minimum index is reached. Thus the extraordinary ray in the case of calcite (negative) has its minimum refractive index 1.4864 when transmitted perpendicularly to the axis, its vibrations being parallel to the axis; and it has its maximum index 1.6583, equal to the ordinary, when it travels along the axis, its vibrations being perpendicular thereto. Hence vibrations parallel to the axis always afford the index 1.4864, and vibrations perpendicular to the axis yield 1.6583 as the numerical value of the index. The opposite is true of quartz (positive), the velocity diminishing and the refractive index increasing up to a maximum of 1.5534, as the direction of transmission recedes more and more from the principal axis until it finally becomes perpendicular to that axis.

The forms of the wave-surfaces of the ordinary and extraordinary rays of a negative uniaxial crystal are represented in vertical section in Fig. 501, and those of a positive crystal in Fig. 502. It will be observed

that in the former case the circle representing the wave-surface of the ordinary ray is inside the elliptic section of the extraordinary wave-surface, giving the whole figure a flattened shape; while in the latter case the circle is outside.

If we draw the ellipsoids of revolution representing graphically the

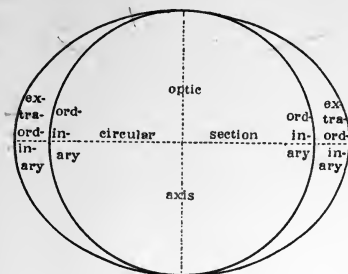


FIG. 501.

Negative Uniaxial Wave-Surfaces.

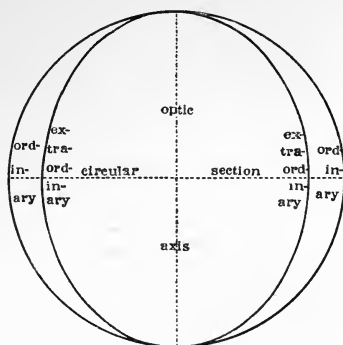


FIG. 502.

Positive Uniaxial Wave-Surfaces.

refractive indices, for a negative and a positive crystal respectively, we obtain an oblate spheroid in the former case, and one standing up like an egg in the latter case, as shown likewise in vertical section in Figs. 503 and 504. Such an ellipsoid of refractive index is the "indicatrix" of Fletcher, which will be discussed more fully in connection with biaxial crystals in the next chapter.

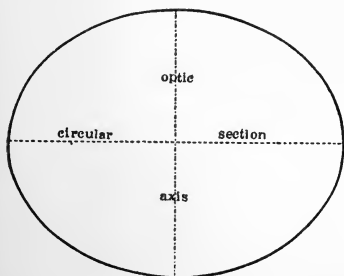


FIG. 503.

Negative Indicatrix.

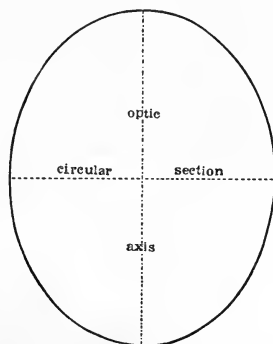


FIG. 504.

Positive Indicatrix.

In each of the cases represented in Figs. 503 and 504 the indicatrix resembles the extraordinary wave-surface.

For the determination of the shape of the wave-surface it is necessary to measure the two velocities of propagation of the extraordinary ray, when it travels respectively parallel and perpendicular to the axis of the crystal. For the ratio of these quantities is that of the two axes of the

ellipse which generates the wave-surface ellipsoid of revolution. As the former velocity is identical with that of the ordinary ray, the ratio in question is:— $V_{\text{ordinary}} : V_{\text{extraordinary}}$, that is, the inverse ratio of the two refractive indices ω and ϵ . Hence, we have only to measure the refractive index of the ordinary ray for any direction, and that of the extraordinary ray at its greatest difference from the ordinary when propagated perpendicularly to the optic axis and, therefore, vibrating parallel to the axis. For it must be clearly remembered that the refractive index corresponds to the vibration direction, not to the direction of propagation of the ray. Thus the whole matter resolves itself experimentally into the determination of the refractive index ω for light vibrations perpendicular to the axis, and of the index ϵ for vibrations parallel to the axis.

This we have seen can be achieved by the use of a 60° -prism cut according to the method of either Fig. 499 or 500. In the case of the prism represented in Fig. 499, when set for minimum deviation, one of the two rays into which the beam separates on entering the crystal vibrates parallel to the prism-edge and therefore to the axis, and thus affords ϵ ; the other vibrates perpendicularly to the axis and to the prism-edge, and consequently affords ω . A Nicol before the eyepiece of the goniometer-spectrometer, when arranged at 90° (assuming the divided circle itself to move with the Nicol and the vernier zero or indicating mark to be fixed at the top of the vertical diameter), that is, with its plane of polarisation (parallel to the 0° - 180° diameter of its divided circle) horizontal and its vibration direction (parallel to the 90° - 270° diameter of the circle) therefore vertical, lets through only the ray vibrating vertically, parallel to the edge of the prism and to the axis, thus only permitting the ϵ -image of the slit to be visible; while at 0° (plane of polarisation vertical and vibration direction horizontal) the Nicol lets through only the other rays vibrating horizontally, perpendicularly to the axis and prism-edge, thus isolating the ω -image. In the case of the prism shown in Fig. 500, the image seen when the Nicol is at 90° (vibrations, as always for 90° , parallel to the refracting edge) corresponds to and affords ω , while that visible when the Nicol is at 0° gives ϵ . At the 45° position of the Nicol in the case of either prism both images are seen, but with only their partial intensities. Their angular separation when arranged for minimum deviation depends, of course, on the amount of the double refraction.

The Nicol of the spectrometer-goniometer No. 1a of Fuess, which the author always employs for refractive index determinations, is arranged to move with the vernier, the circle being fixed; the Nicol is mounted so that the plane of polarisation is parallel to the diameter passing through the zero graduation of the vernier at the end of the radial arm, which moves with the Nicol; the fixed circle about which the arm with its vernier and the Nicol rotates has its 0° - 180° diameter vertical, the zero at the top, the 90° - 270° diameter being horizontal, with the 90° mark at the right. Hence, when the Nicol is arranged so that the arm is horizontal with the zero mark of the vernier opposite the

90° graduation, the vibration direction of the Nicol is vertical, and thus parallel to vibrations of light in the crystal which are parallel to the refracting edge of the crystal prism. The image of the slit which is then transmitted is due therefore to such vibrations only, the image corresponding to horizontal vibrations perpendicular to these being completely extinguished. When the vernier arm is rotated upwards till its zero mark is opposite the 0° graduation, the vibration direction is horizontal. The use of such a Nicol in front of the eyepiece of the spectrometer telescope is invaluable in enabling us not only to isolate the image corresponding to either direction of vibration, for measurement purposes, but also to determine the direction of vibration of the light composing any image.

The determination of both refractive indices ω and ϵ at once may also be achieved by employing the principle of total reflection. For any plate of a uniaxial crystal other than one perpendicular to the axis exhibits two limits of total reflection, corresponding to the ordinary and extraordinary rays. The value of the former is always the same, namely ω , which can thus be got from a plate of any orientation; but the extreme value of the latter is only afforded by the second limit when the plate is parallel to a principal section, that is, to the axis, two rectangular diameters in it then corresponding to the axis and its perpendicular.

RESUMÉ.

Rays travelling along the optic axis (principal axis) of a uniaxial crystal afford only one refractive index, ω , corresponding to vibrations in the circular section of the optical ellipsoid of revolution. Any ray passing obliquely through the crystal and its ellipsoid is resolved into two components. One component is plane-polarised in the principal section (that containing both the direction of the ray and the optic axis) and vibrates perpendicularly thereto along a diameter of the circular section; whatever be the direction of the ray this component, which is conventionally termed the ordinary ray, consequently always gives the same index of refraction, which is symbolised by ω . The other component is an extraordinary ray, and its vibration direction and refractive index differ with the direction of the ray. The difference in refractive index from that of the ordinary ray is most accentuated when the ray is transmitted in the circular section itself perpendicular to the singular axis, its vibrations being then parallel to the axis; and this extreme value for the second refractive index is characteristic of the extraordinary ray and is labelled ϵ .

Hence, the principal practical task connected with the optical properties of uniaxial crystals is to determine the two extreme refractive indices ω and ϵ . Their difference is the measure of the double refraction.

Thus in general, in all uniaxial crystals any ray is divided into:

(1) An ordinary ray, of which the vibrations are always perpendicular to the axis, and perpendicular to a principal section and to the direction of the ray, and of which the plane of polarisation is a principal section.

The wave-front is also perpendicular to the direction of the ray, the wave-surface being a sphere.

(2) An extraordinary ray, polarised in the plane perpendicular to a principal section and parallel to the ray itself, and of which the vibrations occur in a principal section and perpendicular to the ray, and hence parallel to the axis in the special case when the ray is directed along the circular section perpendicular to the axis. This special case is always used for the determination of the refractive indices by a 60° -prism cut as in Fig. 499 or 500. The wave-front is also perpendicular to the direction of the ray in this special case, and also when the ray is propagated along the axis, but generally, for all other directions, it is slightly inclined to the direction of the ray, the wave-surface being an ellipsoid of revolution and not a sphere, as in the case of the ordinary ray.

In a positive uniaxial crystal the ordinary refractive index ω (vibrations perpendicular to axis) is the minimum, and the extraordinary refractive index ϵ (vibrations parallel to axis) the maximum. Example, quartz, SiO_2 , $\omega_{Na} = 1.5443$, $\epsilon_{Na} = 1.5534$.

In a negative uniaxial crystal the ordinary refractive index ω (vibrations also perpendicular to axis) is the maximum, and the extraordinary refractive index ϵ (vibrations also parallel to axis) is the minimum. Example, calcite, CaCO_3 , $\omega_{Na} = 1.6583$, $\epsilon_{Na} = 1.4864$.

CHAPTER XXXIX

CALCITE POLARISING AND DOUBLE-IMAGE PRISMS.

It will be convenient at this stage, after having discussed in detail the passage of light through calcite, as a type of a uniaxial crystal, to describe the construction of the polarising prism devised by Nicol¹ in the year 1828, especially as its use has been so markedly demonstrated in the last few pages.

The Nicol Prism.—This prism of the beautifully perfect form of calcite, carbonate of lime, CaCO_3 , known as "Iceland spar," affords us the most satisfactory means yet discovered of producing a beam of plane-polarised light. Its importance in crystallographic optics is simply incalculable, owing to the absolute transparency and freedom from colour of Iceland spar, and the perfection of the polarisation. Moreover, the "Nicol," as it has come to be familiarly called, is as effective an analyser as it is a polariser, and the dark field obtained with a well-constructed pair of Nicols is practically a perfect extinction of light, and the images of objects polarised suffer no distortion.

To construct a Nicol prism a clear flawless cleavage rhombohedron of calcite, about three times as long as broad, is selected, such as is shown in Fig. 505. It is carefully cut in two halves along a diagonal plane, APCQ in Fig. 505 and Fig. 506, and AC in Fig. 507. The two cut surfaces are polished truly plane, great skill being required on account of the softness of calcite, and cemented together again, in the same position as before cutting, with Canada balsam. The exact direction of the plane of section is a matter of importance. It is parallel to the longer diagonals HE and GF of the end-faces (assuming the four long faces equally developed), almost joins two opposite corners A and C of the rhombohedron, and makes an angle BAC (Fig. 507) of 22° with the long edge of the rhomb AB or CD. The section-plane is also perpendicular to the principal section BAA' (Fig. 505), BA being a long edge and AA' the direction of the optic axis of the crystal; that is, the section-plane is perpendicular to the plane containing both the long edge AB and the shorter diagonal AD of the end-face, as well as the optic axis, which is also the vertical trigonal crystallographic axis.

The natural cleavage end-faces are nearly but not quite also perpendicular to the section-plane, the exact angle being $87^\circ 7'$. For the

¹ *Edinburgh New Phil. Journ.*, 1828, 6, 83; 1839, 27, 332.

edge AB (or the strip-face $e = (101)$ which often symmetrically replaces this edge on calcite) is inclined $70^\circ 53'$ to the natural end-face $r = (010)$ of the rhombohedron (see table of angles of calcite at end of Chapter XXIV.); as the section-plane is inclined 22° to the edge AB, these two angles, together forming the supplement of the angle in question between the section and the end-face, make up $92^\circ 53'$, so that the supplementary angle under discussion is $87^\circ 7'$. It is made 90° , therefore, by grinding away $2^\circ 53'$ in the proper

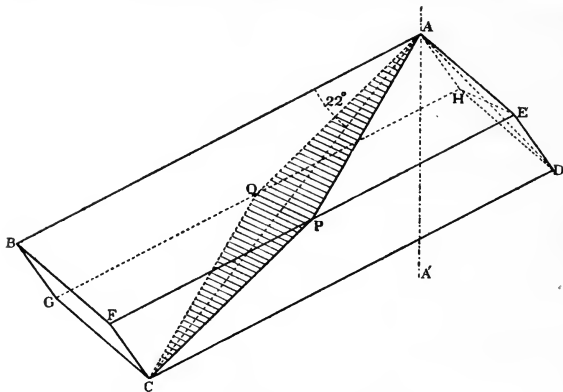


FIG. 505.—Construction of the Nicol Prism.

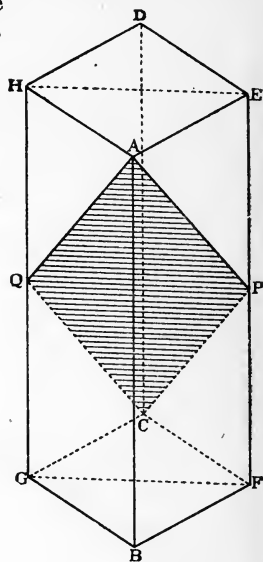


FIG. 506.—Nicol Prism vertically arranged to show another View of the Section-Plane.

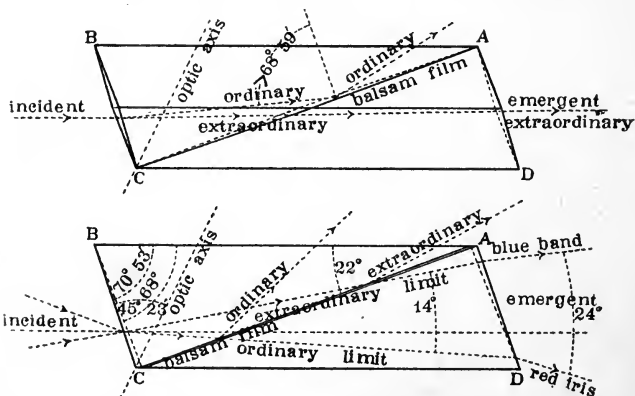


FIG. 507.—The Passage of Light through a Nicol Prism.

direction (thickest part removed at corner A). Indeed the operations are conveniently carried out by first making the ends to have the required inclination of 68° to the edges AB and CD, and then cutting the crystal through at right angles to these ends diagonally from A to C (Fig. 507).

We have seen that the two refractive indices of calcite for sodium

light are $\omega = 1.6583$ (ordinary ray) and $\epsilon = 1.4864$ (extraordinary ray). The refractive index of Canada balsam for sodium light is intermediate between these, namely, 1.548.

Now the index of refraction of the ordinary ray of calcite as compared with balsam is obviously $\frac{1.6583}{1.548} = 1.0712$. If we calculate the corresponding critical angle ϕ for the passage of light from balsam to the denser calcite (as regards the ordinary ray), by finding the angle the sine of which is equal to the reciprocal of this index (see Chapter XXXVI., page 579), we have :

$$\sin \phi = \frac{1}{1.0712}, \text{ and } \phi = 68^\circ 59'.$$

When the incident light enters the Nicol prism at the end-face BC (Fig. 507, upper figure) it is separated into an ordinary and an extraordinary ray, as usual for a uniaxial doubly refractive crystal, both of which rays are bent by refraction as the incidence is not normal, and the ordinary more than the extraordinary ray, corresponding to its higher refractive index. The variable index of the extraordinary ray (the ordinary always being 1.6583 but the extraordinary varying from this down to 1.4864) is about 1.54 for the particular direction which it takes through the prism, practically the same as that of the balsam, through which, therefore, it is readily transmitted. On the other hand the ordinary ray, always possessing the index 1.6583 in calcite whatever the direction, arrives much more bent at the balsam layer, which it strikes at an angle of incidence (measured from the normal to the layer) greater than the critical angle $68^\circ 59'$ just calculated. It is consequently reflected totally to the side of the prism, where it is absorbed by dull lamp-black with which the cork mount is painted. Cork is always used as the mount for calcite prisms on account of the relative softness of the crystal and the general convenience of cork from every point of view. The interior of the cork mount is cut to the shape of the rhomb, and its exterior shape is cylindrical; it is subsequently mounted immovably with cement in a metal tubular mount, which is in turn capable of rotation within an outer brass or gun-metal tube. The rotation is usually indicated on a silver divided circle, either attached to the outer tube or carried by the inner tube as a projecting bevelled flange, and thus moving with the prism itself and its inner metallic and cork mounts; a silver vernier or indicating mark is carried by the other tube. The mode of mounting will be clear from Figs. 518 and 519 (pages 612 and 613), the latter of which shows on the left one of the pair of large Nicol prisms of $2\frac{1}{2}$ inches clear minimum aperture, belonging to the author's projection polariscope. Both these large prisms, arranged as a projection polariscope for parallel light, are shown in Fig. 515, page 609, and one of them also in use in Fig. 518 in an experiment described later in this chapter.

The only light which leaves the Nicol prism is thus that of the extraordinary ray, the vibrations of which are parallel to the principal

section, the plane containing AB, the axis AA', and the shorter diagonal AD of the end-face, that is, the plane of the paper in Fig. 507. The plane of polarisation is, according to the usual convention, perpendicular to this, namely, perpendicular to the plane of the drawing in Fig. 507. The direction of the emergent ray is parallel to that of the incident ray, but a little laterally displaced. The incident ray must not be inclined more than $14^{\circ} 29'$ to the axis of the Nicol, on the side of the acute corner B, in order that total reflection of the ordinary ray shall occur.

On looking through a Nicol prism at an illuminated surface, such as a white wall-screen or a white cloud in the sky, the limitation of the angle of usefulness will be perceived to be somewhere between 15° and 20° . There are two more or less clearly marked somewhat curved boundaries. One is a blue band beyond which the field is dark; this marks the limit where the extraordinary ray also becomes totally reflected at the balsam film. The other limit is the commencement of a series of spectrum-coloured interference fringes, merging eventually into a region of greater brightness; the first interference band marks the limit beyond which the ordinary ray is no longer eliminated by total reflection, and the bright region is that of no apparent polarisation, where both ordinary and extraordinary rays are transmitted. The blue band limit of the extraordinary ray begins to be formed when the incident light is so inclined to the axis of the Nicol that the extraordinary ray strikes the balsam film at an inclination of 7° - 10° . These facts will be rendered clear with the aid of the lower diagram in Fig. 507.

The angular width of the polarised field within the prism is about 14° , which corresponds to a useful field of about 24° in the external air.

The optic axis makes an angle of $45^{\circ} 23'$ with the natural end of the rhomb BC or AD, which brings the axis $44^{\circ} 37' - 2^{\circ} 53' = 41^{\circ} 44'$ from the section-plane. The bisecting line of the internal angle is not parallel to the long edges of the rhomb, hence the origin of the practice of trimming the ends $2^{\circ} 53'$ in the direction which partially corrects this obliquity. The two long edges AB and CD are also often trimmed to the same extent, so that the joint effect is to bring the axis of the Nicol considerably nearer to the bisecting line.

Many variations on the prism of Nicol have from time to time been made, some of which will be referred to directly. In the author's experience, however, none are to be preferred to the original plan of Nicol, when carried out by a master hand such as we fortunately possess in England in Mr. C. D. Ahrens, who has constructed, as far as the author is aware, all the large Nicols in this country. Such large prisms can now no longer be made, the original magnificent deposit of enormous clear rhombs of Iceland spar at Eskifjörður having become exhausted. Such as exist are consequently very valuable. The two described later in this chapter in connection with the author's projection polariscope (Fig. 515, page 609) have a clear aperture of $2\frac{1}{2}$ to 3 inches, and are one of the few original pairs made by Mr. Ahrens. Perhaps the most celebrated of these original pairs of Nicols are those of even larger aperture constructed for the late Mr. Spottiswoode and now in

the Royal Institution. Another pair, of the same large size, nearly 4 inches clear circular aperture, have lately been acquired for the nation, and are now in the Science Collection of the Victoria and Albert Museum, where they are mounted, as shown in Fig. 508, as a horizontal table polariscope. The nearest prism, the analyser, can be rotated from outside the large protective glass case in which they are enclosed, and each of a fine collection of quartzes and other polarising objects, mounted on a large rotating wheel, can be brought in turn as desired between

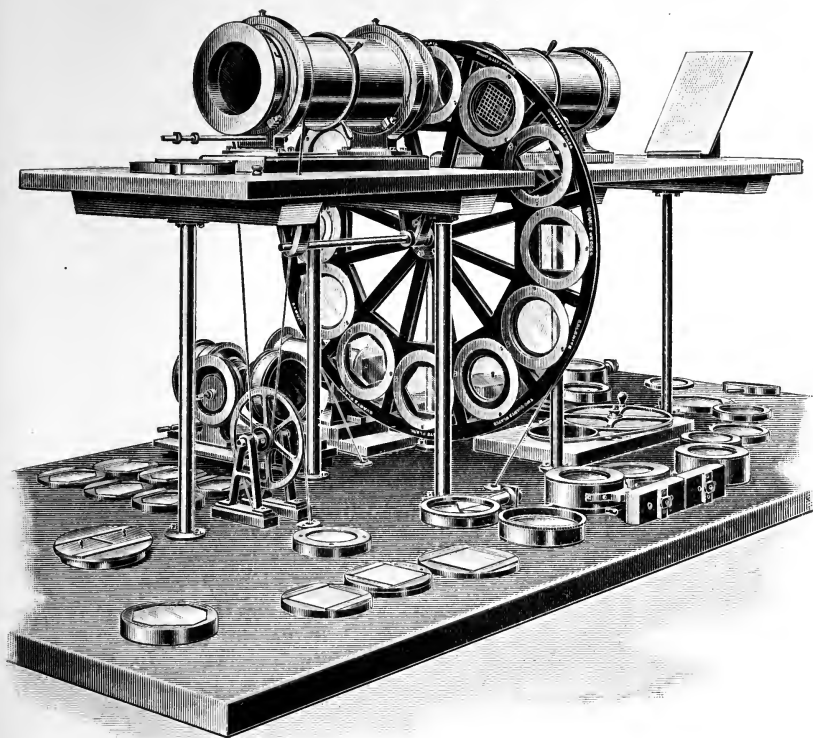


FIG. 508.—Large Nicol Prism Polariscope by Ahrens in the Science Museum at South Kensington.

the polariser and analyser by manipulation of a second large milled head outside the case. The Royal Institution possesses a second pair of Nicols, $2\frac{5}{8}$ inches aperture, formerly also belonging to Mr. Spottiswoode; and also the two somewhat larger historic Nicols of Prof. Tyndall, besides a single very large Nicol formerly belonging to Mr. De la Rue.

There is, happily, plenty of smaller calc-spar constantly being discovered, so that Nicol prisms for all ordinary crystallographic apparatus and the polarising microscope are more or less readily procured, and form the most efficient of polarisers and analysers.

As calcite is a soft mineral, hardness only 4 (see Chapter LV.), the

polished ends of a Nicol prism should never be touched for cleaning purposes with anything but the softest camel-hair brush, and only in the last resort gently rubbed with the very softest of chamois leather. Large Nicols should always be put away with the section-plane vertical, as otherwise thin-film colours are apt to develop in the balsam film.

Trimmed and Flat-ended Nicols.—Some of the more recent Nicols constructed by Mr. Ahrens have been trimmed the other way, the end-faces being made to be inclined 74° instead of 68° to the natural long edges, and the latter have then also been trimmed to follow to the same extent. The effect is to throw back the blue band and to widen the field.

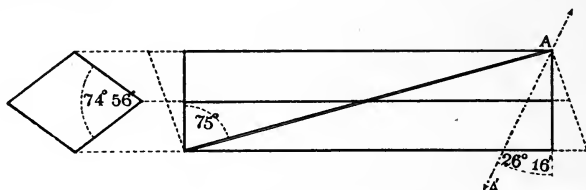


FIG. 509.—Square-ended Nicol Prism.

In order to avoid loss of light by reflection from the oblique end-faces small Nicols are often made "square-ended," that is, with the ends perpendicular to the long edges. The lateral displacement is also lessened. The angle between the optic axis and the end-face is then $26^\circ 16'$, and the section-plane is usually made 75° to the end-face. Each end is a rhombus showing the rhombohedral angle of calcite, $74^\circ 56'$. Such a prism is represented in Fig. 509.

Foucault's Prism.—In this calcite polarising prism a film of air replaces the balsam joint, and one-third the spar suffices. The faces are the natural ones of the cleavage rhomb, as shown in Fig. 510, and the section makes an angle of $58^\circ 7'$ with the basal edge AB and 51° with the end-faces AD or BC. The incident ray i is as usual broken up into an ordinary and an extraordinary ray on entering the crystal; the more refracted ordinary ray o meets the air layer at an angle greater than the critical angle, and so is totally reflected, while the extraordinary ray e is transmitted. The maximum internal angle of separation of the two limits, however, is only $5^\circ 14'$, corresponding to an external field of 8° . There is also a loss of ten per cent of light by reflection at the two uncemented surfaces. The limitation of a convergent beam to only 8° for complete removal of one of the rays is the chief drawback of the prism, which is otherwise very economical of spar. Square-ended Foucault prisms are often made, known as Hofmann's or Soret's prisms, which are useful when very convergent light is not required. Their angular field varies from 7° to 8° .

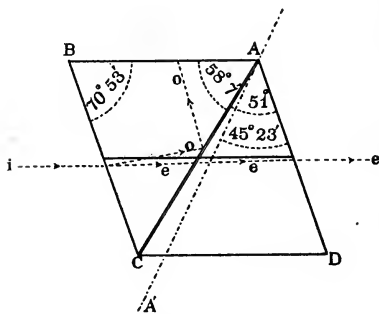


FIG. 510.—Foucault's Prism.

back of the prism, which is otherwise very economical of spar. Square-ended Foucault prisms are often made, known as Hofmann's or Soret's prisms, which are useful when very convergent light is not required. Their angular field varies from 7° to 8° .

Prazmowski-Hartnack Prism.—This jointly devised prism is a rectangular parallelepipedon of Iceland spar, and the section is made from edge to edge instead of diagonally, affording a larger area at which the transmission of the extraordinary ray can occur. In order to throw back the blue band as far as possible the section plane is made perpendicular to the optic axis. As will be obvious from the plan and elevation given in Fig. 511, which also show the shape of the original rhomb from which the prism is

cut, the construction is exceedingly wasteful of spar, and the cost is three times that of a Nicol. It has the advantage, however, of transmitting a much wider angular beam, its width of external field attaining a maximum of $41^{\circ} 54'$ when linseed oil is employed instead of a film of Canada balsam, the angle between the end-faces and the section being then $76^{\circ} 5'$. The prisms of this type usually supplied by the instrument makers have an obliquity of section to end-face of $74^{\circ} 5'$, and afford an angular width of external field of $35^{\circ} 0'$.

Glan-Foucault Prism.—This is a square-ended Foucault prism (with air film) so orientated that the section-plane is a principal section, that is, contains the optic axis. The accompanying plan and elevation given in Fig. 512, showing both the prism and the original crystal, will render the orientation clear. The angle between the end-faces and the section is $39^{\circ} 43'$. These prisms are light and small for their aperture,

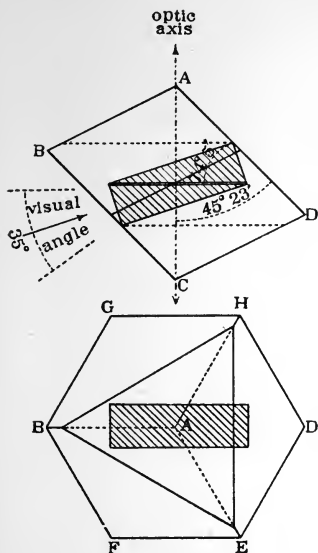


FIG. 511.—Prazmowski-Hartnack Prism.

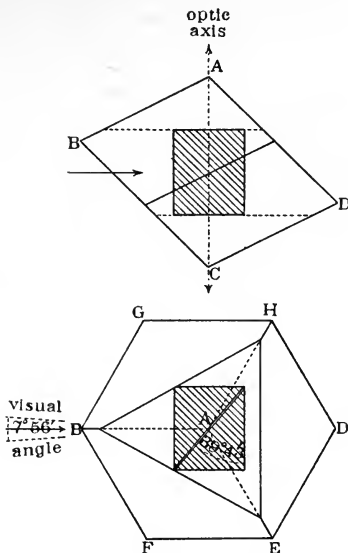


FIG. 512.—Glan-Foucault Prism.

but the useful field width is only $7^{\circ} 56'$, and multiple reflection is apt to occur, as with all air-film prisms, at the section-surfaces.

Thompson Prism.—In this prism devised by S. P. Thompson¹ the section-plane is parallel to a principal section, and the optic axis lying in it is at right angles to the direction of vision. The more recent form of it also has square ends, and is shown in Fig. 513. When Canada balsam is used as cement the angle of field is about 33° , but if linseed oil be used it is increased to $41^{\circ} 50'$. The square-ended form is particularly satisfactory as regards producing a field in which the planeness of the polarisation is as perfect as has yet been obtained. The angle between the film and the end-face is $76^{\circ} 5'$.

Ahrens Prism.—In the year 1886 Ahrens introduced a triple prism of calcite embodying the same theoretically advantageous principles as Thompson's, but consisting of three wedge-shaped pieces cut from the same rhomb. A rectangular block, the

¹ An excellent account of Prof. S. P. Thompson's contributions to our knowledge of polarising prisms will be found in the report of a lecture delivered by him to the Optical Convention of 1905.

long sides of which are 1·8 times the sides of the square ends, is first cut out of a rhomb of calcite, so that the crystallographic axis is perpendicular to one pair of the long faces, the end-faces and the other pair of long faces thus being principal sections. The block is then divided into three wedges as shown in Fig. 514, and after polishing the surfaces are cemented together into the original block with Canada balsam. The ordinary ray is reflected in this prism to both sides, and the useful internal field-angle

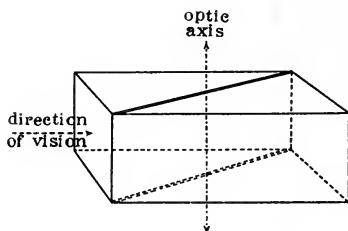


FIG. 513.—Thompson Prism.

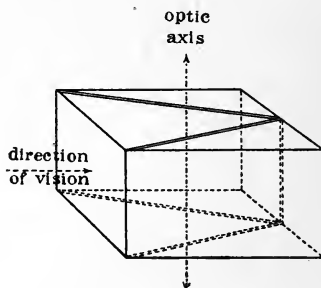


FIG. 514.—Ahrens Prism.

is about 26° , the same as the internal angle of usefulness of Thompson's prism. But it has the advantage that the length for this angle is only half that of a flat-ended Nicol. This prism acts splendidly as a polariser, where alone a large prism is imperative, but not so well as an analyser, as the edge of the middle wedge is apt to produce slight distortion of the images of polarised objects. This disadvantage is more or less got rid of, however, if the end be covered with a cemented cover-glass. The external angle of usefulness of this prism amounts to nearly 60° .

Projection in Polarised Light with Nicol or other Calcite Prisms.—

A large polariser is essential for projection purposes, but a very large angle of usefulness is not indispensable, as the rays from the lantern are preferably parallelised before passing through the polariser, by adjustment of the distance of the condenser of the lantern from the electric arc or other source of light, preferably aided, in order not to have to place the condenser very near to the hot arc, by a convexo-concave lens mounted next to the entrance end of the polariser. The concave lens of the author's polarising Nicol is seen in Fig. 519, page 613, lying against the base; it is mounted in a cap which screws into the receiving end of the prism mount, that having the bevelled divided circle. The inner annulus which holds the lens in the cap may be readily unscrewed and removed, having a milled edge, in the event of the replacement of the lens by a glass plate being desirable. When convergent light is required, the convergence of the beam is always effected subsequently by a converging system of lenses. What is essential in a polariser, however, is a prism large enough to transmit the whole of the parallelised beam, the diameter of which is about two inches, the usual size of objects for the lantern polariscope for parallel light being about $1\frac{5}{8}$ inches in diameter. A large ordinary Nicol prism is the most efficient of all polarisers in the author's opinion, but if such be not available, one of the other forms described may be employed instead. As analyser, a prism affording a wide angular field is necessary, as the rays leave the projecting lens

with considerable convergence; but it is not essential to have a large analyser, as the convergent cone forms its apex about where the prism is introduced. The author finds a large Nicol, however, the companion to the polariser, also the most efficient analyser, when the matter is considered from every point of view, including projections in convergent light and the projection of experiments of all kinds in polarised light. But for many purposes a much smaller prism of wide angle, such as almost any of the forms with flat ends described in the preceding pages, will serve equally well. If, however, the large Nicol analyser is

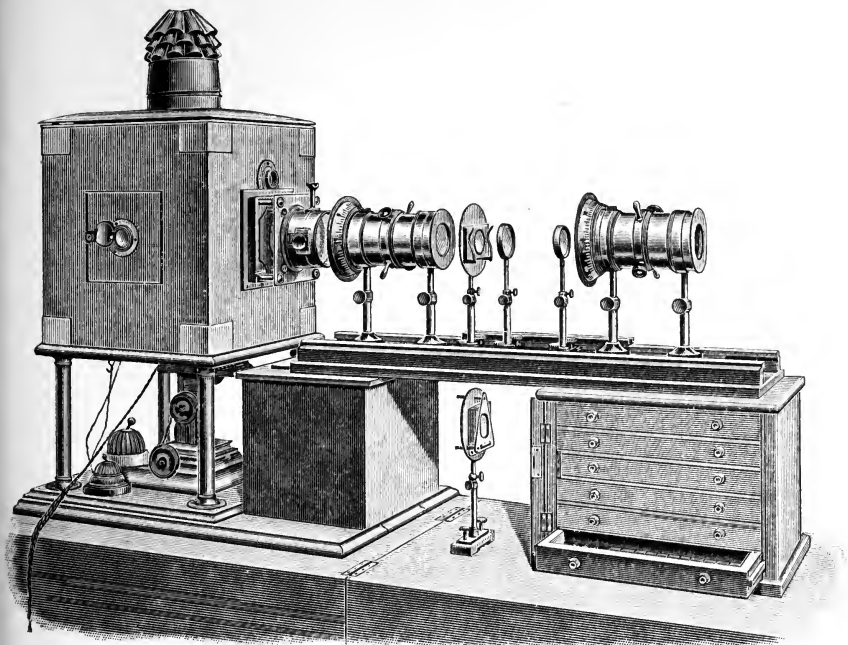


FIG. 515.—Projection Polariscope arranged for Parallel Light.

available, there is no object in replacing it by any of these, and none can surpass it for brilliancy of field.

The author's projection polariscope, as arranged for projections of ordinary objects in parallel light, is shown in Fig. 515.

The lantern, water-cell, and optical bench are as already described and shown in Fig. 497, page 594, but both the large Nicols are now employed, the polariser having its convexo-concave parallelising lens at the end nearest to the water-cell. After the polariser comes the rotating object stage, with aperture of $1\frac{7}{8}$ ths inch diameter, and spring-clips to take the mahogany frames of the standard size ($2\frac{1}{4}$ by 4 inches with clear aperture of $1\frac{5}{8}$ ths inch diameter and rabbet of 2 inches diameter), in which the quartzes, selenites, micas, and crystal objects for the projection polariscope are usually mounted. The objects are cemented with balsam on a glass plate, or between a pair of glass plates, of $1\frac{7}{8}$ ths inch diameter, which just fit in the rabbet, and are secured by

a spring-annulus of brass wire. An alternative stage, not rotating, but having a spring front of such latitude of spring as enables as many as three objects to be held at once, is shown in the recess under the optical bench. It is very useful when besides a polarising crystal it is desired to use a quarter-wave mica plate in front of, and a second one behind the crystal. After the stage come two plano-convex projecting lenses, which form a very effective practically achromatic combination, the first either of 6 inches or 5 inches focus, the latter generally preferable, and the second of $8\frac{1}{2}$ inches focus. The stage and the lenses are each separately mounted on a standard adjustable for height, rotation, and transverse motion, the latter by means of a sliding dove-tailed base moving in a similarly shaped brass guiding bed, mounted in turn on the mahogany longitudinally narrow but transversely long slider in the optical bench. Lastly comes the analysing Nicol. For convergent light work a special set of lenses is used, described in Chapter XLVIII. and shown in Fig. 649, page 794.

The Double-image Prisms of Rochon and Wollaston.—Besides the polarising prisms of calcite described in the foregoing pages, there are two other extremely useful prisms constructed of this important mineral, known as "double-image" prisms, their purpose being to separate the ordinary and extraordinary rays more than can be done with a simple rhomb in the manner described in the first page of the last

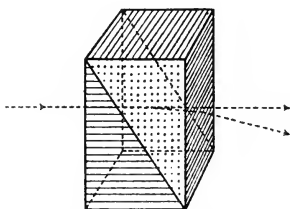


FIG. 516.—Rochon Prism.

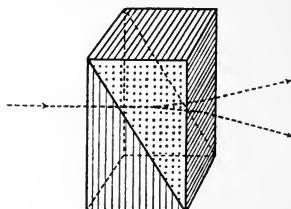


FIG. 517.—Wollaston Prism.

chapter. It was there also shown that if we use a 60° -prism (two truly plane and polished faces inclined at an angle of about 60° being the essentials of such a prism) the two rays are more separated and the divergence increases as the screen is approached on which the two spectra are thrown. If we make the prism angle not so great as 60° , somewhere between 30° and 50° , according to the amount of separation desired, we can neutralise the chromatic dispersion of each image more or less by means of a similar wedge of glass, cemented to the calcite prism by balsam so as to make the whole into a rectangular block. The two images of the aperture in front of the lantern condenser then appear on the screen practically uncoloured. Better achromatism, however, is effected if both wedges are made of calcite, and there are two methods of cutting the second achromatising prism, one of which, that due to Wollaston, gives twice as much separation of the images as the other, that of Rochon, but with less perfect freedom from colour. In both these prisms the essential wedge-prism (the right-hand one in both Figs. 516 and 517) is cut with its refracting edge parallel to the axis of the crystal, the condition for maximum separation. These two types of double-image prisms are diagrammatically shown in Figs. 516

(Rochon) and 517 (Wollaston), the shading indicating the direction of the optic axis.

Rochon's Prism.—In this prism the achromatising wedge has the optic axis perpendicular to the refracting edge, and parallel to the direction of transmission of the light, as shown by the shading in Fig. 516. The rays pass through this wedge, therefore, unaffected, the incidence being normal and the ray passing along the axis of no double refraction. The vibrations thus occur perpendicularly to the axis, and equally in all directions of the circular section around it. In the second wedge double refraction occurs, but the ordinary ray goes through undeflected, while the extraordinary ray is deviated. When the prism is rotated about the direction of the incident ray as axis, the image on the screen due to the extraordinary ray rotates about that due to the ordinary ray, which remains fixed. A Rochon prism used in an actual experiment described on the next page is shown in Fig. 518.

Wollaston's Prism.—The only difference from Rochon's prism is that in Wollaston's the first wedge has the spar rotated a right angle so that the light is incident on the other rectangular face of the wedge, bringing the optic axis perpendicular to the direction of the incident light as well as to the refracting edge. The incident ray is thus divided into two rays immediately in this first wedge, one vibrating parallel to the axis and the other at right angles thereto, the former being an extraordinary and the latter an ordinary ray. The effect is to cause both rays to issue deviated from the second wedge, and thus to afford double the separation of the images afforded by Rochon's prism, as will be clear from Fig. 517. For, the principal planes of the two wedges being at right angles, the ordinary ray received undeflected from the first wedge traverses the second wedge with the velocity and refraction of the extraordinary one, its vibrations being now parallel to the axis, causing that ray (the ordinary) to be as much refracted as the extraordinary one proper, thus doubling the total refraction. The two rays are equally deviated on opposite sides of the normal, and when the prism is rotated about the direction of the normally incident light both images revolve round the centre of the line joining their own centres.

Many uses will be found for double-image prisms during the course of optical work in crystallography, but one of the most instructive and striking experiments may be referred to here, and is illustrated in Fig. 518.

Two images will obviously be thrown on the screen, when either a Rochon or a Wollaston prism is interposed in the beam of light leaving the lens, which focusses on the screen an image of a circular aperture in an opaque diaphragm placed in front of the lantern condenser. These images are composed of rectangularly plane-polarised light. That is, the planes of polarisation of the light composing them are at right angles. Hence, when a Nicol prism is placed between the lens and the double-image prism, as shown in Fig. 518, or indeed anywhere in the path of the beam of light before the deviation has become too great for both rays to get through the Nicol, one image will be found to extinguish at one position of the Nicol in its revolution, and the other at 90° from that position, while midway at the 45° positions both images will be transmitted with partial intensity. Next, when a polarising crystal, such as an evenly polarising film of gypsum (selenite), is introduced between the double-image prism and the Nicol, preferably close up to the former, the two images will be coloured complementarily, with the brilliant colours for which selenite is famous in polarised light,

when the film has been rotated to the position for maximum brilliancy. That the colours are truly complementary, moreover, is absolutely proved at the same time, by the fact that when the two images are made to partially overlap—which is only a matter of choosing an aperture of the convenient size from among the four afforded by the diaphragm—the overlapping portion is quite colourless, being illuminated by brilliant white light. Thus, for instance, a so-called “blue-and-yellow selenite” is a film of such thickness as to afford for a certain position

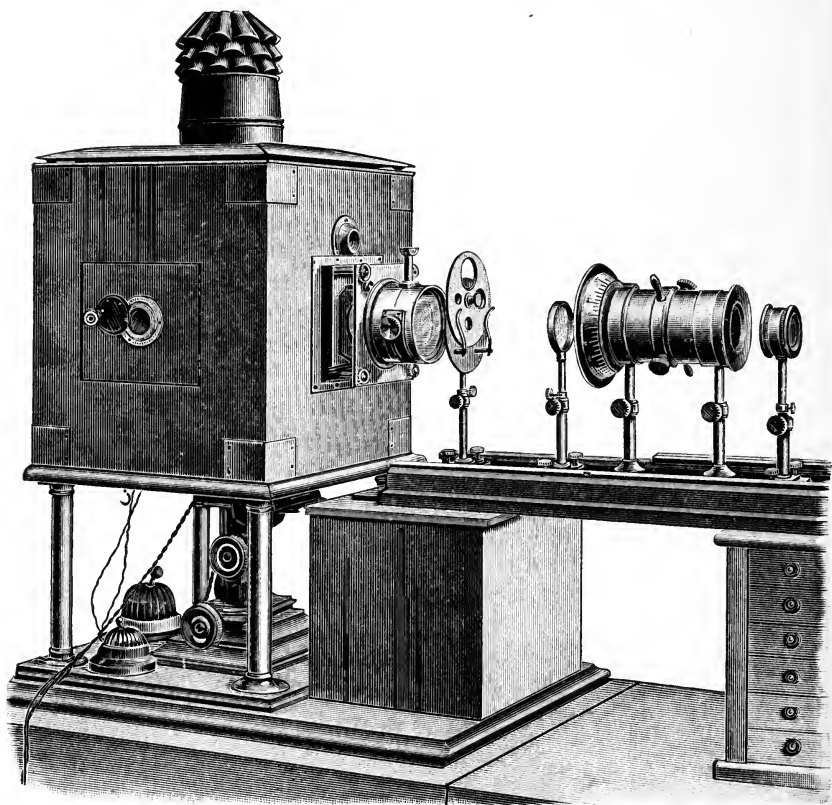


FIG. 518.—Experiment with Double-image Prism, Nicol, and Selenite Film, proving that Complementary Colours make White Light on Blending.

of the Nicol a brilliant blue in one disc and a bright yellow in the other, while the overlapping part is colourless. Similarly, a “red-and-green selenite” produces a red and a green image, with a white lenticular part where the two circular discs overlap. The discussion of the necessary thickness of gypsum, and of the reasons for these phenomena, will be entered into fully in Chapter XLI.

As regards the practical disposition of apparatus for this experiment, that shown in Fig. 518 is particularly convenient. The beam from the lantern condenser passes

first through a thick water-cell carried on the lantern front, in order to remove the heat rays and thus protect the balsam cementing films of the Nicol and double-image prisms from their softening action. Next comes a large rotatable disc-stage with two-inch central aperture, serving either as object stage or, as in this experiment, as signal aperture; as an aperture two inches in diameter is too large for this experiment, and it is desirable to have a ready means of varying the size of the opening, a rotating diaphragm pierced with four apertures of $1\frac{1}{8}$ inch, 1 inch, $\frac{1}{2}$ inch and $\frac{1}{4}$ inch diameter respectively, is attached to it in front, with the chosen aperture, generally one of the smaller ones for ordinary screen distances (the half-inch being especially suitable), concentrically in front of the fixed large aperture of two inches. Next comes the projecting lens, then the large Nicol, and lastly a large Rochon prism, one of the pair shown mounted together on the central stand in Fig. 519. The diaphragm-stage, lens, Nicol, and double-image prism are each adjustable for height, azimuth, and transverse

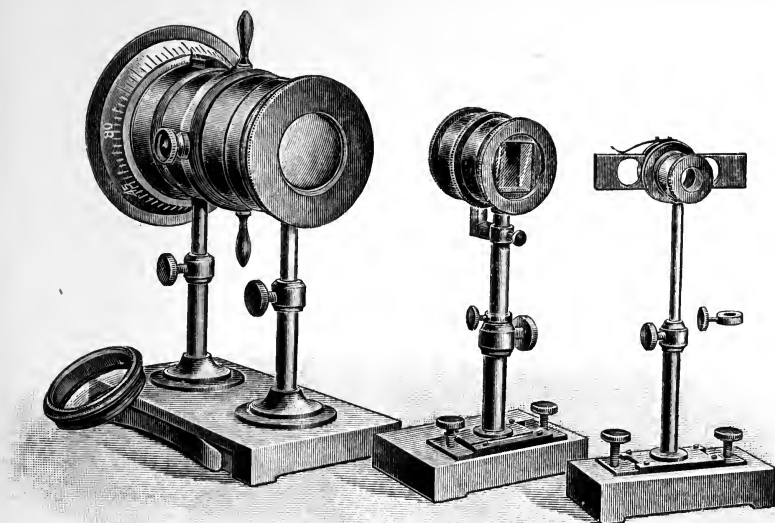


FIG. 519.—Nicol Prism with Concave Parallelising Lens (left), Pair of Rochon Prisms mounted for Huyghens' Experiment (centre), and Wollaston Prism with two Selenites (right).

rectangular position on their separate pedestal stands, which are themselves capable of sliding anywhere along a grooved basal guiding bed, which affords the other rectangular movement, the longitudinal one, necessary for absolutely general adjustment. The mounting of the large Nicol on its stand will be clear from Fig. 519, where it is shown on the left; the transverse adjustment of the two supporting columns is afforded by slots in the base, which are covered by the two broad circular footplates, the columns being rigidly fixed below, after adjustment, by nuts. The selenite, a thin film of gypsum mounted with balsam between two glass plates, is held in a rotating mount with outer milled flange, resembling that of the double-image prism itself except that the tube is much shorter; it fits, like that of the prism, in the same outer tube carried rigidly by the standard, both double-image prism and selenite being thus separately rotatable in this short outer tube. They can also be interchanged as regards their position, if desired, but for this experiment the selenite is fitted in the outer tube at the end nearer to the Nicol; the longer inner-tube mount of the Rochon prism has a similar flange for convenience in rotating the prism.

If desired, the Rochon prism may be replaced by a Wollaston prism in this experiment, when double the amount of separation of the two images on the screen is afforded under similar conditions, and the same size of aperture may therefore be used for a shorter working distance from the screen. Such a Wollaston prism, mounted as just described on an adjustable standard, is shown to the right in Fig. 519. The transversely adjustable base of this standard as shown in the figure, and the standard itself, are typical of a dozen similar ones carrying the lenses and other accessories of the author's crystallographic projection apparatus. The selenite is carried differently in this case, a slide pierced by three circular apertures sliding in a pair of slots in a separately rotatable mount at one end of the outer tube, a spring regulating the tightness of the sliding; the Wollaston prism is rotated by means of a milled flange at the other end. One aperture of the slide is filled with a red-and-green selenite, another with a blue-and-yellow selenite, and the third is vacant, for use when it is not desired to interpose a crystal film.

Huyghens' Experiment with two Double-image Prisms.—A further most instructive experiment requiring a similar pair of double-image prisms affording equal separation of the images, preferably of the Rochon type, is that first performed by Huyghens with a pair of simple rhombs of calcite. It is convenient that the two Rochon prisms should be so mounted that either can rotate separately or be removed altogether, and that a film of gypsum may be introduced between them and also independently rotated. Two such equal Rochon calcite prisms of $1\frac{1}{2}$ inch clear aperture, constructed for the author by Mr. Ahrens, are shown on the central stand in Fig. 519. One of these prisms and the selenite attachment have already been just described, and the second Rochon prism is similarly mounted in a flanged inner tube, rotating by means of the milled flange within an outer tube carried by an adjustable elbow-bracket sliding freely about the standard, and capable of fixation at the exact height which brings the two prisms into line, by means of a fixing screw. This second prism is attached on the selenite side, so that the latter comes between the two Rochon prisms, not quite touching the outer tube of the second one.

A very convenient mode of performing the experiment—which requires no Nicol prism or other accessory than the projecting lens, an aperture-diaphragm, and the water-cell to protect the balsam of the prisms from the heat rays by filtering them off—is shown in Fig. 520, provided the optical front of the lantern is one arranged to permit of the performance of lantern experiments, being unencumbered with any restrictive tubular support for the projecting lens. The form of lantern front adopted by the author, after experimenting with many varieties, is shown in the figure. To the front of the brass plate of the usual lantern-slide carrier-frame there is screwed a very short tube $4\frac{1}{2}$ inches in diameter for the reception of the water-cell, which may, however, be supported independently if desired, as shown later in Fig. 649, and the short tube altogether removed. From the top of this short tube there rises a little upright, terminating above in a stirrup-rest for a rigid tube screwing into a socket on the lantern front above the slide frame; the outer end of this tube carries a tightening collar, to fix in any convenient position an inner tube which slides within it. This adjustable inner tube carries at its outer termination the rectangularly depending arm which supports the projecting objective. The latter consists of an achromatic doublet, composed of two pairs of lenses, of $2\frac{1}{4}$ inches aperture, the two combinations together forming an 8- or a 10-inch objective (it is convenient to possess a pair of objectives of these two focal

lengths, to suit different screen distances); one combination is mounted at each end of the objective tube, which slides within an outer tube carrying a rack, this outer tube in turn sliding with the aid of a driving pinion in the outermost tube carried by the bracket-arm. A large convex metallic screen is also carried concentrically by the outermost tube, in order to shield off any stray light rays. The two double-image prisms on their stand are brought into the clear space between the water-cell and the objective, where there is ample room for proper adjustment. For aperture it is convenient to use the ordinary lantern-slide carrier, with a thick blackened card or brass plate pierced

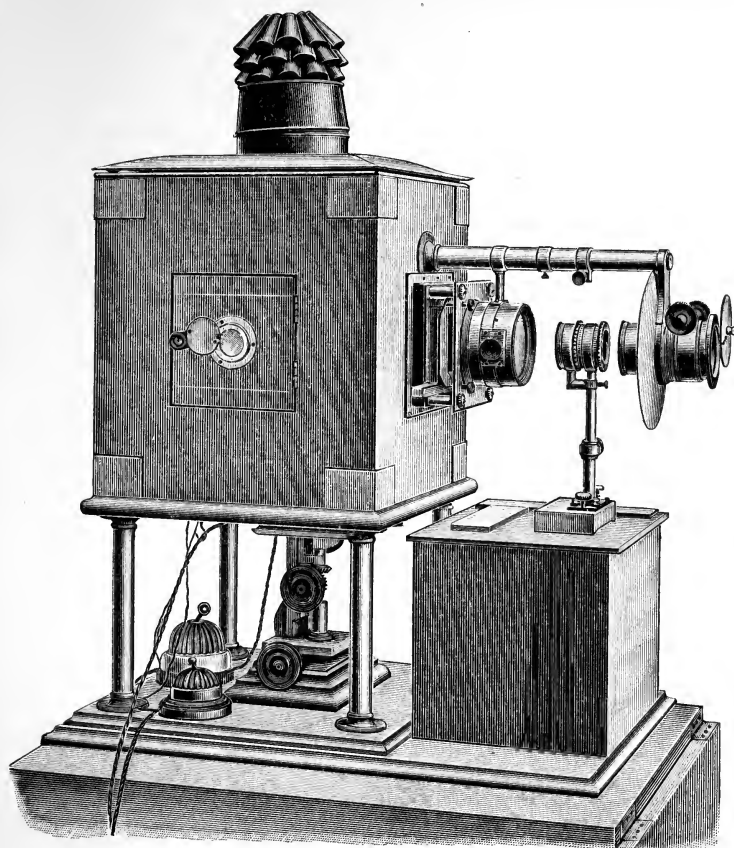


FIG. 520.—Huygens' Experiment as performed with a Pair of Rochon Prisms.

by a cleanly-cut circular central aperture of about half an inch diameter. It is well to have two or three such cards or plates with apertures of larger and smaller diameters, in order that the most convenient size of image may be obtained on the screen.

In performing the experiment, the second double-image prism is first removed, by sliding down the arm carrying it, and the selenite is also removed from the outer tube of the other, thus leaving the first prism alone in operation.

On actuating the lantern two images of the circular aperture are thrown on the screen (the single direct image of the aperture having first been focussed by the objective in the absence of the prism), and when the prism is rotated the image due to

the extraordinary ray will be observed to rotate around that due to the ordinary one. On adding the second prism and rotating it until it is similarly orientated to the first, the two images are observed to be still separated in the same direction—vertical when both prisms are arranged with the optic axis vertical, as indicated by a mark provided on the mount of each—but at double the distance, the two Rochon prisms being of similar angle and equal in every respect. On the least rotation of either prism from this position two other images make their appearance, making four altogether, the two new ones at first very weak, and appearing to start identically from the centre between the other two, and separating more and more so as to form with the others the corners of a rhombus of ever-widening angle, until when the prism has been rotated for 45° all four images are of equal intensity and the angle of the rhombus is also 45° . Rotating further, the first two images begin to disappear, until at 90° they have vanished and there are only two images again of equal intensity, but they are the second pair and arranged along a 45° diagonal. Rotating further towards 135° there are again four images, at 135° arranged as another rhombus of 45° angle; while at 180° there is but a single image, of double the intensity of any brightest pair. In the next half revolution similar phenomena occur in the reverse order. The whole of this

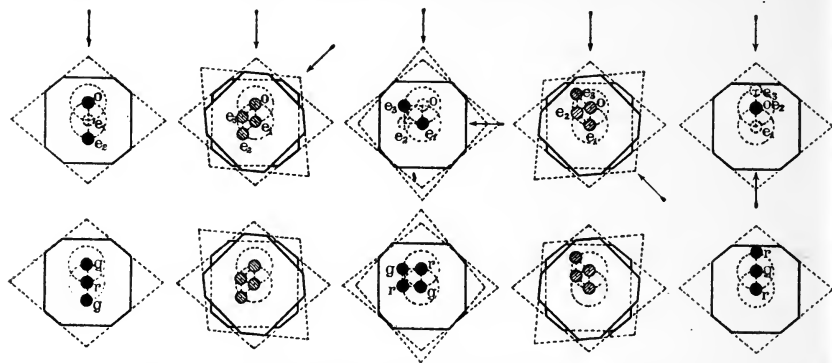


FIG. 521.—Images of a Black Spot afforded by a Pair of Double-image Prisms, one of which is rotated.

interesting behaviour is entirely explained on the assumption that each prism transmits two differently refracted rays, an ordinary and an extraordinary, of which the vibrations occur in two different planes (parallel to the longer and shorter diagonals of the rhomb respectively) at right angles to each other and to the direction of transmission.

When the selenite film is also in position, there are always four images except at 0° and 180° , when all four are in the same vertical line and two of them coincide at the centre. But, except at the 45° positions, when no colour is shown and the appearance is the same as in the absence of the selenite, the images are now coloured, two red or blue and the other two the complementary green or yellow. The use of the selenite is thus to render visible the images, coloured, which are extinguished in the absence of a crystal film. The whole phenomena will be rendered clear by the diagrams given in Fig. 521.

The actual effect drawn is that which was observed when a black circular blot on a sheet of otherwise clean white blotting-paper was viewed directly through the Huyghens' apparatus. A white spot on a black ground answers equally well. The arrows indicate the direction of the optic axis, and the dotted rhombohedral outline indicates the original rhomb from which each prism was cut. The upper row of five figures represents the phenomena without selenite, and the lower row what occurs when a red-green selenite is introduced, the colours being the complementary, when the white

spot on a black ground is used, to those observed when the spot is black on a white ground. The first (top left) figure represents the two images of the spot seen when the prisms are parallel, o being the immovable ordinary image and e_2 the extraordinary. The intermediate dotted ring e_1 indicates the position of the extraordinary image when only one prism is used, the separation then being only half as much. When the selenite is introduced all three images appear, o and e_2 green and e_1 red, as shown in the left lower figure. The second upper figure exhibits the appearance when the front prism has been rotated 45° . There are four images of equal intensity, but only half as strong as the two in the first figure; e_2 has become deflected to correspond to the 45° rotation, e_1 has become visible, and a third extraordinary image e_3 has been formed by resolution out of o , which is thereby diminished in intensity. No change occurs for this position on adding the selenite, no colour being developed. The third upper figure indicates the extinction of o and e_2 , and the brilliant full intensity of e_1 and e_3 , when the front prism has been rotated 90° . When the selenite is added (third lower figure) all four images appear at the corners of a square, o and e_2 red and e_1 and e_3 green. The fourth figures indicate the four half illuminated images for the 135° position of the front prism, and the two last those for the 180° position. Without the selenite there is now only a single image, o and e_2 overlapping, for the front prism now reverses the deflection of the image e_2 by the first prism, and o remains immovable at the centre; e_1 and e_3 are extinguished, but they appear, coloured red, when the selenite is introduced, the overlapping o and e_2 images being both green.

The phenomena may be interpreted very simply as follows.

Two of the four images of the circular aperture or spot occupy the same positions as if only the first prism or thick plate were acting, while the two other images are displaced from these positions along the direction of the principal section of the second prism or plate. If we accept the assumption that the ordinary ray is polarised parallel to the principal section (the vibrations being therefore perpendicular to the principal section), and that the extraordinary ray is polarised perpendicularly to the principal section (its vibrations occurring in that section), then, if we call the ordinary and extraordinary images afforded by the first prism or plate alone O and E , then O becomes O_o and O_e after passing also through the second prism or thick plate, and E becomes E_e and E_o .

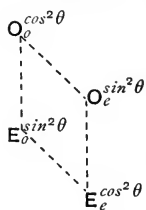
Now the amplitudes of the rays composing the images O_o and O_e stand to each other in the same relationship as the cosine and sine of the angle of rotation θ of one prism or plate with respect to the other, that is, to the angle between the principal sections of the two prisms or plates. Consequently, as the intensity varies as the square of the amplitude of the vibration, the intensities of illumination of the two images vary as $\cos^2 \theta$ and $\sin^2 \theta$. Similarly, the intensities of E_e and E_o also vary as the squares of the cosine and sine of the angle of rotation of one prism or plate with respect to the other. The intensities of O and E are themselves each the half of the intensity of the incident light, neglecting the small loss by reflection and absorption, the latter being almost *nil* in the case of calcite.

Now in the special cases when $\theta = 0^\circ$ or 180° , that is, when the prisms or plates are parallel, only E_e and O_o remain visible; for as $\cos^2 (0^\circ \text{ or } 180^\circ) = 1$ and $\sin^2 (0^\circ \text{ or } 180^\circ) = 0$, the intensities of the images (O_o and E_e) depending on the square of the cosine have their full unit

intensity, while those (O_e and E_o) depending on the square of the sine have no intensity at all, that is, are extinguished. Thus O becomes only ordinarily refracted in the second prism and E only extraordinarily refracted. At 180° the two persisting images O_o and E_e overlap, when the prisms or plates are equally thick, as the extraordinary ray suffers two opposite displacements in the two plates.

When $\theta = 90^\circ$ or 270° , the two principal sections being then perpendicular to each other, only O_e and E_o remain visible, O_o and E_e being extinguished. O in this case becomes only extraordinarily refracted and E only ordinarily refracted in passing through the second prism or plate.

At the 45° diagonal positions of the second plate with respect to the first, as $\cos 45^\circ$ and $\sin 45^\circ$ are each equal to $1/\sqrt{2}$, the intensities of all four images are equal. The positions of the four images at the corners of a 45° rhombus, and their values, may then be graphically expressed by the simple scheme shown in the margin.



CHAPTER XL

THE TRANSMISSION OF LIGHT THROUGH BIAXIAL CRYSTALS.

IN the case of a biaxial crystal, that is, one belonging to the rhombic, monoclinic, or triclinic systems, neither of the two rays into which a beam of light is divided on entering the crystal is an "ordinary" ray, in the strict sense of the term as applied to uniaxial crystals; for neither obeys the law of sines, except when moving along one of the three rectangular axes of the optical ellipsoid. This, however, is always arranged to occur with a 60° -prism prepared, for refractive index determinations, with the aid of the cutting-and-grinding goniometer described in Chapter XLII. For it is quite easy with this apparatus to grind the two surfaces of the prism so that the light is transmitted at minimum deviation along one of the three axes of the ellipsoid, and so that the vibrations of the two rays into which it is decomposed shall occur parallel to the two other rectangular axes. Two such prisms, the refracting edges of which are parallel to two different axes, and the bisecting planes of which are parallel to two different principal planes of the ellipsoid, fulfil these conditions, and together will afford all three refractive indices, and one of them in duplicate. Thus one prism may afford α and β , and the other β and γ , in which case α and γ are furnished once and β in duplicate; or one of the prisms may afford α and γ , instead of either of the two pairs just mentioned, when either α or γ will be obtained in duplicate. Close agreement of the duplicate values will offer an excellent guarantee of the accuracy of the work.

The **wave-surface** of a biaxial crystal is a surface of the fourth degree composed of two shells, one within the other. These two shells touch each other at four points, which are at the bottom of depressions in the outer shell and on the summits of protuberances on the inner one. The depressions or pits are of such a character that a tangent plane to the surface to cover them touches the surface along a circular line of contact. Except for these depressions and protuberances where actual contact occurs the two shells resemble ellipsoids of general triaxial form. Indeed it is common to regard the shells as two separate wave-surfaces. The four points of contact are common points on the two shells where the two "**Optic Axes**" emerge, which cause the crystal to be termed "biaxial," each optic axis passing through the centre and

through two of the points of contact on opposite sides of the centre. In Figs. 522 and 523 an attempt is made to give a correct idea, at first somewhat difficult to grasp, of the real nature of the two shells of

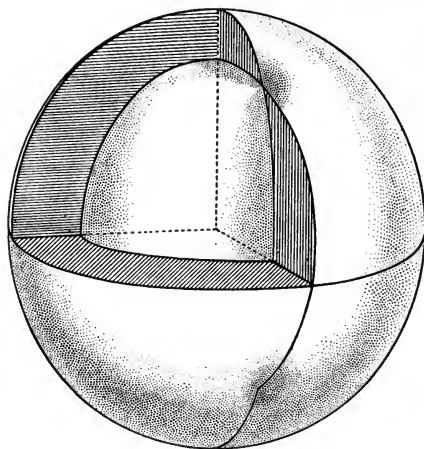


FIG. 522.—Krantz Model of Wave-surface of Biaxial Crystal.

the wave-surface of a biaxial crystal. In Fig. 522 the top-left-front octant of the outer shell has been removed, revealing the inner shell and the nature of the sections by the three rectangular principal planes, the axial planes of the optical ellipsoid. The drawing was made from an excellent model by Krantz, which can be taken to pieces, the three principal section-planes just referred to having been actually cut through. The upper half of another model in plaster is shown in Fig. 523, the two quarters of the outer surface being separated sufficiently on each side of the half of the inner surface to reveal the latter clearly, its protuberances being indicated by the shading. The circular line of contact of the outer surface about the depression with a tangent plane

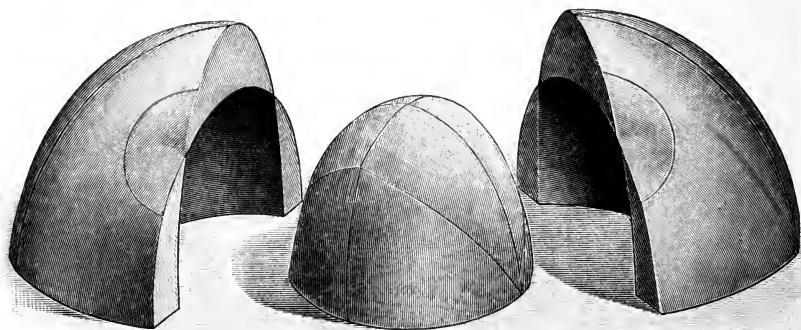


FIG. 523.—Plaster Model of Wave-surface of Biaxial Crystal, showing separated Half of Inner Shell.

is also shown, as well as lines indicating the positions of the chief sections. The three principal section-planes are also diagrammatically represented in Figs. 524, 525, and 526, and their relationship in one octant in Fig. 527. Lastly, in Fig. 528 all three section-planes are shown at once in a manner which will render their relationship clear.

The sections of the wave-surface by each of the three rectangular

planes of the optical ellipsoid consist of an ellipse and a circle, one inside the other in the cases of two of the sections, but one intersecting the other at the four points of contact of the shells in the case of the third section, that by the plane AOC. The lines Op joining these four points p (Figs. 525 and 527) to the centre O are the two optic axes, or,

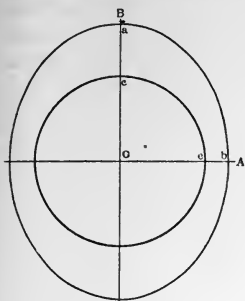


FIG. 524.

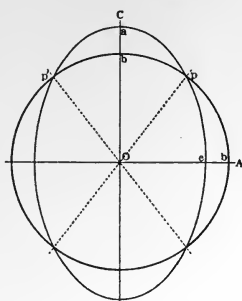


FIG. 525.

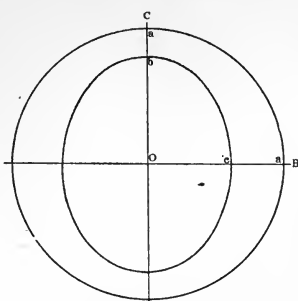


FIG. 526.

Three Principal Section-planes of Wave-surface of Biaxial Crystal.

more precisely, the two “**Secondary Optic Axes.**” The distinction between primary and secondary optic axes will be thoroughly investigated later in the chapter, and it will be sufficient to say here that the distinction is rarely possible to be effected practically, the optic axes observed in the interference figure in convergent polarised light covering in the vast majority of cases both kinds of axes, the angle between the primary and

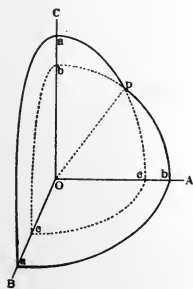


FIG. 527.—Octant of Biaxial Wave-surface showing Principal Sections.

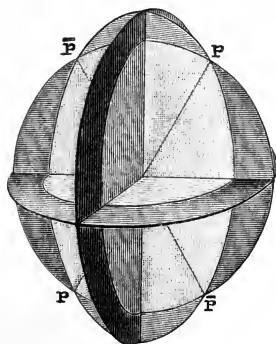


FIG. 528.—Principal Section-planes of Wave-surface of Biaxial Crystal.

the secondary optic axis on each side of the bisecting plane being less than the usual error incident to the determination of the position of the optic axis. It may be useful, however, to turn here for a moment to p. 635, where the discussion occurs and a further diagram, Fig. 536, of the section-plane AOC will be found, in which both kinds of optic axes are elucidated, Omn being the “**Primary Optic Axes**” and Op the secondary, the small angle between them (grossly exaggerated in the figure) being that of the “**Cone of Inner Conical Refraction.**”

It has been necessary in these illustrations to exaggerate the double refraction very grossly in order to render the subject at all clear. To correct the false impression thereby inevitably conveyed, the actual relative dimensions of the two wave-surfaces of a biaxial crystal with exceptionally large double refraction are given in Fig. 529, which represents a section, that containing the maximum and minimum axes of the optical ellipsoid and the optic axes, of the wave-surface of ordinary dextro tartaric acid, $C_4H_6O_6$, which crystallises in the sphenoidal class of the monoclinic system, and is an enantiomorphous substance exhibiting optical activity. Its refractive indices are $a = 1.4961$, $\beta = 1.5359$, $\gamma = 1.6055$, and its double refraction $\gamma - a = 0.1094$. The figure is reproduced from the data obtained by Kohlrausch in an elaborate investigation. The plane of the optic axes, the section-plane, is perpendicular to the symmetry plane $\{010\}$. A typical crystal of tartaric acid has already been illustrated in Fig. 221, page 260. The double refraction

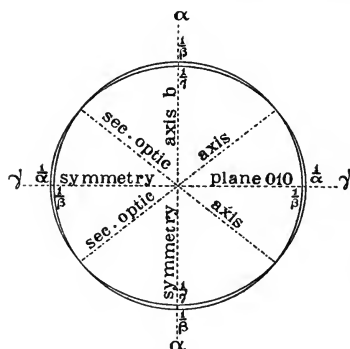


FIG. 529.—Principal Section of Wave-surface of Tartaric Acid.

is positive, which will be shown later to mean that the a and β indices of refraction are closer together than the β and γ indices, and that the direction of vibration affording the maximum index γ is the acute bisectrix of the optic axial angle. This direction lies in the symmetry plane, in the acute angle between the crystallographic axes a and c ; it is represented consequently in the figure by the same horizontal dotted line as stands for the trace of the symmetry plane, and marked γ . The vertical diameter represents the symmetry axis b normal to the symmetry plane; it is the vibration direction which affords the minimum index a . The third rectangular axis of the optical ellipsoid affording the intermediate refractive index β is normal to the plane of the paper. It is easy to see that a beam travelling along the horizontal diameter will be composed of two rays vibrating respectively parallel to the vertical diameter and normally to the paper, their velocities being inversely proportional to a and β , that is to $\frac{1}{a}$ and $\frac{1}{\beta}$. The former consequently gets further from the centre of disturbance than the latter in the same time, and the respective curves are marked with these fractions. Similarly a beam travelling along the vertical diameter is resolved into two rays vibrating respectively horizontally and normally to the paper, with velocities $\frac{1}{\gamma}$ and $\frac{1}{\beta}$. The former is retarded behind the latter in consequence, as shown in the figure. Hence both beams have a component ray vibrating with velocity $\frac{1}{\beta}$, and the section of this

wave-surface is a circle. The other two component rays correspond to the axes $\frac{1}{a}$ and $\frac{1}{\gamma}$ of an ellipse, this being the form of the section of the second wave-surface. The difference in shape from a circle is very slight, but real. The broken-and-dotted lines joining the centre to the points of intersection of the circle and the ellipse are the secondary optic axes, differing by a minute angle from the primary optic axes, as above indicated, both primary and secondary optic axes, however, being here as usual experimentally indistinguishable, coinciding in the optic axes as indicated by the interference figure in convergent polarised light. It will be shown later, however, in this chapter that by special devices the distinction can be experimentally verified and observed.

We saw in Chapter XXXVIII. that in a uniaxial crystal there is one direction, the unique "optic axis," of either maximum (in the negative kind such as calcite) or minimum (in the positive kind such as quartz) velocity of vibration, corresponding to the conversely minimum or maximum refractive index respectively; and that perpendicular to this unique axis there are, lying in the equatorial circular section of the optical ellipsoid, an infinite number of directions of the opposite extreme of vibration-velocity or refractive index.

In a biaxial crystal, on the contrary, we have just seen that the two optic axes do not correspond to either the maximum or the minimum axial directions of the optical ellipsoid or of refractive index, but to a pair of intermediate directions, approximately joining the points of contact of the two shells of the wave-surface to the centre as above stated (these directions being the secondary optic axes), and lying in the principal elliptic section of the optical ellipsoid (as distinct from the wave-surface) the major and minor axes of which are the maximum and minimum axes of that ellipsoid and the directions of extreme refractive indices; the two "optic" axes (more precisely, the primary optic axes) are, in fact, the perpendiculars to the diameters marking the two positions where the radius vector of the principal ellipse is equal to the intermediate axis, the normal to that ellipse, as will be more clearly elucidated later in this chapter.

It is necessary to be quite clear as to the nature of the various ellipsoids that have been referred to. They are (1) the wave-surface, a true ellipsoid in the case of the extraordinary ray of a uniaxial crystal, but modified by the depressions in a biaxial crystal, (2) the vibration-velocity ellipsoid (that of Fresnel), and (3) the indicatrix (of Fletcher). In practical crystallography we do not need to concern ourselves more with the first, the wave-surface, than is adequate to ensure our understanding its nature. It is the "velocity-of-propagation ellipsoid," representing by its axes and its *radii vectores* the propagation-velocity. There are in general two wave-surface ellipsoids, corresponding to the two rays moving with different velocities into which a beam of light is decomposed on entering a doubly refracting crystal. In the case of a uniaxial crystal they are a sphere and an ellipsoid of revolution, but in the case of a biaxial crystal the two shells together form the surface of the fourth

degree already referred to. The second and third ellipsoids, however, are alternative optical ellipsoids, the one the polar reciprocal of the other, which afford us at once a graphical expression of the experimental optical measurements, and are, therefore, of great practical importance. The third, the Fletcher indicatrix, is the most easy to grasp, as it is the direct graphical expression of the value of the refractive index for the same wave-length in the different directions within the crystal. It is now becoming more and more universally employed. The second, the Fresnel vibration-velocity ellipsoid, being the polar reciprocal of the indicatrix, is graphically expressed by plotting the reciprocals of the refractive indices. It represents what, before the advent of the electromagnetic theory of light, was formerly known as the "optical elasticity" or facility of light-vibration in the crystal, perpendicular to the direction of propagation. A ray going along one axis has the velocities of its two component rays represented by the two axes perpendicular to it, for these axes are laid down proportionally to the reciprocals of the refractive indices.

The two Varieties of the Optical Ellipsoid, the Fresnel Vibration-Velocity Ellipsoid and the Indicatrix.—Fresnel was the first to show in the year 1827 that the velocity with which the vibrations of light occur, on the passage of a ray of light through a crystal, could be expressed in general by an ellipsoid, which became, therefore, variously known as the "Fresnel ellipsoid," the "optical vibration-velocity ellipsoid," and formerly as the "optical elasticity ellipsoid." But the idea of optical elasticity became obsolete with the acceptance of the electromagnetic theory of light. The Fresnel ellipsoid will, therefore, be referred to as the vibration-velocity ellipsoid. The relative dimensions of its three rectangular axes are inversely proportional to the refractive indices along those three axial directions, so that the maximum axis, distinguished by the Old English letter α , of the ellipsoid corresponds in direction to that for the minimum refractive index a , and the minimum axis ϵ to that for the maximum index γ , the intermediate axis b corresponding to the direction to which the intermediate refractive index β applies. Also the actual values of these three axes of the Fresnel optical ellipsoid are the reciprocals of the refractive indices, that is: $\alpha = \frac{1}{a}$, $b = \frac{1}{\beta}$, $\epsilon = \frac{1}{\gamma}$. Hence, if we have determined the three refractive indices a , β , and γ , for the three rectangular directions in the crystal which are parallel to the axes of the ellipsoid, and if we express their reciprocals in the manner which has become conventional, so that the intermediate axis = 1, by dividing throughout by b , we have a relative measure of the three axial dimensions of the Fresnel optical ellipsoid in the ratios:

$$(1) \alpha : b : \epsilon, \text{ or } \frac{\alpha}{b} : \frac{b}{b} : \frac{\epsilon}{b} = \alpha' : 1 : \epsilon',$$

$$\text{where } \alpha' = \frac{\alpha}{b} = \frac{\frac{1}{a}}{\frac{1}{\beta}} = \frac{\beta}{a}, \text{ and } \epsilon' = \frac{\epsilon}{b} = \frac{\frac{1}{\gamma}}{\frac{1}{\beta}} = \frac{\beta}{\gamma}.$$

Now an ellipsoid which has axes the dimensions of which are relatively expressed by the refractive indices α, β, γ themselves has been shown by Fletcher¹ to possess certain properties which render it more convenient for use as an expression of the optical properties of crystals than the Fresnel ellipsoid, and has been termed by Fletcher the "optical indicatrix." It is identical with the "indexellipsoid" of MacCullagh. The axes of this ellipsoid may either be directly expressed by the refractive indices themselves, or, in accordance with a similar convention to that referred to in the case of the Fresnel ellipsoid, designed in the cause of simplicity, the intermediate axis β may be taken as unity and α and γ expressed proportionally as follows, by dividing out by β :—

$$(2) \alpha : \beta : \gamma, \text{ or } \frac{\alpha}{\beta} : \frac{\beta}{\beta} : \frac{\gamma}{\beta} = \alpha' : 1 : \gamma',$$

where the axial value $\alpha' = \frac{\text{index } \alpha}{\text{index } \beta}$, and the axial value $\gamma' = \frac{\text{index } \gamma}{\text{index } \beta}$.

It is customary to drop the "dashes" in these abbreviated expressions, and to state the axial ratios of the two optical ellipsoids as $\alpha : 1 : \gamma$ and $\alpha' : 1 : \gamma'$ respectively. By the "axial ratios of the optical ellipsoid," the experimental determination of which is an essential part of the optical investigation of a crystal, and which appear as important optical constants in the tabular presentation of the results of the investigation, we may thus mean either those of the Fresnel ellipsoid of optical vibration-velocity, as given in the final form of the expression (1), or those of the indicatrix, as given in the final form of the expression (2). The latter are rapidly supplanting the former in crystallographic literature, but it is perhaps advisable for the present to express the experimental results in terms of both ellipsoids, or if not, to state clearly to which ellipsoid the constants given refer. The same actual experimental work suffices for both, namely, the determination of the refractive indices for the three rectangular directions of the axes, for these directions are obviously identical for both ellipsoids. The directions of vibration of the rays affording each refractive index are ascertained by means of a Nicol prism, placed in front of the eyepiece of the spectrometer, when observing the two images of the signal-slit afforded at minimum deviation by a 60°-prism, the vibration directions of which are parallel to two of the principal axes of the ellipsoid, just as was described on page 598 with reference to the determination of the refractive indices of uniaxial crystals. Full details will be given in Chapter XLV. of the mode of carrying out the determinations.

An example may be useful to render the mode of calculation of both sets of ratios quite clear. The case of monoclinic ammonium zinc sulphate, $(\text{NH}_4)_2\text{Zn}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$, will serve, the three refractive indices of which for sodium light are: $\alpha = 1.4888$, $\beta = 1.4930$, $\gamma = 1.4994$.

The axial ratios of the Fresnel optical-velocity ellipsoid are :

¹ *Mineralogical Magazine*, 1891, 9, 278.

$$\frac{a}{b} = \frac{\beta}{\alpha} = \frac{1.4930}{1.4888} = 1.0028; \quad \frac{c}{b} = \frac{\beta}{\gamma} = \frac{1.4930}{1.4994} = 0.9957.$$

That is: $-a:b:c = 1.0028:1:0.9957$.

The axial ratios of the optical indicatrix are:

$$\frac{\alpha}{\beta} = \frac{1.4888}{1.4930} = 0.9972; \quad \frac{\gamma}{\beta} = \frac{1.4994}{1.4930} = 1.0043.$$

That is: $-a:\beta:\gamma = 0.9972:1:1.0043$.

It will be observed that 1.0028 is just as much above unity as 0.9972 is below it, and that 0.9957 is as much below as 1.0043 is above unity. This reciprocal relationship between the axial values of the two ellipsoids always holds, and affords a check on the accuracy of the calculations.

The Fresnel Ellipsoid.—The velocity of light varies in a crystal not only with the direction of propagation but with the direction of vibration perpendicular thereto. Thus, let Fig. 530 represent the Fresnel ellipsoid, and O its centre, from which latter an infinite number of radii representing the vibration velocities may be supposed to proceed to the surface of the ellipsoid, their termini, in fact, tracing out the ellipsoidal surface; also let OA, OB, OC be the three rectangular semi-axes, three special cases of such radii, arranged in descending order of vibration velocity. Then the length of the maximum axis OA or a does not represent the velocity of light transmission along the direction OA, but the

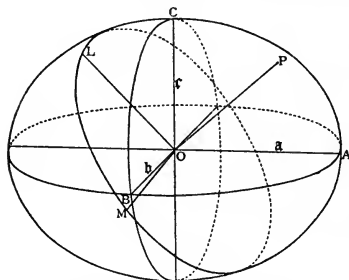


FIG. 530.—The Fresnel Vibration-velocity Ellipsoid.

velocity of the rays which pass through the crystal with the greatest possible velocity and the vibrations of which occur along OA, that is, the velocity of the infinite number of rays propagated and travelling in the plane BOC at right angles to OA. It thus represents all rays polarised in the plane BOC, the direction of vibration being accepted as perpendicular to the plane of polarisation. Similarly, the minimum axis OC or c represents by its length the minimum velocity, that of rays vibrating along OC but transmitted in any direction perpendicular thereto, that is, along any radius in the plane AOB. Likewise, the axis OB or b of intermediate length represents the velocity of light rays transmitted perpendicularly to OB but vibrating along OB, that is, the velocity of rays transmitted along any radius lying in the principal section-plane AOC, and polarised in that plane. Hence, the Fresnel ellipsoid represents by its radii from the centre the velocities of rays vibrating along those radii, the direction of transmission being perpendicular thereto. It is therefore referred to as the vibration-velocity ellipsoid.

Along any one of these principal axes OA, OB, OC it is possible for two rays to travel simultaneously, polarised perpendicularly to each other in the two principal section-planes which intersect in the principal axis along which the rays are travelling, and the velocities and directions

of vibration of which are represented by the two other principal axes perpendicular to the one along which the rays are passing. Thus, for instance, along OA two rays can be transmitted, polarised respectively in the planes AOB and AOC, and vibrating respectively along OC and OB.

Now as the refractive index μ for rays of light vibrating parallel to one of the three rectangular principal axes OA, OB, OC, is the ratio

of its velocity in air V_a to its velocity in the crystal V_c , that is, $\mu = \frac{V_a}{V_c}$,

the refractive index for each of the rays in question on entering the crystal from the air is inversely proportional to the velocity in the crystal. The greater the facility of light transmission in the crystal the less is the difference from the more rapid transmission in air and the smaller

is the value of $\frac{V_a}{V_c}$; and the less V_c becomes the higher does the

refractive index μ work out, so that μ varies inversely as V_c . The three refractive indices, the minimum, the intermediate, and the maximum, for the rays vibrating respectively along OA, OB, and OC being indicated respectively by α , β , and γ in accordance with the usual convention, then OA or a is thus proportional, as already shown,

to $\frac{1}{\alpha}$, OB or b to $\frac{1}{\beta}$, and OC or c to $\frac{1}{\gamma}$. The two rays which can travel

along OA, vibrating perpendicularly to each other parallel to OB and OC, will afford the two indices β and γ ; similarly, the two rays that can travel along OB and which will vibrate parallel to OA and OC, will afford α and γ respectively; likewise, the two rays capable of passing along OC, will be composed of vibrations occurring along OA and OB respectively, and they will consequently afford the two indices α and β .

The double refraction is measured by the difference of the two extreme indices of refraction, $\gamma - \alpha$.

The Fresnel ellipsoid is thus defined by the meaning of its three rectangular principal axes and the three mutually perpendicular axial planes containing them, and this is all that is required for our practical work in determining the optical constants. But the ellipsoid has further important general properties, which at once express the behaviour of light waves transversing any direction in the crystal.

If OP in Fig. 530 be any direction whatsoever in the crystal and its Fresnel vibration-velocity ellipsoid, and if a section-plane perpendicular to it be taken through the crystal, the section of the ellipsoid by this plane being an ellipse, and if further the major and minor axes of the latter be OL and OM, then OL and OM represent the directions of vibration and the velocities of the two plane-polarised rays which can travel through the crystal along OP. The planes of polarisation of the two rays are at right angles to each other, the major and minor axes of the ellipse being mutually perpendicular. This is the fundamental law of double refraction.

The meaning of double refraction will now be quite clear. For it must be obvious that the two rays which can travel along any general direction such as OP, having two different velocities and therefore two different indices of refraction, must have been differently refracted on entering the crystal, that is, their previous paths in the air must have been different. If, therefore, we imagine these two incident rays, at first passing through the air inclined to each other, to close up their angle of inclination until they become identical as regards their direction through the air, their directions on entering the crystal must diverge again. Hence, a bundle of parallel rays derived from the same source must, when incident upon a crystal surface, become divided into two rays on entering the crystal substance, which in general traverse that substance in two more or less inclined directions according to the difference of their corresponding velocities and refractive indices, and the planes of polarisation and directions of vibration of which are perpendicular to each other. In the special case when the bundle is incident parallel to one of the three rectangular axes of the ellipsoid, the two rays into which it breaks up do not diverge, but both continue along the axis, with different velocities, however, and oppositely (rectangularly) polarised.

The major and minor axes of the elliptic section of the Fresnel ellipsoid by any perfectly general plane are termed "singular directions." In the special cases of the circular section of the ellipsoid of revolution of a uniaxial crystal, and the pair of circular sections of a biaxial crystal, every direction in such circular section is equivalent to a singular direction, there being no double refraction. Now there is usually some difficulty in understanding why a ray entering a doubly refractive crystal breaks up into two others only, and why these two should so conveniently vibrate perpendicularly to each other. The reason is that when a vibratory displacement occurs in any direction in the front of a wave it is only those components which are parallel to the singular directions (axes of elliptic section) which are propagated permanently as waves, and their propagation occurs with different velocities, except when the elliptic section becomes a circular section. This fact, which has been proved both experimentally and as the result of profound mathematical analysis, fully accounts for the bifurcation of the ray into two, and two only, perpendicularly vibrating components on entering the crystal.

It must be remembered also that in air the wave-front is perpendicular to the direction of a beam of light, but that in the crystal in general the wave-front is more or less inclined to the direction of propagation, and that the inclinations of the two wave-fronts of the two bifurcated rays in the crystal, to the respective directions of propagation of those rays, are in general different. Similarly two rays travelling along the same path through a crystal and vibrating perpendicularly to each other, but with wave-fronts differently inclined as a rule, on emergence into air bifurcate as two differently refracted rays but with wave-fronts now perpendicular to each of the two different

directions of propagation. These facts will be made quite clear on reference to Fig. 531.

In all the above cases it must be understood that the ray or rays referred to are homogeneous monochromatic ones of definite wave-length, for both the velocity and the direction in general differ with the wave-length.

It must be borne in mind that the wave-front of a beam of light traversing a cubic crystal or other isotropic substance such as glass is perpendicular to the direction of the beam, as in air, and when the beam is cylindrical the plane section formed by the wave-front is a circle. We have just seen, however, that in a doubly refracting crystal the wave-front is in general inclined to the direction of propagation, and the section of the beam by the wave-front will therefore be an ellipse, which will be of different axial dimensions for the two differently refracted bifurcated rays, owing to the different inclinations of the sections to the directions. In the special case, however, of a beam travelling in a crystal along any one of the three axes OA, OB, OC of the optical ellipsoid its wave-front is actually perpendicular to the direction of propagation.

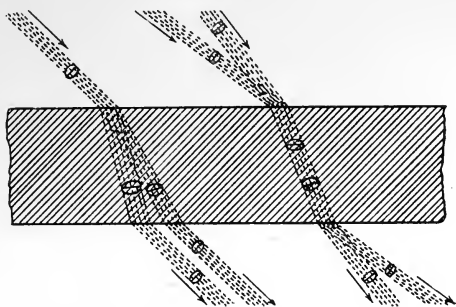


FIG. 531.—Diagram illustrating Different Inclinations of Wave-fronts of two Rays traversing Crystal.

The Fletcher Indicatrix.—The fundamental fact of Fletcher's generalisation¹ is stated by him in the following words:

"The characters of a ray of plane-polarised homogeneous light transmitted within a (crystal) medium are indicated by geometrical characters at a corresponding point on an ellipsoid; the direction of the ray is that of a diameter intersecting perpendicularly the normal drawn to the ellipsoid at the corresponding point; the velocity is inversely proportional to the length of the normal intercepted by the ray; the plane of polarisation is perpendicular to the normal."

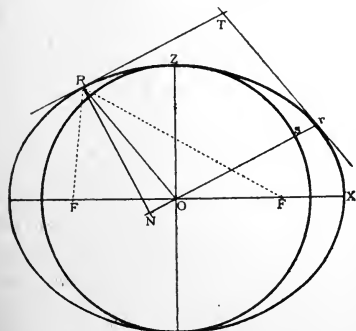


FIG. 532.—Diagram illustrating the Properties of the Indicatrix.

We may illustrate these properties of the indicatrix most clearly by considering first the special case of uniaxial crystals. Let us reproduce in Fig. 532 the wave-surfaces of a negative uniaxial crystal such as calcite, as given in Fig. 501, and draw any radius Or of the elliptic section of the extraordinary wave-surface, also rT the tangent at the

¹ *Mineralogical Magazine*, 1891, 9, 278.

point r , and OR a line parallel to it. Then, the tangent RT at the point R is parallel to Or , and OR and Or are thus conjugate semi-diameters of the ellipse, so that, in accordance with a well-known property of conjugate radii of the ellipse, the parallelogram $ORTr$ is constant and equal to the parallelogram on the semi-axes $OX \cdot OZ$. If now a perpendicular to the ellipse be drawn at R (that is, a line at right angles to the tangent RT), to meet the radius Or produced at N , the parallelogram $Or \cdot RN$ is also equal to $OX \cdot OZ$ (Euclid i. 35). If the value of this rectangle be taken as unity :

$$Or = \frac{OX \cdot OZ}{RN} = \frac{1}{RN}.$$

Hence, the velocity of propagation of any extraordinary ray, expressed by the length Or , is inversely proportional to the length of the straight line which at the point R is perpendicular both to the ellipse (and the spheroid) and to the ray. This straight line RN is the vibration-direction of the ray Or , for the plane of polarisation of the latter is that containing the propagation-direction Or itself and the perpendicular to the section (that is, to the paper).

This is precisely what is implied in the above-quoted statement of Fletcher. For the point R obviously determines both the direction and velocity of propagation, and the directions of vibration and of polarisation of a specific ray corresponding to the point. The point being a general one, that is, capable of being chosen anywhere on the ellipsoid, these geometrical characters at the point define the optical behaviour of any ray travelling within a crystal of uniaxial optical character absolutely. It is only necessary to draw the normal to the surface of the ellipsoid at the point in question, that is, to draw the perpendicular to the tangent at the point (the usual method being to bisect the angle between the lines joining the point and the foci F of the ellipse, the bisecting line being the normal), and then to draw a perpendicular to this to pass through the centre and produce it far enough to meet the ellipse again at r .

Besides RN there will be another normal to the radius vector Or , perpendicular to the plane of the paper at O , and lying in the equatorial circular section, with length equal to OX . It is related to the ordinary ray Os as RN is to the extraordinary Or , for it is the direction of vibration of the ordinary ray just as RN is the direction of vibration of the extraordinary ray, and the velocity of this ordinary ray is inversely proportional to OX .

Thus the relations of the ordinary and extraordinary rays in any direction in a uniaxial crystal can be derived from a single surface, an ellipsoid of revolution, the "indicatrix" or "indexellipsoid," the axis of rotation and the diameter of the equatorial circular section of which behave to each other as do the reciprocals of the velocities of propagation of rays vibrating parallel to and perpendicular to the axis, that is, as the two refractive indices ϵ and ω . For a negative uniaxial crystal, for which

$\omega > \epsilon$, its rotation axis is the smaller, and for a positive one, when $\epsilon > \omega$, the maximum axis is that of revolution.

The above derivation of the indicatrix of a uniaxial crystal is only that of a special case of the general ellipsoid of three axes proportional to the three refractive indices. In this general ellipsoid only the normals to its surface at the six points forming the ends of the three axes pass through the centre. To each of these points there correspond an infinite number of ordinary rays of equal velocity, and the directions of vibration of which lie in the perpendicular principal section-plane. But only three principal sections now occur, instead of the infinite number of such in the uniaxial ellipsoid, namely, those perpendicular to the three principal axes. As the propagation-velocity of the ordinary ray in such a principal section is inversely proportional to the semi-axis perpendicular to it, the length of which is each time different, the rays possess in each principal section a different refractive index. These three indices are the three chief ones α , β , γ , which thus correspond to three kinds of ordinary rays vibrating respectively parallel to the three principal axes.

We can at once construct the "indicatrix" of Fletcher from the three refractive indices α , β , γ determined experimentally for the three axial directions, which latter are common, of course, to the two ellipsoids of Fresnel and Fletcher, and which directions are known either from the morphological symmetry if rhombic, or from experimental determinations by the method of extinctions if the symmetry be monoclinic or triclinic. To construct the Fletcher indicatrix we take the length of the axis OA proportional to the minimum refractive index α , that of OB to the intermediate index β , and that of OC to the maximum index γ , as in Fig. 533.

There are three properties of practical importance possessed by the indicatrix:

(1) The principal axes OL and OM of any section of the ellipsoid—which as in the case of the Fresnel ellipsoid is always an ellipse except in the two special cases of the circular sections perpendicular to the optic axes—are proportional to the refractive indices of the two beams of light the **wave-directions** of which are parallel to the normal to the section OP, and not in general to those of the beams the directions of propagation of which within the crystal are parallel to OP itself. For these wave-directions are the perpendiculars to the wave-fronts, and are only identical with the directions of the rays when the latter both move along one of the three axes OA, OB, OC, that is, when the section is a principal one.

(2) As the principal axes OL and OM of the elliptic section lie in the planes containing the normal OP and the directions of vibration of the two plane-polarised beams, they (the axes) express the refractive

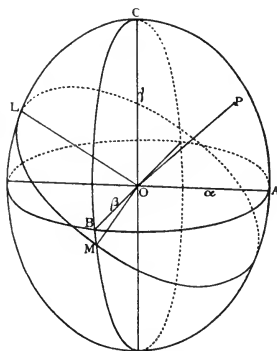


FIG. 533.—The Indicatrix of Fletcher.

indices corresponding to the two beams and also their planes of polarisation.

(3) The difference in length of the two axes OL and OM affords the measure of the double refraction of the section-plate, that is, the birefringence along the direction of the normal OP.

Optic Axes and Median Lines.—The phenomenon of optic axes, so important in practical crystallography, is at once elucidated by the properties of the Fresnel ellipsoid or the indicatrix. It has already been pointed out that the ellipsoid of three unequal rectangular axes, which most generally represents the optical characters of crystals, must of necessity have a pair of circular sections, in certain positions which are dependent on the relations of the lengths of the three axes, but which will in any case be symmetrical to the principal elliptic section, the major and minor axes of which are the maximum and minimum axes of the whole ellipsoid.

For if we examine this principal elliptic section it will be obvious that, between these two extreme radii, in every quadrant of the ellipse there must be a position on the curve where if a radius be drawn to the centre it will be equal to the intermediate axis of the ellipsoid, the normal to the plane of the principal section under consideration. Hence, the section containing these two mutually perpendicular equal radii must be a circle. As the four points in question on the principal ellipse are symmetrically placed with respect to the maximum and minimum axes, they will lie at the ends of two diameters equally inclined to these principal axes, that is, opposite pairs of the four radii are in the same straight line; and the two sections of the ellipsoid, each containing one of these two diameters and the intermediate axis of the ellipsoid perpendicular to them, will be the two circular sections in question which intersect in the intermediate axis.

Now the two normals to these two circular sections of the ellipsoid of general form will clearly be directions of single refraction. For rays transmitted along them will vibrate with equal velocity in all azimuths in the circular section perpendicular to the direction of the rays. These normals to the circular sections are the “**optic axes.**”

This accounts for the fact that crystals of the rhombic, monoclinic, and triclinic systems of symmetry are “**biaxial**”; for these are the systems the optical characters of which are expressed by an ellipsoid of general form.

When two of the axes of the general ellipsoid approach each other in length until they become equal, and the ellipsoid becomes one of revolution, the two circular sections rotate equally about their common diameter, the intermediate axis, until, when the latter becomes equal to the other axis the length of which it is approaching, the angle between the section-planes becomes zero and the two sections coincide, merging into a single circular section, perpendicular to the one principal axis which remains different to the two now equal lying in the circular section. This outstanding axis perpendicular to the unique circular section is then the only one direction in the crystal along which no double refraction

occurs, and the crystal is consequently optically "uniaxial." Thus it is that crystals of the trigonal, tetragonal, and hexagonal systems of symmetry are characterised optically by an ellipsoid of revolution, the axis of which is coincident with the unique trigonal, tetragonal, or hexagonal axis of morphological symmetry, and that they display only a single optic axis, also identical with this same singular direction.

In the case of the optic axis of a uniaxial crystal this direction is truly and absolutely a direction of no double refraction, for rays transmitted along it are free to vibrate in any azimuth perpendicular to it, and do so with absolutely the same velocity in the directions of all the radii of the circular section perpendicular to the axis. Moreover, the wave-front of such a ray is always perpendicular to the direction of propagation of the ray, that is, a tangent to the wave-front is parallel to the circular section; the section of the cylindrical beam of rays itself is also circular and perpendicular to the ray-direction. Further, all rays travelling along the optic axis are ordinary rays, and afford the refractive index ω , corresponding to vibrations in the circular section.

The matter is not so simple as regards the two optic axes of a biaxial crystal, for the two circular sections are not like the uniaxial circular section in being each a principal plane of the ellipsoid; they are merely two ordinary sections of the ellipsoid by two planes symmetrically inclined to one of the principal planes, and intersecting in one of the principal axes of the ellipsoid, the intermediate axis. Hence, the wave-front of a ray travelling along either of the two optic axes of a biaxial crystal, perpendicular to one of these circular sections, will not be perpendicular to the direction of the ray, and the section of the cylindrical beam by the wave-front will have an elliptic and not a circular form. For it has already been shown (page 629) that a perpendicular wave-front affording a circular section of the beam only happens for a biaxial crystal when the ray is propagated along one of the three principal axes. The direction of vibration, however, will, as always, be perpendicular to the direction of the ray, and its plane of polarisation will be perpendicular to the direction of vibration and parallel to the direction of the ray. The inclination of the wave-front to the ray-direction, however, introduces a distinction from a uniaxial optic axis, and although its practical effect is so small that in most cases it is not detected by the ordinary experimental method of locating an optic axis, by means of the interference figure in convergent polarised light, still there are exceptional cases in which it becomes of practical importance, and it is necessary that the real nature of the case should be thoroughly understood. The question must, therefore, be investigated a little more closely.

Let the circular sections of both the indicatrix and the Fresnel ellipsoid be drawn, as in Figs. 534 and 535, and let their normals be N and N' , and n and n' , respectively. As all the radii of the circular section are equal, and in the case of the indicatrix correspond to the refractive index for rays the wave-fronts of which are perpendicular to ON or ON' , it follows that all rays having these wave-fronts have the same

refractive index corresponding to a velocity which is inversely proportional to this index and to the radius OB which represents it graphically. Hence, these rays may be polarised in any plane and will not be resolved into two rays of definite planes of polarisation, and in short behave as if the crystal were singly refractive along these two directions of the normals ON and ON' . These directions are called the “**primary optic axes**,” and by Fletcher more precisely the “**optic binormals**.”

But in the case of the Fresnel ellipsoid, the velocity along any direction is given by the axes of the section perpendicular to it, so that the rays passing along the normals On and On' to the circular sections can only have one velocity, the sections being circles and not ellipses. Hence On and On' are called the “**secondary optic axes**.” The distinction must be remembered that it is here in the Fresnel ellipsoid a question of the ray itself travelling along the normal On or On' , and with

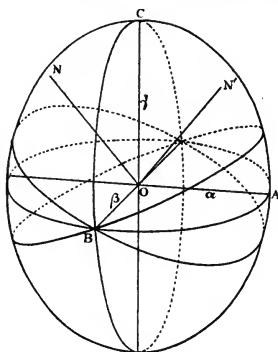


FIG. 534.—Fletcher Indicatrix showing the Primary Optic Axes ON and ON' .

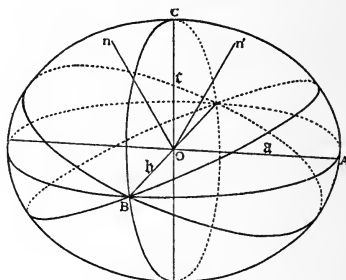


FIG. 535.—Fresnel Ellipsoid showing Secondary Optic Axes On and On' .

a wave-front which may be slightly inclined from the perpendicular to the direction of propagation; whereas, in the case of the indicatrix, it was the wave-front which was perpendicular to the direction ON or ON' and was represented by the circular section, as this was the tangent plane of wave-fronts perpendicular to the binormal. The cases are not similar, because, as regards the case of the Fresnel ellipsoid, two rays may pass in the same direction inside a crystal having different wave-fronts, and also, as concerns the case of the indicatrix, rays having the same direction of wave-front may take slightly different paths through the crystal, that is, their paths may be slightly inclined to ON or ON' .

From the practical point of view, however, this distinction between primary and secondary optic axes is of little importance, for most rays are so nearly perpendicular to their wave-fronts that these two kinds of optic axes appear identical to all practical tests, except such as are specially designed to bring out such a minute difference. For even their accurately determined positions are rarely more than a very few minutes apart. Occasionally, however, exceptions are observed—sulphur

for instance, for which their separation amounts to $3\frac{1}{2}^\circ$ —so that the phenomenon cannot be ignored.

The observed optic axis, either of the two which are indicated in convergent polarised light by the vertices of the two hyperbolic “brushes” in the centres of the innermost optic axial interference rings, is really the axis of a cone of rays of very small angle, passing through the crystal and emerging in air as a cylinder. Those rays the paths of which inside the crystal lie on the surface of this cone are those which alone are singly refracted, and which unite on emergence to form the cylindrical beam. Within the cone, double refraction of a kind known as “Inner Conical Refraction” occurs. Hence, the two optic axes of a biaxial crystal are not entirely of the same character as the single optic axis of a uniaxial crystal. For the unique optic axis of the latter combines in itself the qualities of both primary and secondary optic axes, the two absolutely coinciding. Hence, the singular crystallographic axis of a trigonal, tetragonal, or hexagonal crystal is truly, in the strictest sense, a direction of no double refraction. For both the indicatrix and the Fresnel ellipsoid obviously become equally an ellipsoid of revolution in the cases of uniaxial crystals.

The nature of the **Cone of Inner Conical Refraction** can be best understood by considering the wave-surface. Let OA and OC in

Fig. 536 be the axes of maximum and minimum vibration-velocity of the Fresnel ellipsoid, that is, the major and minor axes of the principal elliptic section. A ray propagated in air along the direction OA would, when travelling inside the crystal, outwards from the centre O towards A, be resolved into two rays vibrating respectively along OC (the direction of minimum vibration-velocity) and OB (perpendicular to the plane of the paper) the direction of intermediate vibration-velocity. When the latter ray had reached *b* the former would only be at *c*. Similarly, rays travelling along OC would be resolved into two inside the crystal, vibrating severally parallel to OA (the direction of maximum vibration-velocity) and to OB again (the intermediate vibration-velocity direction normal to the paper). Consequently, the former ray would have reached *a* when the latter had only reached *b'*. The two positions *b* and *b'*

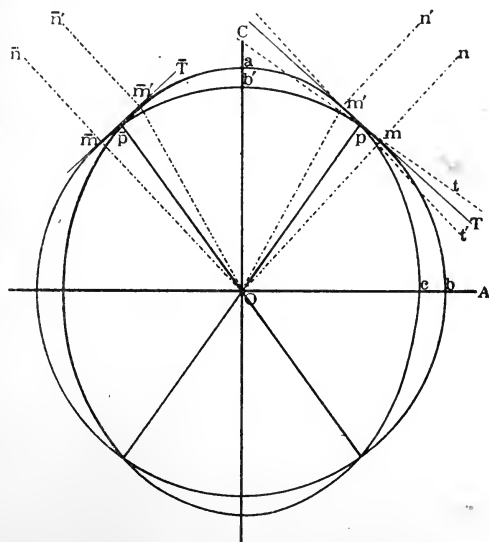


FIG. 536.—Diagram illustrating the Nature of the Cone of Inner Conical Refraction.

Similarly, rays travelling along OC would be resolved into two inside the crystal, vibrating severally parallel to OA (the direction of maximum vibration-velocity) and to OB again (the intermediate vibration-velocity direction normal to the paper). Consequently, the former ray would have reached *a* when the latter had only reached *b'*. The two positions *b* and *b'*

being equidistant from the centre, the section parallel to the paper, of the wave-surface of all rays vibrating parallel to the intermediate axis perpendicular to the paper, is a section of a circle, the radius of which is proportional to the intermediate vibration-velocity. The other curve on which a and c are situated, representing the wave-surface of waves vibrating in the plane of the paper, is an ellipse, with axes Oa and Oc proportional to the maximum and minimum vibration-velocities.

Rays travelling along the line joining the centre O and a point of intersection p of the circle and ellipse would travel with the same velocity within the crystal, because here the vibration-velocity is the same both perpendicular to and in the plane of the paper. These directions Op and $O\bar{p}$ are the "**Secondary Optic Axes**," or "**Axes of Single Ray Velocity**." But the tangents t and t' to the two curves at the point p are obviously differently inclined to Op , and the two rays travelling along Op will consequently be dissimilarly refracted on emergence into air. The two curves, however, have a common tangent T , and as similar arguments apply for rays travelling along OB (perpendicular to the paper) to those which have been used regarding waves propagated along OA and OC , the two curves in Fig. 536 are, as just indicated, really the sections of the two shells of the wave-surface, so that the common tangent T to the two curves is a plan or trace of a common tangent plane to the exterior shell, covering the depression in the latter, and the line of contact of which, as indicated in Fig. 523 illustrating the two shells of the wave-surface, is a small circle on which m and m' are two points, this circle being the base of the cone of inner conical refraction. All rays travelling within the crystal from the centre along the surface of this cone will consequently emerge as a cylindrical beam nm' , parallel to the primary optic axis or the "optic axis" as measured by the use of the interference figure in convergent polarised light. The normal mn to the common tangent at the point m where it touches the circle is obviously a continuation of the radius Om at the point, and is the actual direction of the primary optic axis, the cylindrical beam being not only parallel to it but bounded by it on one side in the plane of the section, the other boundary being the parallel line $m'n'$. This direction of the "**Primary Optic Axis**" Om is also termed the "**Axis of Single-Wave Velocity**," and is identical with the direction of the normal to the circular section of the optical indicatrix, this normal and its fellow on the other side of the bisectrix of the optic axial angle being the "**Binormals**" of Fletcher. These two cylinders of rays nm' and $\bar{n}\bar{m}'$, parallel to the symmetrical pair of primary optic axes, together form the equivalent in biaxial crystals of the single optic axis of uniaxial crystals. But that the two cases are different will be apparent from the fact that there is the inside of the cone to consider, if we desire to go into the ultimate details of the biaxial case. The conditions are grossly exaggerated in Fig. 536 for obvious reasons of clearness, the angle of the cone rarely reaching 2 degrees. If the ordinary case were drawn to scale the angle of the cone would be so small that the two lines Om and

Om' would almost lie in contact throughout their length, with m , p , and m' approximately identical.

When a section-plate is cut out of the crystal perpendicularly to the primary optic axis Omn , and a cylindrical beam of parallel rays is allowed to fall normally upon it, they will travel within the crystal along the surface of the cone, a circumstance from which the term "**Cone of Inner Conical Refraction**" takes its origin.

The above explanation is due to Sir William Hamilton,¹ and its correctness was proved experimentally by Dr. Humphrey Lloyd² in a memorable series of experiments suggested by the work of Sir William Hamilton. A crystal of aragonite was employed, the rhombic form of dimorphous carbonate of lime, CaCO_3 . Aragonite, like the other better-known trigonal form of calcium carbonate, our invaluable calcite, is a beautifully transparent colourless mineral very suitable for the purpose, and particularly so as it possesses a relatively large angle, nearly 2° , of inner conical refraction, combined with an unusually small optic axial angle, the true angle $2Va$ for sodium light being $18^\circ 11'$, and the apparent angle in air $2E$ being $30^\circ 52'$. The mode of carrying out the experiment is illustrated in Fig. 537.

A plate of aragonite $abcd$ a quarter of an inch thick was ground out of a good crystal perpendicularly to the acute bisectrix of the optic axial angle, and a narrow beam of parallel light rays allowed to impinge on it from an aperture in a screen s . The surface of the plate nearest the screen was covered with a thin metallic plate s' , in which also a small aperture was pierced. The parallel beam was so arranged that it passed through the hole in the metallic plate, and was incident on the crystal at O at the semi-optic-axial angle in air, $E=15^\circ 26'$, as measured by the double of the angle of reflection sOr . The angles are all exaggerated in the drawing, as otherwise the cone of inner refraction would not be clearly visible. The rays then passed through the crystal along the surface of this cone of inner conical refraction Omm' , and the cylindrical beam was received on a silver screen s'' . Two images of the aperture were produced on this screen s'' until the adjustment of the plate and of both screens s and s'' was perfectly attained, when suddenly the two images closed up and formed a brilliant ring of light, of unchanging diameter nn' , dependent only on the thickness of the plate. This remarkable result proved that the beam was indeed cylindrical and hollow, and that it corresponded to the light rays proceeding within the crystal along the surface of the cone. The angle of the cone was found to be $1^\circ 50'$, extremely close to the value, $1^\circ 55'$, predicted by Sir William Hamilton.

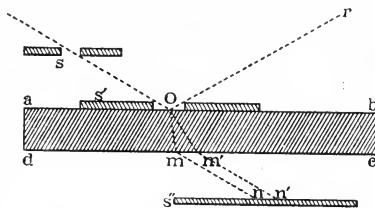


FIG. 537.—Lloyd's Experiment on Inner Conical Refraction of Aragonite.

It is also interesting to observe what occurs when a beam of light traverses the crystal along the direction of one of the secondary optic axes, say Op in Fig. 536. Such a narrow beam of parallel rays should emerge into the air as a conical beam, and does as a matter of fact,

¹ *Dublin Transactions*, 1839, 17, 132, an account of a paper read in 1832.

² *Phil. Mag.*, 1833, 2, 112 and 207.

giving rise to the phenomenon of "Outer Conical Refraction." This was proved by a second experiment of Dr. Lloyd, employing the same crystal plate of aragonite, *abcd* in Fig. 538, as for the experiment just described.

A metal plate perforated by a small hole was laid in this case against each surface of the crystal plate, and the two plates *s* and *s'* were so adjusted that the direction of the line through the crystal joining the two apertures was parallel to the secondary optic axis *Op*. A lamp flame was then adjusted on the *O*-side so that the axis of the cone of rays from it to the aperture at *O* made an angle of incidence of about $15\frac{1}{2}^\circ$ on the plate, the amount of the semi-optic-axial angle. On looking from the other side through the aperture *p* in the plate *s'*, the final adjustments having been carefully made, a bright ring of light was again seen. The cone of outer conical refraction *pqr* had been actually produced, and its basal circle afforded the brilliantly illuminated ring of light observed. On either side of this adjusted position the ring broke up into two images of the flame or aperture *O*. A similar ring of light was produced when the lamp was replaced

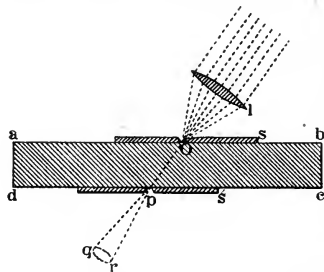


FIG. 538.—Lloyd's Experiment on Outer Conical Refraction of Aragonite.

by a beam of light focussed on the aperture *O* by means of a lens *l*, as shown in Fig. 538, so that the axis of the cone proceeding from the lens to the plate was a continuation of the line *Op*, the aperture at *p* cutting off all but this axial ray *Op*. The aperture *p* requires to be very small in this form of the experiment.

The phenomena of inner and outer conical refraction thus afford the final proof that the theory of the transmission of light through biaxial crystals given in this chapter is correct, and also demonstrate in the most perfect manner the true character of the two optic axes of a biaxial crystal, and their essential difference from the simpler single optic axis of a uniaxial crystal.

The Median Lines, and their Order with respect to the Refractive Indices.—Whichever one of the two extreme axes, the maximum and minimum axes, of the optical ellipsoid (whether the latter be the indicatrix or the Fresnel ellipsoid) bisects the acute angle between the two optic axes, is termed the "First Median Line" or the "Acute Bisectrix of the Optic Axial Angle." The other of these two axes is then obviously the "Obtuse Bisectrix of the Optic Axial Angle" or "Second Median Line." Considering the indicatrix as the more practical ellipsoid for our purpose, as its axes directly represent the relations of the three refractive indices, if the intermediate axis, as represented relatively by its refractive index β , be nearer in relative length to the minimum axis as represented by the refractive index α , than to the maximum axis represented by γ , then the crystal is said to be a "Positive" biaxial crystal; but if β be nearer to γ than to α , then it is a "Negative" crystal. In the former case of a positive biaxial crystal, shown in Fig. 534, the circular sections will be inclined to the plane of the α and β axes at less than 45° , and their normals, the optic axes, will thus lie symmetrically at an angle of less than 45° on each side of the maximum axis γ , which

is, therefore, the acute bisectrix or first median line, α being the obtuse bisectrix or second median line. This will be still clearer from a consideration of the simple Fig. 539. In the latter case of a negative crystal the converse happens, α being the first median line and γ the second, the circular sections lying less than 45° from the axis γ . A very useful fact to remember is that the first median line is always that extreme axis which is furthest in value from β .

In those systems, the monoclinic and triclinic, in which the orientation of the optical ellipsoid is not fixed by the symmetry (except that in the monoclinic system one of the axes of the ellipsoid is identical with the symmetry axis), the median lines will differ in position (except that identical with the monoclinic symmetry axis, if such identity occur) for different wave-lengths of light and different temperatures; in other words the median lines will show dispersion under different conditions. The various modes in which this can occur will be studied in detail in the latter part of the next chapter.

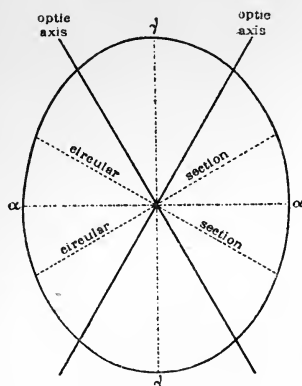


FIG. 539.—Disposition of Optic Axes in Positive Biaxial Crystal.

Resumé of Relations between Morphological Crystal Symmetry and the Optical Ellipsoid.

Uniaxial and isotropic crystals may be considered as special cases of the more general biaxial crystals. For when two of the principal axes of either the indicatrix or the Fresnel ellipsoid become equal, as in trigonal, tetragonal, and hexagonal crystals, an ellipsoid of revolution is produced, the crystal becoming optically uniaxial. The axis of revolution is then not only the principal crystallographic axis, the trigonal, tetragonal, or hexagonal axis in the crystals of these systems, but also the unique optic axis or direction of no double refraction. There will be only two refractive indices, now labelled ω (vibrations perpendicular to the axis) and ϵ (vibrations parallel to the axis) on account of the corresponding rays being ordinary and extraordinary; for two of the three refractive indices corresponding to the general biaxial case are now equal.

When all the axes of the ellipsoid are equal, the latter becomes a sphere, and we have the special case of cubic crystals, with no double refraction in any direction, the isotropic character being accompanied by the display of the highest type of morphological symmetry. All sections being circular, all rays will travel with equal velocity throughout the crystal, and there will be only one refractive index, now generally labelled μ as for isotropic substances in general.

The general case, that of a biaxial crystal, has been shown to apply

to all orthorhombic, monoclinic, and triclinic crystals, the ellipsoid having three different rectangular principal axes, three different refractive indices α , β , γ , corresponding to rays vibrating parallel thereto, and two optic axes. The highest type of symmetry here involved, the orthorhombic, is characterised by identity of the three rectangular crystallographic axes with those of the optical ellipsoid, whether the latter be the indicatrix or that of Fresnel. In the monoclinic system, the ellipsoid is free to rotate about the unique axis of symmetry, which alone is identical in direction with one of the axes of the ellipsoid, which may be any one, but the particular one is definitely determinate for each substance crystallising in this system. The position, as regards rotation about this axis, is definite within certain limits for each substance; but rotation does occur, indicated by dispersion of the two axes of the ellipsoid lying in the symmetry plane, for different wave-lengths of light and for different temperatures. In the triclinic system there are no restrictions as to the relative positions of the morphological and ellipsoidal axes, except, of course, that the three principal axes of the ellipsoid are, in accordance with the very nature of an ellipsoid and in contradistinction to the oblique relations of the crystallographic axes, at right angles to each other as always.

CHAPTER XLI

INTERFERENCE COLOURS OF THIN FILMS DUE TO TWO REFLECTIONS,
AND INTERFERENCE PHENOMENA EXHIBITED BY CRYSTAL PLATES
IN PARALLEL AND CONVERGENT POLARISED LIGHT, DUE TO
DOUBLE REFRACTION.

Newton's Rings and Orders of Colours.—One of the most familiar phenomena in optics is the production of Newton's spectrum-coloured rings by pressing two glass surfaces closely together. It will be remembered that they are best produced either by laying a plano-convex lens of very large radius of curvature by its convex side on a plate of glass, or by placing two circular plates of truly parallel and plane-surfaced glass together, with a marginal annulus of gold leaf between them, and pressing inwards the centre of one of the plates by a screw carried by the supporting annular frame, in order to cause the plates to touch at the centre. In either case the essential thing is a circular film of air the thickness of which increases regularly and excessively slowly from zero at the centre to a perceptible thickness at the circumference. The interference rings, iris-coloured in white light but sharper and black in monochromatic light, may be observed either by reflection or transmission. The former, as it affords the more brilliant spectacle, is usually preferred for demonstration purposes. The reflected and transmitted rings occupy complementary positions, the dark annular bands of the one in monochromatic light occupying the positions of the bright ones of the other. Fig. 540, the reproduction of a photograph

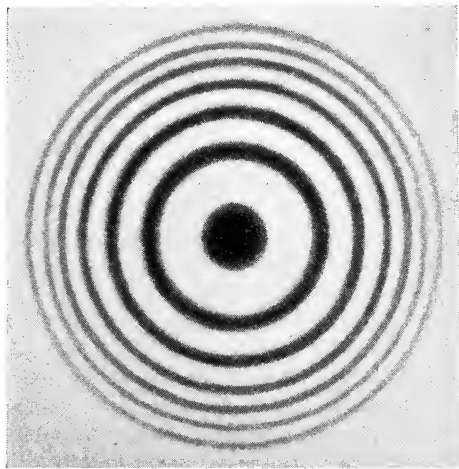


FIG. 540.—Photographic Reproduction of Newton's
Rings as seen in White Light.

of the rings in white light, will remind the reader of their appearance, as far as a photograph in black and white can do so; and Fig. 541 will recall the conditions, the curvature of the curved surface being immensely exaggerated for the sake of clearness.

The rings are equal in area, hence they get narrower as we recede from the centre. When the incident ray enters, it is reflected both

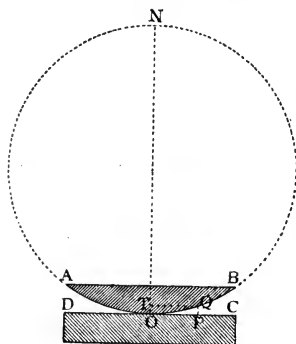


FIG. 541.—Conditions for the Production of Newton's Rings.

from the under surface AOB of the lens or bent plate, say at the point Q, and from the upper surface of the plate COD, say at P, and these two reflected rays will differ in phase according to the thickness PQ of the air-film at the spot, that is, according to the amount of extra path traversed by the ray reflected from the plane surface COD. When the delay is to the extent of half a wave-length, as shown on p. 568 of Chapter XXXV. in discussing the subject of the interference of light waves, total interference and mutual destruction should occur, half-phase difference always producing extinction of the light, assuming it to be strictly monochromatic, that is, composed of vibrations of a

single wave-length. If the wave-length be varied, that is, if the apparatus be illuminated by monochromatic light of different colours in succession, the black rings will alter their position, both as regards their distance from the centre and their mutual distances from each other, becoming closer together or opening out according as the change is made towards the blue or the red end of the spectrum, that is, as the wave-length is shortened or increased. The radius of any ring varies as the square root of the wave-length of the light producing it. Consequently, in white light the dark rings produced for one constituent colour are illuminated by the bright rings of other colours, the most effective and predominant being the complementary colour. Hence, rainbow bands are produced, and each ring is bordered inside by red and outside with violet. After the seventh ring the further ones overlap one another so rapidly that they are invisible in white light, the ordinary effect of white light being afforded. The rings also change their position with alteration of the angle of incidence of the light.

The order of the colours given by Newton as succeeding each other in the several rings produced in white light, starting from the centre is as follows. **First** (central) ring, black (central spot) blue, white, yellow, red; **second** ring, violet, blue, green, yellow, red; **third** ring, purple, blue, green, yellow, red; **fourth** ring, green, red; **fifth** ring, greenish-blue, red; **sixth** ring, greenish-blue, pale red; **seventh** ring, greenish-blue, reddish-white. This succession of colours is universally known as "Newton's Scale of Colours," or "Newton's Seven Orders of Spectra." It is useful to remember it or have ready access to it, so that we may be able to recognise the particular order to which any polarisation or

thin-film colour belongs. Red, in particular, is frequently referred to as "of the second order," "third order," or whatever order the colour under observation belongs to, meaning that it is the same quality of red as is seen in the second, third or other ring of Newton's series of interference fringes.

Measurement of Thicknesses for, and Conditions for Production of, Newton's Rings.—An actual measurement by Schwerd of the thickness of the air-film at the position where the first bright ring in monochromatic red light is produced, surrounding the central black spot corresponding to contact, revealed the fact that it was about the $1/160,000$ th of an inch, which is approximately one-quarter of a wave-length of red light. For the exact wave-length of red C-hydrogen light is, in British measure, $1/38,710$ inch, and that of the red ray of cadmium $1/39,459$ inch.

Now the light rays have to traverse this thickness twice, so that the retardation is $1/80,000$ th of an inch. Hence, the first bright ring corresponds to a retardation of half a wave-length, which we usually associate with extinction. The explanation is a curious and interesting one, namely, that the act of reflection of a light wave in air at the surface of a denser medium, glass in this case, itself alters the phase by half a wave-length. It may be illustrated by the fact that when two particles of different mass collide, if it be the lighter which strikes the heavier the former will not only set the latter in some motion, but will also itself rebound and thus reverse its own motion. This sudden reversal corresponds, in the case of ether particles producing waves of light, to a retardation or acceleration of half a wave-length, $\lambda/2$, that is, to a reversal of phase, on passing from a lighter to a denser medium. If, on the contrary, the heavier particle strike the lighter it continues its direction of motion, carrying along with it the latter, so that no change of phase occurs when light passes from a denser to a rarer medium.

Hence, in the Newton's rings experiment there is no change of phase at the curved upper surface of the two surfaces concerned in the interference, but only at the plane lower surface. Consequently, there is only one such change of phase of $\lambda/2$, which is, therefore, the total retardation due to reflection which requires to be added to that due to the difference of path. Thus it comes about that we have a bright ring at the position where the difference of path alone would correspond to a dark ring, and at the centre, where the difference of path is zero, there being contact, the half wave-length change of phase due to reflection alone operates, the ensuing extinction accounting for the black central spot.

Newton, in a series of experiments with yellow light, found the absolute thicknesses of the film corresponding to successive dark rings to be $2/178,000$, $4/178,000$ inch and so on, when the incidence was 4° . This gives $1/44,500$ inch for the wave-length of yellow light, which is interesting as being the first determination of the wave-length of a light radiation, although Newton himself did not recognise the fact.

Each dark ring thus corresponds to a further difference of path (twice the difference of thickness, as the space is traversed twice) of a

whole wave-length, the additional half wave-length due to reflection being constant throughout. The difference of thickness of the air-film for each successive dark ring is thus half a wave-length. These facts will be clear from the following table, which shows the thicknesses and retardations for each dark and bright ring in their regular order, as far as the third in each case, starting from the centre.

CONDITIONS FOR PRODUCTION OF NEWTON'S RINGS.

Order of Ring.	Thickness of Air-film.	Retardation.		
		Due to Difference of Path.	Due to Reflection.	Total.
Central black spot .	Zero	0	+	$\frac{\lambda}{2}$
First bright ring .	$\frac{\lambda}{4}$	$\frac{\lambda}{2}$	+	λ
First dark ring .	$\frac{\lambda}{2}$	λ	+	$1\frac{1}{2}\lambda$
Second bright ring .	$\frac{3\lambda}{4}$	$\frac{3\lambda}{2}$	+	2λ
Second dark ring .	λ	2λ	+	$2\frac{1}{2}\lambda$
Third bright ring .	$\frac{5\lambda}{4}$	$\frac{5\lambda}{2}$	+	3λ
Third dark ring .	$\frac{3\lambda}{2}$	3λ	+	$3\frac{1}{2}\lambda$

The thickness t of the film for the production of any dark ring, the radius r of the ring, and the radius of curvature R of the curved surface (of the lens or curved plate) are connected in the following manner by the two formulæ numbered (1) and (2), which can readily be elucidated from Fig. 541. If ON be that diameter of the circle of curvature of the lens which is normal to the plate, and a line TQ be drawn parallel to the plate from the point Q where the light is incident on the curved surface, so that $TQ=r$, and $OT=QP=t$, then, by a well-known property of the circle (Euclid, iii. 35) :

$$(TQ)^2 = NT \cdot TO ;$$

that is,

$$r^2 = NT \cdot t, \text{ or } t = \frac{r^2}{NT}.$$

But when R is very large NT may without sensible error be taken as equal to $2R$, and it may certainly be so taken for a curved surface so nearly a true plane as is required for the production of Newton's rings ; hence,

$$(1) \quad t = \frac{r^2}{2R}.$$

Now it will be shown in the next section that the condition for interference (extinction) is :

$$t = \frac{n\lambda}{2\mu \cos \phi},$$

a formula which thus affords another expression for t , and in which λ is the wave-length in the medium of the film, in this case air, ϕ is the angle of incidence on the second surface relevant to the interference, n is the number of the ring taken consecutively from the centre, and μ is the refractive index of the material of the

film. As the latter in the case of Newton's rings is air, $\mu=1$. Hence, for the production of the black ring n we must have:

$$(2) \quad \frac{r^2}{2R} = \frac{n\lambda}{2 \cos \phi};$$

or, if r_n be the radius of the ring numbered n ,

$$r_n^2 = \frac{n \cdot \lambda \cdot R}{\cos \phi}.$$

When the incidence is normal $\phi=0$, and $\cos \phi=1$, so that the formulæ (1) and (2) simplify respectively as under, t_n being the thickness of the air-film at the place where the ring n is produced:

$$(1) \quad t_n = n \frac{\lambda}{2}, \text{ and } (2) \quad r_n^2 = n\lambda R.$$

Hence the squares of the radii of the dark rings are proportional to n , which is represented by the natural numbers 1, 2, 3, 4, and so on. At the centre there is interference for all colours; consequently, a black spot is produced.

Colours of Thin Films.—If a transparent parallel-surfaced plate be so thin that the retardation of the rays reflected from one surface, behind those reflected from the other surface, is so slight as to fall within the limits corresponding to the first seven rings of Newton, the whole plate will exhibit the particular colour to which such retardation corresponds in the case of the production of the rings. In Fig. 542 let ABCD represent such

a film, say of blown glass, soap solution, or turpentine floating on water, and IP a ray incident at the angle θ on the upper surface APB of the film, where part is reflected and part refracted at the angle ϕ towards S. At the second surface CSD of the film reflection occurs again towards Q, and at Q also from the recovered first surface; but here refraction also

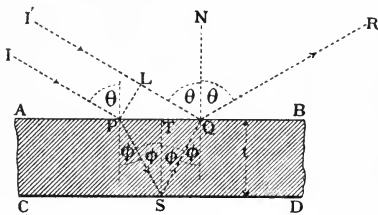


FIG. 542.—Diagram explanatory of Colours of Thin Films.

occurs, the refracted ray escaping towards R. But a directly reflected ray QR, incident at Q in the direction I'Q, will also reach the observer's eye together with this refracted ray QR derived from the internal reflection of the first ray IPS at S; that is, two rays, one of which has traversed the film and the other has not, will reach the eye of an observer looking along the direction RQ. They will be in a position to interfere with each other if their difference of path should be approximately half a wave-length or an odd multiple of this, and the effect will be visible by the production of colour of one of Newton's orders when the retardation does not exceed the seven wave-lengths corresponding to the seventh ring. Their difference of path may be found in the following manner.

Draw PL perpendicular to IP and also the normals QN and TS to the two surfaces. Then the difference of path of the two rays is $PS+SQ-LQ$. Now $PS+SQ=2SQ$, and if the thickness of the film be t :

$$(a) \quad 2SQ = \frac{2ST}{\cos \phi} = \frac{2t}{\cos \phi}.$$

Also $LQ = PQ \cos LQP = PQ \sin \theta$, and $PQ = 2TQ = 2TS \tan \phi = 2t \cdot \tan \phi$.

Hence, (b) $LQ = 2t \cdot \sin \theta \tan \phi$.

But the part (b) of the path, LQ, is in air, while the part (a), PSQ, occurs in the film, the relative velocities being afforded by the ratio of the refractive index μ for the material of the film to that of air, which latter is unity. Hence, the value of (a), as the progress of this ray will be prolonged, must be multiplied by the refractive index of the film μ . We then have for the total difference of path d of the two rays:

$$d = (a) \cdot \mu - (b) = \frac{2t\mu}{\cos \phi} - 2t \cdot \sin \theta \tan \phi.$$

As $\mu = \frac{\sin \theta}{\sin \phi}$, and therefore $\sin \theta = \mu \sin \phi$, we may substitute this latter for $\sin \theta$ and thus obtain:

$$\begin{aligned} d &= \frac{2t\mu}{\cos \phi} - 2t\mu \cdot \sin \phi \tan \phi = 2t\mu \left(\frac{1}{\cos \phi} - \sin \phi \tan \phi \right) \\ &= 2t\mu \left(\frac{1}{\cos \phi} - \frac{\sin^2 \phi}{\cos \phi} \right) = \frac{2t\mu}{\cos \phi} (1 - \sin^2 \phi) = \frac{2t\mu}{\cos \phi} \cos^2 \phi, \text{ that is,} \\ (c) \quad d &= 2t \cdot \mu \cdot \cos \phi. \end{aligned}$$

Thus we arrive at the simple expression (c) for the difference of path d between the two rays proceeding together from any point of a thin film to an observer's eye, the one ray by direct reflection at the surface and the other having also traversed the film.

But there is a further consideration affecting the relative phases of the two, namely, as to what occurs at the limiting surfaces of the lighter air and the denser film. It has already been shown that a change of phase of $\lambda/2$ occurs at the limiting surface when a light wave in air meets with a denser medium, and is reflected at the surface of the latter; while no such change of phase occurs when the conditions are inverted, the amplitude of the vibration only being affected without reversal of its direction. Hence, there is only one such change of phase of $\lambda/2$ due to the limiting surfaces, as in the case of Newton's rings, namely, on the part of the directly reflected ray I'QR. We have to add, therefore, to the difference of path d just found a further $\lambda/2$, which may be equally indifferently considered as a retardation or an augmentation, the interference effect of an exact half wave-length being the same for both, namely, conversion of light to darkness or *vice versa*. Hence, the total retardation of the penetrating ray behind the directly reflected ray is

$$2t\mu \cos \phi + \frac{\lambda}{2},$$

when t is the thickness of the film, μ its refractive index, ϕ the angle of incidence on the second surface of the film, and λ is the wave-length of the light in air.

When t becomes infinitely small the two waves will differ by the phase change of $\lambda/2$ alone, and extinction will occur, as at the central black spot in Newton's rings. It will be obvious that the film will dissipate, disrupt, or burst before this infinite thinness is attained, but just before it does so the brilliant colours give place to a dark gray, corresponding to just beyond the beginning of the first order of Newton's scale.

Extinction will also occur if the total retardation is any odd number of half wave-lengths, that is, if

$$\text{Total retardation} = 2t\mu \cos \phi + \lambda/2 = (2n + 1)\lambda/2,$$

or simply, when

$$\text{Retardation due to difference of path} = 2t\mu \cos \phi = n\lambda.$$

That is, the thickness of the film for total interference, extinction, must be

$$t = \frac{n\lambda}{2\mu \cos \phi}.$$

It has been essential to render this condition for interference in thin films perfectly clear in order that we may thoroughly grasp the nature of the colours exhibited by doubly refractive crystal plates; for the retardation behind the other, of one of the two rays into which such a crystalline plate divides the light, gives rise to phenomena in polarised light of a similar character in many respects to the colour changes displayed in ordinary light by thin films of liquids or singly refractive transparent solids.

The fact will doubtless have been made apparent that a parallel-surfaced film merely affords over an extended surface the phenomenon of light, colour, or darkness (extinction) exhibited at any one annulus of equal thickness of the plano-concave lenticular film employed in producing Newton's rings. The consideration of these phenomena of ordinary light in a film of gradually varying thickness will also assist us greatly in understanding the important action of the quartz wedge in polarised light, to be studied later in this chapter.

Colours afforded by Crystal Plates in Parallel Polarised Light.—

When a parallel-faced plate cut or ground out of a cubic crystal free from internal strain, or a naturally tabular crystal of this system, is examined between the crossed Nicol prisms of a polariscope, the rays being approximately parallel (that is, the special "converging system" of lenses usually provided with a polariscope not being used), it behaves like a plate of glass or other non-crystalline transparent isotropic substance. The field remains dark, and does so also when the crystal is rotated in its own plane. But if a similar plate belonging to a doubly refractive crystal, that is, one belonging to any other crystal system than the cubic, be examined and rotated in its own plane in the dark field, it will show light in certain positions of azimuth, unless, if it be uniaxial, it happens to be a section cut perpendicularly to the optic axis or, if it be biaxial, it is normal to one of the two optic axes. The light will be white or coloured according to the relation between the thickness of the plate and the amount of the double refraction. During a complete revolution of the plate in its own plane, the polariscope being provided with a rotating stage, the field of view will become four times light and four times dark, light and darkness succeeding each other alternately at equidistant intervals of 45° . That is, the positions for darkness will be at right angles to each other, and fairly sharply defined, within half a

degree of $90^{\circ} 0'$ even to an unpractised eye. These positions of darkness occur in the case of a uniaxial crystal when the trace of that principal section of the optical ellipsoid of revolution which contains the normal to the plate becomes parallel to the plane of polarisation of either of the crossed Nicol prisms. This is the law of Malus. In the case of a biaxial crystal the extinction directions are parallel to the traces of the planes bisecting the angles between the two planes containing the two optic axes and the normal to the section. This is the law of Biot. In the simple special case of a uniaxial crystal plate cut parallel to the axis, the extinctions occur whenever the direction of the optic axis is parallel to either of the vibration planes of the crossed Nicols; in the special case of a biaxial crystal-plate cut perpendicularly to any one of the three principal axes of the optical ellipsoid, extinction occurs whenever the vibration directions of the Nicols are parallel to the other two axes of the ellipsoid to which the plate is parallel.

Whether the light which appears at the intermediate positions, the maximum intensity being at 45° from the positions of extinction, is white or coloured depends on the relation just referred to between the thickness and the double refraction. For the production of colour in ordinary light by a thin film is closely paralleled by the display of brilliant colour by crystal plates in polarised light. We have seen that in reflection from the two surfaces of a thin film one ray, that which penetrates to the second surface before reflection, is retarded behind the other, reflected directly from the first surface, by double the thickness of the film which it had twice to traverse, the interference thereby produced being the cause of the colour. In the case of a crystal plate, however, the interference is brought about by the retardation of one of the two rays, into which a doubly refracting crystal divides the light transmitted by it, behind the other, owing to the difference in velocity of the two rays, the path traversed not being necessarily different, although usually it is slightly so owing to different refraction being the accompaniment of different velocity.

When the doubly refracting crystal plate is of such a thickness that the retardation of one of the rays behind the other, due to both difference of velocity and difference of path, is such that the two rays emerge differing by a half wave-length, we naturally expect interference to occur, extinction if the light be monochromatic, but chromatic interference if white light be used, and assuming that the planes of polarisation are compatible with interference.

In reality, however, when we examine such a plate in the dark field of the polariscope—the planes of polarisation of the Nicols being thus crossed, and the plate being arranged as regards azimuth so that its directions of vibration (traces of principal sections of optical ellipsoid or of the Biot bisecting planes, the “extinction” positions) are at 45° to the planes of polarisation or vibration of the two Nicols, the positions for maximum transmission of light—we find that interference occurs when the two rays differ in phase by a *whole* wave-length, instead of by a half

wave-length as expected. The fact is most clearly revealed in monochromatic light, for instead of the anticipated light of the colour used we get darkness when the difference of phase is one, two, three or up to seven, whole wave-lengths, the intermediate odd half wave-lengths of difference corresponding to the transmission of the coloured light. In white light we get brilliant colour instead of darkness for whole-wave retardation.

The reason for this reversal of our expectations is due to the fact that the Nicol analyser itself introduces, when crossed to the polarising Nicol, a change of phase of half a wave-length, like the act of reflection in the case of thin films, and this $\lambda/2$ requires to be added to the retardation of one ray behind the other brought about in traversing the crystal.

The polarising Nicol provides us with a beam of plane-polarised light, vibrating perpendicularly to the plane of polarisation of the Nicol. The latter was shown in Chapter XXXIX. to be parallel to the longer diagonal of the end faces of the prism. Supposing this to be horizontal, the plane in which the vibrations occur (parallel to the shorter diagonal) will be vertical, assuming the light to be traversing the prism in the proper manner, parallel to the long edges of the rhomb, and that the polariscope is horizontally arranged. On now introducing a doubly refracting crystal plate, with its rectangular directions of vibration (parallel to two of the three axes of the indicatrix if the crystal be biaxial and the plate be cut perpendicularly to the third axis, or in general, those indicated by the laws of Malus and Biot) parallel and perpendicular respectively to the direction of vibration of the light leaving the polariser, the plane-polarised beam will obviously be transmitted, its vibrations being parallel to the former of the two possible planes of vibration in the crystal. The analysing Nicol will also clearly permit the beam to pass through to the observer or the screen, if it be also arranged with its plane of vibration parallel to the same direction, but will cut off the light altogether if crossed at right angles thereto. Thus the dark field afforded by the crossed Nicols alone is unaffected by the introduction of the crystal plate between them if the vibration directions of the rays possible to be transmitted by the crystal, in accordance with the law of Malus for a uniaxial crystal and that of Biot for a biaxial one, are parallel to the two rectangular vibration directions of the Nicols. This is the foundation of the "extinction" mode of determining the positions of the axes of the optical ellipsoid, which will be practically described in Chapter XLIV., and as regards methods invoking the use of the polarising microscope, in Chapter LII.

An excellent example of a doubly refracting crystal plate for verifying these facts, and those of the next few paragraphs, is a cleavage one of gypsum (selenite), known to opticians as an "even selenite." Thin plates, almost films, of this mineral afford particularly brilliant colours, owing to the beautifully transparent and colourless nature of selenite. The cleavage occurs parallel to the symmetry plane of the monoclinic crystal, so that the plate is perpendicular both to the

crystallographic symmetry axis and to one of the axes of the optical ellipsoid, as will be more fully explained later in this chapter (see Fig. 543, p. 655). The other two rectangular axes of the ellipsoid consequently lie in the plane of the plate, and are the directions of "extinction." Two varieties are to be purchased, if it be not desired to cleave the necessary film for oneself (which, however, is advisable in order to gain the necessary most useful experience), namely, one which affords green and red complementary colours when the Nicols are parallel and crossed respectively and the selenite is placed with its planes of vibration at 45° to those of the Nicols, and another giving yellow and blue under the same conditions.

When we rotate the doubly refracting crystal plate between the crossed Nicols we observe thus four positions at 90° from each other in which the dark field is restored, the planes of vibration of crystal and Nicols being identical. At the intermediate 45° positions the maximum of light is observed to be transmitted, and less and less on each side as the 90° positions are approached. When the plate is thick the light is white, although it may be "white of the higher orders" of Newton when the thickness is not much greater than corresponds to the production of seventh order colours. But if the plate be thin enough, such as in the case of an "even selenite," to correspond to a retardation of anything under seven wave-lengths, colour is produced of the same type as one of Newton's orders, the particular colour depending on the thickness for any one substance. Thus, when a red-green even film of selenite is used the colour will be brilliant red with crossed Nicols and the plate at 45° to them, and if a blue-yellow film be employed a bright greenish-blue will be seen when the film is arranged at 45° to the crossed Nicols.

The α and γ axes of the indicatrix are the two which form the rectangular vibration directions of a cleavage plate of selenite parallel to the symmetry plane of the monoclinic crystal, and the normal to the plate is the β axis of the indicatrix. When the crystal plate is at the 45° position or thereabouts the vibrations, occurring in the vertical plane, of the light received by the plate from the polariser, all of which are in the same phase, are resolved by the crystal into vibrations occurring in its own two planes of vibration, and these two rays are again similarly resolved by the analyser into vertical and horizontal vibrations, but each separately. For their velocities are different, and thus the phases are altered with respect to each other. Those two component rays (one of each velocity) which are vibrating parallel to the analyser are then permitted to escape to the observer or the screen, while the other pair vibrating perpendicularly thereto are both extinguished. Hence, two originally identically similar rays reach the eye or screen; they are of maximum intensity when the plate is exactly at 45° , and are vibrating in the same plane parallel to that of the analyser. But one of them has got behind the other during its passage through the crystal plate, the two, in general, thus differing in phase and being capable of interfering and of producing colour, when white light is used and the retardation of the one behind the other does not exceed seven wave-lengths.

When the analyser is arranged with its plane of vibration parallel to that of the polariser, the crystal plate, whenever its vibration directions (axes of indicatrix) are at the 45° position to the Nicols, shows the complementary colour to that exhibited with crossed Nicols, namely green in the case of a red-green selenite, and yellow with a blue-yellow gypsum film. At the parallel positions of either of the two rectangular vibration directions of the plate with the common plane of the two parallel Nicols, that is, at 0° , 90° , 180° , and 270° , white light uncoloured is seen, the ordinary bright field corresponding to parallel Nicols.

Hence, if the crystal plate be arranged to begin with so that its two rectangular extinction directions (α and γ axes of indicatrix in the case of gypsum) are at 45° to the plane of vibration of the polarising Nicol, and if the analyser be then crossed to the polariser, a colour is produced, red or blue as we have seen in the case of gypsum. On rotating the analyser 90° , that is, until its plane of vibration is parallel with that of the polariser, the complementary colour is afforded, green or yellow when the plate is an adequately thin one of gypsum. Between the two positions, when the analysing Nicol has only been rotated 45° , it being immaterial on which side of zero the rotation occurs, white light uncoloured is transmitted, similar to the ordinary bright field of the parallel Nicols. For, the crystal plate having then a plane of vibration in common with the analyser, that ray thus vibrating passes through the latter unhindered, while the other rectangularly vibrating ray (of the two rays into which the crystal resolves the polarised beam) is simply extinguished, being unable to pass through the analyser. Consequently, in a complete rotation of the analyser, with a film of even selenite arranged with its vibration planes at 45° to the polariser, we observe red or blue twice, namely, at 90° and 270° from the plane of the polariser, green or yellow twice, at 0° and 180° with respect to the polariser, and no colour at the four intermediate positions of 45° , 135° , 225° , and 315° to the polariser.

These somewhat complicated phenomena require to be carefully followed, understood, and remembered, as they are the basis of all the immense assistance which we derive from the polariscopical examination of a crystal. It is obvious that a crystallographic polariscope must be provided with a rotating divided stage for the crystal, as well as with rotating Nicols, each of the latter provided with a divided circle and an indicator or vernier, the one on the mount within which it rotates and the other on the rotating inner tube which actually carries the Nicol.

It will doubtless prove useful to record in concise tabular form the phenomena occurring in the typical case of an "even" film of red-green selenite. It must be remembered, of course, that the crystal itself is absolutely colourless, the term "red-green" being merely the conventional one to indicate a thin plate of gypsum which affords red under crossed Nicols and green under parallel Nicols.

PHENOMENA AFFORDED IN WHITE PARALLEL LIGHT BY A GYPSUM (SELENITE) FILM, FOR DIFFERENT RELATIVE POSITIONS OF POLARISER, CRYSTAL, AND ANALYSER.

Plane of Vibration of Polariser.	Plane of Vibration of Analyser.	Relations of Nicols.	Positions of Vibration Planes of Gypsum.	Phenomenon observed.
Vertical,	Horizontal, —	Crossed, +	$45^\circ, \times$	Red colour
Vertical,	$45^\circ, \diagdown$ or \diagup	\diagdown or \diagup	$45^\circ, \times$	Colourless
Vertical,	Vertical,	Parallel,	$45^\circ, \times$	Green colour
Vertical,	Horizontal, —	Crossed, +	Vertical and horizontal, +	Dark field
Vertical,	$45^\circ, \diagdown$ or \diagup	\diagdown or \diagup	Vertical and horizontal, +	Half light
Vertical,	Vertical,	Parallel,	Vertical and horizontal, +	Bright colourless field

The red and green referred to in the last column are strictly complementary. For brilliant colours the retardation of one of the two rays should be less than 4λ and preferably not more than 2λ for very brilliant colours.

The interference of the two components above referred to under crossed Nicols causes total extinction if monochromatic light be used, and if the section-plate of the crystal be of such thickness that one of the two rays on leaving the plate is a whole wave-length before or behind the other. For the additional half wave-length introduced by the analyser then makes the total effective retardation $3\lambda/2$, and the effect of this is practically the same as $\lambda/2$. A total retardation of $5\lambda/2$, due to two whole wave-lengths' difference on leaving the crystal together with the analyser's $\lambda/2$, and also successive retardations of $(2n+1)\lambda/2$, that is, an odd number of half wave-lengths, also cause extinction. Generally therefore, the section-plate appears dark—when placed with its vibration directions at 45° to those of the crossed Nicols and monochromatic light is used—whenever its thickness is such that the two rays rectangularly vibrating within the crystal and starting therein together in the same phase, differ on emergence by a whole number of wave-lengths, the extra change of phase of $\lambda/2$ due to the analysing Nicol causing this to correspond to darkness instead of the bright field. The monochromatic colour will be transmitted by plates of the intermediate thicknesses.

That the colours thus afforded by gypsum or any other doubly refracting crystal, when its vibration directions are at 45° to the crossed and parallel Nicols respectively, are truly complementary, is readily proved by means of a double-image prism, of either the Rochon or Wollaston type described in Chapter XXXIX. (p. 610), employed as analyser instead of a Nicol, and projecting the phenomena on the screen.

The experiment has already been fully described in Chapter XXXIX. and illustrated in Fig. 518 (p. 612). Supposing a red-green selenite to be employed, two images of the circular aperture in front of the lantern, a red disc and a green one, are seen on the screen, more or less overlapping each other, and this lenticularly shaped superposed portion is quite colourless. Such a size of aperture should be selected from the four apertures of the rotating diaphragm shown in Fig. 518 as will give, with the particular double-image prism employed (whether of the Rochon or Wollaston type), an overlapping of the two images of about a third of each disc. If a blue-yellow even selenite be interchanged for the red-green one, the overlapping part will again be quite white.

As the Nicol, the selenite, and the double-image prism are all independently rotatable, the relative positions can readily be found for which the brightest red and green or blue and yellow discs are simultaneously afforded, when the overlapping lenticular-shaped part will be absolutely white, in accordance with the fact that truly complementary colours produce white light.

The explanation of the phenomena observed when the crystal plate is placed with its two possible vibration planes, parallel to the axes of the optical indicatrix, at 45° to the 0° and 90° planes of the crossed or parallel Nicols is as follows. The light from the polariser, vibrating in one plane, is resolved—along these two 45° directions of the axes of the optical ellipsoid, which have been shown in Chapter XL. to be the only possible directions of vibration within the crystal—into two rays which, after traversing the crystal with different velocities, emerge one retarded behind the other by an amount which depends on the double refraction, the thickness of the plate, and any difference in the length of the path traversed in the plate owing to difference of refraction. They do not yet interfere, however, for they are vibrating in perpendicular planes. The analyser sifts them, however, and again resolves each of these 45° sets of vibrations into 0° and 90° vibrations, and then permits to pass only that component of each which is vibrating parallel to its own vibration direction. The pair of components thus allowed to pass, as they are vibrating in the same plane, are pre-eminently in a condition to interfere in accordance with their difference of phase and, as regards quality of the light, with the total number of wave-lengths by which they are retarded, that is, in accordance with the order of Newton to which they correspond. Also, the analyser resolves the pair in opposite directions, introducing thereby the half wave-length change of phase already referred to as due to the action of the analyser.

Two facts cannot be too emphatically stated. The first is that **for interference to occur the vibrations must be in the same plane**; if they are not, in general elliptic vibrations result from the combination of two rays polarised in different planes, and in the special case when the planes are at right angles, the amplitudes equal, and the phase difference a quarter of a wave, vibrations in a circular orbit are produced. The second fact is that **it is the function of the analyser to reduce the two perpendicular sets of vibrations derived from the crystal to the same plane**, and thus render them capable of interfering.

When white light is used, the colour which is observed is the product of the residual vibrations of different wave-lengths left after the vibrations of a particular wave-length have been extinguished, the complementary colour to the one having that wave-length naturally prevailing. Hence also is explained the complementary nature of the colours afforded when the analyser is respectively crossed and parallel to the polariser, as will be more precisely shown from experimental data later in this chapter (page 657). **For the intensity in any angular position θ of the analyser is complementary to that in the perpendicular position $\theta + 90^\circ$** , the sum of the two intensities being equal to that of the incident light after deducting any loss due to absorption and reflection. Hence the rotation of the analyser by 90° changes either the intensity or the tint into the complementary.

The intensity of the colour depends, as we have seen, on the number

of the order of Newton to which the total retardation corresponds, which in turn depends both on the thickness of the plate and the amount of the double refraction. The colours of the second order are most brilliant, hence the advantage of a retardation of about two wavelengths only.

Both monoclinic gypsum (selenite, hydrated calcium sulphate, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$) and trigonal quartz (silica, SiO_2) are brilliantly polarising mineral crystals, the former in the thin cleavage films which have already been so much referred to, and the latter both in thin plates such as those of ordinary rock sections (which are usually of 0.04 millimetre thickness or thereabouts) as well as in thicker section-plates up to seven millimetres thick; for although the double refraction is about the same for both minerals, $N_{\lambda-\alpha} = 0.0092$ for selenite and $N_{e-\omega} = 0.0091$ for quartz (positive uniaxial), the latter in addition possesses the property of rotation of the plane of polarisation, which introduces further colour phenomena when the thickness increases up to the limit of seven or eight millimetres, and the plate is cut perpendicularly, or nearly so, to the optic axis. The fact of the brilliant polarisation of quartz in thin films, however, due to the double refraction, is of great importance in petrology, for it is characteristic of the mineral to show in such sections first and second order colours, and without twinning in the vast majority of cases; hence quartz is the most easily recognised of all mineral crystals under the polarising microscope. The latter is essentially a microscope carrying also a polarising and an analysing Nicol prism, the former below and the latter above the object stage; Chapters LI. and LII. will be devoted to its construction and use. Crystals having feebler double refraction than gypsum show brilliant colours in thicker plates, a greater distance requiring to be traversed to effect the same retardation. On the other hand, crystals of much higher double refraction than selenite, such as calcite (negative uniaxial), one of the most highly doubly refractive of substances, for which $N_{\omega-\epsilon} = 0.1719$, only show brilliant colours in films excessively thin, ordinary thin plates exhibiting white of the higher orders, or only very feeble and impure colours containing much white light.

When the section-plate is thicker than corresponds to a retardation of 7λ , white light of the higher orders is observed instead of colour. When rotated between crossed Nicols, therefore, such a doubly refracting crystal plate merely becomes four times dark (at the positions where its two rectangular vibration directions coincide with those of the Nicols) and four times light (at the 45° -diagonal positions); this is the ordinary case observed in determining extinction angles already referred to on p. 649.

The conditions in the very frequently occurring and typical case of gypsum (selenite) will be rendered clear by Fig. 543, which shows the shape of the section-plate (cleavage film) parallel to the symmetry plane $b = \{010\}$ of the monoclinic crystal, the crystallographic axes a (inclined) and c (vertical), and those of the optical indicatrix (γ the first median line and α the second median line). There are also indicated on the figure

the positions of the poles of the three pairs of principal faces of the orthozone perpendicular to the section, which bound the latter, namely, the ortho-pinakoid $a = \{100\}$, the basal pinakoid $c = \{001\}$, and the primary orthoprism $d = \{101\}$. An actual crystal of gypsum has already been represented in Fig. 397 (p. 422).

The dotted vertical and horizontal lines may be taken as representing the planes respectively of the polarising and analysing Nicols. The vibration planes, extinction directions, of the crystal are represented by the strong continuous lines, the first and second median lines. The light leaves the polariser and enters the crystal vibrating, let us say, parallel to the vertical dotted line. A vibration direction (first or second median line, the γ or α axis of the indicatrix)

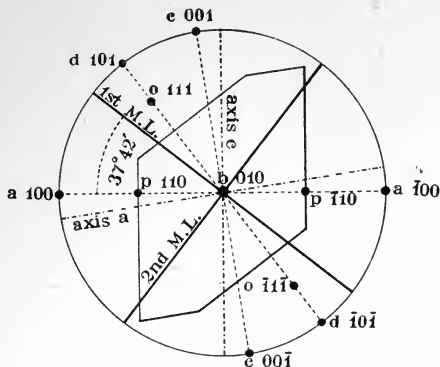


FIG. 543.—Optical Characters of Gypsum Cleavage Film.

of the crystal not being coincident with this, but, say, at 45° thereto, the plate having been rotated to bring them so, the rays divide in the substance of the crystal, into two rays vibrating parallel to the first and second median lines respectively; for rays along each of these directions are the resultants of two components vibrating vertically and horizontally. The horizontal component in each case is parallel to the analyser, when the latter is crossed to the vertical polariser, so on reaching the analyser the two rays vibrating parallel to the two median lines are resolved back again into their vertical and horizontal components, and the latter component is in each case transmitted by the analyser. Hence, the crystal plate, if adequately thin for the difference of path between these two transmitted horizontal component rays to be only two or three wave-lengths, appears brightly coloured owing to the interference of the two rays; it appears less brightly when thicker than corresponds to 3λ but not so much as affords 7λ difference, while if thicker than corresponds to 7λ it appears brightly illuminated with white light, on the dark field of the crossed Nicols in each case, the maximum colour or white light illumination appearing when the median line directions are at exactly 45° to those of the Nicols. The thickness of plate required to give the same colour will be the same for all such cleavage plates of gypsum, all being parallel to the symmetry plane of the monoclinic crystal.

In general, the thickness of a crystal plate required to give the same colour will be the same for all sections of the same substance cut in the same direction. But it will be obviously different for differently orientated section-plates, as the difference of refraction, and therefore of velocity, of the two rectangularly vibrating rays will vary.

Practical Use of a Sensitive Gypsum Plate.—A very thin even

selenite film which gives in the dark field of the polariscope the red of the first order is particularly useful for determining the character of the double refraction and orientation of specific axes of the indicatrix of a microscopic or other thin crystal; that is, it enables us to distinguish which of the two directions of extinction in a doubly refracting crystal plate corresponds to the lower, and which to the higher, refractive index. For when the field of the polariscope, using crossed Nicols, is filled with the red tint, if such a crystal be introduced in addition, say on the stage of the polariscope or polarising microscope, the selenite being inserted in a slot in the optical tube provided for the purpose, the colour must be altered within the limits of the crystal boundary. By the nature of the new colour thus produced in this part of the field we can ascertain whether the crystal is adding to or subtracting from the action of the gypsum. When the red is depraved to the faint yellow or gray of the first order it must be obvious that the crystals are acting in opposite senses, while if the red be enhanced to the blue or bright yellow of the second order the crystal section must be adding its effect to that of the gypsum. Indeed, the crystal may even be so very feebly doubly refractive as to give when alone in the dark field only the gray or other early tints of the first order even in thick plates, yet it will produce a marked effect on the red selenite field.

The directions of the two axes α and γ of the optical indicatrix of the gypsum are, of course, known and marked on its margin, and the extinction directions of the crystal will have been determined. A rise in order of tint on introducing the crystal so that its axes and those of the gypsum are parallel consequently means that the crystal is arranged with respect to the gypsum so that corresponding axes, greater or smaller, are in coincidence, producing superposition effects; whereas a fall in tint indicates that the smaller axis or direction of smaller refractive index of the one is parallel to the greater axis or index direction of the other.

Relations of Double Refraction and Thickness of Plate to Retardation.—The relations between the difference of path d , the thickness of the crystal plate t , and the double refraction at any point corresponding to which the two refractive indices concerned are μ_1 and μ_2 , are derived as follows. The number of vibrations occurring in the thickness t for the ray with the wave-length λ_1 is t/λ_1 , and for the ray with wave-length λ_2 it is t/λ_2 ; the difference in the number of vibrations is thus $t/\lambda_1 - t/\lambda_2$, and the difference of path d , referred to the wave-length in air λ_0 , is:

$$d = \left(\frac{t}{\lambda_1} - \frac{t}{\lambda_2} \right) \lambda_0.$$

Now the refractive indices μ_1 and μ_2 of the two rays in relation to air are:

$$\mu_1 = \frac{\lambda_0}{\lambda_1}, \text{ and } \mu_2 = \frac{\lambda_0}{\lambda_2}.$$

Substituting μ_1 and μ_2 for these values we have:

$$d = t (\mu_1 - \mu_2).$$

That is, the difference of path is equal to the thickness of the plate multiplied by the difference of the two refractive indices (the effective double refraction) at the point in question.

Relation between Thickness of Crystal Plate and Order of Interference Colour.—The wave-lengths of the visible spectrum (see table, p. 566, Chapter XXXV.), lie between 0.0007950 millimetre, the wave-length of the dark red rubidium line δ in the extreme red, and 0.0003934 millimetre, that of the solar violet calcium line K. When a doubly refracting crystal plate is so thin that the difference of path between its two rectangularly vibrating rays is less than 0.000100 mm. no colour is produced in the polariscope, but the gray of the first order, which becomes deeper and deeper as the plate becomes thinner, until just before the extremity of possible thinness is reached it is black. With a path-difference of 0.000100 mm., $\frac{1}{3}$ th of a red wave or $\frac{1}{4}$ th of a violet one, a feeble lavender tint makes its appearance. From this to a thickness affording a path-difference of 0.000200 mm. a pale violet white or bright gray prevails, then a bluish, and subsequently a greenish, white appears, until at 0.000300 mm. path-difference we have a bright yellow, that of the first order; at 0.000400 mm. the bright orange of the first order appears, and for 0.000500 mm. a bright red, the red of the first order, which order terminates conventionally just below a thickness corresponding to a path-difference of 0.000600 mm. The most brilliant part is the red of the first order, the maximum brilliance corresponding to a path-difference of 0.000530 mm. Now this is the wave-length of the bright green of the ordinary spectrum, and the red is here displayed predominating at a maximum in the case of a crystal film because the complementary bright green of the wave-length just mentioned is extinguished between crossed Nicols (the vibration planes of the crystal being at 45° to the Nicols) by the interference of the two rays having such a difference of path. When the Nicols are parallel this bright green itself is actually observed, composed of vibrations near 0.000530 mm. wave-length, which are transmitted while the red is extinguished.

The violet which is produced at the conventional end of the first order, about 0.000600 mm. is known as the "sensitive violet," or "transition tint," inasmuch as it passes with a very slight change of difference of path on one side (first order) or on the other side (beginning of second order) into red or blue respectively. The first order spectrum thus consists in their respective sequence of gray, lavender gray, bluish gray, greenish white, yellowish white, yellow of the first order, orange of the first order, red of the first order, and transition violet.

The blue of the second order is most brilliant for a difference of path of 0.000660 mm., the green of the second order for 0.000800 mm. difference, orange and yellow of the second order respectively for 0.000850 and 0.000900 mm., and finally the brilliant red (most brilliant part of the whole series of spectra) of the second order for a difference of path of 0.001060 mm.; this is double that (0.000530 mm.) for the first order red and about 2λ for the bright spectrum green near E of the solar spectrum, which is consequently extinguished under crossed Nicols,

leaving the complementary red to predominate. We have next a transition violet of the second order for a difference of 0.001200 mm., and then the beginning of the blue of the third order, for a thickness of plate corresponding to a difference of path of 0.001130 mm., and so on through the third and subsequent orders, until eventually we reach first the white of the higher orders and finally pure white light.

Fox's Wedge.—One of the most useful objects to possess, in order to familiarise oneself with the more important earlier orders of Newton, is an artificial stepped wedge of mica, made up of twenty-four superposed films of this very readily cleaved mineral, the wonderful cleavage of which was discussed in Chapter XXVII. The films are cemented, by Canada balsam

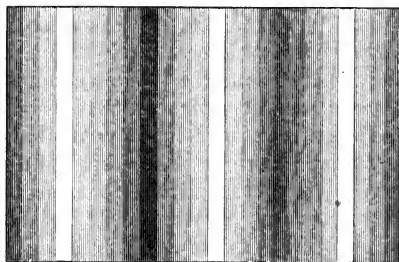


FIG. 544.—Fox's Wedge in Polarised Light between Crossed Nicols.

in benzene, one after another on the top of each other on a circular glass plate of the regulation size, $1\frac{7}{8}$ inch diameter, for projection polariscopical objects; they are so arranged, step-wise, that each does not quite cover the one previously laid down, being one-sixteenth of an inch shorter, leaving a parallel-edged strip of that amount uncovered as shown in Fig. 544. The films are all such as afford one-eighth of a wave retardation for sodium light, indeed they are cut from the same $\frac{1}{8}$ th-wave film, that is, one of the two rays transmitted by this doubly refracting crystal film is retarded behind the other to the extent of one-eighth of a wavelength. The mode of selecting such a $\frac{1}{8}$ th-wave film, and the theory of its behaviour in polarised light will be discussed later in the chapter. Such a wedge was first constructed by C. J. Fox, and hence bears his name. When placed in the dark field of the polariscope, between crossed Nicols, it exhibits the first three orders of Newton's rings in eight graduated strips for each order. The films are cut out of the large $\frac{1}{8}$ th-wave sheet so that the angle of the strips is 45° to the β and γ axes of the optical indicatrix. The second band gives a quarter-wave difference of phase, and the fourth band a half-wave difference, hence an accurately constructed wedge of this kind is a valuable help in constructing quarter or half-wave mica plates, which we shall see later are of great use in crystallographic optics.

Instead of arranging the mica so that parallel narrow rectilinear strips possess successively greater and greater thickness, the layers may be arranged concentrically, so that the strips have a circular annular form, increasing in thickness as the centre is receded from, as shown (with more rapid gradations) in Fig. 545. A reproduction of the effect exhibited in Newton's rings, by crystallographic double refraction, but with the colours in

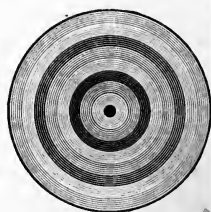


FIG. 545.—Mica Circles of Graduated Thickness in Polarised Light.

definite narrow annuli instead of graduating more imperceptibly one into the other, is thus afforded.

Concave Gypsum or Quartz Plates.—The crystallographic reproduction of Newton's rings may, however, be effected with the natural gradation very readily and conveniently for projection purposes, by grinding one side of a parallel plate of a doubly refractive crystal slightly concave, the concavity being of spherical curvature of large radius (the larger the greater the double refraction) like the lens in the ordinary Newton's rings experiment. A film of gypsum (selenite) or a plate of quartz serve the purpose best. Fig. 546 is a photographic reproduction of the rings produced, with the projection polariscope shown in Fig. 515 (p. 609), by such a concave selenite film placed on the object stage, the film being cut circular concentrically to the centre of the depression, about an inch and a quarter in diameter, and mounted with balsam between two circular glass plates of the standard size ($1\frac{7}{8}$ inch diameter) to fit the usual mahogany carrier-frames. The Nicols being crossed and their vibration-directions at 45° to those of the concave crystal plate, the effect on the screen is very beautiful and might well be mistaken for a projection of

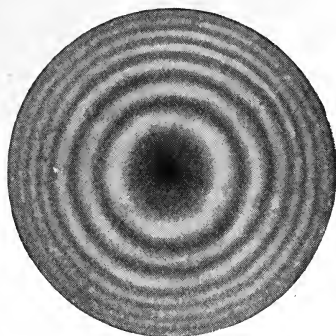


FIG. 546.—Concave Gypsum in Parallel Polarised Light.

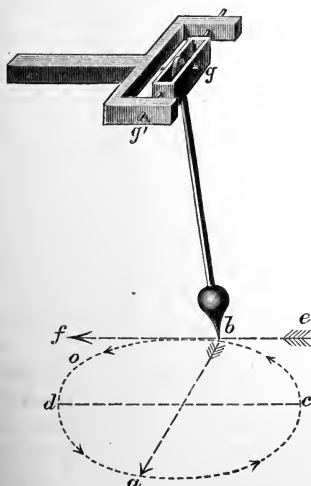


FIG. 547.—Pendulum Experiment illustrating Production of Circular Orbit.

Newton's rings in the ordinary manner, except that the intensity of the colours is probably greater than is commonly obtained on the screen with the latter. When a quarter-wave mica plate, to be described in the next section, is also placed with it on the stage facing the analyser, and the latter is rotated, the rings move out from, or inwards towards, the centre, according to the direction of rotation.

The Quarter-Wave Plate.—If a pendulum be suspended from a rectangular pair of gymbals, g and g' being their axes, so as to be quite free to swing in any plane, as in Fig. 547, the path of its bob, if swinging on one gymbal g only, will be ab ; but if the swinging occur only on the other g' , the path will be cd . Imagining these to be two planes of polarisation at right angles to each other, and the bob a vibrating ether particle, it is easy to see that if, on arriving at the position b at the end of a vibration along ab , when it is at rest just before starting on its return journey to complete the vibration, it receive

an impulse from another source to move parallel to cd , that is, along ef , the composition of the two forces will drive it into a circular orbit bod . Such an impulse parallel to the plane cd is given by a wave movement which is one-quarter of a wave behind that along ab . For the latter has moved a half-wave from a to b , and is just ready to start back with adequate force to propel it to a again, the whole wave comprising also this return to a ; while the former must have travelled along a half-wave from d to c and then have reached half-way along its way back to d , that is, it must have completed three quarters of its whole double swing, in order to affect the particle b and be in the position to impel it to f . It is clear, then, that at b the second impulse must be a quarter of a wave behind the first impulse in order to propel the particle into the circular orbit.

This simple mechanical experiment with a pendulum illustrates the composition of two rectilinear vibrations into a circular vibration by the use of a quarter-wave plate. Equal rectangular vibrations in the same phase compound into one plane vibration inclined at 45° to the original planes. When the phase differs by half a wave the vibrations of the resulting single plane-polarised ray are also at 45° to the original planes, but this diagonal plane is 90° removed from the 45° -plane just referred to, which causes the complementary colours to be shown to those afforded by the latter resultant. But if the equal rectangular vibrations differ in phase by a quarter of a wave, we now see that a circular orbit results. Intermediate differences of phase afford elliptic orbits.

A half-wave plate consequently reverses all polarisation effects. It makes the bright field dark, or the dark field to become illuminated, and on its addition into the system changes any colour produced with a polarising object into the complementary colour. A quarter-wave plate inserted at 45° to the crossed Nicols, instead of giving colour as a thicker plate would do, gives a bright field with a feeble tint, and the bright field remains so when either of the Nicols is rotated, the tint merely changing from slightly yellowish gray to faintly bluish gray, for positions of the Nicols 90° apart. If sodium light be used the field remains equally illuminated throughout the revolution of the Nicol.

When the experiment is a projection one, and the issuing light is passed through a double-image Wollaston or Rochon calcite prism, a circular aperture in front of the lantern condenser being used as the object projected, as already described in Chapter XXXIX. and illustrated in Fig. 518 (p. 612), the two images of this aperture thrown on the screen will always be equally brilliant. That the light is, however, polarised is proved by inserting an additional polarising object, when such colour as it shows on the screen when the quarter undulation plate is temporarily withdrawn will be changed by a quarter of the Newtonian order which it is exhibiting, when the quarter-wave plate is again inserted in position.

The most convenient substances of which a quarter undulation plate may be made are gypsum (selenite) and mica, on account of the ease with which they cleave into thin plates. Muscovite mica answers the purpose best of all, for this mineral may very readily be cleaved into

plates thinner than sheets of paper, and approximately perpendicular to the acute bisectrix of the optic axial angle.

It has been shown in Chapter XXVII. that although muscovite mica is monoclinic, the axial angle β is within a few minutes of 90° , and that the acute bisectrix of the optic axial angle is inclined at less than 2° from the normal to the plane of cleavage, the basal pinakoid $c = \{001\}$; this bisectrix is the axis a of the indicatrix, as mucovite mica is negative in double refraction. The interference figure in convergent polarised light consequently appears apparently symmetrical to the centre of the field of the polariscope, as if the mineral were rhombic. This fact assists us greatly in preparing our quarter undulation plate. For if we select a plate sufficiently thin to show the first ring as a complete ellipse-like lemniscate, as shown in Fig. 548, enclosing both the optic axial brushes, the two rays of light travelling through the plate normally to it will be vibrating in planes which are perpendicular to each other, parallel to the β and γ axes of the indicatrix, and in phases differing by one-quarter of a wave, for light of the middle part of the spectrum, such as sodium light. This will be fully explained a few pages further (page 673).

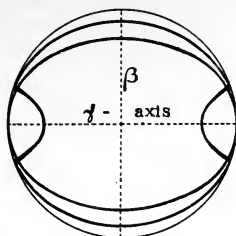


FIG. 548.—Interference Figure in Convergent Polarised Light given by a Quarter-Wave Mica Film.

A good even film, as thin as is available, should be selected from the sheets sold in "books" by the mica dealers; this should then be split further by the careful insinuation of a needle point. After a dozen or more films have been so procured, they should be placed one by one between a couple of glass plates to keep them flat, and examined in parallel light in the polariscope. The particular film required, when inserted with the axes of the indicatrix at 45° to the planes of the crossed Nicols, instead of exhibiting colours when the analyser is rotated, like thicker plates do, will remain unaffected if sodium light be used, and almost so when white light is employed, merely changing as already stated from a faint bluish shade in one part of the revolution to a pale fawn colour in the other. And when the convergent system of lenses is added to the polariscope it will afford, under crossed Nicols, the figure with one complete ellipse-like ring surrounding both optic axes shown in Fig. 548, when the plate is at 45° to the Nicols; when parallel to the crossed Nicols the brushes will join up to form the usual biaxial rectangular cross, the single ring remaining unchanged, and indicating the quarter-wave retardation.

Moreover when a Noremberg "doubler" is available as described in Chapter XLIV. and shown in Figs. 608 and 609, and the film is laid on the silvered mirror at the base of the apparatus in the right azimuth, the quarter-wave retardation is doubled to a half-wave; the Nicol analyser adds another half-wave as already explained, and thus the total effect of a whole wave retardation is obtained, so that the light ought to be fully transmitted with the Nicol crossed to the polariser-plate.

When the Nicol is arranged parallel to the polarising plate, the violet transition tint should be given, thus affording an effective test for a quarter-wave film.

The light leaving the quarter-wave crystal film is thus circularly polarised, so that the resolution once more into two plane vibrations by the analyser may occur equally well at any part of the circular orbit;

one of these resolved rays is then alone transmitted by the analyser, that parallel to its own vibration plane, while the other will obviously be extinguished. Absolute equality of light as the analyser is rotated is only possible, even with a very accurately chosen quarter undulation plate, for the one monochromatic wave-length used in making the selection, usually sodium light, other wave-lengths on each side being slightly admitted, giving rise to the slight fawn and bluish tints observed in white light, at complementary positions 90° apart during the rotation.

The quarter-wave mica plate should be marked to indicate the direction of that axis (γ) of the indicatrix which joins the positions of the two optic axes. One of its chief uses is to determine the sign of the double refraction of a crystalline substance, before the prolonged operations of the determination of the refractive indices are undertaken; for a characteristically different effect on the interference figure given by a positive and by a negative crystal in convergent polarised light is produced by the introduction of a quarter-wave plate between the crystal plate and the analyser, as will be fully described in Chapter XLVII. If the quarter-wave plate, therefore, be required for this purpose, for use with the observing polariscope or polarising microscope, it is cut into an elongated oblong shape, with the long sides parallel to the line joining the optic axes, and is mounted with balsam between two similarly shaped truly plane and parallel-surfaced glass strips, about 6 centimetres long and 11-12 millimetres wide. The shape of the plate is thus an adequate indication of the direction of the γ axis of the indicatrix passing through the optic axes.

If the quarter-wave plate be required for lantern polariscopical projection work a larger film is used, and mounted between two circular glass plates of the usual size, $1\frac{7}{8}$ -2 inches diameter, for insertion in the rabbet of a standard mahogany mount 4 by $2\frac{1}{4}$ inches in size, the circular aperture of which, that of the containing rim of the rabbet, has the regulation diameter of $1\frac{5}{8}$ inches. The composite disc of quarter-wave film and glass is held in position in the rabbet of 2 inches diameter by a ring of brass wire with free ends acting as a spring. The disc should be rotated into the position in which the quarter-wave plate is required, before pressing the wire ring home, and it is advisable that the periphery of the disc should be engraved with a dot or other mark at both ends of the diameter joining the positions of the optic axes.

As a matter of convenience it is a great advantage to have two such quarter-wave plates for projection purposes, one mounted with this marked diameter parallel to the length of the frame, and the other with it at 45° thereto, a position in which it is much required. As two quarter-wave plates are frequently simultaneously used in projection experiments, the possession of a second film is, in fact, essential anyhow. The special stage to take both these quarter-wave plates at once if desired, with a third polarising crystal or other object plate between them, all three mounted in their mahogany frames, is shown under the optical bench in Fig. 515 (p. 609).

When the quarter-wave plate is arranged with its vibration directions,

the indicatrix axes β and γ (the latter the diameter joining the optic axes just referred to and the former the perpendicular diameter), parallel and perpendicular respectively to the plane of vibration of the polarising Nicol no effect is produced, the plane-polarised beam passing through unaffected. But when a crystal plate with vibration directions at 45° precedes the quarter undulation plate, the vertically or horizontally vibrating beam from the polariser is resolved by the latter into two diagonally (45° to the vertical and horizontal) vibrating rays, and the quarter-wave plate arranged vertically and horizontally again resolves these. Thus the use of a quarter-undulation plate is generally only effective when its planes of vibration are at 45° to those or that of the preceding object, whether it be a polarising crystal or the Nicol polariser itself. And as the preceding polarising object will sometimes have its planes of vibration vertical and horizontal and sometimes diagonally arranged at 45° , according to the nature of the experiment, it is very convenient to have two quarter-wave plates always ready, one mounted diagonally and the other parallel to the frame edges, so as to be always prepared for either case.

The Quartz Wedge.—It has been shown in Chapters XXIII. and XXXVIII. that quartz is a positive uniaxial substance belonging to the trigonal system, having a refractive index for the ordinary ray of $\omega = 1.5443$, and for the extraordinary of $\epsilon = 1.5534$, the amount of the positive double refraction being thus $\epsilon - \omega = 0.0091$. This amount is relatively small, only one-nineteenth of that of negative calcite (for which $\omega - \epsilon = 0.1719$). The ordinary ray travels in quartz, as in all uniaxial crystals, along the singular axis, the unique optic axis, and vibrates in the circular section perpendicular thereto, so that ω corresponds to vibrations perpendicular to the axis, and ϵ to vibrations parallel to the axis.

When a section-plate is cut, ground, and polished from a prismatic quartz crystal along the direction of the prism, parallel to the optic axis, vibrations in the direction of its length will correspond to ϵ , the greatest refractive index as in the case of the mica quarter undulation plate, while those perpendicular to the length and therefore to the axis will correspond to the minimum index ω . When the section is made to taper slightly, that is, wedge-shaped, as represented in elevation at *a* in Fig. 549—the two faces being true planes meeting in an edge which is perpendicular to the length of the wedge, and inclined at a very small angle to each other, and prepared so carefully that the greatest possible tenuity shall be attained at the thin end—such a wedge of quartz will exhibit in the dark field of the polariscope arranged for parallel light, and best when the wedge-length is arranged at 45° to the crossed Nicols, several of the orders of spectra of Newton's scale, starting from the thin end, as shown diagrammatically at *b* in Fig. 549. The wedge may, in fact, be imagined to be a strip, radiating from the centre, cut out of the concave quartz disc (a plate one side of which has been ground concave) referred to on p. 659, except that both surfaces are now true planes slightly inclined; or it may equally well be considered as a strip

cut out of the Newton's ring apparatus itself, except that the tapering film is here quartz instead of air, and that the second surface is a true plane instead of being a curved surface of large radius. Also the interference is due to the difference of velocity of the two rays produced by double refraction, instead of being due to reflection from the two surfaces of the wedge. The effect of the regularly increasing thickness is to increase the retardation of one ray, the slower extraordinary ϵ ray vibrating parallel to the length of the wedge, behind the other, the ordinary ray ω vibrating parallel to the edge of the wedge.

The wedge requires to be very carefully cut to show the first black band, owing to the brittleness of quartz and the consequent difficulty of attaining great tenuity with the mineral. Only one, cut by Messrs. Steeg and Reuter, out of several in the possession of the author, exhibits the band to perfection. Usually the wedge begins at the thin end by

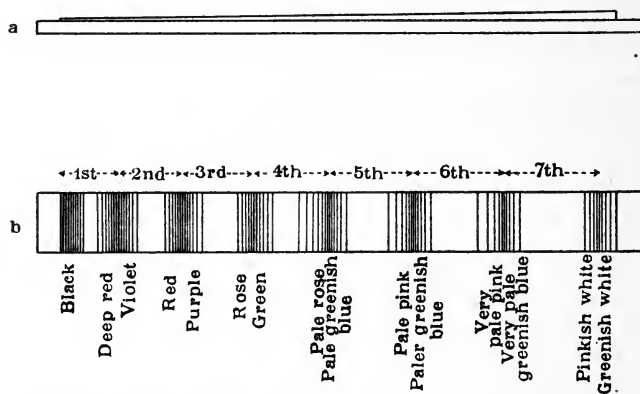


FIG. 549.—The Quartz Wedge, in Section at *a*, and the Seven Orders of Spectra afforded by it in Parallel Polarised Light at *b*.

showing the faint gray and then the yellow of the first order spectrum. The junction of the first and second orders is a strikingly brilliant band coloured deep red on the thinner side and deep blue on the thicker, with dark violet or purple between. Each successive junction-band of two successive spectra becomes less striking than the one before it; the second appears red on the thinner side and purplish blue on the thicker, without the deep purple centre; the third is rose pink on one side and green on the other, and so on, with accession of more and more white light, until the end of the seventh order is reached, when the tint is almost pure white. The succession of spectra is diagrammatically represented and the order of colours stated in the lower part *b* of Fig. 549.

Obviously, the smaller the angle of the wedge the more extended are the spectra, and a wedge which only just shows the seven orders in about two inches of length is perhaps the most useful of all, although for some purposes one showing only the first three orders in this length is very

convenient. It is advisable to have at least one such quartz wedge of the same elongated dimensions as the quarter-wave mica, namely, 6 centimetres by 11 to 12 millimetres. It is mounted on a glass plate of the same width but a little longer, and in the case of the thinnest showing the black band the quartz is covered with a similar thin glass plate to avoid injury to the fine edge, the mounting being in all cases with Canada balsam.

The use of the quartz wedge in connection with optic axial interference figures will be discussed in Chapter XLVII.

Crystal Plates in Convergent Polarised Light.—It is a very natural step now from the colour phenomena of crystal plates, wedges, and concave plates in parallel polarised light to those exhibited when the light rays are made to converge. A further factor is now introduced, namely, different obliquity of the rays in the various parts of a conical beam of light. Hollow cones of ever-widening angle, one outside another, succeed each other as the axis of the cone is departed from, corresponding for a specific thickness of plate and for light of a definite wave-length to differences of phase between the two rays travelling along the surface of each cone, with velocities varying differently with the angle of the cone, of whole wave-lengths, that is, to λ , 2λ , 3λ , and so on, of retardation.

Let us take the simplest case first, that of a plate of a uniaxial crystal cut perpendicularly to the trigonal, tetragonal or hexagonal axis. We have seen that along this axis there is no double refraction, while in all directions at right angles thereto there is a maximum of double refraction, and that the optical ellipsoid is one of revolution. In the diagram at *a* in Fig. 550 OC represents the optic axis, and the three cones one outside another may be taken to represent conical surfaces along which, for a section-plate AB of some convenient thickness and for monochromatic light of some specific wave-length, the difference of phase between the ordinary and extraordinary rays ω and ϵ —due to progressive difference of velocity between the two rays as the axis of no double refraction is left and the maximum difference for rays travelling in the circular section is approached, and also as the thickness of plate traversed increases with the obliquity—is respectively λ , 2λ , and 3λ . Such a section-plate of the crystal will cut the cones in circular sections α , β , γ , shown in perspective at *a* and in plan at *b* in Fig. 550, these concentric circles becoming closer as the centre is receded from, as in the case of Newton's rings. At the centre, where there is no double refraction, the ordinary and extraordinary rays are composed of

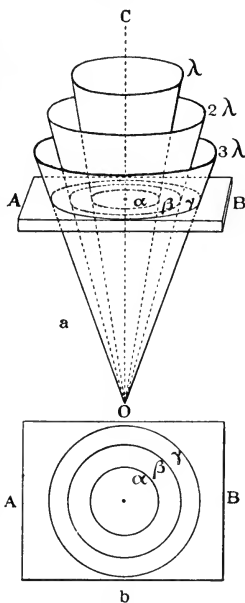


FIG. 550.—Passage of a Conical Beam of Light through a Uniaxial Crystal Plate perpendicular to the Axis.

vibrations both sets of which are perpendicular to the axis, their direction of propagation, and parallel to the circular section of the optical ellipsoid; they travel, therefore, with the same velocity, so that the waves will be in the same phase. They correspond to the same refractive index ω , and are thus identically similar, except that their vibrations may occur in any azimuth of the circular section, there being no tendency to vibrate in one principal section (plane containing the axis) rather than another. After traversing the plate along the surface of the cone λ , however, and more still at the cones 2λ , 3λ , and successive ones, a difference has manifested itself, due both to the fact that double refraction has revealed itself, involving different velocity and refractive index of the two mutually rectangularly vibrating rays into which the light is divided, and that with increasing distance from the centre the rays have longer and longer distances to travel in which such retardation can develop. One ray vibrates still in the circular section and perpendicularly to the principal section containing the axis and to the ray, and affords the index ω ; the other vibrates at right angles thereto in the principal section and perpendicularly to the direction of propagation, and with refractive index more and more removed from ω in the direction of ϵ as the centre is receded from. The retardation of the one behind the other (whether ω or ϵ is the slower depending on which is the greater refractive index) is equal to exactly one wave-length λ at any point on the circle of intersection α of the first cone by the exit surface of the plate, to 2λ at any point on the second circle β in Fig. 550, to 3λ on the third circle γ , and so on.

Consequently, when the crystal plate is observed between crossed Nicols in a conical pencil of rays, produced by a strongly converging lens-system, these circles should appear bright and the intervening annuli dark. But we have to remember that the analysing Nicol, as explained on page 649 in connection with the occurrences in parallel light, also introduces a change of phase of half a wave-length, so that the circles corresponding to whole wave-lengths of retardation of one ray behind the other really appear dark in the monochromatic light employed, and the intervening spaces bright. The centre is not only a black spot, as in Newton's rings, but is also the centre of a black rectangular cross radiating through the circular rings, marking the directions of the planes of vibration of the Nicols. For both at the central part where there is no double refraction, and in the cross, the dark field prevails, the rings having their maximum clearness in the 45° diagonal positions.

If the circular rings α , β , γ shown in Fig. 550 were produced in yellow sodium light, those for red light would be wider apart, and those for blue light closer together, the distance varying regularly with the wave-length according to a law similar to that already explained for Newton's rings. If ordinary white light be employed, the circles will represent the colour which is extinguished, and therefore along such circles the complementary residual colour will predominate. Hence the black rings of monochromatic light become circular spectra in white light, just as in the case of Newton's rings.

For a biaxial crystal very similar considerations apply. The two optic axes will be, within the limits of error appertaining to the determination of the optic axial angle, lines of no double refraction, but the curves of λ , 2λ , 3λ , and so on, of retardation will now be of oval section, corresponding to the lower symmetry of rhombic, mono-

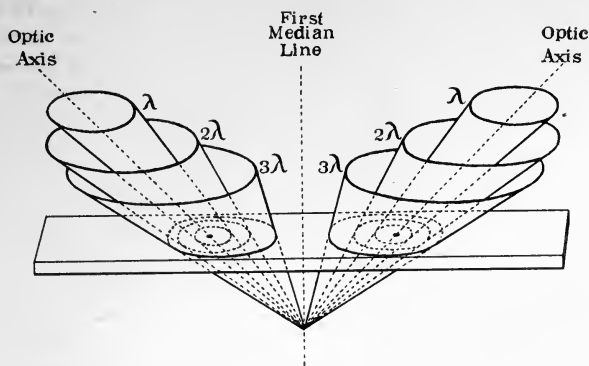


FIG. 551.—Passage of Convergent Polarised Light through a Biaxial Crystal Plate perpendicular to the First Median Line. Cones of Equal Retardation about the Two Optic Axes.

clinic, and triclinic crystals. The double refraction along any line in each case is measured by the difference between the axes of the section of the indicatrix perpendicular to it. Fig. 551 will help to render the case of a biaxial crystal clear. The sections of the two series of cones of equal retardation about the two optic axes, by a plate perpendicular to the bisectrix of the acute angle between the latter, are indicated in dotted lines.

Before passing to the further consideration of the more complicated biaxial phenomena, another mode of considering the uniaxial case is instructive.

Suppose $abcd$ in Fig. 552 to be a section of a uniaxial plate perpendicular to the optic axis CC' . The section of the plate shown in the figure is a principal section, as it is parallel to the unique optic and singular crystallographic axis. Assuming the Nicols crossed and the particular section $abcd$ arranged at 45° to the vibration directions of the Nicols, and that monochromatic light is employed, a central ray in the converging bundle passing along CC' from C' to C , will suffer no double refraction, and the field in the centre will consequently appear as if no crystal plate were there, namely, dark. But a ray such as ef , inclined to CC' , will be more or less doubly refracted, the plane polarised light from the polarising Nicol being resolved into two rays fg and fh , an ordinary and an

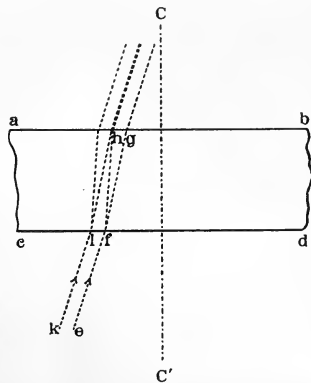


FIG. 552.—Passage of Convergent Polarised Light Rays through a Uniaxial Crystal Plate perpendicular to the Axis.

extraordinary ray, vibrating at different rates, one (the extraordinary) in the section $abcd$ and the other (the ordinary) in a plane at right angles to the section. At h , however, the oppositely vibrating component of an adjacent ray kl will also emerge, and as all the rays on emergence follow a direction parallel to the original ray these two rays will proceed from h in company, but without interference owing to their planes of vibration being at right angles. The analysing Nicol again resolves each of these and permits the two components, one from each, which are vibrating parallel to its own vibration direction to pass, extinguishing the other two. As the two component rays thus transmitted by the analyser vibrate in the same plane they are in a condition to interfere, which they do in accordance with their relative phases and the total retardation of one behind the other, due to difference both of path and of velocity of vibration. When this amounts to half a wave-length, instead of the extinction which occurs at the centre where the difference is *nil*, light will appear, and the first dark circular ring will be formed at the locus of points where the retardation amounts to a whole wave-length, and so on, light recurring for annuli corresponding to odd half wave-lengths and darkness to whole wave-lengths of retardation, the ordinary phenomena of interference being thus reversed owing to the half wave-length of difference of phase introduced by the action of the analyser.

Similar points around the axis CC' , where components of the ordinary and extraordinary rays interfere to the same extent, must obviously lie on a circle, and such circles for which the difference between the pairs of interfering rays is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, and so on, must appear as bright rings in concentric succession, coloured according to the tint of the monochromatic light used; for the double refraction constantly increases as we

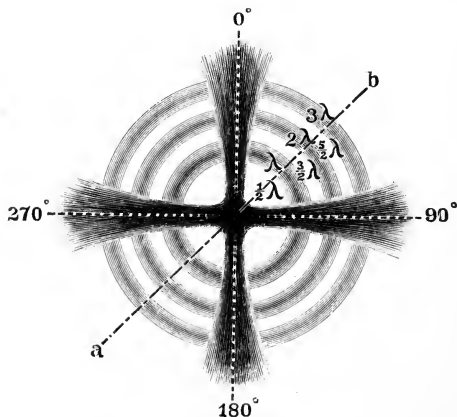


FIG. 553.—Interference Figure of Concentric Circles and Dark Cross afforded by Uniaxial Crystal Plate perpendicular to the Axis in Convergent Polarised Light.

recede from the centre and approach the maximum at right angles to the axis. The intervening annuli will be dark, the maxima of darkness occurring for circles of exactly λ , 2λ , 3λ , and so on, as at the centre where the difference is zero.

Hence, a figure of concentric circles of alternating light and darkness is produced, as shown in Fig. 553, where ab is the plane of the section $abcd$ of Fig. 552, the crystal plate itself being assumed to be parallel to the paper; the central point is that of emergence of the optic axis, and the directions 0° - 180° , 90° - 270° are those of the vibration planes of the polarising and analysing Nicols respectively.

What occurs in the section ab 45° to the planes of the Nicols, or the analogous one 45° on the other side at right angles to ab , occurs similarly but with less intensity in all other directions, as there is symmetry round the optic axis; hence, we obtain the concentric circular rings of maxima and minima of light. There is a gradual transition from light to darkness, so the figure in monochromatic light is a soft one without sharp circular lines, the central umbra of each dark band being bordered on each side with a penumbra, fading away into the bright ring adjoining, which also varies in intensity up to a central most brightly illuminated circular line, softly falling off in brilliance on each side.

The black cross, having arms sharp near the centre but spreading out and shading off near the extremities, is produced because the intensity of the light is due to the sum of the two amplitudes of the two component rays which reinforce or interfere, and this is a maximum at the two 45° diagonal positions and a minimum in the 0° - 180° and 90° - 270° directions, the Nicols being assumed to be crossed as throughout the above discussion of the phenomena.

The distance of the rings from each other for one and the same wave-length diminishes as the centre is receded from. On varying the wave-length the separation of the rings alters; they close up nearer to each other with shortening wave-length, as the blue end of the spectrum is approached, and open out when the wave-length is increased, as the red end is approached.

When white light is employed the extinction of a particular colour at any point, on account of the difference of path of the two rays for that colour amounting to a whole wave-length, leaves the complementary colour predominating, and so spectrum-coloured rings are produced instead of black ones. The cross, however, remains black.

Increase of the thickness of the plate of the same substance causes the rings to close up, less obliquity being required to afford the same relative retardation and length of path of the two rays on passing through the crystal; while diminution of the thickness causes their expansion, a greater obliquity being obviously required to afford the same relative retardation and length of path through the crystal.

Careful reproductions of actual photographs, taken by the author, of the interference figures in convergent polarised yellow light afforded by plates of two uniaxial crystals, those of trigonal calcite and hexagonal apatite, cut perpendicularly to the optic axis, are given in Figs. 554 and 555 on the adjoining Plate I. The crystal plate was very thin in the case of calcite, and very thick in the case of apatite, the former being a substance of very large double refraction ($\omega - \epsilon = 0.1719$ for Na light) and the latter one of small birefringence ($\omega - \epsilon = 0.0044$ for Na light). Fig. 555 is also similar to that afforded by a slightly thicker plate of calcite than that used for the production of Fig. 554. The two Nicols were crossed in each case, and their vibration directions were vertical and horizontal respectively when the photograph was taken for Fig. 554, but they were rotated 45° to these directions for Fig. 555. The arms of the dark cross thus follow these vibration directions of the Nicols.

Rotation of the section-plate does not alter the uniaxial figure. This will be clear from the interference figure of apatite in Fig. 555 of the adjoining plate; for the figure was photographed with the crossed Nicols simultaneously rotated to the diagonal positions as just stated, which is equivalent to a rotation of the plate for 45° . The effect is

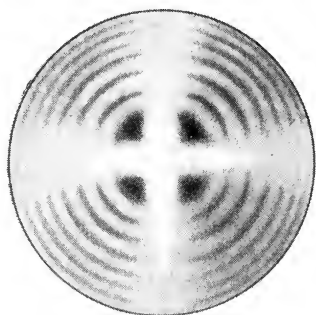


Fig. 556.—Interference Figure afforded by Calcite with Parallel Nicols.

merely to rotate the cross round 45° , its arms following the directions of vibration of the crossed Nicols. The figure itself is identical, and the only difference is as regards the position of the cross with respect to the edges of the page.

But rotation of the analysing Nicol to parallelism with the polariser causes the complementary phenomena to be produced, namely, in monochromatic light a bright cross, and black rings where under crossed Nicols bright rings had appeared, and in white light a white cross and complementarily coloured rings to those appearing with crossed Nicols. The reproduction of a photograph in white light is given in Fig. 556.

A Plate of a Biaxial Crystal cut perpendicularly to an optic axis affords very similar phenomena, but the rings or curves of equal retardation are not circles, although they often resemble circles very closely, as in the case of gypsum, the figure for which has already been shown in Fig. 493 (p. 589); they are really ellipse-like curves of the kind known in analytical geometry as lemniscates. The black cross is now only represented by a single bar, which, moreover, is only straight when the section-plate is arranged with the plane of the optic axes (one of the three principal planes of the optical indicatrix) parallel to one of the vibration directions of the crossed Nicols; in other positions the bar becomes a hyperbolic curve with expanded sails but sharply outlined vertex, the latter passing through and being pivoted on the site of the optic axis. The ellipse-like rings one outside another are symmetrical to the rectilinear direction of the straight black bar, but in the direction of the bar the pivot position occupied by the hyperbolic vertex, marking the optic axis when the plate is rotated, is nearer to the ring on one side than on the other, the relations being as the differences of the intermediate refractive index β from the two extreme indices; on one side the distance between the pivot position and the ring will be proportional to $\gamma - \beta$, and on the other to $\beta - \alpha$.

The first (innermost) dark ring in monochromatic light corresponds to a retardation difference, of one of the two rays which interfere, behind the other, of one wave-length, λ . The second ring corresponds to a difference of two wave-lengths, 2λ , the third to 3λ , and so on. As explained for a uniaxial plate, these whole-wave-length differences correspond to darkness rather than to light on account of the additional half-wave-length change of phase introduced by the action of the analyser.

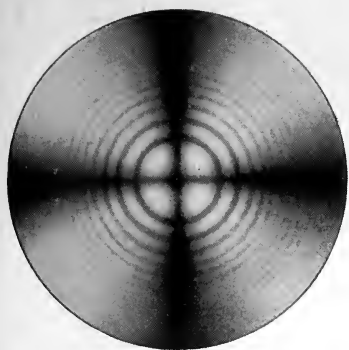


FIG. 554.—Uniaxial Figure given by Thin Plate of Calcite (Trigonal) perpendicular to the Axis. Vibration Planes of Nicols Vertical and Horizontal.

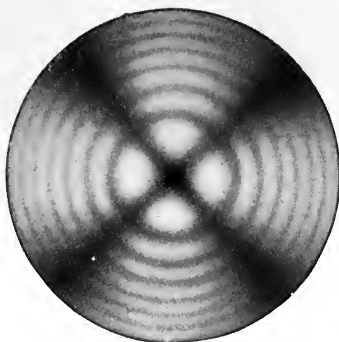


FIG. 555.—Uniaxial Figure given by Thick Plate of Apatite (Hexagonal) perpendicular to the Axis. Vibration Planes of Crossed Nicols arranged at 45° .

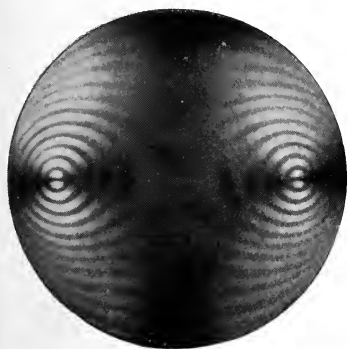


FIG. 557.—Biaxial Figure given by Cleavage Plate of Mica (Monoclinic) perpendicular to the First Median Line. Nicols Vertical and Horizontal.

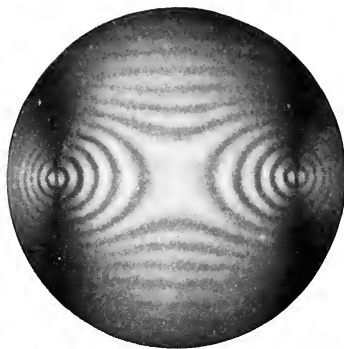


FIG. 558.—Figure given by the same Mica Plate when the Crossed Nicols have been simultaneously rotated 45° .

REPRODUCTIONS OF DIRECT PHOTOGRAPHS OF INTERFERENCE FIGURES
AFFORDED BY CRYSTAL PLATES IN CONVERGENT POLARISED YELLOW
LIGHT.



When the section-plate is cut perpendicularly to the bisectrix of the acute angle between the optic axes both sets of rings appear in the field, one on each side of the centre, the black bar of the two joining up to form a continuous straight line running through both optic axes, when the plane of the latter is arranged parallel to a plane of one of the crossed Nicols; it is crossed at the centre by the rectangularly transverse bar corresponding to that of the cross of the uniaxial figure, but relatively very much broader and less black and sharp than the one joining the optic axes, which is very clearly defined at and between the optic axes and until it gets near the margin of the field.

The figure afforded by a fairly thick cleavage plate of muscovite mica is shown in Fig. 557 (the third figure on Plate I.), which is the reproduction of a direct photograph by the author. It has already been explained that the cleavage plane of this variety of monoclinic mica is within 2° of being exactly at right angles to the acute bisectrix of the optic axial angle, which latter is sufficiently large, about 75° in air, to show the axial "eyes" very well separated in the field of the polariscope.

When the section-plate is rotated, the black cross of a biaxial figure opens out into two hyperbolic curves or "brushes," like the one just described as being shown, passing through the single optic axis, by a plate cut perpendicularly to one axis only, the vertex of each hyperbola being sharp and pivoted on the position of emergence of the optic axis. This will be rendered clear by Fig. 558 (the fourth figure of Plate I.), which is a reproduction of another photograph taken when the crossed Nicols had been simultaneously rotated (as is possible with the "Dick" crystallographic microscope constructed by Swift) for 45° , that is, to the diagonal positions. Such simultaneous rotation of the Nicols is obviously equivalent to the rotation of the plate, and is frequently much more convenient.

The effect of actually rotating the plate itself, however, the Nicols remaining fixed in their horizontal and vertical fixed positions, is shown in two stages in Figs. 559, 560, and 561. Fig. 559 represents the original position, the plane of the optic axes being parallel to the plane of vibration of one of the Nicols. Fig. 560 is the figure afforded when the plate has been rotated $22\frac{1}{2}^\circ$, and Fig. 561 when the rotation amounts to 45° . When the rotation has attained the midway 45° -position the figure is suitable for the measurement of the optic axial angle, by successive placing of the vertices of the brushes to the cross-wires, by rotation of the plate about a transverse axis perpendicular to the plane of the optic axes. For the vertical spider-line becomes a tangent to the vertex in each case, and if a pair of vertically parallel spider-lines are employed,

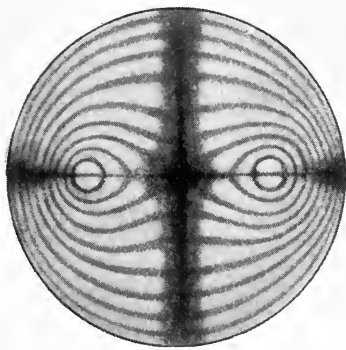


FIG. 559.—Biaxial Figure. Vibration Directions of Plate and Crossed Nicols Parallel.

as shown in Fig. 562, adjustable at a distance apart just greater than the thickness of the brush at the vertex, the latter can be delicately adjusted between them. The two vertices are preliminarily adjusted

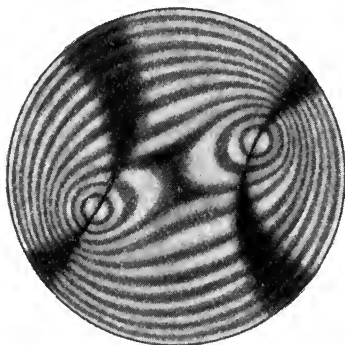


FIG. 560.—Biaxial Figure after rotation of Plate for $22\frac{1}{2}^\circ$.

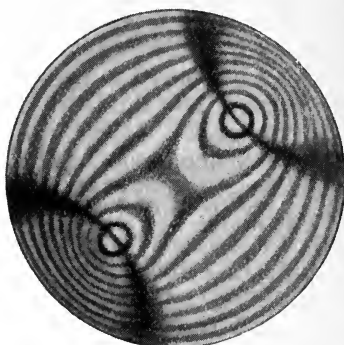


FIG. 561.—Biaxial Figure after rotation of Plate for 45° .

so as to be bisected by the single horizontal spider-line, and to remain so on rotation of the section-plate. A detailed description of the mode of measurement will be given in Chapter XLVII.

With plates of different thickness the optic "brushes" remain constant, except that the sharpness of their vertices becomes diminished as the plate becomes thinner than that showing several complete rings round each axis, which is the figure of suitable sharpness for the purpose

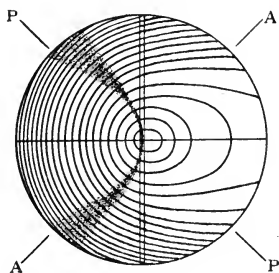


FIG. 562.—Adjustment of Optic Axial "Brush" to Spider-lines.

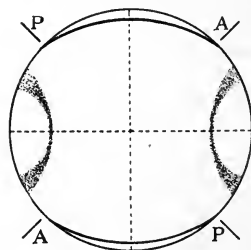


FIG. 563.—Figure given by Mica affording $\frac{1}{8}$ Wave Retardation.

of measurement of the optic axial angle. The rings rotate with the plate, maintaining their relative positions. The number of complete rings round each axis depends on the thickness of the plate, as will be clear from what has already been stated; the thicker the plate, the more numerous, smaller, and sharper the rings, as the number of complete wave-lengths of retardation, corresponding to complete rings round each axis, increases with the thickness. Thus, for instance, four plates of muscovite mica of graduated thicknesses afforded the interference figures shown in Figs. 563, 564, 565, and 566. The first (Fig. 563), the thinnest plate,

was a film for which the difference of path within the crystal was such as to bring about a retardation of one ray behind the other amounting to one eighth of a wave; no complete ring is shown, the first lemniscate being of ellipse-like character enveloping both axes, near and partly outside the margin of the field. The figure afforded by a quarter undulation plate of mica has already been given in Fig. 548 (p. 661), and comes next in order, showing one entire lemniscate surrounding both axes and a second curve partially visible outside it. The second figure now shown, Fig. 564, was afforded by a still thicker plate corresponding to half a wave-length of retardation; no complete ring is again visible, but there are six lemniscates now in the field, the innermost of which is bending in at the centre to form the approach to a loop. The third (Fig. 565) cleavage film is again thicker, and exhibits one whole wave-length of retardation, the loop of the innermost lemniscate being just complete, that is, each optic axis is now surrounded by one complete ring. The fourth (Fig. 566) is the thickest plate, and shows exactly two wave-lengths of retardation of one ray

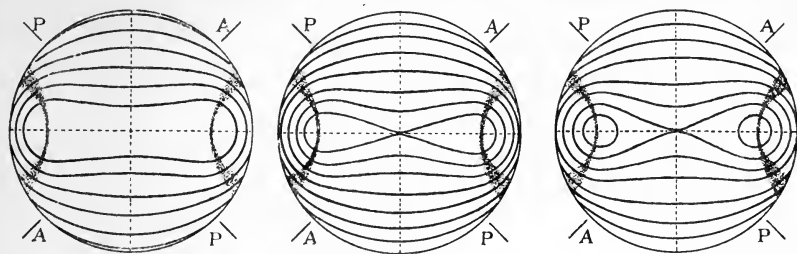


FIG. 564.—Figure given by Mica affording $\frac{1}{2}$ Wave Retardation.

FIG. 565.—Figure given by Mica affording a Retardation of one Wave-length.

FIG. 566.—Figure given by Mica affording a Retardation of two Wave-lengths.

behind the other, two complete rings being present round each axis. The drawings were made for sodium light, so that the wave-lengths referred to are those for light of the average wave-length of the D-lines of the spectrum, 0.0005893 millimetre. They were made for the crossed 45° positions of the planes of the polariser PP and analyser AA.

For the same thickness, the number of rings will obviously depend on the amount of the double refraction, $\gamma-a$, the rings becoming smaller, sharper, and more numerous as the double refraction increases. Also, when monochromatic light is used, say that of the spectroscopic illuminator described in Chapter XLIII., and the various colours are employed in succession, the rings will be observed to close up and become more numerous as the red is receded from and the blue end of the spectrum approached, that is, with shortening wave-length.

Outside the last pair of complete rings the next lemniscate is of almost but not quite complete loop form, and the next one or two are still of this character, but the loop is gradually straightening out, after which the lemniscates become more and more ellipse-like and approach nearer and nearer to a circular curvature and become closer together, as the margin of the field is approached.

In white light the figure is similar, but composed, as already stated, of spectrum-coloured rings, loop-lemniscates, and ellipse-like lemniscates. It differs essentially, however, from a uniaxial figure, inasmuch as the optic axial angle is different for each wave-length, the succession of colours being almost altogether dominated by this fact, and by the dispersion of the bisectrices, shortly to be referred to. Indeed the phenomenon of dispersion of the optic axes for different wave-lengths goes so far in the cases of "crossed axial-plane dispersion" that the optic axes are situated in one principal plane of the optical ellipsoid for red light, and in a second perpendicular principal plane for blue light. Hence, the simple succession of colours, as in Newton's rings, observed with uniaxial crystals, is very considerably modified and affected by this essential difference of the conditions. Also, the rings clearly visible in white light cease with the seventh, further spectrum coloured rings being obliterated by white light of the higher orders, whereas in monochromatic light a very large number of rings are visible, only limited in fact by the extent of the field.

It must be remembered that the angle between the optic axes as observed in air, generally denoted by $2E$, is larger (on account of the difference of refraction) than the true optic axial angle within the crystal, which is represented as $2V_a$, the suffix a indicating that the symbol refers to the acute angle between the optic axes.

Effect of Symmetry on Biaxial Figure—Phenomena for Rhombic Symmetry.—The biaxial interference figure of a rhombic crystal in either white or monochromatic light is truly symmetrical about its two rectangular di-meters, that is, about the central line of the black bar joining the optic axes and the central line of the transverse bar between the axes. For these two rectangular lines are the intersections of the section-plate (itself parallel to one of the principal planes of the optical ellipsoid and to a crystallographic axial plane) with two of the principal planes of the optical ellipsoid, which are identical with two of the crystallographic axial planes. For, the crystallographic axes and those of the ellipsoid being identical in direction, for all wave-lengths of light and at all temperatures, from the very essence of rhombic symmetry, and the acute and obtuse bisectrices or "median lines" of the optic axial angle being identical with two of those axes, there can be no dispersion of the median lines in the case of a rhombic crystal. The only possibility of dispersion is that common to all biaxial crystals, namely, that of the optic axes themselves, the angle varying with the wave-length and the temperature. The only colour complication with rhombic crystals, as regards their biaxial interference figure in white light, is, therefore, that due to this cause.

Figs. 567 and 568 on the adjoining Plate II. are reproductions of the author's photographs of the interference figures shown by a plate of rhombic aragonite cut perpendicularly to the acute bisectrix of the optic axial angle. Fig. 567 was produced when the plane of the optic axes was parallel to one of the crossed Nicols, and Fig. 568 when the Nicols were rotated 45° simultaneously, which is equivalent to the rotation of the crystal plate in its own plane for 45° . The apparent optic axial angle in air $2E$ of aragonite for sodium light is only $30^\circ 52'$, this mineral

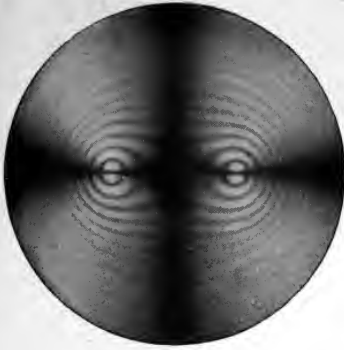


FIG. 567.—Figure given by a Plate of Aragonite (Rhombic) perpendicular to the First Median Line. Nicols Horizontal and Vertical.

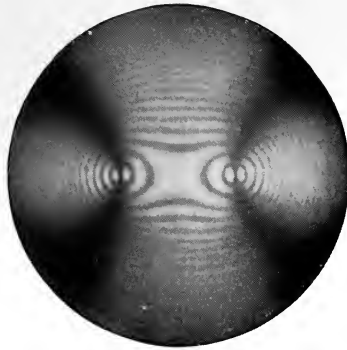


FIG. 568.—Figure given by same Plate of Aragonite when the Crossed Nicols have been simultaneously rotated 45° .

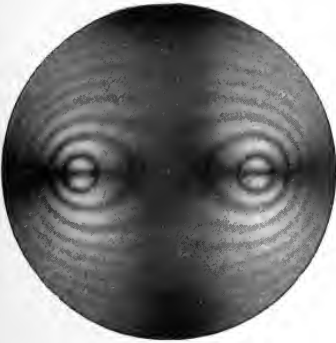


FIG. 578.—Figure given by a Plate of Borax (Monoclinic) perpendicular to the First Median Line. Nicols Horizontal and Vertical.

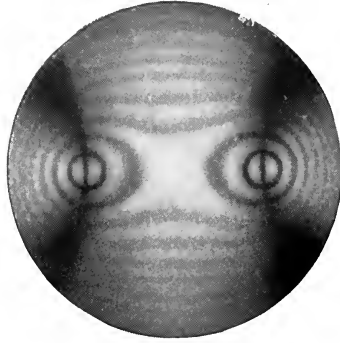
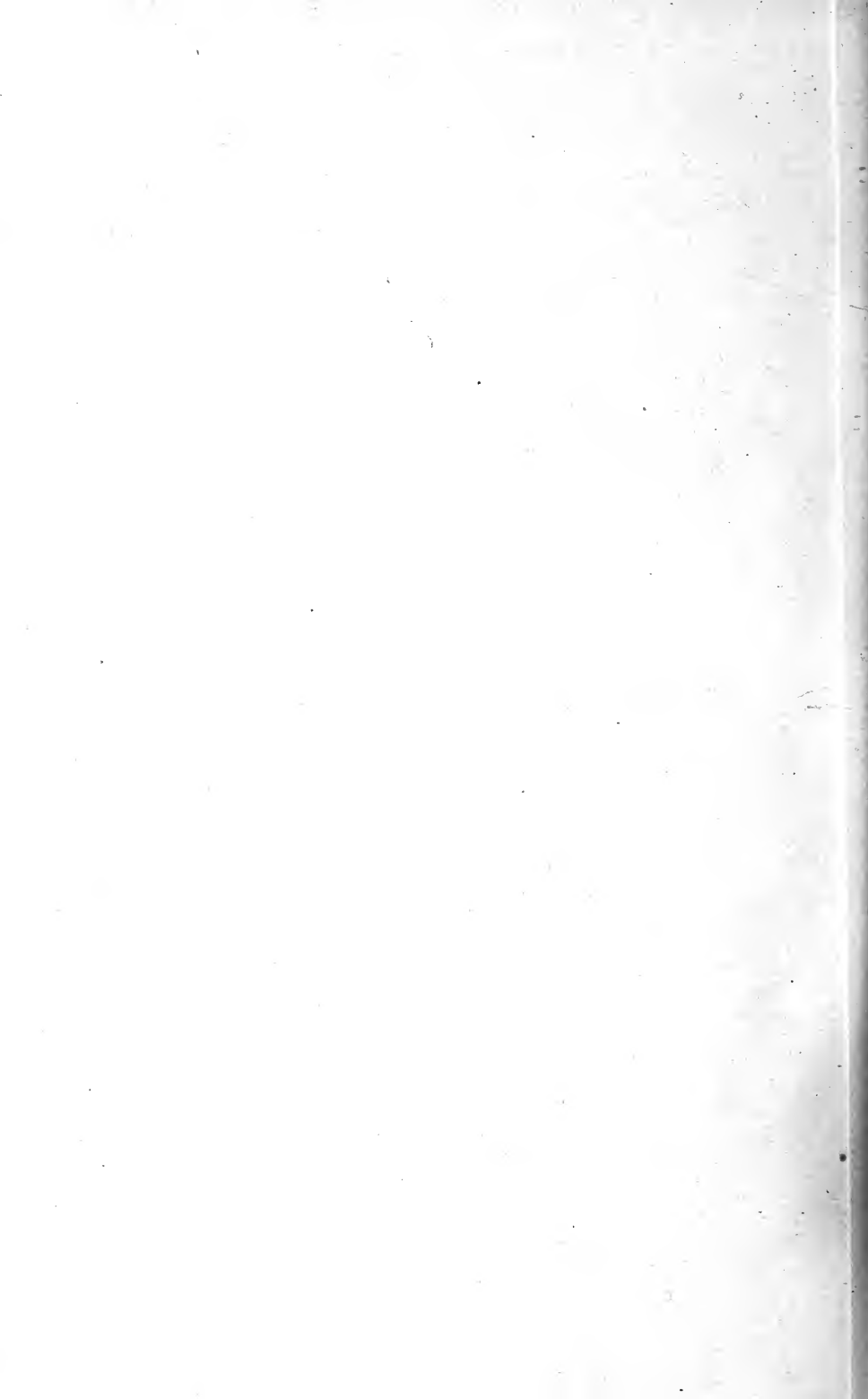


FIG. 579.—Figure given by same Plate of Borax when the Crossed Nicols have been rotated together for 45° .

REPRODUCTIONS OF DIRECT PHOTOGRAPHS OF INTERFERENCE FIGURES
AFFORDED BY CRYSTAL PLATES IN CONVERGENT POLARISED YELLOW
LIGHT.



being distinguished for its small optic axial angle. The dispersion for the two ends of the spectrum is very small, not exceeding a degree.

As an instance of accurately measured dispersion of the optic axes, the true optic axial angle $2V_a$ of potassium sulphate, which has been shown to be orthorhombic in symmetry in Chapters IV. and VIII., is $67^\circ 25'$ for red lithium light, and diminishes gradually and regularly as the spectrum is traversed until for greenish-blue F-hydrogen light it has become reduced to $67^\circ 7'$. This is a normal case of small dispersion, but there are other cases, even in the same rhombic series of alkali sulphates, in which the dispersion of the optic axes is very much more considerable. Thus rubidium sulphate exhibits an optic axial angle of $28^\circ 30'$ for red Li-light and $47^\circ 30'$ for violet G-light, a difference of 19° .

The effect of change of temperature on these two typical cases of small and large dispersion of the optic axes is still more striking. On heating the section-plate of potassium sulphate perpendicular to the first median line to 100° above the ordinary temperature, namely, to 115° , the apparent angle of the optic axes in air, $2E$, increases by 7° , corresponding to an increase of the true angle $2V_a$ of 4° . On the other hand for a similar change of temperature the optic axial angle of rubidium sulphate changes so rapidly, and in the direction of diminution, an approach of the axes towards each other, that their separation becomes reduced to zero for each wave-length in succession, beginning with the red, and a uniaxial figure, cross and circular rings, is temporarily produced, at 40° for red Li-light, and eventually even for violet G-light at 65° . After this the rectangular cross opens out again and the optic axes re-separate, but along the direction at right angles to their former one. This interesting phenomenon is called "crossed axial-plane dispersion," as it involves the crossing from one principal plane of the optical ellipsoid to another at the critical point when the uniaxial figure is formed. It is brought about in other cases by a change of wave-length alone, at the ordinary temperature, as in the case of brookite, the rhombic form of titanium dioxide, TiO_2 , the apparent optic axial angle $2E$ for red light being 55° and for yellow light 30° , both in the plane $\{001\}$, while that for green light is 33° in the perpendicular plane $\{010\}$. In all these remarkable changes in rhombic crystals, however, the median lines remain fixed, for all colours and temperatures, identical with the crystallographic axes and those of the optical ellipsoid, although in certain of the more extreme cases, where the optic axial angle begins by increasing with heat, what was the first median line becomes the second as the crossing point is approached. Indeed in the very extreme case of cæsium selenate, within 250° of temperature all three axes of the optical ellipsoid, all three crystallographic axes, become the first median line in turn.

In white light, the colour phenomena only involve pronounced broadening of the spectrum rings and the conversion of the dark brushes also into spectrum hyperbolæ, when the crossing of the axial planes is not reached. But when the latter occurs, the colour phenomena are specifically extraordinary and exceedingly beautiful, concave spectrum bands symmetrical to the axial rectangular directions, being essential

features. A special account of these cases will be given in Chapter XLVIII., and the general explanation of the whole subject of crossed axial plane dispersion discussed.

Dispersion of the Median Lines.—In the monoclinic system an additional phenomenon is introduced, namely dispersion of the bisectrices. For only one axis of the optical ellipsoid is now fixed by the symmetry, that which is normal to the unique plane of symmetry of the system. Hence, as at least one of the bisectrices of the optic axial angle must lie in the symmetry plane—in which rotation of the two axes of the ellipsoid lying in that plane is the general rule whenever the physical conditions, temperature or wave-length of illuminating light, are changed—the position of that bisectrix may be anywhere in the symmetry plane, and will vary under different physical conditions; indeed, when the invariable symmetry axis happens to be the intermediate axis of the ellipsoid, both bisectrices will lie in the symmetry plane, and will be equally capable of variation or “dispersion” in that plane, retaining always, of course, their mutually perpendicular relationship. Three distinct cases or types of such dispersion of the median lines are afforded by the possibilities of the conditions in the monoclinic system, and are conveniently known as “inclined,” “horizontal,” and “crossed” dispersion of the median lines. In these three cases the symmetry axis b of the crystal is respectively the third (that perpendicular to the first and second), second, and first median line.

Case 1, Inclined Dispersion of Both Median Lines.—The two optic axes and the bisectrices of the acute and obtuse angles between them in this case lie in the symmetry plane, so that both bisectrices are equally dispersed in that plane, the invariable b -axis being perpendicular to their plane. Hence, if as usual there is also dispersion of the optic axes, one

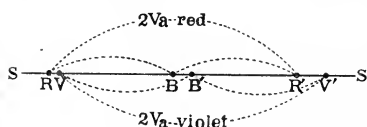


FIG. 569.—Inclined Dispersion of the Median Lines.

of them will show more dispersion than the other, and as a matter of fact the dispersion of the median lines is measured by observing this difference of dispersion of the optic axes. For if, in Fig. 569, SS represent the symmetry plane, perpendicular to the paper, B the position of the acute bisectrix (first median line) for red light and B' that for blue, then, when the optic axial angle is larger for violet than for red, as may be conveniently expressed by the symbol $\rho < \nu$, the angle for red will be RR' and that for violet VV' . The dispersion of the left axis is then obviously RV , and that of the right axis $R'V'$, and BB' , the dispersion of the first median line, is the mean of the two, namely, $\frac{RV + R'V'}{2}$; for both axial movements are in the same direction, as

represented in Fig. 569. When the two movements are in opposite directions, that is, when the two axes for red are either both within or both without those for violet, the dispersion of the acute bisectrix is equal to half the difference of the two optic axial movements, and in the direction

of the larger movement. The mode of carrying out the measurement will be given in detail in Chapter XLVII.

Gypsum, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$, monoclinic hydrated calcium sulphate, the remarkable crossed-axial-plane dispersion of the optic axes of which for different temperatures will be dealt with in Chapter XLVIII., is also noteworthy as exhibiting for change of temperature of less than 100° inclined dispersion of the median lines of nearly 6° . In an actual experiment of the author's the first median line moved in the symmetry plane, which is the plane of the optic axes for temperatures up to 105.5°C ., towards the crystallographic axis c to the extent of $5^\circ 41'$ between 10° and 105.5°C . The optic axis which moves fastest exhibits the wider rings. It is for this temperature of 105.5° that the crossing of the axial planes occurs for this same crystal, with production of the uniaxial cross and circular rings.

As illustrated markedly by the extreme case of inclined dispersion exhibited by gypsum, the two series of rings round the two optic axes display a difference in their shape and size, which is more or less

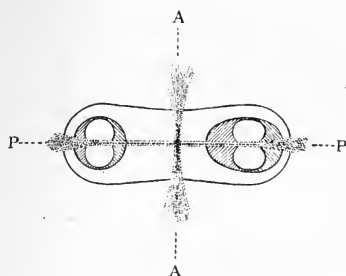


FIG. 570.

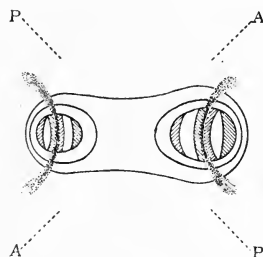


FIG. 571.

Shapes of Optic Axial Rings and Distribution of Colour in Cases of Inclined Dispersion.

accentuated according to the amount of the inclined dispersion. The innermost ring round the one axis is often nearly circular, while that around the other is elliptical. Figs. 570 and 571 will render this clear, for the two positions in which the section is respectively parallel and at 45° to the planes of polarisation of the Nicols. The dotted line PP represents the direction of vibration of the polarising Nicol, and AA that of the analysing Nicol. It will also be seen from the figures that the distribution of colour about the rings when white light is employed is symmetrical to the horizontal line (parallel with PP in Fig. 570), the trace of the symmetry plane.

Case 2, Horizontal Dispersion of the First Median Line.—In this case the plane of the optic axes is perpendicular to the symmetry plane, and the second median line is the immovable symmetry axis b of the crystal; but the first median line lies in the symmetry plane, and is, therefore, capable of dispersion in that plane. Regarding the interference figure, as usual, through a section-plate cut for sodium light perpendicularly to this acute bisectrix, and with the planes of vibration

of the Nicols parallel to those of the crystal, the figure is observed in white light to be laterally symmetrical, but not so above and below the horizontal axial line, as will be clear from Fig. 572. The next illustration, Fig. 573, shows the laterally symmetrical but vertically unsym-

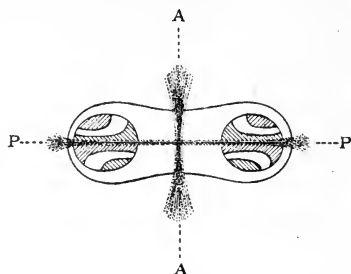


FIG. 572.

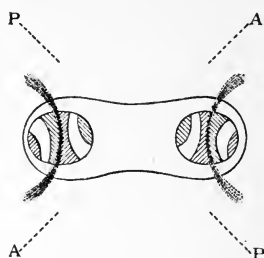


FIG. 573.

Shapes of Optic Axial Rings and Distribution of Colour in Cases of Horizontal Dispersion.

metrical arrangement of colours at the diagonal 45° -position of the plate. On illuminating with the spectrum colours in succession, and adjusting the section first so that for one extreme, say red, the horizontal bar of the figure is coincident with the horizontal cross-wire, as at RBR' in Fig. 574, this bar will be observed to travel parallel to itself

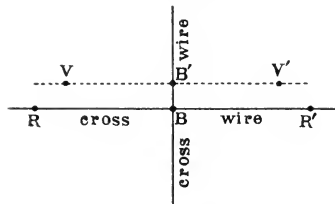


FIG. 574.—Horizontal Dispersion of the Median Lines.

vertically up or down, the optic axes remaining symmetrically placed with respect to the vertical line, the trace of the symmetry plane, but at an alternating angle, say VV' for violet light; the bisectrix B thus moves in the symmetry plane to B' , and the dispersion consists of the displacement of the horizontal bar RBR' to $VB'V'$, assuming the plate to be parallel to the Nicols, while the vertical transverse bar remains on the vertical cross-wire. The character of the dispersion is, therefore, said to be "horizontal." The interference figure in white light is correspondingly unsymmetrical to the horizontal cross-wire, but symmetrical to the vertical wire.

An excellent example of this type of dispersion is afforded by sanidine feldspar. Muscovite mica is also another example, reproductions of photographs of the figures given by which in the parallel and 45° -diagonal positions and in yellow light have already been given in Figs. 557 and 558 on Plate I. facing page 670.

An example of the mode of measuring this type of dispersion will be given in Chapter XLVII.

Case 3, Crossed Dispersion of the Second Median Line.—In this case the b axis of symmetry is the acute bisectrix, and is thus immovable, while the obtuse bisectrix lies in the symmetry plane and is dispersed therein. The plane of the optic axes is perpendicular to the symmetry plane. The interference figure in white light is unsymmetrical to any

line, but symmetrical to the centre, where the symmetry axis identical with the first median line emerges, the optic axes themselves being dispersed but not the median line.

If the circle in Fig. 575 represent the symmetry plane, and RR' the optic axes for red light, then these will be displaced for violet light to a position such as VV' , likewise symmetrical to the centre but along a line inclined to the horizontal more or less according to the amount of the crossed dispersion of the second median line. The kind of distribution of colour (centro-symmetrical) will be clear from Figs. 576 and 577.

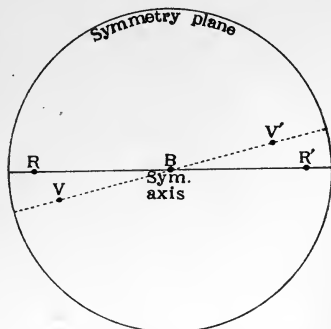


FIG. 575.—Crossed Dispersion of the Median Lines.

An excellent example of this kind of dispersion of the median lines is afforded by borax, $\text{Na}_2\text{B}_4\text{O}_7 \cdot 10\text{H}_2\text{O}$, reproductions of direct photographs of the interference figures of which in yellow light and for the parallel and 45° -diagonal positions are given in Figs. 578 and 579, the two lower figures of Plate II. facing p. 674. These

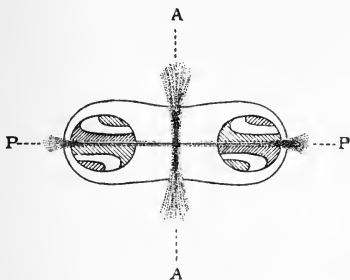


FIG. 576.

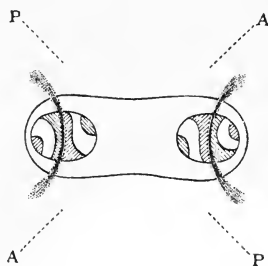


FIG. 577.

Shape of the Optic Axial Rings and Distribution of Colour in Cases of Crossed Dispersion.

photographs, as in the similar cases of those of mica given in Figs. 557 and 558 of Plate I. facing p. 670, being taken for a particular colour of light, do not show the distribution of colour, but give a clear idea of the shape of the rings. The colour scheme, however, is well shown in Figs. 576 and 577. The optic axial angle in air 2E for borax in sodium light is $59^\circ 23'$, an angle intermediate between that of aragonite ($30^\circ 52'$) illustrated in Figs. 567 and 568 on Plate II. facing p. 674, and that of mica (75°) as photographed in Figs. 557 and 558. These three pairs of figures thus illustrate a typically graduated series of optic axial angles of biaxial crystals, as seen in air.

The mode of measuring the amount of crossed dispersion will be described in Chapter XLVII.

Hyperbolic Interference Curves.—A plate of a uniaxial crystal cut parallel to the optic axis usually affords interference curves of hyperbolic character. Fig. 580 reproduces photographically the effect

afforded by a plate of quartz so cut. A thinner plate of calcite parallel to the axis also yields curves almost exactly like those in Fig. 580; a calcite plate of the same thickness as a quartz plate always shows the hyperbolæ more closely congregated together. In Fig. 581 is shown an interference figure also composed of hyperbolic curves, afforded by a

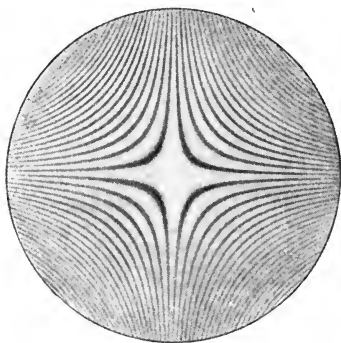


FIG. 580.—Hyperbolic Curves given by Plate of Quartz parallel to the Axis.

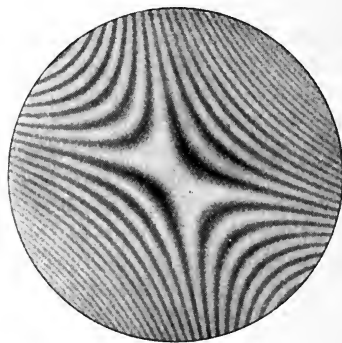


FIG. 581.—Hyperbolic Curves given by Plate of a Calcite Twin cut $22\frac{1}{2}^\circ$ to Optic Axis.

plate of calcite cut from a natural twin at an angle of $22\frac{1}{2}^\circ$ to the optic axes. This figure is more extended along the direction of one of the planes of the Nicols than along the other, whereas Fig. 580 is symmetrical to the centre. Both figures were obtained when the plate was arranged

with its planes of vibration at 45° to the planes of the crossed horizontal and vertical Nicols.

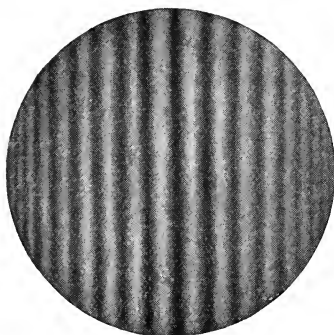


FIG. 582.—Savart's Bands afforded by Two Plates of Quartz cut 45° to Axis, and crossed at 90° .

Savart's Bands.—When two plates of quartz or calcite are cut at 45° to the axis, and then crossed and cemented with balsam at right angles to each other, the composite plate exhibits the parallel bands shown by photographic reproduction in Fig. 582. They are known as "Savart's Bands," as Savart employed the device for the construction of an analyser, the double plate affording an extremely delicate test for small amounts of polarisation of light.

For only a very slight amount of polarisation in the light reaching such an analyser at once renders the bands visible.

CHAPTER XLII

THE CUTTING AND GRINDING GONIOMETER.

THIS instrument, with which all the section-plates and 60° -prisms employed in the author's investigations have been prepared, is shown in Fig. 583. It was devised in order to replace by a method of precision the difficult, wearisome, and at the best only approximate, current method of grinding by hand, upon a slightly convex plate of ground glass lubricated with oil or a solvent for the crystal substance, the section-plates and prisms of the relatively soft and fragile crystals of artificial preparations which are required for the determination of the optical constants. It is equally capable of preparing such surfaces, however, on the harder crystals of naturally occurring minerals.

It is possible by means of it to cut or grind and polish a truly plane surface in any desired direction in a crystal accurately to within three minutes of arc, an amount of possible error which would exercise no appreciable influence on the values of the optical constants. This result may be achieved in a small fraction of the time hitherto required by hand-grinding, and, owing to the provision of a delicate arrangement for suitably modifying the pressure with which the crystal bears on the grinding plane, with only the very slightest risk of fracturing even a friable crystal. An automatic time-saving device is also provided, by which a second surface may be ground parallel, with a like degree of accuracy, to the first. Moreover, there is a special apparatus for preparing the two faces of a 60° -prism with only a single setting on the crystal holder. The section-plates and prisms of non-perishable substances furnished by the instrument possess the further advantage of being so highly polished as to enable them to be employed in most cases directly, without cemented cover-glasses, for the determination of the refractive indices and optic axial angle. The values of these constants derived from prisms and section-plates prepared with this instrument are thus no longer only approximate but precise.

The instrument consists essentially of five parts: (1) an accurate goniometer of the suspended type with horizontal circle, provided with (2) adjusting movements for the crystal, furnished with divided circles and of special construction; with them are combined (3) a crystal-cutting gear and (4) a separate crystal-grinding and polishing gear, (5) an

apparatus for controlling the pressure with which the crystal bears on the lap, and (6) a variety of crystal holders specially constructed for the different objects in view, and affording considerable means of preliminary adjustment.

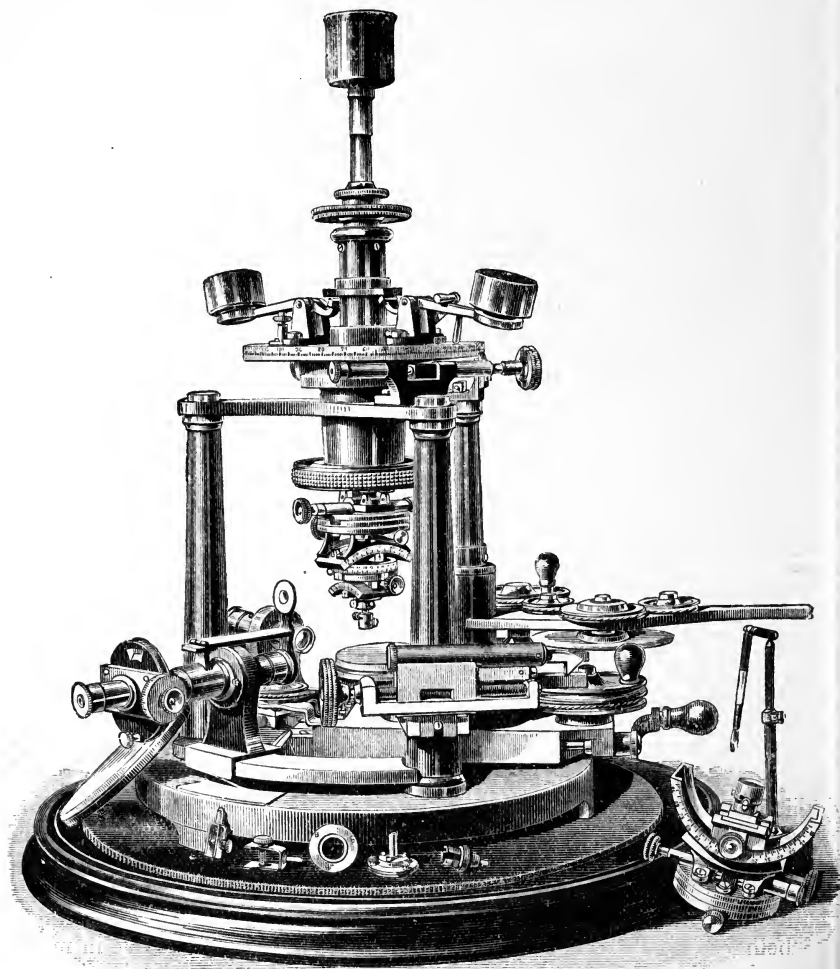


FIG. 583.—The Cutting and Grinding Goniometer.

The goniometer circle and adjusting movements are supported at the convenient height above a heavy circular base by three stout pillars and a triangular cross-plate, the latter lettered *a* in the section given in Fig. 584. The central bearing cone *b* for the various axes is part of the same casting with the cross-plate, and within it rest in succession three movable inner axes. The construction of these will be rendered clear by the section, Fig. 584, to which all the letters in this description refer. The

first *c*, which may be rotated by a milled ebonite wheel *d*, carries at its upper end the horizontal circle *e*, which is directly divided into half-degrees and reads with the aid of a vernier to single minutes. It is provided with a fine adjustment *f*, and fixing screw *g*. The milled head of the fine adjustment screw is indicated at *h* in Fig. 584 and is seen in front in Fig. 583.

The bore of the circle axis *c* is cylindrical, and within it the second movable axis *i* is capable of vertical motion only, rotation being prevented by a rib and groove. This axis *i* is designed entirely for the purpose of modifying the grinding pressure between crystal and lap.

The whole weight of the axis and all that it carries may be wholly or partially counter-balanced by a pair of levers *k*, the fulcrum supports *l* of which rest upon the circle plate *e*. Their inner and shorter curved arms terminate in blunt knife edges which press upwards against a collar *m* carried by the axis *i*, and which is adjustable to any convenient height; their power-arms are weighted by leaden weights *n*, which together counterpoise the whole weight. One of the levers, the right one in Fig. 584 and left one in Fig. 583, may be thrown out of action by raising a capstan screw *o*, thus leaving half the weight of the axis only in operation. The author generally employs the axis so, manipulating the free lever with the left hand, lightly holding the counterpoise *n* between the thumb and two fingers, and manipulating it according to the "feel" of the grinding. A spur *p* prevents the axis being raised inconveniently high. In the cases of very soft artificial crystals the screw *o* can be lowered and thus both levers rendered available in reducing the pressure. On the other hand, when hard mineral crystals are being dealt with, the weight on this axis *i* can be increased by placing small shot or weights in a brass cup carried at the upper end of the axis, its stem fitting over the cylindrical cover-cap *v* indicated in section in Fig. 584, as shown in position in Fig. 583. These arrangements thus enable the weight above the crystal to be varied during grinding or polishing to any desirable extent. The use of the left hand on the left lever enables the most delicate control to

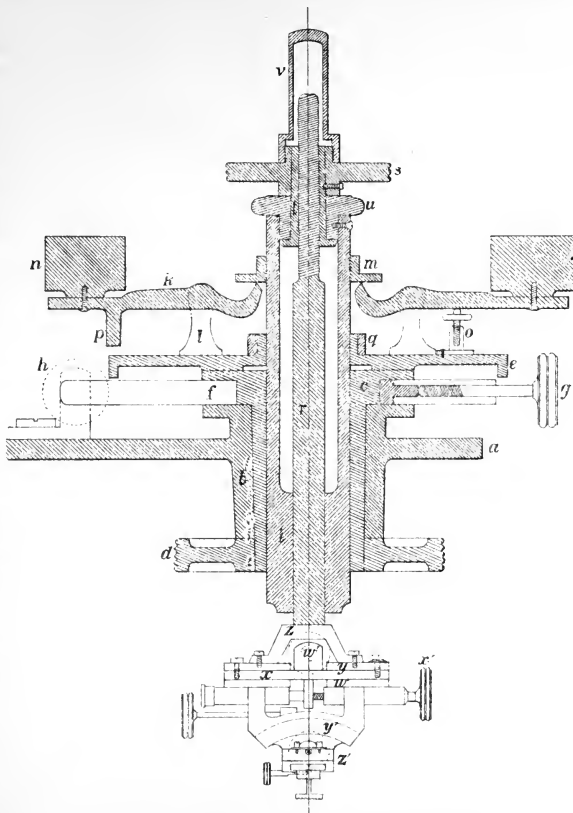


FIG. 584.—Section of Axial System of Cutting and Grinding Goniometer.

be maintained over the pressure on the lap, and after a little practice it is an extraordinary occurrence that a crystal should break or crumble under the operations. The axis i can be fixed rigidly to the circle (the latter being previously fixed by the clamping screw g), by means of the gripping collar g , the screw of which is tightened up by a key provided. This fixation is necessary during cutting operations. During adjusting operations it is not necessary to keep the axis i up by this fixation, a little locking hook being simply thrown over the longer arm of that lever which is usually manipulated by the left hand, but rotated to the right in Fig. 583.

Within this second axis i slides, also without rotation owing to keying, the third axis of steel r , which carries at its lower end the crystal and its means of adjustment. The upper portion of this axis is tapped with a fine screw thread, and the axis can be raised or lowered by a milled-headed wheel s , which manipulates the driving-nut t . The latter rotates without vertical motion within the solid head u of the hollowed axis.

The crystal-adjusting and centring apparatus is carried at the lower end of the inner steel axis r , and consists of the usual two circular adjusting segments and pair of horizontal centring movements, but is constructed more strongly than usual. The centring is attained by the movement of two circular discs w and x , about each other by means of a pivot near the periphery, and of these two about a third y in a similar manner, which latter is rigidly fixed to the lower end of the steel axis r by means of the bridge z . The movements are maintained steady by a slot and screw with large head in each case, near the periphery at the opposite side of the centre to the pivot. The end of the centring screw w' or x' presses against a short vertical piece fixed to the central disc x , and passing through a central hole in the outer disc w or y , contact between screw and upright being maintained by a spring piston pressing on the other side of the upright. The two movements are arranged at right angles to each other. The adjusting movements are the two circular movable segments y' and z' , rotated over strong cylindrical guiding beds by a couple of tangent screws with milled heads, arranged, like the movements, at right angles to each other, particular care having been taken to set both at exactly $90^\circ 0'$. The screw which rotates y' is shown to the left in Fig. 584. The movements differ from those of an ordinary goniometer in being provided with silver scales divided directly to degrees, on the faces of the guiding beds, and with silver indicator-marks to correspond on the movable segments. The tenth of a degree of movement may readily be set, and a twentieth ($3'$) estimated.

Besides this pair of circular motions of which the two planes are fixed at right angles, an alternative set is provided, shown both in position in Fig. 583 and in Fig. 585, in which the lower movement may be adjusted at any desired angle to the upper movement, by means of a horizontal circle with silver scale, also divided directly into degrees, carried by the upper movement, the lower movement carrying on a similar circle plate four indicating marks on silver 90° apart. The lower movement may be clamped to the upper one in any position by means of a double tightening screw, which fixes the indicating disc to the circle, and which can be manipulated from either side by means of a key supplied.

The crystal is to be attached to the small cross-grooved disc of the crystal-holder by hard and rapidly-setting opticians' wax. It is to be adjusted so that the zone of faces perpendicular to which it is desired to grind a surface, if such a zone be developed, is parallel to the vertical axis. When a zone of this character is not present, as will happen in monoclinic and triclinic crystals, a prominent one of convenient and accurately known position with respect to the surface to be ground is adjusted, and with the aid of the two circular movements—arranged either at 90° , or at any other desirable angle if the crystal be triclinic—the calculated necessary rotations can be given to bring the plane in question within the crystal parallel to the grinding lap. For all ordinary cases up to and including monoclinic crystals, the author prefers to use rectilinearly fixed movements, and in order that the desired plane shall be adjusted for grinding or cutting it is usually only necessary to rotate one of the movements for some particular

angle—indicated by the extinction determinations—after the prominent zone has been accurately adjusted, and after one face has been set either parallel to one of the movements or at some definite angle with it. Provision for setting any face exactly parallel to a circular motion is provided in a series of special crystal-holders of graduated sizes to suit different sized crystals, which permit of the requisite amount of rotation after the holder is fixed in its socket. Two different ones are shown in position in Figs. 583 and 585 respectively. The cross-grooved attaching-plate forms the bottom of a hollow cylinder, which is rotatable about a solid cylindrical core, and capable of fixation in the adjusted position by two screws passing through slots of adequate length to permit of over 45° of adjustment. A very small glass plate, cemented on the side of the adjustable segment strictly parallel to the plane of its circular movement, is used to obtain a reflection of the collimator signal-slit, and for the allocation of this image to the cross-

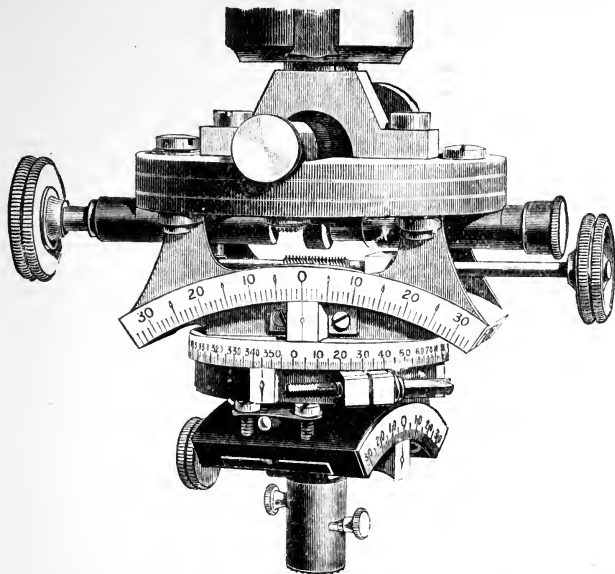


FIG. 585.—Special Crystal-adjusting Movements of Cutting and Grinding Goniometer.

wires of the telescope the circle reading has once for all been determined, thus enabling the setting of the face to the same reading by means of the adjusting cylinder to be accurately attained. In Fig. 584 one of the unadjustable holders is shown. In addition, three gripping holders are also provided, shown resting on the wooden basal plinth of the instrument in Fig. 583, over which fits the protective glass shade. All are padded within their prongs with cloth; two are merely of the pincette type, while the third is triply split, and the three prongs are tightened when gripping a crystal by working down an annular nut upon them, the upper part of the stem being tapped with a screw thread. All the crystal-holders fit by means of a central peg into an axial hole in the lower adjusting segment, and can be fixed there by a tightening collar (Fig. 583) or fixing screw, the lowest visible in Fig. 584.

For grinding the parallel surface of a section-plate, after the first surface has been prepared, a further special crystal-holder is provided, consisting of two parts screwed together, one being like a shallow cap and the other a disc with central peg. The crystal is mounted by its already ground surface on a circular disc one centimetre in diameter of truly parallel glass, with hard Canada balsam dissolved in benzene. This

is laid, when the balsam is quite dry and hard set, in a concentric rabbeted receptacle for it in the cap-like part of the special holder, with the crystal projecting through. This part is then fitted to the other, which consists of a thick metallic disc one inch in diameter; one surface of this disc is a true plane, and the other carries in its centre and perpendicular to the true plane the usual insertion peg, and also an enveloping cover-plate very similar to the cap. The disc fits nicely into the circular depression in the cap, and the latter bears three little screw-tapped uprights at equidistant positions on its marginal flange, which fit through holes in the cover-plate, so that when little milled nuts are screwed on the three screws the two parts are firmly fixed together, with the glass plate pressed into full contact with the true plane. This true plane should previously have been adjusted—when placed in position with the peg (which has a groove fitting a pin in the socket, to prevent rotation and enable it always to regain a similar position on insertion) in the socket at the bottom of the crystal-adjusting apparatus—so that it is absolutely parallel to the grinding lap. This is readily ascertained by lowering it close to the lap and adjusting till an evenly thin line of light is seen between the two in all positions on rotation of the crystal axis. It will then recover this parallel position on reinsertion after attaching to it the lower part containing the section and its glass-plate mount by means of the three screws. As the rabbeted annulus holding the glass-plate is very thin, a section may be ground which may, if required, be almost as thin. The cap is made of very hard white metal in order that the thinness of section-plate may be as extreme as is ever required, in the cases of crystals of high double refraction. The second surface thus ground is bound to be parallel to the first, as the glass-plate on which it is mounted is parallel-surfaced, and the surface of the true plane, itself parallel to the grinding lap, is in absolute contact with it. The disc with its cemented crystal may be removed from time to time during the grinding without disturbing the adjustment, so that grinding may be stopped when the section is of the desirable thickness to enable it to exhibit small rings in its interference figure, the condition required for measurement of the optic axial angle; or whenever any other desired thickness, to suit some other condition or experiment, is attained.

In order to prepare a 60° -prism, that is, two surfaces inclined at 60° , by one setting of the crystal on the wax of the holder, it is obviously necessary to rotate the crystal for 60° on each side of the particular principal optical plane which has been adjusted vertically to the grinding plane, with the aid of the goniometrical arrangements provided on the instrument, and to which optical plane the two required surfaces are to be symmetrical. The two cylindrical movements of the ordinary adjusting apparatus already described admit of 35° of rotation on either side of the vertical axis in each case; this amount is ample for all the purposes of the preparation of section-plates, and also for the preparation of a 60° -prism by separately setting the direction of each required prism-face, by rotation of 30° from the plane perpendicular to the bisecting plane, which can usually be as readily adjusted as the bisecting plane itself. But it repays to render the adjusting mechanism more cumbersome in order to achieve both surfaces by one setting; hence, a special alternative adjusting apparatus is provided for this purpose. It is shown in Fig. 583 in the immediate foreground, to the right.

A duplicate centring arrangement is given with it, so that it is only necessary to detach the whole of the ordinary apparatus from the bottom of the steel axis r , by unscrewing the four screws attaching the upper centring disc to the bracket (z in Fig. 584), these screws in the latest form of the instrument (Fig. 583) having milled heads and also capstan holes, for more rigid fixation by a steel lever provided. The new attachment^o is exactly like the ordinary movements down as far as the upper fixed cylindrical segment and its divided silver scale reading 35° on each side of the indicator-zero. But the movable segment sliding within it is made double the usual size, that is, of rather more than 150° . On one face, immediately below the fixed scale, it bears at the centre an indicating mark on silver for that upper fixed scale, while on the other face it bears a

silver arc graduated to 75° on each side of the centre. This large movable segment is only movable as regards the 35° of adjustment, and is usually only moved for the few degrees required for the adjustment of the bisecting plane of the proposed prism; it is thus the fixed segment as regards the large rotation for the two faces to be ground. Within it, underneath, the usual lower adjusting movement of the ordinary kind slides to the large extent of 75° on each side already indicated, the plane of the two movements being in this case permanently fixed at 90° in order to save further unnecessary complications. The adjusting movement of the large-segment about the upper fixed one is effected in the usual manner, by tangent screw, for the 35° of its path on each side, indicated by the upper ordinary scale. The lower rectangular movement is similarly supplied with a tangent screw. But the large amount of sliding of the carrier of this lower movement for the grinding of the two prism-faces, within the large new segment, is effected by hand, and fixation at any required position with reference to the large silver arc can be brought about by a milled-headed screw-clamp on the opposite side to the index and lower tangent screw.

In using the apparatus, the crystal is attached by the minimum of wax to the smallest of the adjustable cylindrical crystal-holders, so that the proposed bisecting plane of the desired 60° -prism is arranged vertically, as nearly parallel to the goniometrical axis as possible, and parallel to the lower tangent screw. The azimuth adjustment of the crystal-holder is very useful in attaining this. Exact adjustment of the plane in question is then perfected with the two tangent screws, the clamping screw being fixed with the indicator on the large arc at zero. The screw of the clamp is then unloosed and the slider with the lower movements and the crystal moved round 60° on one side, as indicated by the silver index against the large arc, and the clamp again fixed. The first surface may then be ground and polished. The clamp is then again loosed, the lower part of the apparatus rotated to 60° on the opposite side of the centre, the clamp fixed, and the second surface ground and polished.

The goniometrical telescope and collimator are arranged in the horizontal plane at the convenient height for observing the crystal. They are movable over circular guiding arcs, the centre of which lies in the vertical axis of the instrument. This adjustment can be perfectly attained once for all by means of three adjusting screws passing through collars round the optical tubes into the corresponding bearing supports; a strong annular spring is confined between the two annuli in each case and presses them asunder, while the three adjusting screws at 120° intervals draw them together against the force of the spring, which is so strong as to afford a very rigid adjustment.

A combined goniometer- and micrometer-eyepiece is provided, due to the suggestion of Dr. H. A. Miers; it has two fixed spider-lines at 90° , one adjusted exactly parallel to the vertical axis of the goniometer and the other horizontal, and a third spider-line which is both rotatable and capable of movement parallel to itself in the focal plane. The fixed lines are attached in the central aperture of a circle-plate 7 centimetres in diameter fixed round the optical tube of the eyepiece, and which carries near its periphery a circle divided into degrees. The movable spider-line is arranged in the aperture of a micrometer-box carried in front of a similar circular plate, fitting closely to the first plate and partially enveloping it with a milled flange so as to be rotatable about it. A window through this front plate, seen above the micrometer in Fig. 583, allows the circle to be read through it, the inner edge of the window being bevelled and engraved with a vernier, by which the circle can be read to minutes. The movable spider-line is fixed to the traversing frame of the micrometer, at the focus of the Ramsden eyepiece, which slides in a short front tube. By a suitable device the fixed lines are brought also into the focal plane and almost in contact with the movable line. The traverse of the latter and its frame is recorded by a drum on the right, divided into 100 parts, the reading being by an indicator alongside. The two circles can be clamped together, when desired, by the fixing screw seen below the micrometer in Fig. 583.

These arrangements on the telescope enable the instrument to be used not only for its specific purpose, but also for the study of small movements of the image of the collimator signal-slit, which in this instrument is of the kind shown in Fig. 15 (p. 36), reflected from a crystal face during the growth of the crystal in a cell of mother-liquor placed on the grinding table, and due to disturbance of the thermal or other conditions of the solution. In fact this instrument serves for a most useful variety of other purposes than that for which it was primarily designed, particularly for the determination of refractive indices by the total-reflection method, as will be found described in Chapter XLVI., and is a most valuable acquisition to a crystallographic laboratory.

The removable lens in front of the telescope objective is capable of being thrown into position as usual for the purpose of converting the telescope into a low-power microscope, or a duplicate may be placed in front of the eyepiece and made to travel for some distance in front of the latter along the optic axis, in order to follow any signal-image right up to the image of the crystal face reflecting it. The pair of lenses are mounted for this purpose on the cross-bar of a T-piece, the stem being hinged to a short upright from a tube sliding without rotation over the main optical tube. This arrangement is of considerable value in tracing the images derived from vicinal faces; such vicinal faces may be actually studied during the growth of the crystal, by the method employed by Miers (see Chapter XXX.), to whose suggestion this accessory is also due.

Just below the plane of the optical tubes the **horizontal grinding table** or "lap" is supported, in a frictionless bearing carried by an adjustable dovetailed slider moving over a correspondingly dovetailed guiding bed. Ten interchangeable laps of different materials are provided, to suit the grinding and polishing of crystals of every degree of hardness. There are three glass laps, two of which are respectively very finely and moderately finely ground, while the third is polished clear; there are four metallic laps, of iron, gun-metal, hard white-metal, and pewter respectively, for use with emery, putty-powder, rouge or rottenstone and water; another lap is of boxwood, for use with putty-powder in polishing, and a further polishing lap is of hard optician's wax, mounted in a metallic tray, for use with rouge as in polishing glass surfaces; the tenth is an emery wheel.

The surface of each of these laps is made a true plane, and any one can be fitted on the permanent supporting table of the same diameter ($4\frac{1}{2}$ inches) at a moment's notice, fitting by three pegs into corresponding holes in the table, and being simply locked there by rotating a little lever under the table, which brings a locking-plate into grooves in the pegs. The glass laps are employed chiefly with the crystals of artificial salts, one of the two ground-surface laps being used for grinding, lubricated with "brick" oil (olive oil in which a red-hot brick has been plunged, which appears to destroy certain clogging constituents), or in some cases only with the moisture of the breath; the clear plate-glass lap is used for polishing, lubricated in a similar manner.

The lap is rotated by a driving gear arranged equally on each side of the axle so as to diminish strain, the driving pulley being on the right and a friction pulley on the left. The grinding is best under control when the driving pulley, which is provided with a suitable handle, is rotated by hand, each revolution producing two revolutions of the grinding lap. The driving pulley is, however, provided with two grooves, the upper of which is intended for the reception of a band from an electric motor.

Concentric grooving of the lap is avoided, and more or less equal wear ensured, by the mounting on the slider, which is made to traverse the dovetailed bed resting on the circular iron base of the instrument by rotation of a long driving screw gearing with a thread in the bed and manipulated by a winch handle at the right-hand end. The inner end of the screw being unflanged, the slider and all that it carries can readily be removed altogether after driving outwards to the end of its path, in order to afford

more room when the cutting apparatus is in use. The permanent table on which the laps are interchangeably mounted is provided with the means of adjustment by three strong screws, which level its axle base-plate with respect to the slider, in order to be able to bring the table exactly perpendicular to the vertical goniometrical axis. The friction pulley is also adjustable by means of slots in its bearing-bracket, through which its fixing screws pass, so as to be able to tighten the driving band. Any friction of the band may also be entirely avoided, as both driving and friction pulleys are so mounted that they may be slightly tilted if desired, three-screw adjustments being provided for the purpose.

An adjustable guard-screen is also provided to protect the observer and the optical parts of the instrument from splashing during the cutting or grinding of hard crystals. It is very seldom required, however, as splashing rarely occurs with brick oil as lubricant, in the cases of the small crystals generally operated on for crystallographic-optical investigations. It is never required for use with the glass laps and artificial chemical crystal-preparations.

The **cutting apparatus** is carried at a height which brings the diamond-fed cutting disc of soft iron, 4 inches in diameter, to exactly the same level as the grinding surface of the lap. It is separately mounted on a rigid horizontal arm pivoted on the back pillar of the instrument, so that it can be rotated out of the way during grinding operations. It is further supported when required for use upon an adjunct of the right-front pillar. The cutting disc is supported between two stout broad washers carried by the frictionless axle, and is driven by an independent gear also mounted on the arm, consisting of a driving pulley and two friction pulleys on opposite sides of the axle. The driving pulley is given in duplicate, the second one being below on the same axle, and intended for motor driving when desired. The arm on which the whole apparatus is carried is bent inwards towards the crystal, in order to advance more conveniently the cutting diamond-edge of the iron disc to the crystal. Three interchangeable cutting discs are provided, one of the usual thickness of a lapidary's cutter, and two somewhat thicker and therefore more rigid. Their edges are charged with diamond dust (a diamond being crushed in a steel mortar to provide it) by smearing a mixture of oil and the dust round the edge and then forcing it into the soft iron by holding a piece of agate against the disc during its rotation. After one or two repetitions of this treatment the cutting edge of the disc becomes charged with enough diamond to last for years. The edges of all the discs had first been carefully squared made perpendicular to the surfaces of the discs).

The support for the arm on the front pillar is a fitting which is instantly removable, by sliding upwards out of a dovetailed grooved bearing, the latter forming a permanent part of the pillar; the fitting is always thus removed during grinding operations. The sliding dovetail, which forms the back portion of the removable fitting, slides immediately into position in the recess, a stop arresting the fitting when at the right height. The attachment includes not only a supporting fork for the arm (somewhat like a tuning fork placed horizontally), but also the apparatus for directing and controlling the cutting. The latter consists of a horizontal traversing bed and slider, manipulated by the large milled head of the traversing screw seen in front of the centre in Fig. 583. The arm is attached to the slider by being gripped between a small spring piston and a spring-hinged wedge-shaped hook in front of the fork; the hook is pushed out of the way by the arm in entering the fork, but is caused to slip back behind the arm by the spring pressing it down from above. This gripping arrangement is carried by a much larger piston, retained by a strong spring in a cylinder fixed to and above the slider. The possibility of undue pressure being developed between cutter and crystal, owing to injudiciously rapid rotation of the traversing screw, is thus avoided; for the spring gives way before the pressure becomes dangerously great, the piston being pulled out of the cylinder. Two inches of traverse are permitted by the length of the fork and of the traversing bed of the slider, an amount ample to permit of the cutting

through of a crystal nearly an inch in diameter, a size never exceeded and rarely approached in work of the character contemplated with the instrument.

This form of cutting apparatus is found to work admirably. There is no tendency to jamming when once the operator has become familiar with the rate at which the milled head of the traversing gear can most advantageously be rotated with the left hand, while the driving wheel is manipulated with the right; and the cutting is more perfectly under control than when the pressure of the cutter is brought about by a spring or a weight hanging over a pulley.

In order to cut through a crystal along a specific plane, the latter is first adjusted parallel to the plane of the optical tubes, and therefore to both the cutting disc and grinding lap, by the goniometrical means already described. The crystal is then lowered by rotation of the milled head at the top of the instrument until it is at a convenient height for cutting, in which position it is fixed by first clamping the circle to the fixed cone by the milled-headed screw forming part of the arrangement for fine adjustment of the circle (at the back of the circle in Fig. 583, *g* in Fig. 584), and then preventing any movement of the balancing axis *i* by tightening the collar *q* immediately above the circle, by means of the key supplied. Cutting is then proceeded with, commencing the rotation of the cutter and its traversing very slowly at first, and gradually increasing the speed according to the "feel" of the cutting, the diamond wheel being lubricated with brick oil. If by inadvertence a jam should ever occur, the slightest reversal of the motion of the traversing screw will instantly release the cutter. The speed should be materially reduced towards the finish, in order that the end of the crystal may be cut off cleanly right up to the furthest edge. A hard crystal such as a topaz (hardness 8) may be cut through in a very few minutes, and the cut surface is so smooth that very little grinding is required, which may be at once proceeded with on the gun-metal or hard white-metal lap, with the finest washed flour emery, made into a paste with water or brick oil. A thick glass disc an inch in diameter is provided for use as a "bruiser," to work down the emery paste to an even consistency on the lap before grinding. The cutting apparatus is rotated out of the way to the back, and the front-pillar attachment altogether removed, in order to afford ample room. The pressure between crystal and lap is controlled as already described, by manipulation of the left counterpoised lever with the left hand, and the path of the crystal on the lap changed by moving the slider along its bed more or less from time to time. During grinding, the axis *i* is, of course, quite free, having been released after the cutting. Polishing may occur exactly as carried out by opticians, with any of the polishing laps already mentioned, which provide for all the current methods of work, except that the crystal is fixed while the polisher revolves.

A further useful accessory is a small adjustable stand, shown to the right in Fig. 583 behind the prism-preparing adjusting movements, to which may be fitted either an oblique caoutchouc wedge, for keeping the emery paste moving into the path of a hard mineral crystal during grinding, or a camel-hair brush, to keep the path in front of a softer crystal of an artificial chemical preparation free from particles of crystal and well supplied with the lubricant.

The glass laps are exclusively used for the preparation of surfaces on such artificial crystals and no grinding or polishing powder is used; also nothing but brick oil or the moisture deposited by breathing on the cold lap is used as lubricant, whichever is found most suitable by experience with the particular substance in hand. The surfaces are in these cases finished on the plate-glass clear lap, as a rule with only a trace of moisture or brick oil on it. Such surfaces, when the crystal is not one of an efflorescing or otherwise deteriorating substance, are usually beautifully polished, and section plates and prisms provided with such surfaces can at once be used for the purposes of measuring the optical constants. If the surfaces rapidly deteriorate they are protected by cementing over them, with hard balsam in benzene miniature plates of

cover-glass, which had previously been optically selected, by reflection tests, as truly plane-surfaces.

In concluding this description of the cutting and grinding goniometer, it may be stated for the benefit of workers only with artificial chemical preparations, for whom the cutting apparatus is unnecessary, that a smaller model of the instrument has also been constructed by Messrs.

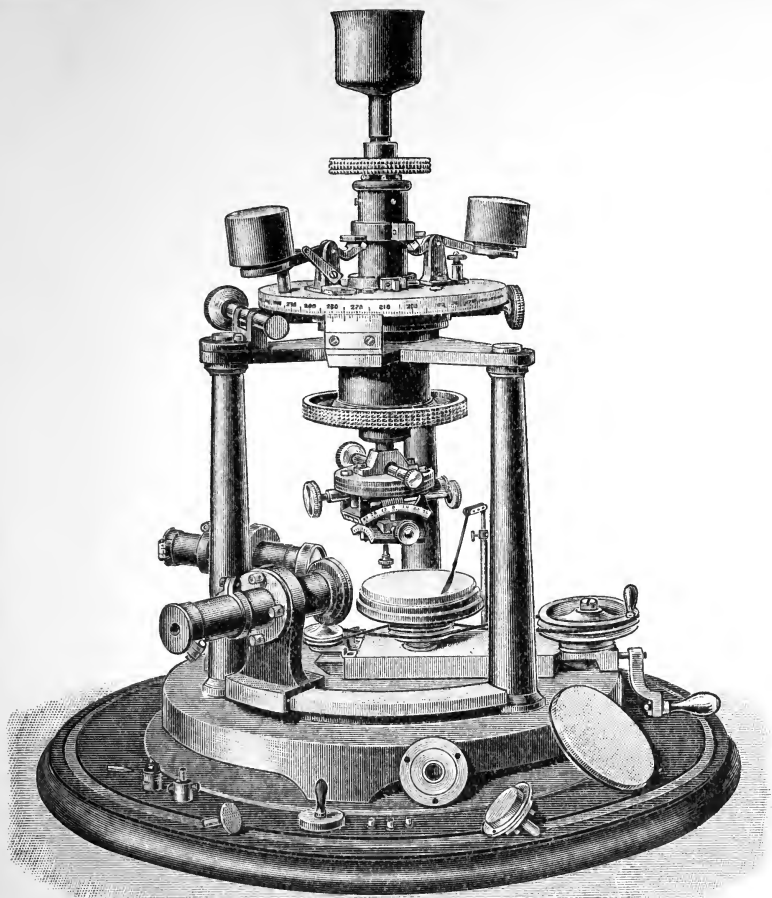


FIG. 586.—The Crystal-grinding Goniometer.

Troughton and Simms, the makers of the larger model, without the cutting gear, at a considerably less cost. It is shown in Fig. 586. The grinding apparatus and all the rest of the instrument is exactly as described in the preceding pages. This is a remarkably handy and efficient little instrument, and is of the utmost utility to the chemical crystallographer.

The Wülfing Crystal-grinding Apparatus.—An ingenious little apparatus, shown in Fig. 587, for grinding an orientated surface on a crystal has been devised by Wülfing¹ and is constructed by Fuess.

It consists of a small tripod of which the table *a* is a thick plate of aluminium, a somewhat flat isosceles triangle in shape, and cut away in the middle. Two of the feet *b* and *c* are steel screws, and the third is formed by the crystal itself, cemented to a holder *d* of gun-metal, which is one of four interchangeable holders, of different shape at the lower end, the other three *e*, *f*, *g* being shown in front in Fig. 587. The selected one of the four is fixed in position by the milled nut *h*, which gears with a screw thread cut near the upper end of the holder, the middle cylindrical part of which fits in the boring through the table at the angle formed by the two equal sides. One of these holders *e* terminates simply in a flat normal surface, while the other three

and 60°, *f* being 20° and 50°, and *g* being 30° and 40°, in order to facilitate the cementing of the crystal in the approximately desired position with respect to the grinding surface.

A surface is first ground on the crystal approximately to the eye in the desired direction, with reference to some well-developed crystal face, and the holder *d*, *e*, *f*, or *g* on which it is mounted is then gripped about its broad cylindrical end by the fitting *k*, a stout pincette supported at right angles by a bracket-plate carrying a central peg, by which it is next attached to the crystal-adjusting ap-

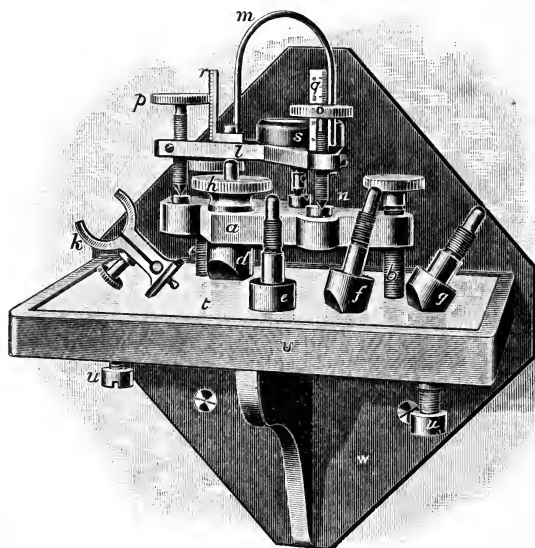


FIG. 587.—Wülfing's Crystal-grinding Apparatus.

paratus of the Fuess No. 2*a* goniometer instead of an ordinary crystal-holder. The real angle with the crystal face, and thus the error, is then measured on the goniometer. The crystal-holder is then detached from the pincette and fitted to the tripod, and over the latter is placed a second tripod *l* provided with a lifting loop *m*, in such a manner as to rest on three steel supports, the tops of screws, two of which are driven through little projections from the two equal sides as at *n*, and the third driven through the middle of the basal side of the triangle. One of these screw heads has a flat top, another a conical countersunk central depression, and the third a V-groove, so that this second tripod *l* may rest on the main tripod table *a* in the approved method for stable three-point contact.

Two of the legs of this tripod are screws of half a millimetre pitch with sharp conical points and large milled heads *o* and *p*, divided on their flat tops into 12 parts, every second of which is numbered to count as 10, so that the circle corresponds

¹ *Zeitschr. für Kryst.*, 1890, 17, 445; *Jahrbuch für Min.*, 1902, 2, 1.

to 60 divisions. Each of these divided heads rotates almost in contact with a vertical scale, *q* or *r*, of half millimetres, to measure the number of revolutions of the screw. The third leg is fixed, after once for all being adjusted to a convenient length, and immediately above it on the table top of this second tripod lies a circular spirit level *s* of $2\frac{1}{2}$ centimetres diameter. This second tripod *l* is so constructed that each complete rotation of either of the graduated screws brings about an inclination of 1° , the 60 divisions of the screw head thus corresponding to minutes.

With the help of these calibrated adjusting screws and the spirit level of the upper tripod on the one hand, and the adjusting screws of the lower tripod on the other, it is possible with a little thought and a few trials to correct the crystal surface to the desired direction. The whole apparatus may be mounted on the ground-glass crystal-grinding plate *t*, fitting over three levelling screws *u* in the horizontal table *v* of a cast-iron bracket *w* screwed to the wall of the room or other rigid vertical support.

When it is not possible to incur the expense of the author's cutting and grinding goniometer, or even of the smaller grinding goniometer, which enables the desired surface to be cut or ground and polished right away to the degree of accuracy of a very few minutes of arc, this inexpensive method of approximations of Wülfing may well be resorted to.

Cutting, Grinding and Polishing Apparatus of Steeg and Reuter.

—This well-known firm of Homburg, so justly celebrated for its preparation of accurately orientated sections of the crystals of naturally occurring minerals, and of those of artificial preparations when of some size, for the study of optic axial interference figures and other crystallographic purposes, constructs excellent crystal cutting and grinding apparatus, with approximate orientation devices, for work on the larger scale. The following four illustrations show some of their most useful machines.

Fig. 588 represents a grinding and polishing apparatus somewhat akin to Wülfing's.

It enables a surface to be ground at any desired inclination to either one or two crystal faces, by means of adjustments with the aid of two graduated circles, and is also of tripod form, with spirit level. The arrangement will be clear from the figure. The grinding plate is supported on a strong table, provided with levelling screws both for the tripod itself and for the plate. The polishing disc is so arranged that silk, felt, or chamois leather can be stretched over it by means of a ring and tightening screws. The grinding laps supplied with the apparatus are of iron, ground glass, optician's wax, and pewter. Emery of two finenesses for grinding and polishing respectively are also supplied, together with tripoli powder and polishing rouge. A grinding apparatus of the nature of a lapidary's wheel, suitable for larger crystals, is shown in Fig. 589.

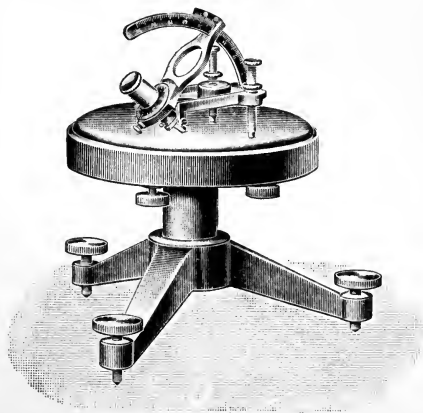


FIG. 588.—Steeg and Reuter's Crystal-grinding Apparatus.

In Figs. 590 and 591 are shown hand and foot machines for both cutting and

grinding larger crystals. To either may be fitted a goniometrical orientating and adjusting apparatus for the crystal, although it is actually shown only in Fig. 590 in

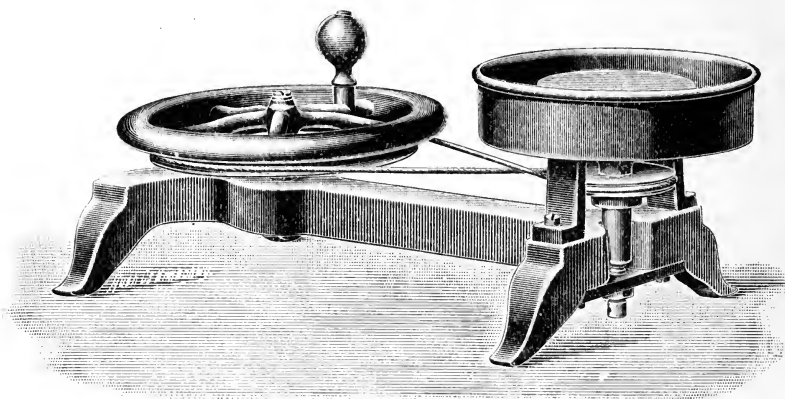


FIG. 589.—Steeg and Reuter's Grinding Apparatus for Large Crystals.

connection with the vertical hand-driven machine ; it enables the crystals to be rotated and adjusted in three rectangular directions, the three movements being graduated.

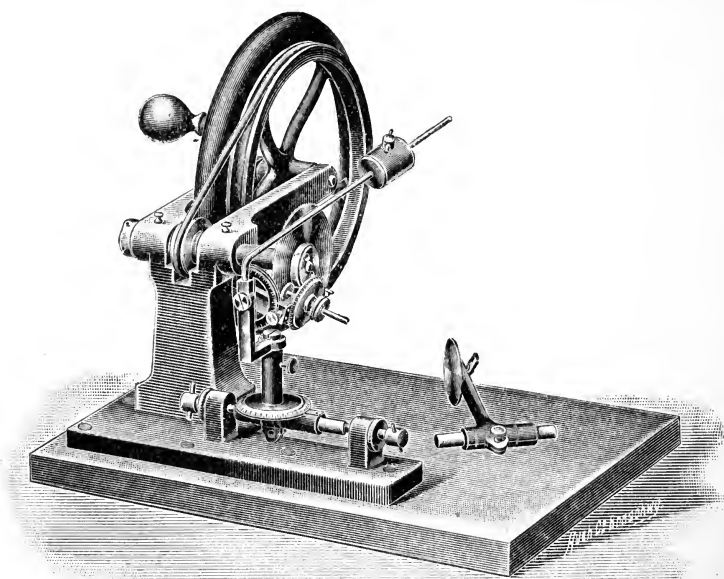


FIG. 590.—Steeg and Reuter's Hand-cutting and Grinding Apparatus with Goniometrical Adjustment.

The foot machine, Fig. 591, possesses both cutting and grinding apparatus, and a further development of this machine has recently been brought out by the firm for

motor driving, electric power being now so universal and highly convenient. This latest apparatus is shown in Fig. 592.

The construction of all these forms of crystal cutting and grinding apparatus will be clear from the figures. The foot and motor-power machines are equally capable of preparing the thin sections of crystalline rocks for microscopic investigation, or the sections or 60°-prisms of mineral or other crystals for more purely crystallographic purposes. Perhaps the instrument for foot driving shown in Fig. 591 combines the greater number of advantages for the latter purposes, with which this book is more particularly concerned. The cutting disc is at *d*, and the grinding lap in a protective well at *m*.

The feeding gear, to which the orientating apparatus shown in front in Fig. 590 may be attached instead of the simpler one actually shown, is indicated by the letters *b* and *c*, the latter showing the weighted lever by which the pressure of the crystal against the cutting disc (which must rotate downwards in front) can be modified. A surface is first ground on the crystal, as approximately correct in direction as is attainable by the eye, with the aid of the grinding lap *m*, which should be rotated from left to right, and moist emery. The crystal is then cemented by this surface with shellac on to the disc *a*, both disc and crystal as well as the shellac being warmed to the necessary extent to melt the latter. If the crystal will not

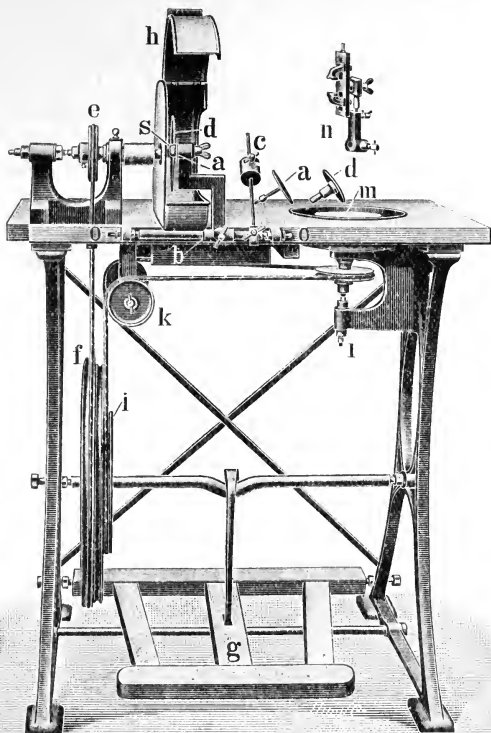


FIG. 591.—Steeg and Reuter's Foot Machine for Crystal Cutting and Grinding.

withstand this temperature, however, hard balsam in benzene must be used as cement, and then an interval of a day or two allowed to intervene for proper hardening to occur. The disc *a* is subsequently fitted by its perpendicular handle in the crystal-holder, an elbow from *b* or the alternative one of the orientation-fitting shown in Fig. 590, this latter fitting enabling the correct orientation of the crystal to be attained. The feed apparatus to which the crystal-holder is attached is then adjusted for the proper desired thickness of section, the weight *e* arranged to afford a convenient pressure, and cutting proceeded with, the cutting disc being properly fed with moist emery during the process, the receptacle at the bottom of the protecting screen *h* containing a supply for the purpose, which is applied to

the disc from time to time by means of a spoon provided. A set of six cutting discs is provided, three of which are diamond-fed for use with hard crystals, and three for use with emery alone when softer crystals are in hand. The usual varieties of laps are also provided, including one or two of glass, more or less ground. The arrangement shown at *n* is for the gripping of larger crystals or pieces of rock. After cutting off the end of the crystal, leaving the section cemented to the glass plate, the cut surface is ground and polished on *m*, the section removed by warming or dissolving the cement with a solvent, recemented by the newly polished second surface, and the first surface then reground and polished truly parallel to the second

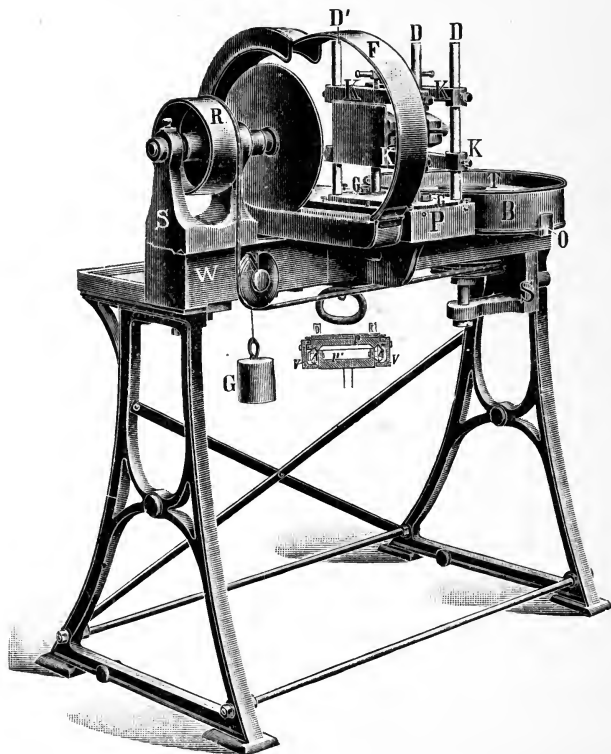


FIG. 592.—Steeg and Reuter's Motor-driven Crystal Cutting and Grinding Machine.

surface. The section is then finally warmed or dissolved off the disc *a* and mounted on a glass plate or otherwise in whatever manner is most convenient for the purpose in view.

In the motor-power machine, Fig. 592, the mandril *S* of the cutter, the splash-guard *F*, the grinding lap *T* with its guard *B*, and the apparatus for the support of the crystal *P*, are mutually adjustable with respect to each other. The crystal-holding apparatus *DD'K* shown in the figure is chiefly intended for large crystals or rock pieces, and when smaller crystals are being dealt with it is replaced by the orientating apparatus shown in Fig. 590, which is attached by the two screws *G* to the same feed-apparatus *P*, the details of which are shown in the inset *PP'* beneath the table. The movement is one of sliding of a grooved carrier over four rollers fitting the grooves, the adjustable angle-brackets *V* serving to direct the movement accurately.

CHAPTER XLIII

THE PRODUCTION OF MONOCHROMATIC LIGHT.

THE most satisfactory source of monochromatic light for practical crystallographic purposes is the continuous spectrum itself, filtered through a selecting slit, of fine opening, which permits a band of not more than a two-hundredth part, and preferably only a three-hundredth part, of the visible spectrum to escape. It should be produced from a powerful origin, best of all from the electric arc, by means of a single refracting prism, a grating affording insufficient light. The two refracting surfaces of the prism should be truly plane, and the prism itself of highly refractive and dispersive glass, but perfectly colourless so as to transmit the violet end of the spectrum without absorption. The objectives and optical tubes should also be of wide aperture, in order to transmit as much light as possible, which permits of the desirable narrowing of the exit slit.

A form of apparatus which admirably fulfils these conditions has been in use by the author since the year 1893, and was described to the Royal Society in February 1894.¹ It is shown in Fig. 593, and also in Fig. 594 in actual use with the electric lantern as source of light, to illuminate the polariscopical goniometer in a determination of the optic axial angle of a crystal.

It is essentially a very compact spectroscope, constructed to transmit as much light as possible. The two optical tubes *a* and *b* (see Fig. 594 for letters) are exactly alike, each carrying an adjustable slit *c* at one end and an achromatic lens *d* of nine inches focal length and two inches diameter at the other, the former being carried in an inner draw-tube so that the objective may be accurately adjusted to focus the slit by means of a rack and pinion *e* which effects the movement of the draw-tube. The pair of lenses forming each objective are separated by an air-film or narrow cell, in order to minimise the passage of heat rays. Either optical tube may be used as telescope and the other as collimator. That which is chosen as collimator (*a* in Fig. 594) is complete as already described. The other *b* chosen for use as the telescope is at once converted into such by screwing on to a tapped annulus, which is carried by both tubes in front of the slit, a cap *f* (lying on the base in Fig. 594) carrying a short tube in which slides an eyepiece. Three interchangeable eyepieces are provided (the two others being *g* and *h*), magnifying respectively two (*g*), four (*f*), and six (*h*) diameters. When one of the optical tubes is left in its simple condition, and the other is fitted with one of these

¹ *Phil. Trans.*, 1895, A, 185, 913.

eyepieces, the two tubes are precisely those of an ordinary, although very compact and stout-tubed spectroscope. The instrument is, in fact, used as such—with its collimator slit nearly closed but the telescope slit in front of the eyepiece opened wide, serving merely as a diaphragm aperture—for observing the Fraunhofer lines and employing them (with the telescope slit narrowed down again) for the purpose of the calibration of the instrument for the delivery of light of a number of definite wavelengths, or for the observation of the bright lines of sodium, lithium, thallium, and hydrogen for the like purpose. The two optical tubes are carried in a similar manner on counterpoised arms *i*, adjustable and capable of fixation by milled-headed clamping screws *j* at any relative positions about the circle *k*.

The prism *l* employed on this instrument is a large one of 60° angle in order to receive all the light emanating from the objective of the collimator, the refracting faces being $4\frac{1}{2}$ by $2\frac{1}{2}$ inches. This fine prism has also the exceptional double advantage

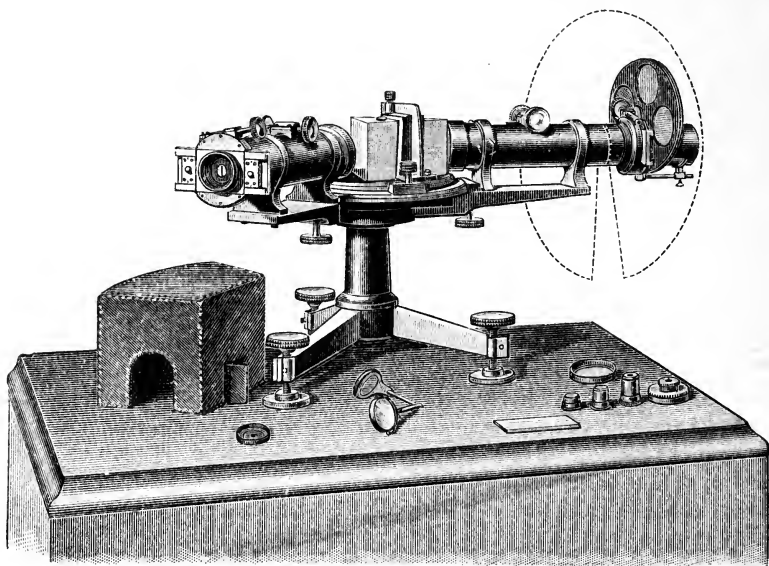


FIG. 593.—The Spectroscopic Monochromatic Illuminator.

of affording unusually large dispersion, nearly twice that of ordinary flint glass, while being perfectly colourless and unabsorptive. A Hilger constant-deviation prism may be used instead, however, although experience shows that it does not transmit as much light as the author's prism, nor does it yield as much dispersion; also, it is not so free from colour and absorptive effect. It is, however, a very convenient form of prism, the deflection being uniformly 90° , and for most purposes it serves quite well. When a Hilger drum calibrated directly in wave-lengths is used in connection with it the arrangement, shown later in this chapter in Fig. 595, is particularly convenient. As Messrs. Chance have been unable to quite reproduce the glass of the author's prism, a Hilger constant-deviation prism, with calibration drum, may be recommended as the next best substitute, and the instrument as thus constructed is now furnished by Messrs. Hilger.

The prism *l* is mounted on a rotating divided circle *k*, reading directly to half-degrees, and to single minutes with the aid of the vernier *m* attached to the fixed table *k'*. It is provided with a fine adjustment *n* extending for 7° , which is just ample to enable the whole spectrum to be traversed past the exit slit (that of the tube used as telescope,

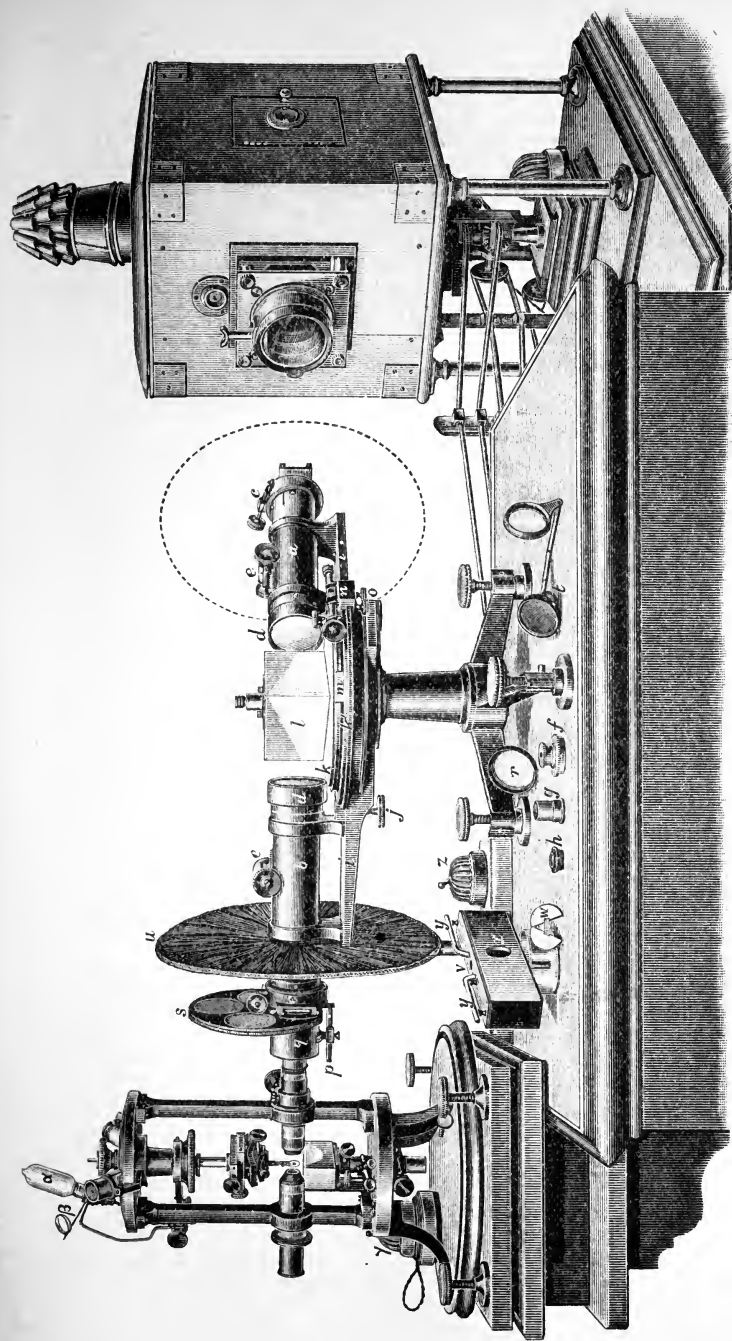


FIG. 591.—The Monochromatic Illuminator in Use with the Optic Axial Angle Goniometer.

the eyepiece being removed and the slit closed down to the same fine opening as that of the collimator). When the Hilger drum is employed, it simply replaces this fine adjustment, which is readily detachable, the screw and spring-piston portion being attached to the fixed circular plate k' , and the projection moved by the screw and held between it and the piston being part of an attachment to the circle k , both attachments being by a pair of milled-headed screws, o in the case of the lower one, the vernier plate or circle in each case being gripped by a bracket through which the screws pass.

When in use for the purpose of producing monochromatic light the eyepiece and its screw-cap attachment are removed from the telescope, and replaced by an annulus bearing a similar screw thread and carrying at right angles a horizontal rod p of square section, over which fits a little slider carrying above a wide but very short tube q in which in turn slides the carrying frame of a ground-glass plate. Two such interchangeable plates are provided of different finenesses of ground surface, ready mounted in their own annular carrier-frames; one is shown resting on the base against the tripod at r . The finer is preferable for use when refractive indices are being determined, and the coarser when optic axial angles are being measured. The use of this ground-glass screen is to diffuse slightly the linear beam of monochromatic light streaming through the exit slit, the beam being the narrower the finer the slit and more perfect the monochromatism. After such diffusion the light is capable of evenly filling the field of any optical instrument, such as a refractometer or polariscope, which is brought in front of the ground-glass screen, when the slit of the collimator is illuminated by the converging beam of brilliant light from the condenser of the electric lantern.

It is found with such an arrangement—the two optical tubes being fixed at the suitable angle found by experiment, and the prism free to rotate—that there is a particular angle of setting of the two optical tubes with respect to each other and the prism, near but not quite that for minimum deviation for any ray of the visible spectrum (a wave-length in the ultra-violet being really set for minimum deviation), at which, by rotation of the prism the whole of the visible spectrum may be caused to pass the exit slit of the second optical tube. Both optical tubes are clamped when this position is found, and remain always afterwards fixed. The prism is also clamped to the circle by means of the milled-headed screw working through the top of a bracket fixed to the circle plate, most of the pressure being taken up by a three-armed convex plate lying on the prism to protect it from the screw. If the Hilger constant deviation prism be used, the optical tubes are fixed at 90° once for all.

Any stray white light, or coloured light of other than the desired wave-length, due to internal reflections from the glass or inner tube surfaces, may be cut off completely by introducing a screen of either ruby-red, signal-green, or cobalt glass, or a combination of these two latter glasses, between the exit slit and the ground-glass screen. A couple of rotatable discs s , each furnished with four circular windows arranged like a quatrefoil, three of which contain these coloured glasses while the fourth in each case is left an open aperture, are supported in a dovetailed fitting above the slit box; the bevelled bracket carrying the axle on which the discs rotate and which slides in the dovetail is so arranged that when pushed home the discs are suspended so that any window may be brought by rotation of the disc exactly in front of the slit, with the centre of the window in the optical axis. The two discs are each separately rotatable about their common axis carried by the bracket. Hence, either the two blank windows can be brought opposite to the slit, when the monochromatic light streaming from the latter goes through unaffected, or any one glass window, or any combination of two differently coloured glass windows, may be arranged in the path of the emerging rays, and be thus used as light filters. This accessory proves very useful in the case of the investigation of crystals showing great change of optic axial angle with different wave-length of the illuminating light; for the presence of stray white light blurs the sharpness of the interference figures, and in extreme cases destroys the figures

altogether or renders them useless for measurement purposes. The accessory is not required for the middle part of the spectrum about the yellow.

On the base-board of the instrument as seen in Figs. 593 and 594, several accessories are shown, besides the eyepieces and the second ground-glass screen already alluded to. One is an adjustable mirror (t in Fig. 594), carried by an annulus gearing with the screw thread on the tapped circular rim in front of the slit of either optical tube. This enables sunlight to be directed on the entrance slit when desired, from a heliostat or otherwise. There is also a velvet-covered shade (shown in Fig. 593) for the prism and objective ends of the optical tubes, which enables the apparatus to be used in daylight as well as in a darkened room. A circular screen is also shown at u in Fig. 594 and indicated in dotted lines in Fig. 593 near the exit end of the second optical tube. This is a wire fitting covered with radial folds of dark green silk cloth, and serves to screen off any stray lantern rays from the observer's eye. A similar screen is usually also fitted near the entrance slit end of the first optical tube, as indicated by the dotted ellipse in Fig. 594, in order to prevent such rays getting directly at the prism or second objective. Another little accessory shown on the base in Fig. 593, is an elongated, rectangular, bevelled-edged, ground-glass plate, the same size as the jaws of the slit. Its purpose is to act as a screen for the reception and direct inspection of the spectrum, the slit-jaws being removable by unfastening a simple locking device and sliding them out, when the bevelled glass slip can be pushed into the dovetailed recess, thus left, instead.

Calibration of the Instrument for the Exit of Monochromatic Light of Specific Wave-lengths.—To calibrate this illuminator so as to know what divisions of the circle correspond to the issue of light of known wave-lengths, the entrance slit is illuminated by sunlight reflected from the mirror fitting, the slit being closed down to a very fine line by means of the adjusting screw, which moves each jaw independently and oppositely, so as to maintain the centre of the slit constant. The middle power eyepiece f is fitted to the other optical tube instead of the ground-glass diffuser, and the Fraunhofer lines examined as in an ordinary spectroscope, at first with the exit slit opened wide to reveal a considerable portion of the spectrum, but subsequently closed down for the placing of any Fraunhofer line to its centre.

The opening of the exit slit may be so narrow that when focussed by the eyepiece the two jaw edges appear only slightly further apart than is necessary to permit the sodium D-lines to be seen between them, the jaws serving as a couple of spider-lines; the D-lines (or any other line adjusted) appear parallel and symmetrically placed between them. If the optical tubes have not yet been fixed in their proper positions, they are arranged at first in the usual well-known way for minimum deviation of the middle part of the spectrum. The jaws of the exit slit may be removed altogether if desired, so as to obtain a full aperture during this review of the spectrum, for the locking device referred to enables removal to be accomplished without deranging their adjustment when replaced again. Moreover, even if not removed, as the fine adjustment of the slit is arranged to move both jaws equally, any spectrum line adjusted between them remains approximately so when their opening is varied, even to the extent of full aperture.

The telescope is then moved so as to pass through the spectrum from red to blue and as far as a position at which the last traces of visible spectrum have just disappeared; it is then fixed there and the collimator likewise fixed, a wave-length in the ultra-violet being thus set for minimum deviation. On rotation of the prism and its circle in either direction the whole of the colours of the spectrum in succession will move past the vertical diameter of the field. That one of the two directions is then chosen, for which the greater loss of light by reflection from the receiving surface of the prism occurs for the red end, as the illuminating powers of the different parts of the spectrum are then more nearly equalised. The entrance slit being set to a very fine opening as above mentioned, the definition of the Fraunhofer lines is still excellent

when focussed by the rack and pinion of the telescope. The fine adjustment for the divided circle is then set so as completely to carry out this movement of the whole spectrum, about 7° , past the centre of the field.

The slit jaws are next replaced, or if they had been simply opened wide they are closed up to the fine opening already specified, through which when the prism circle is rotated to the right position the two solar D-lines may both just be seen clearly focussed between the two jaw-edges. The reading of the circle is then recorded. This reading should afterwards be confirmed for the bright sodium lines by means of a sodium flame. The prism circle should then be rotated and other prominent solar lines adjusted and their circle readings recorded. The readings for the red, greenish-blue, and violet hydrogen lines H_α , H_β , and H_γ , at C, F, and near G of the solar spectrum will be included among these, and they should next be confirmed by means of the bright lines afforded by a hydrogen Geissler tube. It is advisable also to determine the readings for the red, green and blue lines of cadmium and for the green line of mercury, with the aid of Geissler tubes, as these rays are now much used in optical work. Similar readings should also be taken for the red lithium line and the green thallium line, when flames coloured by the salts of these two metals are produced just as for sodium light. Moreover, from time to time it is well to check the readings of the circle for the issue of the three metallic lines from the centre of the exit slit, certainly always before any original investigation is undertaken. For this purpose the author has always at hand a mahogany box, lined and pyramidally terminated above (as a chimney) with tinned iron, and furnished with a large glass window which can be placed opposite the entrance slit of the illuminator, with a door at the back, and with a perforated base raised a couple of inches so as to admit air freely beneath. A Bunsen lamp is placed within, the gas-tube connection being passed through one of the basal perforations, and a set of three stout platinum wires terminated by platinum boats are arranged at the height of the flame on a brass standard which is rotatable from without by means of a lever handle projecting from under the raised base; so that when the three salts are placed in the boats any one of the latter can be rotated into contact with the flame, and thus all three metallic lines can be rapidly reviewed and their situations verified by rotating the prism to the graduations corresponding to them.

If on thus reviewing each of the three lines it should not prove to be quite central between the jaw-edges of the exit slit, as focussed simultaneously with the lines by the eyepiece, the prism circle should be adjusted until it is, and the exact reading noted. As the jaws of the slit have occasionally to be opened a little wider when very small or not perfectly transparent crystals are under investigation, and then subsequently nearly closed again for use with good crystals, these variations, although the jaws are arranged to move equally on each side of the centre, may cause minute differences of circle readings to occur, which require to be observed. Such differences, however, have never been found to exceed $2'$, and the minute correction is the same for all the lines. In the case of the sodium D-lines, even the lowest-power eyepiece separates them, and the middle power which is generally used shows them an apparent millimetre apart. The reading for sodium light is taken when they are symmetrically placed between the two slit-jaws, the separation of which is about double that of the two lines.

In order that the readings for all wave-lengths may be known, and the prism-circle set for any wave-length at any time, the results of the calibrations are graphically expressed on curve-paper, taking circle readings for ordinates and wave-lengths for abscissæ; the curve joining the points on it thus graphically set forth is then drawn with a free hand. It is then only necessary, in order to procure light of any

wave-length whatsoever, to set the circle to the reading indicated by the curve as corresponding to this wave-length, and to illuminate the entrance slit with the powerful white light from the lantern, when monochromatic light of the desired wave-length will stream through the exit slit, and, the eyepiece in front of it being removed and the ground-glass diffuser added instead, will form a brilliant patch on the diffusing screen, adequate to illuminate any observing instrument.

If the Hilger constant-deviation prism and wave-length drum are employed, the actual wave-lengths themselves of the various spectrum lines adjusted between the slit-jaws are marked on the drum directly. The calibration in this case is very conveniently done with the aid of the arc spectrum of copper, the copper lines affording an adequate number of fixed wave-lengths in themselves to enable most others to be obtained by interpolation, and they can always afterwards be verified as regards the sodium, lithium and thallium lines.

This monochromatic illuminator is especially useful when some exceptionally interesting phenomenon is observed—the crossing of the optic axial planes and production of a temporary uniaxial figure by a biaxial crystal for instance—with light of a wave-length intermediate between the wave-lengths corresponding to two spectrum lines, say somewhere in the green between sodium and thallium light, and for which it is desired to know the exact wave-length. It is only necessary to take the circle reading and consult the curve, or if the Hilger drum is in use to take its indication, in order at once to ascertain the required wave-length.

The author's instrument was constructed by Messrs. Troughton and Simms, and has proved most efficient, having the great advantage of the exceptional dispersion of its magnificent 60° -prism. Similar instruments have been supplied to the Mineral Department of the British Museum and to the Manchester School of Technology, which are almost equally effective, although the dispersion is not quite so great, on account of the difficulty in reproducing the original glass.

The Hilger Wave-length Spectroscope adapted as Monochromatic Apparatus.—As already mentioned, the author's 60° -prism and circle fine adjustment may be replaced by a Hilger constant-deviation prism and wave-length drum. As many ordinary spectrosopes have now been supplied by Messrs. Hilger fitted with these arrangements, it may be useful to state that they can at once be adapted for the production of monochromatic light by simply replacing the two optical tubes by a pair with the author's special fittings, and which are shorter and stouter in order to afford a wider objective aperture and to transmit a great deal more light. Such an instrument, one quite recently constructed by the firm of Hilger under the author's supervision, is shown in Fig. 595. The wave-length drum is also shown enlarged in Fig. 596 and a plan of the constant-deviation prism, showing the path of the light rays, in Fig. 597. The indication of the wave-length on the drum is made by an indicator projecting from the fixed frame and which bears a narrow rib below, gearing in a helical slot in the cylindrical surface of the drum.

The prism is of the shape which will be clear from Fig. 597. It may be considered as built up of a couple of 30° -prisms and a right-angled totally-reflecting prism, and it is from the hypotenuse of the latter that the internal reflection occurs. The path of the rays is indicated by the thick dotted line and the arrows, I being the incident ray and R the finally refracted ray. As the light rays pass through the 30° -prisms at right angles to one of the imaginary planes (the internal one) forming the 30° -angle in each case, and parallel to the imaginary base which is perpendicular to that plane, the refracting effect is as if the rays had traversed a 60° -prism parallel to the base, that is, at minimum deviation,

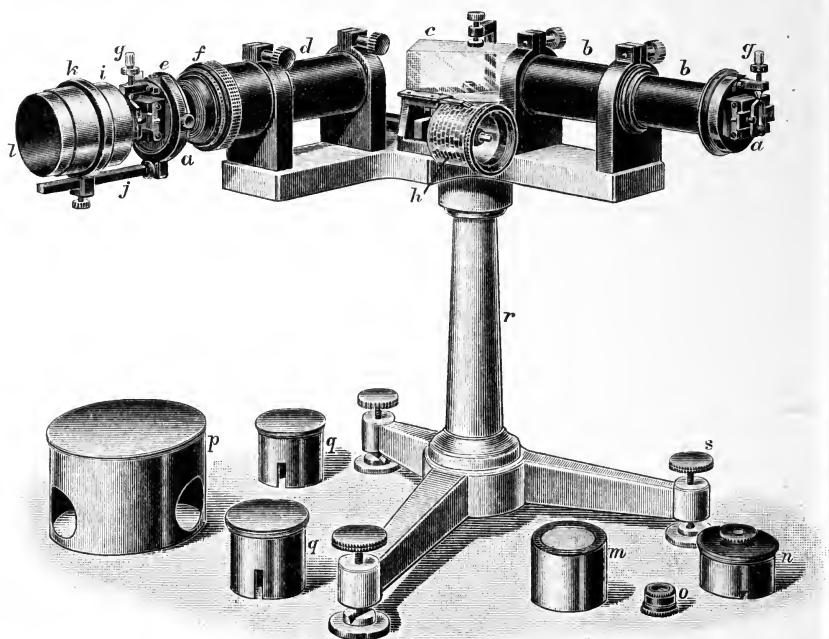


FIG. 595.—The Monochromatic Illuminator with Constant-deviation Prism and Wave-length Drum as constructed by Hilger.

and had emerged with the refracted ray making an angle of 90° with the incident ray. Moreover, the deviation remains constant at 90° for all wave-lengths, rotation of the prism by the wave-length drum simply bringing successive colours thus deviated past the centre of the field, the telescope and collimator being fixed at right angles to each other. This instrument has been thoroughly tested by the author, and proves very efficient. Although the intensity of the monochromatic light afforded by it is not quite as great as that delivered by the Troughton and Simms instrument, the purity of the spectrum is greater, and it is quite unnecessary to employ the coloured glass filters. The two slits *a* are of identical construction, the jaws moving equally, on each side of their closed position, when manipulated by the adjusting screw *g*, seen

above the slit. It is convenient to employ the optical tube seen to the right *b* in Fig. 595 to receive on its slit the converged beam of light from the lantern, as the wave-length drum *h* can then be more readily manipulated by the right hand than when the other tube is so used. After preliminary adjustment of each slit to the focus of its respective objective, the final focussing of the spectrum to the exact plane of the exit slit (the spectrum lines being then focussed by the eyepiece simultaneously with the jaws of the exit slit) is best performed entirely on the second optical tube *d* carrying the exit slit and eyepiece; it is achieved by means of a milled rotatable focussing ring *f*, which acts as a driving nut replacing the ordinary rack and pinion and affords a very steady fine adjustment. The eyepiece is carried in a detachable cap *n*, seen lying on the table to the right, which slips on to the exit slit end of the optical tube when required for observation or calibration of the spectrum lines; two interchangeable eyepieces are supplied (the second being shown at *o*), adjustable in the cap so as to focus the slit, the one clearly dividing the two sodium lines and the other separating them to the extent of an apparent millimetre. When the exit slit is opened wide and the eyepiece is in position the instrument is a most efficient wave-length spectroscop.

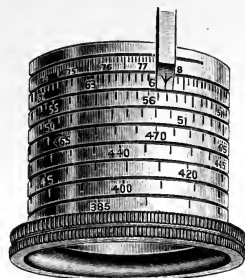


FIG. 596.—Hilger Wave-length Drum.

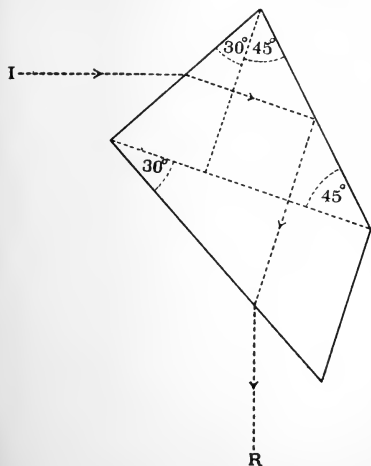


FIG. 597.—Construction of Constant-deviation Prism.

Ordinarily, when the instrument is in use as a monochromatic illuminator, the eyepiece is replaced by the diffusing-tube fitting *i*, capable of sliding along a bar *j* by means of a collar *k* carrying below a slider; the bar is carried by an annulus *e* attachable by two milled-headed screws. The short sliding tube is provided with two alternative movable inner tubes *l* and *m* (the latter shown lying on the table next to the eyepiece) each carrying a ground-glass screen, one being more finely ground for refractive index determinations and the other coarser for optic axial angle observations. The constant-deviation prism *c* is provided with a metallic cover *p*, pierced with two apertures opposite the objectives of the optical tubes. It is shown resting on the table to the left. The other two objects on the table are a pair of cover-caps *q* to protect the slits when the instrument is not in use. The whole instrument is mounted on a stout pedestal *r*, and tripod provided with levelling screws *s*.

A monochromatic illuminator is now also constructed on these lines by Fuess and is shown in Fig. 598. In this instrument, the author's ground-glass diffuser is replaced by a lens, on the recommendation of

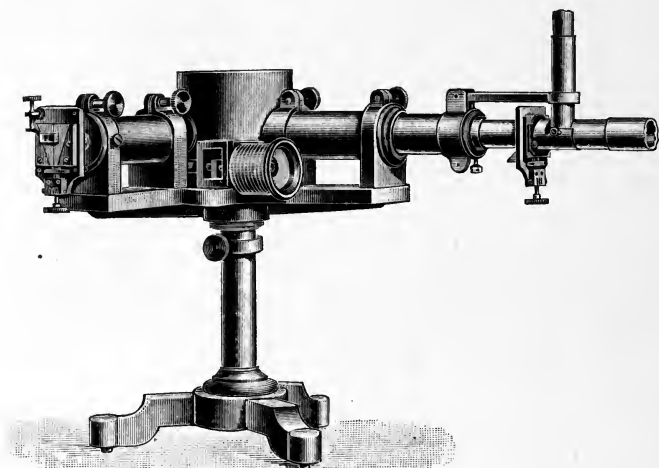


FIG. 598.—Fuess-Wülfig Monochromatic Illuminator.

Wülfig. The author's earlier experiments with lens diffusers led him to discard them for the ground-glass diffuser, which is infinitely more satisfactory provided a brilliant source of light, the electric arc, is available.

CHAPTER XLIV

THE DETERMINATION OF EXTINCTION ANGLES BY THE STAUSCOPE.

THE most efficient form of polariscope for the accurate determination of extinction angles is the "Stauroscope" belonging to the "Universal Crystal Apparatus" of von Groth, constructed by Fuess of Berlin. The polarising microscope may be used for preliminary or approximate work, as will be fully described in Chapter LII. ; but for accurate measurement no instrument has yet proved so satisfactory as the Fuess stauroscope. The optical tube which carries the large polarising Nicol is the same as that of the optic axial angle goniometer of the universal apparatus, but without the convergent system of lenses, and the same Nicol analyser is employed with both, but a different optical tube and other fittings are employed with it.

The stauroscope is shown in Fig. 599, partially in section, and later in Fig. 601 in position during a determination in monochromatic light, with the aid of the spectroscopic illuminator. It is essentially a polariscope arranged for parallel light, having wide optical tubes, large lenses for the parallelisation of the beam of light, and a large Nicol prism polariser, in order to transmit the maximum of light. Besides a Nicol analyser of smaller size, provided with a very small eye-hole to avoid parallax, and covered with a small slightly enlarging concave lens, the second or analysing tube is provided with a Calderon double calcite plate, a half-shadow compound plate resembling a bi-quartz, for enhancing the sensitiveness of the determination of the exact positions of extinction in the crystal plate.

From a strong base *a*, between the two claws of which an adjustable mirror *b* is carried, chiefly for use in white light observations, rises a column *c* of triangular section, carrying two sliders *d* and *e*, the lower *d* being capable of fixation at any convenient height, by means of the milled-headed clamping screw *f*, and the upper *e* being adjustable at any height by rack and pinion, the latter manipulated by the milled head *g*. Each slider carries one of the two wide optical tubes, which have a common axis, supposing the apparatus to be standing on its base as in Fig. 599. The lower tube *h* has an inner tube *i* carrying a doubly convex lens *j* at its lower end, and a similar one *k* at its upper end, the pair together furnishing a parallel beam of light, all of which has passed through the polarising Nicol *l*, in which the rays cross, and which has a minimum aperture of $1\frac{1}{4}$ inches. The inner tube carries a clamping ring *l'*, which has a V-shaped projection fitting in either of two corresponding notches 45° apart in

the lower edge of the outer tube. The ring is clamped on the inner tube in such a manner once for all that when the V fits one of the notches the plane of polarisation of the Nicol is arranged symmetrically to the stand, and corresponds to the zero mark of the vernier, which is fixed at the outer end of that diameter of the circle which is parallel to the line passing symmetrically between the claws of the stand.

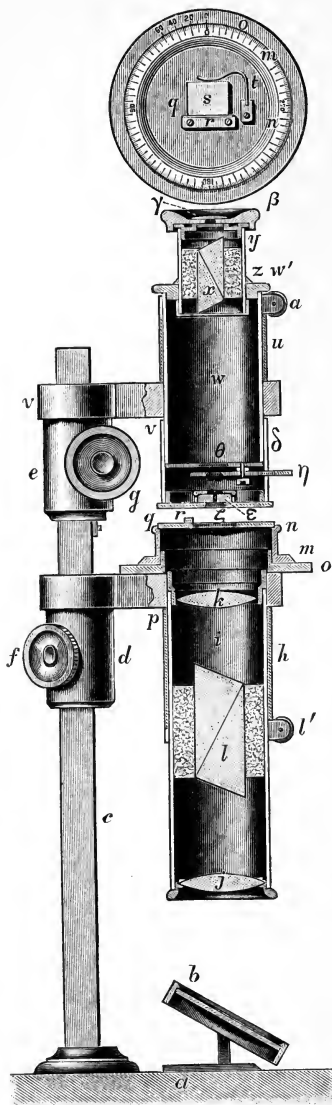


Fig. 599.—The von Groth Stauroscope as constructed by Fuess.

About the upper end of the outer tube is carried the rotating divided circle *m*, as the silver bevelled flange of a rotatable cap or short tube *n*, which carries a special arrangement for the support of the crystal section-plate, which latter is cut or ground parallel to the symmetry plane when the crystal is monoclinic. The bevelled graduated silver limb *m* of the circle rotates over a silvered flange *o*, solid with the supporting ring *p*, which is carried at the end of the short arm proceeding from the slider *d*; the outer tube is screwed directly into this ring *p*, and the silvered flange *o* bears the vernier graduations on the front part of its horizontal surface. The circular table *q* forming the top of the cap is detachable, but when replaced always takes up the same position again, owing to its having a notch fitting about a pin in the annulus rim. The table is perforated by a rectangular central aperture, along one of the longer edges of which a straight strip of hard metal *r* is carried, the edge of which abutting on the aperture is a truly straight line, and the narrow surface is bevelled slightly out of the exact vertical plane so as to make the important upper straight-edge slightly overhang.

In some instruments this edge is very slightly curved, in order to afford two-point contact only to the plane edge of the mount-plate presented to it; the line joining the two points, or the straight-edge itself when no curvature is given to the edge, is arranged so as to be exactly parallel to the right and left 90°-270° diameter of the circle, when the zero graduation is opposite the zero of the vernier, as shown in the separate upper part of Fig. 599, and also in Fig. 600. Against the straight-edge (or the two points of the curved edge) an especially truly planed and polished normal edge-face of a rectangular glass plate *s* is maintained pressed, during the determination of extinction, by a spring *t*, as shown in the upper part of

Fig. 599. It is on this plate that the crystal-section is mounted with hard balsam dissolved in benzene. The normal edge-face in question of the glass plate is 2 millimetres wide, corresponding to the thickness of the plate, and is polished truly at 90° 0' to the broad plate surfaces, so as to afford an excellent image of the goniometer

signal, the other three edge-faces being left rough-ground. The polished edge of the plate is marked in the middle near the position for the crystal with a deep round niche, which enables the light rays to get at the crystal faces during the measurement on the goniometer of the angles between the plate surfaces and crystal faces of reference.

The crystal is mounted on this glass plate so that one of the original natural faces—belonging to the ortho-zone if the crystal be of monoclinic symmetry, normal to the ground pair of faces parallel to the symmetry plane—is approximately parallel to this reflecting glass face. The angle between the two—for except by chance absolute parallelism cannot be attained by hand setting—can then be determined by actual measurement on the goniometer, and thus be used as a correction for the true position of the crystal face with respect to the direction of the 90° - 270° circle graduation. Any slight error of setting of the straight-edge itself to the 90° - 270° graduation is corrected by reversal of the glass plate and crystal, the latter then being below the plate and projecting through the aperture of the stage, and by taking a second series of stauroscopic observations for this position. The error is eliminated in taking the mean of the two observations, as well as any error of setting of the 90° - 270° and 0° - 180° diameters of the circle to the planes of vibration of the Nicols.

The upper optical tube consists of an outer tube u carried by the annular termination of the arm v proceeding from the slider c . Into the upper part of this tube slides an inner one w , terminating above in a thick flange w' with truly plane upper surface of silver, on which a radial zero mark is engraved. Within this tube at its upper extremity fits the analysing Nicol x , the supporting tube y of which carries about one-third from the bottom a bevelled silver circle z , fitting fairly tightly over the silvered top of the flange w' . The Nicol is thus rotatable and its position with respect to the zero mark indicated by its circle. Immediately below the flange a tightening collar a is fixed, bearing a V-projection capable of fitting in a corresponding notch in the upper rim of the outer tube u , and the collar is tightened at such a position that the zero mark then corresponds to that of the vernier of the circle plate o of the lower optical tube. Above the analyser there fits loosely a detachable shallow cap β , pierced at its centre by a circular aperture of only 3 millimetres diameter, and closed by a small concavo-convex lens γ . This is essential, for in order to obtain a sharp extinction the position of the eye must be maintained perfectly central. The lens also affords a slight enlargement of the field and of the crystal placed on the stage-plate.

When this inner tube w is pushed down into position in the outer tube u , with its collar projection properly fitted in the notch, about three-quarters of an inch of it projects below the annulus v of the supporting arm. Over this slips tightly from below a further short tube or lower cap δ , forming a detachable continuation of the outer tube. The lower end of this cap carries the stauroscopic plate ϵ for enhancing the sensitiveness of the determinations of extinction. In the author's instrument it is a Calderon double plate of calcite. It is constructed by cutting a rhomb of calcite along the shorter diagonal of the rhombohedron, the plane of cutting being thus a principal section containing the optic axis; from each half a wedge is then ground, leaving the cut surface as one face of the wedge in each case; the two wedges then are cemented together by their other ground surfaces, after the latter have been polished, the thick end of one wedge being cemented to the thin end of the other, one wedge being thus rotated for 180° . A plate is then cut from the compound block thus obtained, at right angles to the plane of cementation, so that when the plate is regarded normally the junction plane is seen in plan as a fine line. When placed between crossed Nicols, with this dividing line parallel to the vibration plane of one of them, such a plate shows an equal amount of extinction in each half, that is, the slight shade in each half is the same; for each is equally slightly removed, by the angle of the wedges, from the principal-section plane. If a crystal plate be interposed,

on its glass plate laid on the stage at the top of the lower optical tube, that is, at ζ , the two halves will at once show a marked difference of colour tint or brightness, unless the extinction directions of the crystal coincide exactly with those of the Nicols. Hence, when such a difference appears, the crystal and stage must be rotated along with its divided circle until equality of tint or intensity is again restored. This angle of rotation is the difference between the setting of the crystal on the plate, which has been goniometrically determined with reference to an actual face of the crystal, and the extinction direction. The angle of extinction with reference to the face in question is thus the sum or difference, according to the direction of rotation, of the angles found respectively on the stauroscope and the goniometer.

In order to screen off as much extraneous light as possible a rotatable diaphragm carrying four apertures of different sizes is placed at η , so that the circular aperture of the fixed diaphragm θ can be more or less constricted at will, and the crystal thus made to occupy most of the visible field. The inner tube w is slotted for half an inch at its lower end, to take a pin screwed so as to project through the tube δ of the cap, at such a position as brings the dividing line of the Calderon double-plate parallel to the plane of vibration of one of the crossed Nicols.

Thus when all is in position the conditions are as shown in Fig. 600. The crystal edge of reference c , the plan or trace of an existing face, is either parallel to the straight edge ab and to the 90° - 270° graduation and to the plane of vibration of the corresponding Nicol, or makes a known (measured) angle therewith.

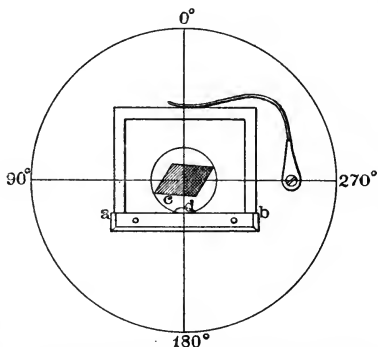


FIG. 600.—Conditions in the Stauroscopic Determination of Extinction.

Before carrying out a determination of extinction, the polarising Nicol is arranged once for all so that its vibration direction is as nearly as possible parallel to the line joining the centre of rotation of the circle to the 0° mark of the fixed vernier o .

The analyser is then arranged with its vibration direction perpendicular to this position, so that the maximum darkness of field is produced when the Calderon double-plate is removed and nothing interposes between the two Nicols. The cap containing the Calderon plate is then replaced in position at the lower end of the tube w , so that the pin fits in the slot and is pushed home to the end of the latter. Two of the rectangular glass plates should, the day previously, have had the pair of ground and polished crystal section-plates mounted on them, near the ground-away niche d (Fig. 600), for then when the plate is placed in position against the straight-edge, and the diaphragm-aperture of suitable size adjusted, the crystal will appear centrally in the field of view on looking through the analyser, as shown in Fig. 600. The angle a made by the crystal face, which has been arranged approximately parallel to the truly plane glass edge-face, with the latter, should then be measured on the goniometer, just before placing it on the stauroscope stage.

Before thus placing it, moreover, the adjustment of the Calderon double-plate should be looked to, and the fact verified that the two halves are absolutely equally in half shadow, that is, equally only very slightly lighted, almost dark, under the crossed Nicols. After having done this efficiently once, it will be found on subsequent occasions that it is only the analyser which is not absolutely at its zero graduation if the lighting of the Calderon plate is not evenly distributed.

On now placing the glass plate and its mounted crystal section-plate in position on the circle-cap, against the straight-edge, and sliding it slightly, if necessary, while pressed against the straight-edge by the spring, until the crystal appears centrally situated in the field, the crystal will be seen to be sharply divided off into a bright half and a dark half, by the line of junction of the two halves of the Calderon double-plate, unless the position of the crystal is by chance adjusted near to that for extinction. On rotation of the circle, which involves that of the whole cap and crystal stage, the difference of brightness or tint of the two crystal halves will be either still further accentuated or diminished, according to the direction of rotation. The direction for diminution being then known, the circle is turned in that direction until equality of depth of grayness or absolute match of colour tint is shown in the two halves, no difference whatever being apparent between them. This position, with a good crystal, is capable of detection in the author's instrument with certainty to 5 minutes of arc. It is useless to expect to pass beyond this degree of accuracy, or even to attain so sharp a determination in the case of a poorer crystal, with any known form of sensitiveness enhancer, as the result of a single determination. Hence, half a dozen different settings should be made to the extinction position, and the mean taken, when a somewhat closer approximation will be reached.

When the position of extinction has thus been found, what has really been achieved is that the two perpendicular directions of extinction (for the same character of extinction occurs at 90° to the position found) have been ascertained with reference to the planes of vibration of the Nicols, assuming that the 0° - 180° and 90° - 270° diameters of the circle respectively have been set truly parallel to the planes of vibration of the Nicols, which, however, is not the fact, for this setting can at the best have been only approximately achieved experimentally. Moreover as regards defining the extinction position crystallographically, with respect to the reference face, the angle of rotation on the circle from the 0° - 180° diameter, that is, from the zero of the circle as read by the fixed vernier, gives the extinction direction with respect to the straight-edge and reflecting edge-face of the glass mount-plate, and not with respect to the reference face. But as the small angle α enclosed between the glass edge-face and the crystal face arranged approximately parallel to it has previously been measured on the goniometer, it is only necessary to subtract it from or add it to the mean stauroscopic extinction angle just found, according as the angle through which the crystal has been rotated includes this small angle or is adjacent and

additional to it, in order to arrive at the extinction angle with respect to the reference face. No mistake in applying this correction can be made if the direction of the correction angle is first of all carefully observed, and the graduation of the circle appreciated which really corresponds to the setting of the reference crystal-face itself (rather than the straight-edge and glass edge-face) to the zero position. Indeed, to avoid the possibility of any such mistake in the sign of the correction, it is not a bad plan to so mount the crystal that the reference face makes a decided visible angle with the edge-face of the glass mount-plate. Another precaution to remember is that, in measuring the correction angle, the glass plate must be adjusted truly horizontal on the goniometer axis, assuming the reference face to be one in the ortho-zone of a monoclinic crystal, which is thus perpendicular to the section-plate parallel to the symmetry plane, and to the glass mount-plate. For other more complicated cases special instructions will presently be given.

In order to eliminate finally the possible error in setting the plane of the polarising Nicol parallel to the diameter containing the zero of the vernier, the glass plate is then reversed, so that the crystal is below it instead of above it, but with the reflecting edge-face of the glass plate still pressed by the spring against the straight-edge. A second series of half a dozen adjustments to extinction should again be made, and the corresponding readings recorded and their mean taken. This is then independently corrected for the setting of the crystal on the mount, the small angle α being added or subtracted as the case demands. The mean of the two values thus obtained, for the reversed positions of the crystal above and below the plate respectively, is then taken as the true extinction angle ϕ for that particular crystal.

Precisely similar procedure is adopted for the observations with the second (duplicate) crystal plate, and the mean of the results for the two crystals, which should not differ by more than a very few minutes, is accepted as the true extinction angle for any plate of this crystalline substance having this orientation. When the crystals are monoclinic, the result is thus to afford the directions of extinction in the symmetry plane, the section-plates having been ground parallel thereto; hence, they are really the directions of the two axes of the optical ellipsoid lying in the symmetry plane.

Now as the two perpendicular extinction directions—in such a plate parallel to the symmetry plane of a monoclinic crystal or to a principal plane of the optical ellipsoid of a triclinic crystal, that is, the directions of two of the three principal axes of the optical ellipsoid—are dispersed for different wave-lengths of light, it is always advisable, and imperative when this dispersion exceeds 5° , that the observations should be made in monochromatic light. Usually it suffices to employ sodium light for extinction determinations, the most intense one procurable being used, such as is afforded by a fan-tail Bunsen burner into the broad and high flame of which several little platinum boats containing a bead of a sodium salt are introduced, or any other of the many

laboratory arrangements now available for procuring an intense sodium flame. Better still, however, is the monochromatic illuminator described in the last chapter, with which the determinations should be made for red C- and greenish-blue F-hydrogen light as well as for sodium light, the light of all these three wave-lengths being supplied directly from the illuminator, using the electric lantern as source of light. A convenient mode of supporting the stauroscope horizontally in front of the ground-glass diffuser of the spectroscopic illuminator is shown in Fig. 601. Excellent illumination is obtained by this mode of procuring the

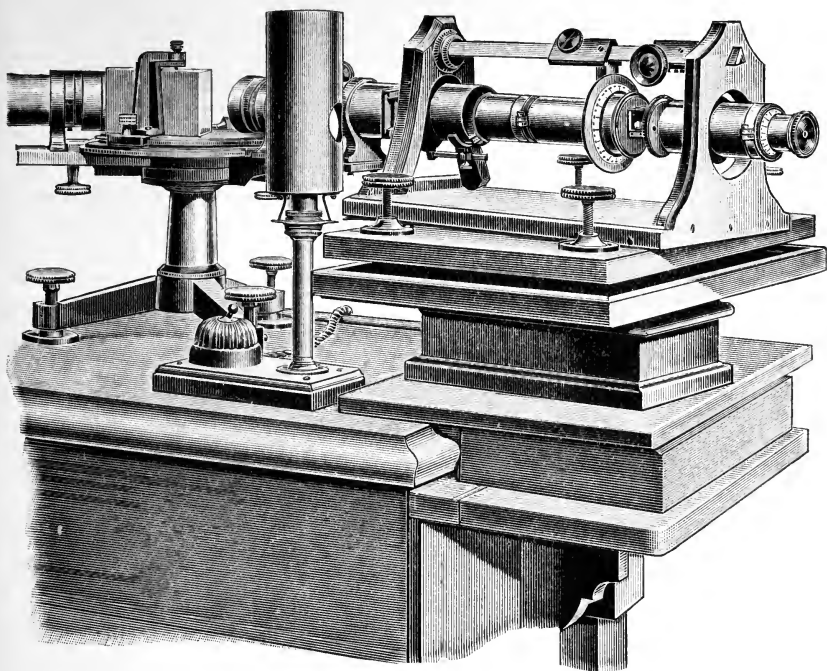


FIG. 601.—Determination of Extinction Angles in Monochromatic Light.

necessary monochromatic light, for not only these three, but any other desired wave-lengths whatsoever.

The mode of carrying out such an observation of extinction will now be illustrated by a concrete example, that of ammonium magnesium sulphate, the substance taken in Chapter XVIII. as an instance of characteristic monoclinic symmetry. Sodium light here suffices as the dispersion of the axes of the optical ellipsoid is extremely small.

Determination of Orientation of the Axes of the Optical Ellipsoid of Ammonium Magnesium Sulphate, $(\text{NH}_4)_2\text{Mg}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$.—The section-plate parallel to the symmetry plane $b = \{010\}$ is represented in Fig. 602. Its edges were formed principally by the two pairs of parallel faces $a = (100)$ and $a = (\bar{1}00)$, and $c = (001)$ and $c = (00\bar{1})$, the

latter forming the long edges, and also subsidiarily by smaller faces of $r' = (201)$ and $r = (20\bar{1})$. Strips of these six faces were left along the margin of the plate, and as they all belonged to the ortho-zone parallel

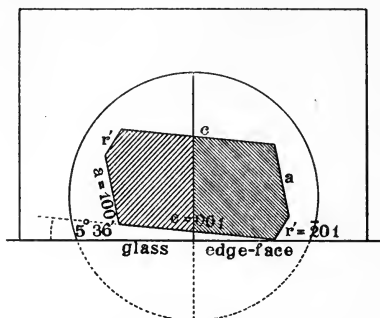
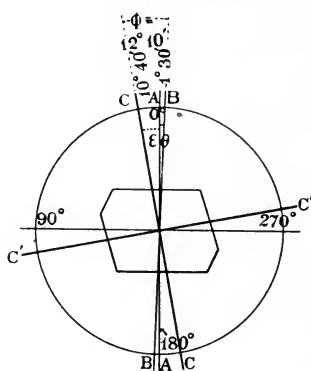


FIG. 602.—Section-plate of Ammonium Magnesium Sulphate as seen in Stauroscope.

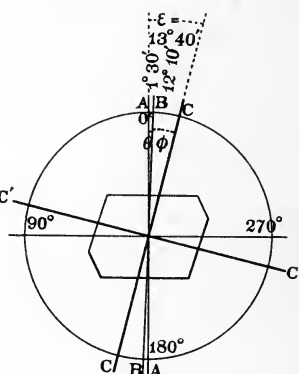
to the symmetry axis b (the normal to the plate) they were all perpendicular to the plate. The face $c = (001)$ afforded a particularly good single reflection of the goniometer signal, and hence was chosen as the reference face. The crystal was mounted with hard balsam dissolved in benzene, so that this face made a small visible angle with the reflecting edge-face of the glass mount-plate. After drying for a day to harden the balsam cement this angle α was measured on the goniometer. The stauro-

scopic measurements were then made, first with the mount laid on the stage with the crystal above, and afterwards with the mount



Crystal
above

FIG. 603.



Crystal
below

FIG. 604.

Conditions in Determination of Extinction Directions of Ammonium Magnesium Sulphate.

reversed and the crystal below. The results were as under. Figs. 603 and 604 graphically express the conditions in the two series of observations.

Angle of inclination α of c -face to glass edge-face, $5^\circ 36'$.

DETERMINATIONS OF EXTINCTION ANGLE.

Crystal above glass.		Crystal below glass.	
	4° 56'		8° 6'
	5 6		8 8
	5 6		7 54
	4 52		8 4
	5 6		8 14
	5 12		7 58
Mean	5° 3'	Mean	8° 4'
+ α	5 36	+ α	5 36
Apparent Extinction Angle 10° 39'		Apparent Extinction Angle 13° 40'	
referred to normal to		referred to normal to	
$c = \{001\}$.		$c = \{001\}$.	
True (Mean) Extinction Angle, 12° 10', with respect to normal to $c = \{001\}$.			

This extinction angle of 12° 10' made with the normal to the basal plane, $c = \{001\}$, lies in the direction behind the normal, as will be clear from Fig. 605, which is the type of figure which should be drawn to express graphically the results for every substance investigated.

The difference between the two results for the reversed positions of the crystal, 10° 39' and 13° 40', namely, 3° 1', is double the difference between the setting of the plane of polarisation of the polarising Nicol and the zero diameter of the fixed vernier circle. The correction angle itself is thus 1° 30'. That is, the true circle reading for one of the vibration directions of the two Nicols (that of the analyser when crossed to the polariser with maximum darkness of field) is 358° 30' instead of 360° 0' (the same as the zero mark 0° 0'), when the zero marks of the fixed vernier and rotatable circle are opposite each other. This error of 1° 30' in the placing of the zero to the plane of vibration is thus eliminated by actual experiment, but when it is known it may be used directly as a correction to the apparent extinction angle. It is much better, however, to eliminate it thus experimentally, for it varies slightly with the setting of the analysing Nicol for production of the most perfect dark field, which can of course only be achieved within the limits of the usual few minutes. The correction α of 5° 36' for setting of the crystal plate on the mount is clearly shown in Fig. 602, but is ignored in Figs. 603 and 604 to save unnecessary complication, the crystal being assumed in these latter figures to be set with its c -face parallel to the edge-face of the glass mount.

In Figs. 603 and 604 the direction A is that of the 0°-180° diameter

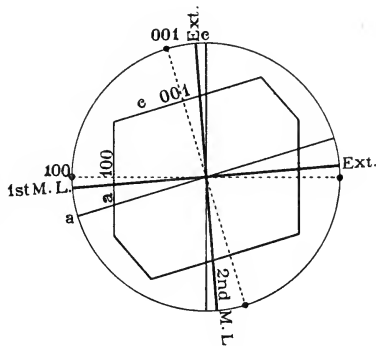


FIG. 605.—Extinction Directions in Plate of Ammonium Magnesium Sulphate Parallel to Symmetry Plane.

of the circle. B is the true direction of vibration of the analyser and perpendicular to the vibration direction of the polariser, when the 0° reading of the circle is opposite the 0° of the vernier (which is immovably fixed with respect to the polariser). C and C' are the extinction directions in the crystal. The correction angle of $1^\circ 30'$ for the true position of the Nicols is θ ; ϵ is the apparent extinction angle, and is the angle actually read off, corrected by adding or subtracting, in this case the former, the angle α between the reference crystal face and the glass edge-face. Lastly, ϕ is the true extinction angle, and is the sum of θ and ϵ in the case of Fig. 603 with the crystal above the plate, but the difference of ϵ and θ in the case represented by Fig. 604, with the crystal reversed underneath the glass plate.

A second section-plate parallel to the symmetry plane, ground from another crystal of ammonium magnesium sulphate derived from a different crop, afforded the value $12^\circ 5'$ for the true extinction angle of ammonium magnesium sulphate in the symmetry plane, with respect to the normal to the same face $c = (001)$.

Hence, the author has accepted the mean of the two values derived from these two different crystals, $12^\circ 8'$, as the true extinction angle, that between the a axis of the optical ellipsoid (indicatrix), the second median line, and the normal to the basal plane $c = (001)$, the direction being behind the normal and near to the vertical axis c of the crystal, as shown in Fig. 605.

This will be the normal method of determining the extinction directions of a monoclinic crystal, for it is rare to find absolutely no faces developed in the ortho-zone, suitable for reference faces. Occasionally, however, it will happen that there are no such faces, and with triclinic crystals more frequently, although the plate investigated is often very conveniently ground perpendicular to a pair of well-formed faces. One ought to be prepared to meet such cases of greater difficulty, and the following method of dealing with them is the simplest and most satisfactory.

Suppose Fig. 606 to represent the glass mount-plate— g being the upper surface of the plate and g_1 the truly plane-polished edge strip—with such a monoclinic crystal section-plate parallel to the symmetry plane mounted on it, of which c is the upper ground and polished surface and c_1 is the face of a principal form inclined to two axes, a prism face such as (110) or (011) , for instance. The angle $c:c_1$ is measured on the goniometer, the edge cc_1 being adjusted for the purpose, and the little niche in the centre of the glass edge gg_1 has been cut to admit the light rays more fully for the purpose, for if the crystal plate be thin the glass edge otherwise gets in the way. During this measurement it will probably be noticed that the goniometrical signal-image from the glass surface g is coincident with that from c ; it should be so if the crystal plate has been truly ground parallel-sided and

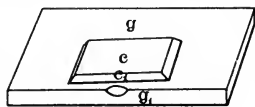


FIG. 606.

evenly cemented with only a thin film of balsam. But unless the edge cc_1 has been set truly parallel to the edge gg_1 during the cementing, the images from c_1 and g_1 will not be in the same zone $[cc_1]$, that from g_1 appearing above or below in the field when the images from c and c_1 are adjusted to pass through the intersection of the spider-lines on rotation of the goniometer axis. The object is to find the mutual inclination of the two zones $[cc_1]$ and $[gg_1]$, for this is the same as that between the crystal and glass edges cc_1 and gg_1 ; this correction angle is lettered α as usual. Let the measured crystal angle $c:c_1$ be γ . We have first to find the angle δ which the rays reflected from c_1 make with the plane of the reflections from g_1 , c , and g . Then Websky and Liebisch have shown that the desired small angle α can be calculated at once from the formula :

$$\sin \alpha = \frac{\sin \delta}{\sin \gamma \sqrt{2}}.$$

A very simple way of measuring δ is to set the telescope of the goniometer at exactly 90° to the collimator, by first placing it in the same straight line, so that the signal seen directly through the telescope and collimator is adjusted to the crossed spider-lines, and then rotating it 90° ; we then know the angle of incidence, which will obviously be 45° . The ordinary Websky signal-slit is replaced by the alternative signal consisting of a small central circular hole, and the crossed spider-lines of the telescope are arranged as in Fig. 607, at 45° to their usual position, by rotation of the eyepiece for 45° ; a second V on the collar of the eyepiece tube at 45° to the one usually employed, and fitting to the same usual notch in the outer tube, facilitates this adjustment. The images from c and c_1 , the adjusted faces of the crystal, will still pass through the centre of the field on rotating the circle and the crystal-holder which it carries, one of them, say c , being shown adjusted at the centre in Fig. 607. But g_1 will pass either above or below the centre, as shown passing above in Fig. 607, its path being dotted because c and g_1 will obviously not be visible in the field at once, owing to the considerable angle between them. If we adjust the g_1 image first to one spider-line and then to the other, and take a circle reading for each such adjustment, the difference of the two readings will be δ . For it is half the observed angle of sweep of the ray between the adjusted position of the image g_1 on the right and that on the left, a reflected ray moving through twice the angle of movement of the surface reflecting it, which latter is that recorded by the circle readings. If we draw a perpendicular cN to the line of passage g_1Ng_1 , then, the angles at g_1 and c being 45° ,

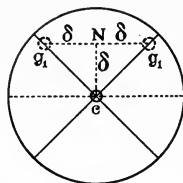


FIG. 607.

$$g_1N = \frac{1}{2}g_1Ng_1 = cN = \delta.$$

That is, cN is the desired angle δ of deviation of the rays out of the zone $[cc_1]$, and is at once afforded by the difference of the two readings

for the adjustment of g_1 to the two spider-lines as shown in Fig. 607. Hence, as we now know both δ and γ from direct measurement, we can calculate α by means of the formula.

As an aid to the calculation the following table has been prepared by Liebisch¹ and von Groth² in which the values of α are given for every 10' of δ up to 3°, beyond which it is not required, and 5° or 10° of γ between 20° and 90°.

VALUES OF THE STAUSCOPIC CORRECTION ANGLE α .

	$\gamma=20^\circ$.	$=25^\circ$.	$=30^\circ$.	$=35^\circ$.	$=40^\circ$.	$=50^\circ$.	$=60^\circ$.	$=70^\circ$.	$=80^\circ$.	$=90^\circ$.
$\delta=0^\circ 10'$	0° 21'	0° 17'	0° 14'	0° 12'	0° 11'	0° 9'	0° 8'	0° 8'	0° 7'	0° 7'
20	0 41	0 33	0 28	0 25	0 22	0 18	0 16	0 15	0 14	0 14
30	1 2	0 50	0 42	0 37	0 33	0 28	0 24	0 23	0 22	0 21
40	1 23	1 7	0 57	0 49	0 44	0 37	0 33	0 30	0 29	0 28
50	1 43	1 24	1 11	1 2	0 55	0 46	0 41	0 38	0 36	0 35
1 0	2 4	1 40	1 25	1 15	1 6	0 55	0 49	0 45	0 43	0 42
10	2 25	1 57	1 39	1 27	1 17	1 5	0 57	0 53	0 50	0 49
20	2 45	2 14	1 53	1 39	1 28	1 14	1 5	1 1	0 58	0 57
30	3 6	2 31	2 7	1 51	1 39	1 23	1 13	1 8	1 5	1 4
40	3 27	2 47	2 21	2 3	1 50	1 32	1 22	1 15	1 12	1 11
50	3 48	3 4	2 36	2 16	2 1	1 42	1 30	1 23	1 19	1 18
2 0	4 8	3 21	2 50	2 28	2 12	1 51	1 38	1 30	1 26	1 25
10	...	3 38	3 4	2 40	2 23	2 0	1 46	1 38	1 33	1 32
20	...	3 54	3 18	2 53	2 34	2 9	1 54	1 45	1 41	1 39
30	...	4 11	3 32	3 5	2 45	2 18	2 2	1 53	1 48	1 46
40	3 46	3 17	2 56	2 28	2 11	2 0	1 55	1 53
50	4 1	3 30	3 7	2 37	2 19	2 8	2 2	2 0
3 0	4 15	3 42	3 18	2 46	2 27	2 15	2 9	2 7

As an example of the use of the table, and in proof of the accuracy of the method, the case of a monoclinic crystal may be quoted, in which the calculated angle could be actually measured and confirmed, the angle between c and c_1 being 90°, an excellent face in the ortho-zone perpendicular to the symmetry plane being developed. The actual angle between c_1 and g_1 was measured directly on the goniometer, as usual when such a face perpendicular to the plate is present, that is, in the same zone with g_1 when the glass plate and crystal section-plate are horizontal, perpendicular to the goniometer axis. This angle $c_1 : g_1$ was found to be 2° 6'. The crossed spider-lines were then rotated 45°, by rotating the eyepiece so as to fit the second V of its collar into the notch of the outer telescope tube, and the Websky signal replaced by the hole signal; the mount-plate with its crystal was next arranged vertically, and the image from g_1 (for either c_1 or g_1 may be so centred) adjusted to pass through the intersection of the crossed spider-lines on rotation of the goniometer axis with the crystal-holder. The image from c_1 was then immediately above that from g_1 , just sufficiently in the field to be able to be placed in succession to the two diagonally crossed spider-lines, where they nearly reached the margin of the circular field. The angle passed through on rotating the circle and crystal-holder

¹ *Zeitschr. für Kryst.*, 7, 304.

² *Physikalische Krystallographie*, 4th ed. (1905), p. 734.

between these two positions was $2^{\circ} 55'$, which was thus the value of δ . Looking out from the table the angle a corresponding to $\delta = 2^{\circ} 55'$ and γ (the angle between the two crystal faces c and c_1) $= 90^{\circ}$, we find it half-way between $2^{\circ} 0'$ corresponding to $\delta = 2^{\circ} 50'$ for $\gamma = 90^{\circ}$, and $2^{\circ} 7'$ corresponding to $\delta = 3^{\circ} 0'$ for $\gamma = 90^{\circ}$, that is, $2^{\circ} 4'$. This is only $2'$ removed from the measured value $2^{\circ} 6'$, and thus verifies the table by direct experiment.

The polarising microscope with rotating divided stage and rectangularly crossed spider-lines at the eyepiece focus, can be used very conveniently for preliminary determinations of extinction, the reference face of the crystal being set to either of the cross-wires, one of which corresponds to the zero graduation of the rotating stage-circle, the Nicols being crossed. A special chapter, LII., will be devoted to the use of the crystallographic microscope, including its use as stauroscope, and other forms of sensitiveness-enhancing plate than the Calderon double-plate will there be dealt with. The methods which have been devised for determining the extinction directions in very small, even microscopic, crystal plates will also be treated of, including the use of such results in ascertaining the positions of the optic axes and the axes of the optical ellipsoid, even when derived from section-plates or thin crystals of any orientation whatsoever. For the purposes of the original investigation of substances the crystals of which can be procured of a size as large as that of a pin's head or larger, somewhat larger in preference, say that of a small pea, the stauroscope as described in this chapter is the best instrument to employ, in accordance with the methods here recommended.

Norremberg's Doubler.—A simple form of polariscope designed originally by Norremberg, although superseded for accurate measurement by the stauroscope just described, has such useful properties in qualitative and demonstration work, particularly in its original form for the examination of thin films such as those of mica and gypsum, as to merit special notice before concluding this chapter. It combines a Nicol prism, used as analyser, with a bundle of glass plates as polariser. It is shown in Fig. 608 in its original form, and in Fig. 609 as constructed in its modern form by Fuess. The outer mirror a of the two hinged mirrors a and b of the Fuess apparatus, or the similar one constructed by Steeg and Reuter, at the base of the apparatus, is an ordinary silvered mirror, by which the light rays can be directed at the polarising angle on to the second mirror b , which consists of the bundle of glass plates, from which the polarised light is reflected upwards into the instrument. The lower of the three sliders c carries a short tube d terminating above in a rotating divided stage e for the crystal object, below which is the lens for parallelising the light rays, and which can be supplemented by a convergent system, including a hemispherical condensing lens just under the object plate, when convergent light is required. The longer optical tube f , supported by the arm g projecting from the top of the vertical column h , carries a lens system which acts either as a low-power microscope for parallel light, or, with the addition of a convergent system

similar to the lower one, for collecting the rays converged by the latter, and enabling the interference figures in convergent polarised light to be clearly viewed. The Nicol prism analyser *k* is carried at the top of a further short flanged tube *l*, in which it is rotatable, and which is pierced by a slot *m* for the insertion of a quartz wedge or a quarter undulation mica plate. The instrument includes a very wide angle of field, both optic axes of crystals possessing apparent optic axial angles $2E$ of as much as 130° being completely visible with their ring systems.

In the earlier form in which Norremberg invented it, shown in Fig. 608, the silvered mirror *a* is laid on the basal box *b* of the apparatus,

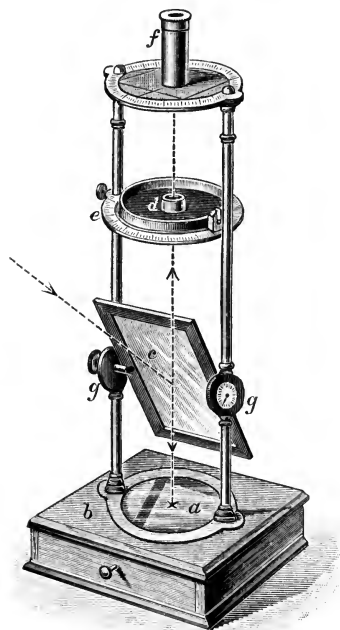


FIG. 608.—Norremberg's Doubler.

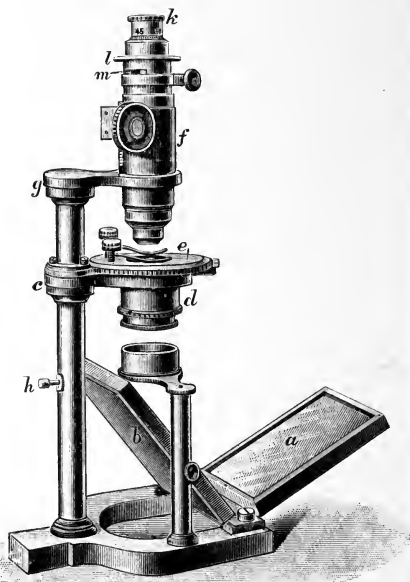


FIG. 609.—Modern Form of Norremberg's Doubler as constructed by Fuess.

and the bundle of plates is replaced by a *single* plate of glass *c* carried in a rectangular frame. The rays of light are arranged to fall on this plate at the polarising angle, $56\frac{1}{2}^\circ$, and so as to be reflected perpendicularly downwards to the mirror, from which they are reflected upwards to the object *d* on the stage *e* above the glass plate, and through the Nicol prism analyser *f* to the eye, passing through the glass plate on the way. The latter thus acts like the glass plate of the Gauss ocular or Becker fitting shown in Fig. 13 (page 34).

A small divided circle *g* is arranged at each end of the axle of the frame carrying the glass plate, in order to indicate the angle of tilt. When the crystal plate under investigation is laid on the stage, the

polarisation phenomena of an ordinary polariscope are afforded. But when it is laid on the silvered mirror, or for greater convenience on a second stage inserted between the mirror and the inclined glass plate, the rays of light then traverse it twice, both in going to and in returning from the mirror, and the phenomena observed are those of a plate of the crystal of double the thickness, thus giving the name "doubler" to the instrument. Provided the crystal does not fill the whole field, two images of it will be seen, a larger one, the directly viewed image, and a smaller one reflected from the mirror, and this latter will show effects as if the plate were of twice the thickness.

A simple addition to the polarising microscope of an inclined (rotatable) glass plate fitting, such as is used for the illumination of opaque objects under the microscope, converts the latter into a very efficient "doubler."

CHAPTER XLV

THE DETERMINATION OF THE REFRACTIVE INDEX OF CRYSTALS BY THE 60°-PRISM METHOD, AND OF MOLECULAR REFRACTION.

By far the most satisfactory and accurate method of determining the refractive index or indices of a crystal is that of the 60°-prism, provided such a prism, if the crystal be cubic in symmetry and therefore singly refractive, or the necessary pair of prisms, if the crystal be doubly refractive, can be cut or ground out of the crystal. This can always be done, unless the crystals are only of microscopic size, by means of the cutting and grinding goniometer described in Chapter XLII. and shown in Fig. 583, employing the special crystal-adjusting apparatus for use in the preparation of 60°-prisms there described. Even soft and brittle crystals can be successfully ground into prisms with this most useful apparatus, with one setting of the crystal on the crystal-holder, and the author has frequently prepared satisfactory prisms out of crystals no larger than the head of a pin. Crystals of the size of a small pea are the most suitable and convenient, when available. But even much larger crystals may be employed, and correspondingly larger prisms prepared, when the crystal is an internally homogeneous and perfectly transparent one.

Isotropic (cubic) Crystals.—When the crystal is one belonging to the cubic system, and therefore isotropic, a single 60°-prism will afford the unique refractive index μ , and it may be orientated anyhow in the crystal. The case is the simple one of an isotropic substance described on p. 582, a cubic crystal or glass, for instance, behaving similarly in this respect. But in the original investigation of an apparently cubic substance, the careful observer would prepare at least three such 60°-prisms, and orientate them so that the bisecting plane was parallel to each of the three pairs of faces of the cube in turn, in order to confirm that the refractive index is really identical for the directions of all three principal axes of the crystal. A fourth 60°-prism, cut obliquely, should also be ground, in order to establish finally the fact that whatever be the direction in which the light travels through the crystal, the refractive index corresponding is the same.

Any and all of these prisms, when arranged for minimum deviation, will yield only a single image of the "Websky" signal-slit of

the goniometer-spectrometer, when the latter is illuminated by monochromatic light, or a single corresponding spectrum when white light is employed, just as in the case of a glass prism; the angle of the minimum deviation of this single image, for each of the six usual wavelengths of light, together with the accurately measured value of the angle of the prism—as determined by the ordinary goniometrical method of measurement of the angle between two faces, after having adjusted the refracting edge between those two faces with the aid of the goniometrical centring and adjusting movements—afford all the data necessary for the calculation of the unique index of refraction. Moreover, the image remains constant and permanent when a Nicol prism is attached in front of the eyepiece and rotated, the only difference of intensity observed being that due to polarisation by reflection at the incident face of the prism, the reflected rays being more or less polarised in the plane of incidence, and the refracted beam reaching the eye in the absence of a Nicol being partially polarised perpendicularly to the plane of incidence.

The direction of vibration of the refracted polarised light is thus parallel to the plane of incidence, the horizontal plane when the prism is set up as usual on its triangular base. Consequently, when the vibration direction of the Nicol is also horizontal these vibrations get through unaffected, but when the Nicol is rotated a right angle so as to bring its vibration direction vertical they are extinguished, so that the ordinary refracted light only (that larger part which had never been polarised) gives rise to the vertically polarised light produced by the Nicol. Thus it is that the light received from the prism, when the Nicol is also introduced in front of the telescope eyepiece, appears less intense when the Nicol is at 90° (vibration direction vertical) than when it is at 0° (vibration direction horizontal).

Uniaxial Crystals.—If the crystal be of tetragonal, hexagonal, or trigonal symmetry it will possess two extreme refractive indices, the ordinary ω corresponding to light vibrating along all directions perpendicular to the principal (trigonal, tetragonal, or hexagonal) axis, the optic axis, and the extraordinary ϵ corresponding to light vibrating parallel to the optic axis. Which of the two is the greater index depends on the sign of the double refraction; in positive crystals it is ϵ , and in negative ω .

We have to remember the following salient facts as regards the refractive indices of uniaxial crystals. Rays travelling along the optic axis, the principal crystallographic axis, afford only one refractive index ω , corresponding to vibrations in the circular section. Any ray passing obliquely through the crystal and its imaginary optical ellipsoid of revolution is resolved into two component rays. One of these, the so-called ordinary ray, is plane-polarised in the principal section (that containing both the direction of the ray and the optic axis) and vibrates perpendicularly thereto along a radius of the circular section. Whatever be the position of the ray this component consequently always gives the same index of refraction, namely, ω . The other, the extraordinary ray, differs in direction of vibration and in refractive index with the

direction of the ray, and the difference from the refractive index ω of the ordinary ray is at its maximum corresponding to the extreme value ϵ , the quantity to be determined, when the ray is transmitted in the circular section itself and vibrates parallel to the optic axis. The practical task is thus to determine ω and ϵ . Their difference is the measure of the double refraction.

A single prism will enable both ω and ϵ to be determined, provided it be cut or ground either so that the refracting edge is parallel to the optic axis, or else so that the refracting edge is perpendicular to the optic axis and the plane bisecting the refracting angle of the prism is parallel to the axis. That is to say, the bisecting plane of the prism must contain the optic axis, either as the refracting edge itself or as the perpendicular to that edge. These two varieties of prisms have both been fully described in Chapter XXXVIII. describing the passage of light through uniaxial crystals, and illustrated in Figs. 499 and 500 (p. 595).

Such a prism, when arranged for minimum deviation, and when the goniometer-spectrometer is illuminated by monochromatic light, instead of yielding a single image of the Websky signal-slit as in the case of a cubic crystal or glass, affords two such refracted images (see Fig. 612, p. 731), which are the more separated the greater the amount of the double refraction and the greater the angle of the prism. The double refraction is rarely so large as to prevent both being simultaneously visible in the field of view of the telescope. When the Nicol prism is in position in front of the eyepiece of the latter, as it always should be when prisms of doubly refracting crystals are under investigation, it will be found that one of the two images extinguishes absolutely when the Nicol is arranged with its plane of vibration at 0° , and the other also perfectly when the Nicol is rotated a right angle to its 90° -graduation. Both images will be visible at the 45° intermediate position of the Nicol. The image which is in view when the Nicol is at 90° is usually weaker in intensity than the other which is seen at its maximum intensity when the Nicol is at 0° , owing to the light entering the crystal being partially already polarised at the reflecting surface, apart from the polarisation due to double refraction, with vibrations in the horizontal plane of incidence. This somewhat weaker 90° -image is due solely to the double refraction and corresponds to vibrations parallel to the vertical refracting edge of the prism. The stronger one which is visible at the 0° -position of the Nicol corresponds to light vibrating perpendicularly to the edge but also in the bisecting plane. For the light passing through the prism parallel both to the imaginary triangular base and to the imaginary symmetrical third face (it being obviously unnecessary to cut these surfaces), as is always the case when the prism is arranged for minimum deviation, divides into two rays, both vibrating in the plane at right angles to the direction of transmission, the one along the vertical principal axis (optic axis) of the crystal and the other along the direction perpendicular thereto. That is, the two rays vibrate respectively parallel to the two axes of the optical indicatrix; one of these is the optic axis and the other may be anywhere in the circular section of the indicatrix, the latter being an ellipsoid of

revolution in the case of a uniaxial crystal. The former ray vibrating parallel to the optic axis gives ϵ , and the other vibrating parallel to the circular section gives ω . Which of the two corresponds to light vibrating parallel to the refracting edge of the prism depends, of course, on which of the two alternative modes of cutting the prism has been adopted. The author's practice is to prepare six such prisms, three of each mode of cutting, in order to eliminate, in taking the mean values of all the results, all possible experimental or structural errors, which usually only affect one or two units in the fourth place of decimals.

The image which is nearer to the direct reading of the signal-slit, that is, which is the less deviated, corresponds obviously to the smaller refractive index, ω in the case of a positive crystal and ϵ in the case of a negative one; the more deviated image corresponds to the larger refractive index, ϵ for a positive and ω for a negative crystal. The angle of minimum deviation δ is the only variable, for α the angle of the prism is the same for both. From the formula given in Chapter XXXVI. (page 583) for the calculation of the refractive index μ , namely,

$$\mu = \frac{\sin \frac{\alpha + \delta}{2}}{\sin \frac{\alpha}{2}},$$

it will be clear that the index varies directly as δ , which only occurs in the numerator.

When the first mode of cutting the prism has been adopted, that of Fig. 499, the refracting edge being parallel to the optic axis, the image which is alone visible when the Nicol is at 0° is formed by rays vibrating perpendicularly to the optic axis and to the refracting edge of the prism, and affords ω . The other weaker image seen alone when the Nicol is at 90° is afforded by rays vibrating parallel to the optic axis and to the refracting edge, and therefore gives ϵ .

When the second mode of cutting has been adopted, that of Fig. 500, the 0° -image is formed by rays vibrating parallel to the optic axis, and corresponds to ϵ . The 90° -image on the other hand is afforded by rays vibrating parallel to the refracting edge, as usual for a 90° -image, and also perpendicularly to the principal axis, and gives ω .

In order to arrive at a final result free from all possible slight errors, whether due to the crystal or to the measurements, it is advisable to prepare six prisms of each substance investigated, three on each plan. The three prisms in each case should be orientated in three different positions about the optic axis, so that the value of ω may be obtained for three different radii of the circular section. By so doing not only is the accuracy enhanced but absolute proof is obtained that the refractive index along different equatorial azimuths is really identical, and that, therefore, the equatorial section of the optical indicatrix is in truth a circular one.

When the crystal is tetragonal two of the three positions should be those of the lateral rectangular axes, and the third one of the 45° -interaxes. When the crystal is hexagonal two of the positions should

be those of two adjacent 60° -hexagonal axes, and the third one of the 30° -interaxes. The mean of all six results for each index may be confidently taken as the true value. The determination should in every case be made for the usual six wave-lengths of light.

Biaxial Crystals.—If the crystal belong to the rhombic, monoclinic, or triclinic systems it will possess three refractive indices, namely, α the lowest expressing the refraction along the minimum axis of the optical indicatrix, now an ellipsoid of general form having three different rectangular axes, β the intermediate index representing the refraction along that rectangular axis of the optical indicatrix which possesses an intermediate length, and γ the highest refractive index denoting the refraction along the maximum axis of the indicatrix. A 60° -prism, the bisecting plane of which contains two of these three principal axes of the optical ellipsoid, and the refracting edge of which is parallel to one of these two axes, will afford two of the three indices directly. Hence two prisms may be ground which between them will afford all three indices of refraction and one in duplicate. But a complete investigation should include the preparation of six such prisms, two affording α and β , two furnishing β and γ , and two others yielding α and γ . Each index is thus obtained four times, and the mean value may be taken as the truth when the individual values are concordant. When the crystal is rhombic, the directions of the axes of the optical ellipsoid are known from the symmetry. If the crystal be monoclinic, the orientation of the ellipsoid will have been ascertained by determinations of extinction with the aid of section-plates parallel to the symmetry plane. If triclinic, the more difficult determination of the position of the ellipsoid will have been achieved by making a number of extinction observations on a variety of section-plates or tabular crystals, and by the examination of the optic axial interference figures in convergent polarised light.

The preparation of a 60° -prism from a rhombic crystal is especially simple. The zones of faces developed will usually enable the desired principal crystallographic plane, identical with a principal plane of the optical ellipsoid, and which is to be the bisecting plane of the prism, to be adjusted at once, as also the direction of the crystallographic and optical axis which is to be the direction of the refracting edge of the prism. The case of a monoclinic crystal is nearly as simple, the rotation of the optical ellipsoid in the symmetry plane being the only additional complication, and the setting of this can be readily carried out with the aid of the graduated adjusting movements provided on the cutting-and-grinding goniometer, including the use of the special 60° -prism-preparing adjusting apparatus. The plane of symmetry of the monoclinic crystal can usually be very readily set vertically, parallel to the axis of the goniometer and parallel also to the plane of the prism-cutting 120° -adjusting-segment; it can then be rotated through the extinction angle by means of the upper ordinary adjusting segment, so as to bring one of the two axes of the optical ellipsoid lying in the symmetry plane vertical, after which both faces of the 60° -prism can at once be ground in succession, without detaching the crystal after the

grinding of the first surface, by merely rotating the large segment for 60° first on one side and then on the other side of the adjusted position.

In the case of a triclinic crystal, when once the much more difficult task of discovering the orientation of the optical ellipsoid is achieved, the knowledge can at once be applied and experimentally translated on the cutting-and-grinding goniometer, and the necessary prisms cut or ground almost as readily as for a monoclinic crystal.

The cutting-and-grinding goniometer thus enables the necessary 60° -prisms to be prepared for crystals of every variety of symmetry. It is consequently an invaluable instrument, providing as it does the means of directly determining the primary optical constant, refractive index, with ease and accuracy. It would be waste of time, in the author's opinion, to dilate on other very much less accurate means of obtaining prisms of artificial or mineral crystals. Occasionally prisms of 55° - 65° angle are naturally available by the prismatic formation of the crystal itself, a couple of suitable faces of a prominent zone being especially well developed at an angle not too far removed from 60° . When the angle exceeds 65° there is danger of total reflection occurring within the crystal, and non-emergence of the refracted rays. Less than 55° brings down the accuracy somewhat. It is not a very frequent occurrence to find such a pair fortunately placed, in relation to other faces present, as to permit of light passing through in the correct position for minimum deviation, but whenever it does happen, and the faces of the prism are such as yield excellent images of the signal, such a prism may well be used. Every effort should be made, however, to obtain at least the smaller grinding goniometer illustrated in Fig. 586 (p. 691), and if the larger apparatus shown in Fig. 583 (p. 682) can be procured, one is then in the position of being as well prepared for the cutting of the prisms of hard mineral crystals, as for the grinding of those of softer artificial chemical preparations. So fundamental is this constant, refractive index, as regards crystal optics, that the acquirement of one of these instruments is, in the author's opinion, as important as the acquisition of a good goniometer itself.

The actual determination of the refractive indices for a number of different wave-lengths of light is the same in all cases, and a description of the method of carrying it out will now be given.

Measurement of the Angle of the Prism.—This is carried out precisely like the measurement of an ordinary crystal angle, on the goniometer which is being used as spectrometer, the crystal being mounted, however, with hard optician's wax instead of the usual more pliable goniometer wax, on one of the usual crystal-holders of the simple kind. The adjustment of the crystal-prism for the angular measurement serves also to adjust it for the subsequent determination of minimum deviation. The author always employs the large Fuess No. 1a goniometer, described in Chapter XXVIII., and shown to the right in Fig. 610, for refractive index determinations, but the ordinary crystal-measuring Fuess goniometer No. 2a serves almost equally well, as shown to the left in Fig. 611. If, on examination of the prism during its adjustment for the measurement of the angle between the two ground and polished surfaces, the latter should be found insufficiently polished to afford a clear refracted spectrum or pair of spectra, and also in all cases of deliquescent or

rapidly alterable crystals, the two surfaces may be covered with correspondingly small pieces of microscope cover-glass, selected, after testing their reflections on the goniometer, for true planeness of surface. The selected discs should be cut up carefully with a diamond into the desired small rectangular or other conveniently shaped pieces, and two of them cemented over the refracting surfaces with hard Canada balsam dissolved in benzene. In this case the crystal must be allowed to remain in position on the crystal-holder for at least 24 hours, for the balsam to dry and set hard, after verifying that the glass plates have been properly attached and that the angle is not appreciably altered thereby, and that good refracted images of the signal-slit are now obtained. Provided great care has been taken to select good cover-glasses reflecting perfect single images of the Websky slit, the author has found no error to be introduced by the use of such cover-plates, at any rate not greater than one or possibly two units in the fourth place of decimals of the refractive index, and this is entirely eliminated when the mean of the four values for each index is taken, and is anyhow less than the differences exhibited between the values yielded by different prisms of the same orientation, which often amount to three or four or even more such units. In the great majority of cases, however, an adequate polish will be able to be given to the ground surfaces, by finishing with the polished glass polishing lap, moistened with either a trace of brick oil or, in many cases of artificial salts not very soluble in water, with the moisture of the breath.

Determination of Minimum Deviation.—Having adjusted the crystal-prism and measured its angle in the usual manner with the aid of the white-light goniometer lamp, it is next arranged for minimum deviation. The collimator and the circle being fixed, the telescope is rotated until it is in line with the collimator, and the Websky signal-slit is then directly viewed through the telescope and its image adjusted to the crossed spider-lines. This "direct reading" is then read off on the circle and recorded. The crystal axis and the telescope are next rotated, independently of the circle, until they are in the position for probable minimum deviation, the telescope being some 30° to 40° away from the direct reading and the crystal-prism so arranged that light from the collimator shall be refracted through it parallel to the imaginary third side and base of the prism and enter the telescope, as in Fig. 489 (page 582) in Chapter XXXVI. The image of the Websky slit broadened into a spectrum, or the pair of such spectra if the crystal be other than a cubic one, will then be seen on looking through the telescope after a few trials in slightly varying positions of either crystal or telescope, and should then be adjusted absolutely for minimum deviation; when thus adjusted the slightest rotation of the crystal prism in either direction will cause the image or images to move further away from the direct reading.

The whole apparatus as used by the author for the determination of minimum deviation is shown in Fig. 610. It consists essentially of the spectrometer-goniometer, the monochromatic illuminator, and the electric lantern as source of light. Two accessories are left out, for the sake of showing the essential parts more clearly; one is a folding screen of thick cardboard, covered on the outside with dark red cloth and on the inside with black velvet, which surrounds the goniometer on three sides and effectually screens off all stray light from the lantern, the room being otherwise dark; the other is a white-light lamp, an electric glow lamp surrounded by a brass cylindrical shade, the latter pierced by a window the same diameter as the collecting lens of the illuminating tube of the goniometer (a continuation of the collimator) and at exactly the same height. This little lamp stands at the right-front corner of the base-board of the spectroscop, together with a switch for it (see Figs. 601 and 614, in which it is shown). It will be observed that the lantern and spectroscop are arranged on a large table in the convenient positions for the illumination of the slit by the converging rays from the lantern condenser, filtered from the heat rays through a circular water cell two inches thick fitting in a short tubular receptacle for it in the lantern front. The spectroscop is

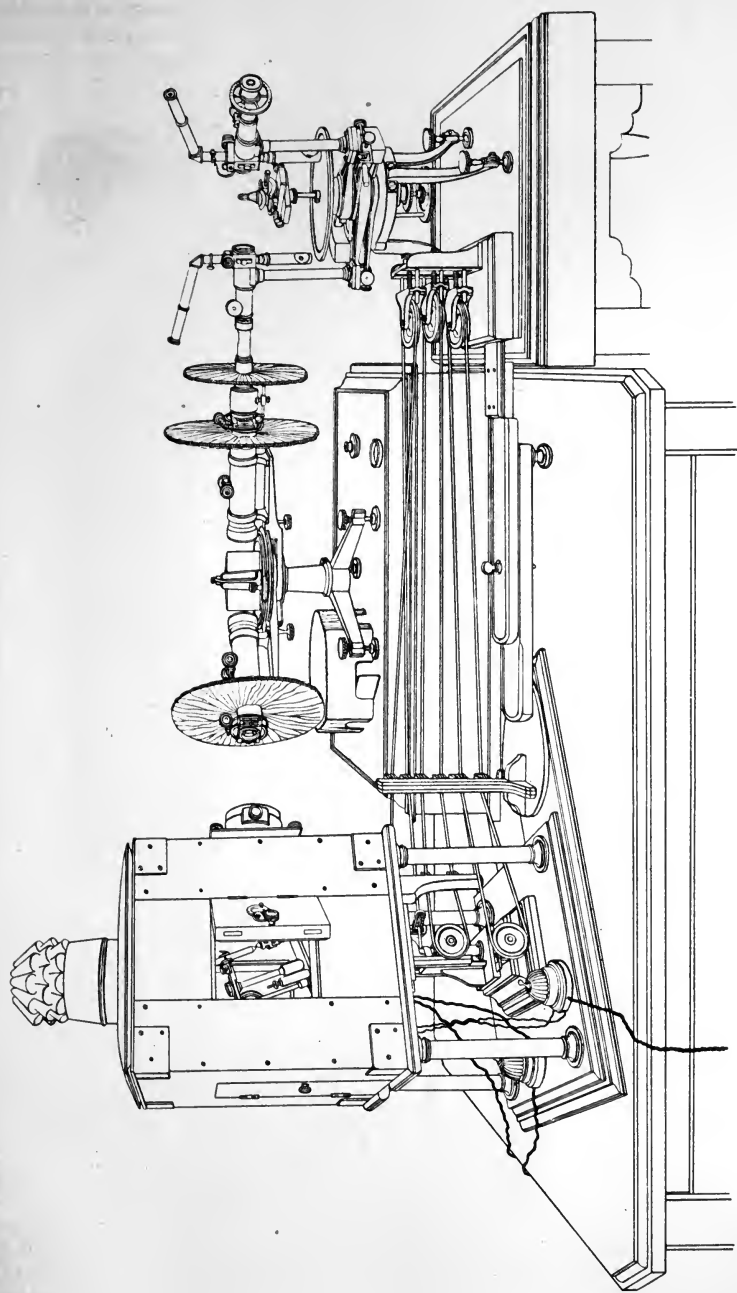


FIG. 610.—The Fuess No. 1a Goniometer, the Spectro-monochromatic Illuminator, and the Electric-arc Lantern, arranged for the Determination of Refractive Indices.

arranged to come up to the right-hand edge of the table, to which the goniometer is also brought up, so that the illuminating tube of its collimator may just fit into the diffusing tube in front of the exit slit of the spectroscope. This is

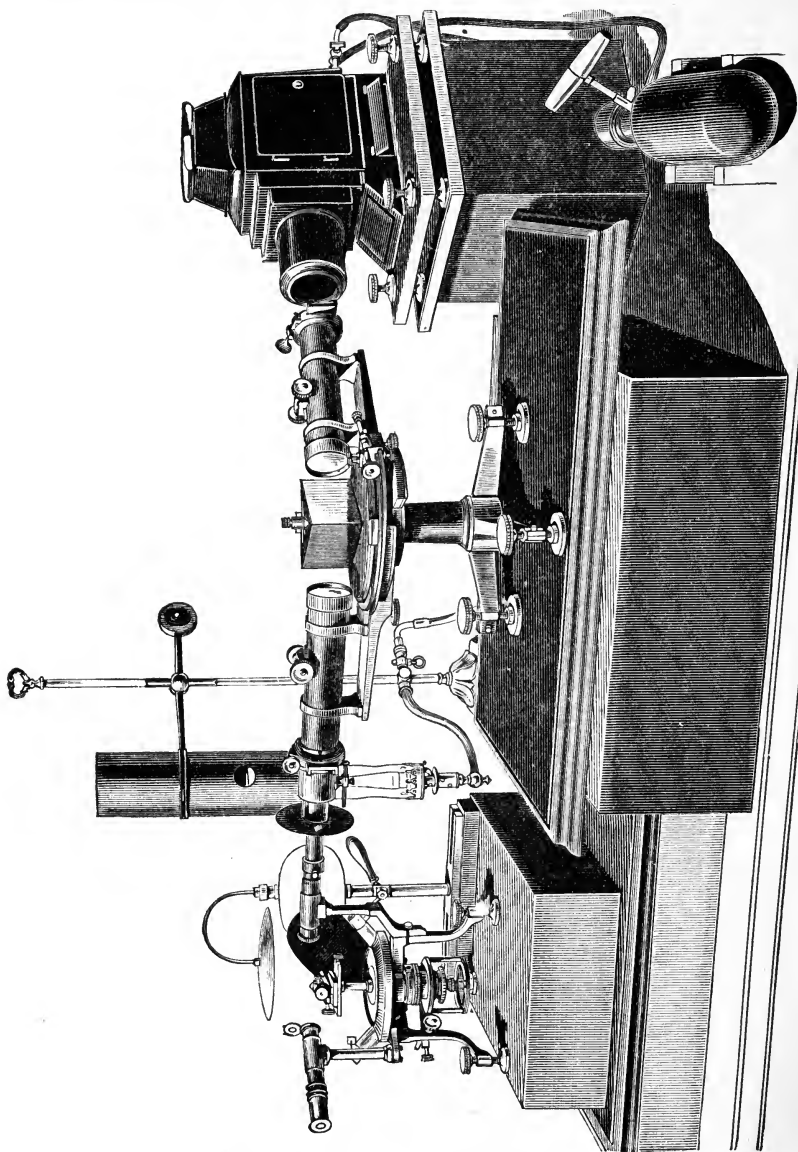


FIG. 611.—The Fuess No. 2a Goniometer, Monochromatic Illuminator, and Limelight Lantern, arranged for Refractive Index Determinations.

a very convenient mode of working, and particularly so inasmuch as it permits of the very ready use of the pulley-gear adjustment for the lantern arc. It is especially desirable that such a means should be available of absolutely centring the electric arc, for although the Brockie-Pell or Oliver self-feeding electric arc-lamp so nearly

maintains a constant position, still in refractive index work it is a great advantage to be able to give it just the last touch of adjustment while actually observing the illuminated images of the Websky slit. This the pulley-gear enables the observer to do with his left hand, while his right is free for the goniometer, and the maximum light can be obtained at any critical moment without taking the eye from the telescope of the latter or rising from the observing stool.

Another arrangement, which the author has of late adopted, is shown in Fig. 614, which illustrates the determination of refractive indices at higher temperatures. The lantern is mounted on its usual stand for projection and all other purposes, and the spectroscope also on its own stand, which is the same basal plinth as is shown in Fig. 610, but mounted on a special four-legged pedestal on castors. All the observing instruments in the author's laboratory have lately been so mounted on separate stands, all of exactly such a height as brings the optic axis of the instrument to the same level as that of the projection axis of the lantern. Hence at any moment the spectroscope can be brought up to the lantern, and fed with the rays from the electric arc, and any instrument in the room can then be brought up to the diffusing tube of the spectroscope and fed immediately with a stream of monochromatic light.

In Fig. 611 is shown a simple way of determining refractive indices with the No. 2a goniometer, and the lime-light as source of light with which to feed the monochromatic illuminator. Very good work is possible with even this arrangement. An incandescent gas mantle is also employed instead of the white-light electric glow lamp, it being assumed that no electric current is available.

Whatever mode of disposition of the three essential parts of the apparatus for determining refractive indices is adopted, the following is the method of work. The finer of the two ground glass diffusing screens is used in the diffusing tube in front of the exit slit of the spectroscope, to diffuse laterally the monochromatic ribbon of light issuing from the latter, and the condensing lens of the illumination tube forming the prolongation of the collimator of the goniometer is brought almost close up to the ground glass surface, shaded by the short tube carrying the latter. Even a very small crystal prism, having only faces a square millimetre in area, when perfectly transparent (and no other should ever be selected if possible) yields monochromatic images of the Websky slit of great intensity when the electric arc is the source of light, even although the opening of the exit slit (as well of course as of the entrance slit) is so fine as only to transmit the three-hundredth part of the visible spectrum. The pair of such brilliantly coloured images yielded by a prism of a doubly refractive crystal form magnificent objects in the field of view, changing their colour and their positions as the prism circle of the monochromatic illuminator is rotated so as to permit light of the various wave-lengths in succession to pass through the exit slit. They are shown in Fig. 612, as seen separated at an average distance in the field of the telescope, one of them adjusted to the vertical spider-line for minimum deviation.

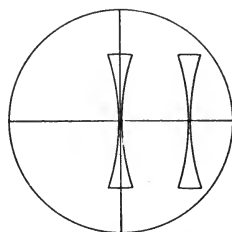


FIG. 612.—The two Images of the Websky Signal-slit afforded by a 60°-Prism of a Doubly Refracting Crystal.

When the double refraction is small, and the two images are consequently close up together, the setting of one of the images to minimum deviation will usually suffice for both; but when the images are fairly widely separated, corresponding to a larger amount of double refraction, the two images require to be independently set for minimum deviation. Hence, it is best to be on the safe side and always to set each image for each wave-length employed truly to minimum deviation, or to verify the setting if no further adjustment is required. A mere trace of further adjustment is, as a rule, all that is required to make it perfect after the setting has already been achieved for the previous measurement of the position of the other image or for another wave-length.

The process of ascertaining the minimum deviation for the various wave-lengths, say the six already specified as being conveniently spaced (red Li-light, bright red C-hydrogen, yellow Na, green Tl, greenish-blue F-hydrogen, and violet hydrogen near G), is very simple, consisting of five operations.

(1) The prism circle of the monochromatic illuminator is set to the reading for the particular wave-length it is desired to commence with, which may most conveniently be that corresponding to sodium light, as affording the most intensely illuminated images and being more or less centrally situated in the spectrum.

(2) The Nicol prism in front of the eyepiece is set to its 45° -graduation, which permits both images of the Websky signal refracted by the crystal (assuming the latter to belong to one of the doubly refracting systems) to be seen and approximately adjusted; the Nicol is then rotated to its 0° -graduation, its vibration plane being then horizontal, the image corresponding to which has already been pointed out to be slightly more intense of the two images afforded by such a prism. This image, corresponding to light vibrations perpendicular to the refracting edge and in the bisecting plane of the prism, will now be seen at full intensity, while the other, corresponding to vibrations parallel to the edge of the prism and to the 90° -setting of the Nicol, will be extinguished. This 0° -image is then finally perfected as regards its setting to minimum deviation, that is, so that it is symmetrical to both spider-lines of the eyepiece, the vertical spider-line bisecting it longitudinally so that the central very narrow part of the Websky image is practically coincident with the line, a minute amount only of the image being here visible equally on each side, as shown in Fig. 612. The slightest rotation either clockwise or anti-clockwise of the crystal axis of the goniometer should cause the image to move away from this setting for minimum deviation, and in the same direction, that of larger deviation, this adjustment of the telescope, and of its spider-line to the image, being the nearest that can be achieved to the direct-reading position. When the adjustment is thus perfect, the circle is read, and the reading recorded as that for minimum deviation for this wave-length and this image, corresponding to vibrations along the known axis of the optical indicatrix. Its difference from the direct reading of the Websky slit, when telescope and collimator were in the same straight line, is the minimum deviation angle required.

(3) The Nicol is next arranged at its 90° -graduation, and similar operations are carried out on the second (slightly fainter) image then transmitted, the vibrations of which are parallel to the refracting edge of the prism, and to that axis of the optical indicatrix parallel to which the edge has been cut.

(4) Leaving this 90° -image adjusted, the circle of the spectroscopic illuminator is then rotated to the graduation corresponding to the next wave-length for which observations are to be made, say red C-hydrogen light, and the image placed at minimum deviation for this, by making use of the fine adjustment of the goniometer telescope, the circle and collimator remaining fixed as all through these determinations of minimum deviation. The amount of fine adjustment given with the telescope of the No. 1a goniometer is ample to enable all the images to be brought to the cross-wires by its use, after one has been set to minimum deviation, except in cases of extremely high dispersion and very large difference of the two refractive indices. After also next rotating the Nicol back to its 0° -position and taking the minimum deviation reading of the other image when set to its minimum deviation, the spectroscope circle is again moved on and set for the issue of the next wave-length, say Li-light, for which a similar pair of minimum deviation readings are taken. In like manner readings are subsequently taken for the positions of the two images at minimum deviation for settings of the spectroscope circle for the issue of Tl-light, F-light, and violet hydrogen light. It may also occasionally be desired to supplement these readings by those for other wave-lengths.

(5) At the conclusion of the set of observations of minimum deviation for all the six or more wave-lengths on this side of the direct reading, the telescope is rotated,

first to the position of the direct reading itself, the circle being read to confirm and verify this reading, and then to the symmetrical position on the other side of the direct reading; the crystal axis is also correspondingly rotated till the light is incident on that face of the prism which had formerly been the face from which the light had emerged, and a duplicate of the whole series of operations carried out. The mean of the values of minimum deviation obtained on the two sides of the direct reading, for each of the two images corresponding to the two vibration directions, and for each wave-length of light, is taken as the true angle of minimum deviation for that vibration direction and wave-length, any error of setting to minimum deviation, or in taking the direct reading, being thereby eliminated. There should never be more, at the very most, than three minutes of difference between the corresponding angle of minimum deviation for the same index-image and wave-length for the two sides of the direct reading. In the vast majority of cases the values will be identical, or at most differ by one minute of arc.

The mode of concisely setting down the observations may be best illustrated by the record of an actual determination of the refractive indices of a doubly refracting substance, say of ammonium magnesium sulphate, the substance taken in Chapter XVIII. as the example of monoclinic symmetry. On the left-hand page of the note-book the simple record of the observations is conveniently given, and on the right-hand opposite page the corresponding calculations are worked out, in accordance with the formula :

$$\mu = \frac{\sin \frac{\alpha + \delta}{2}}{\sin \frac{\alpha}{2}}.$$

Instead of the general symbol of the refractive index μ , however, the specific symbols of the three refractive indices α , β , γ will of course be given.

Determination of Refractive Indices of Am-Mg Sulphate, $(\text{NH}_4)_2\text{Mg}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$.—The prism employed was ground with a refracting angle intended to be 64° (32° each side of the bisecting plane), a favourable angle for this salt. On goniometrical measurement after grinding and polishing on the cutting-and-grinding goniometer the angle actually proved to be $64^\circ 2'$. The refracting edge of the prism was parallel to the axis γ of the optical indicatrix, the first median line or acute bisectrix of the optic axial angle. Vibrations parallel to the refractive edge are afforded with the Nicol at 90° .

The other vibration direction was that of the β axis of the indicatrix, the symmetry axis b of the crystal. Vibrations parallel to this direction are transmitted by the Nicol when arranged at its 0° -graduation.

EXPERIMENTAL DATA.

Angle of Prism, $\alpha = 64^\circ 2'$. Direct reading of slit : $193^\circ 57'$.

Circle Readings for Minimum Deviation.

Light.	Telescope on left of Direct Reading.		Telescope on right of Direct Reading.	
	Nicol at 0° (β).	Nicol at 90° (γ).	Nicol at 0° (β).	Nicol at 90° (γ).
Li	$232^\circ 20' \cdot 5$	$232^\circ 53' \cdot 5$	$155^\circ 34' \cdot 5$	$155^\circ 1' \cdot 5$
C	$232 \quad 22$	$232 \quad 56$	$155 \quad 32$	$155 \quad 0$
Na	$232 \quad 37$	$233 \quad 10$	$155 \quad 17$	$154 \quad 44$
Tl	$232 \quad 52 \cdot 5$	$233 \quad 25 \cdot 5$	$155 \quad 2 \cdot 5$	$154 \quad 29 \cdot 5$
F	$233 \quad 11$	$233 \quad 42$	$154 \quad 45$	$154 \quad 12$
G	$233 \quad 37$	$234 \quad 10$	$154 \quad 19$	$153 \quad 46$

EXPERIMENTAL DATA (*continued*).*Angles of minimum deviation, δ .*

Light.	For Nicol at 0°, β image.			For Nicol at 90°, γ image.		
	Left.	Right.	Mean.	Left.	Right.	Mean.
Li	38° 23'·5	38° 22'·5	38° 23'	38° 56'·5	38° 55'·5	38° 56'
C	25	25	25	59	57	58
Na	40	40	40	39 13	39 13	39 13
Tl	55 ·5	54 ·5	55	28 ·5	27 ·5	28
F	39 14	39 12	39 13	45	45	45
G	40	38	39	40 13	40 11	40 12

CALCULATIONS.

$$\frac{\alpha}{2} = 32^\circ 1'. \quad \text{Log. } \sin 32^\circ 1' = \bar{1} \cdot 72441 = a.$$

Light.	For β -index (Nicol at 0°).			For γ -index (Nicol at 90°).		
	$\frac{\alpha+\delta}{2}$	Log. $\sin \frac{\alpha+\delta}{2} = b$.	$b - a = \text{Log. } \beta$.	$\frac{\alpha+\delta}{2}$	Log. $\sin \frac{\alpha+\delta}{2} = b$.	$b - a = \text{Log. } \gamma$.
Li	51° 12'·5	$\bar{1} \cdot 89178$	$\bar{1} \cdot 16737$	51° 29'	$\bar{1} \cdot 89344$	$\bar{1} \cdot 16903$
C	13 ·5	89188	16747	30	89354	16913
Na	21	89264	16823	37 ·5	89430	16989
Tl	28 ·5	89339	16898	45	89505	17064
F	37 ·5	89430	16989	53 ·5	89589	17148
G	50 ·5	89559	17118	52 7	89722	17281

RESULTS FOR REFRACTIVE INDICES β AND γ .

Light.	β .	γ .
Li	1·4702	1·4758
C	1·4705	1·4761
Na	1·4731	1·4787
Tl	1·4756	1·4813
F	1·4787	1·4842
G	1·4831	1·4887

A similar series of observations with a second prism, ground to afford α parallel to the refracting edge and β for the other index, yielded the following values, from which will be seen the kind of concordance of the values of the common index, β in the case of this pair of prisms, which may be expected from complementary prisms prepared with the aid of the cutting-and-grinding goniometer.

VALUES OF α AND β FROM SECOND PRISM.

Light.	α .	β .
Li	1·4688	1·4701
C	1·4692	1·4705
Na	1·4719	1·4730
Tl	1·4742	1·4755
F	1·4773	1·4784
G	1·4817	1·4829

The maximum difference between the two values of β for any wave-length is 0·0003, which only occurs once, the differences in four cases being only 0·0001 or nil.

Besides these two prisms four others were also employed, two pairs, each pair yielding all three indices, as in the case of the pair quoted. It will be of interest to give the final values of the three refractive indices derived as the final mean values for all six prisms, from which the relation of the values derived from this one pair to the results of the whole series of observations will be seen.

REFRACTIVE INDICES OF AMMONIUM MAGNESIUM SULPHATE
DERIVED FROM THREE PAIRS OF PRISMS.

Light.	α .	β .	γ .
Li	1·4685	1·4701	1·4756
C	1·4689	1·4705	1·4760
Na	1·4716	1·4730	1·4786
Tl	1·4740	1·4755	1·4811
F	1·4771	1·4786	1·4842
G	1·4814	1·4831	1·4888

The maximum difference of any individual value from the final mean of all is again only 0·0003, which occurs four times. In all other cases it is less, and in seven cases the values are identical. This is a quite typical series of observations, and indicates how satisfactorily the 60°-prism method works when the prisms are prepared with the aid of the cutting-and-grinding goniometer.

As regards the angle of the prism, 60° is an average convenient size. It will be observed, however, that the angle was 64° in the example quoted. This value, exactly 32° on each side of the bisecting principal plane of the optical ellipsoid, is especially convenient with substances like ammonium magnesium sulphate, of weak double refraction. For the whole spectrum still clearly emerges, and the accuracy is enhanced. On the other hand in cases of exceptionally high refraction an angle somewhat less than 60° is safer, and as low as 55° is best in extreme cases, owing to the danger with a prism of 60° of the critical angle of total reflection being exceeded in the interior incidence on the second face of the prism, with possibility of non-emergence of at least a part of the spectrum. In cases of very low double refraction the angle may be increased even up to 68° or 70° without such danger. Whatever the angle, however, it must be bisected by a principal plane of the optical ellipsoid, that is, in grinding the prism the crystal must be rotated for an equal amount on each side of that plane in order to grind the two refracting surfaces. These considerations as to the favourable angle must be remembered in deciding as to the possibility of the use of a naturally occurring prism, as already referred to on page 727.

Occasionally, in the cases of rhombic substances the crystals of which are too small for convenient cutting, and of which there are no naturally formed prisms of 50° - 70° symmetrical to a principal plane of the ellipsoid, it may happen that a natural prism not exceeding 35° in angle may be formed by two predominating faces on one or more of the crystals, the one a pinakoid face parallel to a principal plane of the ellipsoid and the other a face of a prismatic or domal form also parallel to the same axis of the ellipsoid as the pinakoid face, the refracting edge being thus parallel to that axis. Such a case, shown in Fig. 613, can be utilised for the determination of those two of the three refractive indices the vibrations of which are parallel to the refracting edge and to the axis of the ellipsoid perpendicular to the latter, lying also in the pinakoid face. The only conditions are that the light shall be incident normally on the pinakoid face and that the refracting edge shall, as usual, be adjusted parallel to the goniometer

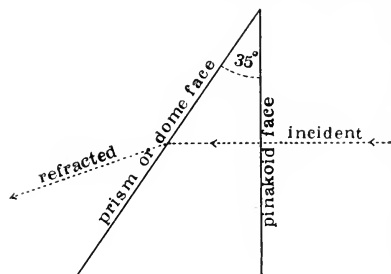


FIG. 613.

axis. The prism then behaves as one half of an ordinarily cut 60° -prism, the rays traversing it parallel to the imaginary third side (perpendicular to the pinakoid face) and base. The measured angle of this prism and the measured angle of minimum deviation for any wave-length, will be half of α and δ respectively, as will be clear on comparing Fig. 613 with Fig. 489 (p. 582), and from the obvious facts that the prism angle is half that of the normal case and that the light rays are refracted once instead of twice symmetrically, no refraction occurring at the pinakoid face with normal incidence. Hence to get the α and δ of the ordinary formula we double the measured angles throughout.

To set such a prism with its pinakoid face absolutely normal to the incident rays from the collimator, it is only necessary to set the telescope and collimator at exactly 90° angle with each other (setting them first in line and adjusting as for the direct reading, then rotating the telescope 90°), to adjust to the spider-lines the image of the signal reflected from the pinakoid face, and then to bring the collimator 45° round nearer to the telescope, when its axis will be normal to the face in question. Both collimator and prism must then remain fixed during the observations of minimum deviation.

Determination of Refractive Indices at Higher Temperatures.—This can be carried out very satisfactorily with the No. 1a Fuess goniometer and its crystal-heating arrangement described in Chapter XXVIII. Full details were given in that chapter of the mode of measuring crystal angles at temperatures up to 200° C. by use of this apparatus, and the measurement of the angle of the prism is carried out in the same manner as that of an ordinary crystal angle. One or two limitations, however, are imposed by the experimental conditions at higher temperatures, compared with the greater goniometrical freedom at the ordinary temperature, when the heating bath and supports are out of the way.

The telescope will usually be arranged to point to that side window of the air bath which is inclined at the smaller angle of 40° to the direct-reading position. For this window permits the refracted images of the signal derived from the prism set to minimum deviation to be viewed by the telescope, while the window at 55° , which is used for ordinary crystal-angle measurements, only does so in the cases of very highly

refractive substances. That window which best suits the amount of refraction preliminarily observed to be possessed by the crystalline substance under investigation must obviously be used, and in most cases this will be the one at 40° . The crystal requires to be particularly carefully adjusted on the special movable-button crystal-holder, on which it is gripped either by the two pins or by an improvised clip cut out of platinum foil to suit the shape of the crystal-prism and held between the jaws of the miniature vice carried by the button. The adjustment of the crystal and of the bath, as regards both centring of the crystal within it and the positions of the windows, must be carried out so that both the reflected and refracted images of the Websky signal-slit of the collimator can be viewed equally well through the telescope, and are not cut off by the bath wall adjoining the window. Moreover, it is only convenient to work on one side of the direct reading, there being only one window at 40° and one at 55° , one on each side. Care must be taken also in the centring of the crystal-prism, that the thermometers shall not be touched by the latter or its holder on rotation, as this would upset the adjustment.

With these limitations the process is as described in Chapter XXVIII. as regards the measurement of the angle of the prism, and as described in the previous part of this chapter as regards the determination of the angles of minimum deviation for light of the usual six wave-lengths. The whole apparatus as actually arranged during a series of observations is shown in Fig. 614.

It is convenient and saves time to proceed in the following order. The collimator is fixed and its direct reading taken by the telescope, the circle being also fixed. The crystal-prism is then adjusted to minimum deviation at the ordinary temperature, and the telescope rotated round to the necessary position for viewing the refracted images at minimum deviation, and clamped there, so that the middle of the spectrum corresponds to the middle of the range of the fine adjustment. The Nicol in front of the analyser is conveniently arranged at its 45° -position during these preliminary adjustments, so that both refracted images can be seen at once. Before proceeding to the measurement of the prism angle the Nicol is arranged at 90° , for which position the reflected light is a maximum, such light as is polarised by reflection from the incident face of the prism vibrating parallel to the refracting edge, and therefore being transmitted by the Nicol when its vibration plane is at 90° ; whereas more or less extinction, complete when the angle of incidence happens to be the polarising angle, of the reflected light occurs when the Nicol is at 0° . When it has been arranged that the two reflected images from the two prism faces, and also the refracted images, are properly visible in the telescope by rotation of the crystal axis, using ordinary white light, the reflected images from the two faces of the prism can be brought and accurately adjusted to the spider-lines and the readings for the angle of the prism taken. For this purpose it is necessary to loosen the circle and fix it to the crystal axis instead. The telescope and collimator are left clamped to the fixed stand and not touched during the measurement of the prism angle. Hence the same value of the circle for the direct reading can be recorded when proceeding to the determination of minimum deviation, and also the telescope is ready at the position for seeing the images at minimum deviation. The whole goniometer and its accessories are then arranged in front of the spectroscopic monochromatic illuminator, as shown in Fig. 614, the illumination tube of the goniometer just entering the diffusing tube of the spectroscope, until the

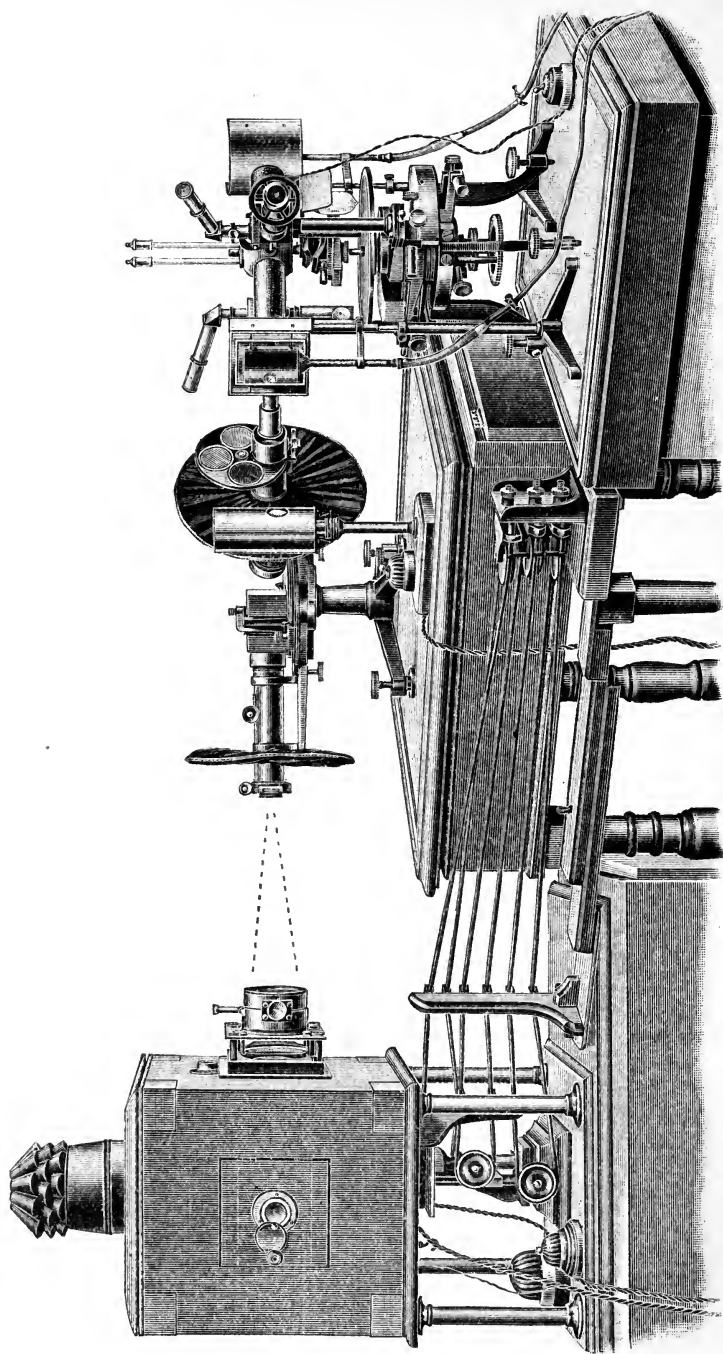


FIG. 614.—Arrangement of Apparatus for the Determination of Refractive Indices at Higher Temperatures.

condensing lens nearly touches the ground glass screen, the finer of the two screens being used.

The circle of the spectroscope is arranged first for the exit of light of the wave-length of the sodium rays, and the images are again reviewed and the measurement of the angle of the prism in sodium light repeated, in order to be quite sure that the adjustments have not been impaired by moving the goniometer. The electric arc should, of course, be regulated with the aid of the pulley gear so that the images are illuminated with their maximum intensity. The prism is then again set for minimum deviation, with the Nicol at first at 45° so as to view both images, supposing the crystal to be a doubly-refracting one; a couple of measurements of minimum deviation, with sodium light, for each refracted image, with the Nicol at 0° for the one and at 90° for the other (the vibrations being respectively parallel and perpendicular to the refracting edge), may then be carried out, in order to verify that the values for this wave-length for the ordinary temperature, the reading of the thermometer for which is noted, are identical with those already obtained. Greater confidence is hereby imparted to the subsequent comparison of the indices at the ordinary and the higher temperature.

The heating can then be proceeded with, and after constancy at the required temperature has been attained the observations are repeated, as complete a set being taken as possible, beginning with the angle of the prism in sodium light and the angles of minimum deviation for the same wave-length, and going on to include minimum deviation determinations for the other usual wave-lengths, at any rate for Li-light, C-light, Ti-light and F-light. G-images are usually too feeble as seen through the glass windows of the bath, owing to the loss by repeated reflection. It is important to finish with a final couple of readings for the positions of the reflected images from the two faces, in order to be assured that the angle of the prism has remained constant during these observations of minimum deviation at the higher temperature. If it has not, another set must be taken, going on if necessary until such is the case. When the precautions for the procuring of constancy of the temperature are taken, which were recommended in Chapter XXVIII., the angle of the prism will be found to have remained likewise constant.

Many artificial crystals, those of our example ammonium magnesium sulphate for instance, which contains six molecules of water of crystallisation, will not permit their refractive indices to be determined with safety at temperatures approaching near 100°C. , owing to loss of water and ensuing opacity, unless their ground and polished surfaces are covered with balsam-cemented glass plates, of the miniature kind already described. But it is quite possible to obtain satisfactory determinations with covered faces, provided the very minimum of hard balsam in benzene be employed to cement them on to the faces of the prism, and that it has been allowed to dry and harden for several days before the determination is attempted. Such glass plates do not usually move during an observation at a temperature as high as 80° , although the balsam of course softens, and the fact that the values of the prism angle, obtained before and after the determinations of minimum deviation at this temperature, are identical is a guarantee of no movement. If any movement should occur, the observations must be repeated till none is observed. Such cases of difficulty are usually overcome with patience and a little extra trouble, and satisfactory observations generally obtained, even although it may be necessary to re-cement the plates with less balsam (the thinnest possible film being the *desideratum*) and to wait a few days longer for hardening.

In the case of the prism of ammonium magnesium sulphate affording β and γ , for which the ordinary temperature results were quoted on page 734, the following results were obtained for the temperature of 70° :

REFRACTIVE INDICES OF AMMONIUM MAGNESIUM SULPHATE AT 70° .

Angle of Prism before Observations of Minimum Deviation, $63^\circ 57'$.

Light.	Minimum Deviation Angles.	
	β .	γ .
Li	$38^\circ 8'$	$38^\circ 39'$
C	$38 10$	$38 41$
Na	$38 24$	$38 56$
Tl	$38 38$	$39 11$
F	$38 56$	$39 29$

Angle of Prism after Observations of Minimum Deviation, $63^\circ 57'$.

Light.	Refractive Indices derived from above.	
	β .	γ .
Li	1.4684	1.4737
C	1.4688	1.4741
Na	1.4712	1.4767
Tl	1.4736	1.4792
F	1.4766	1.4823

It will be observed that these values are about 0.002 (ranging from 0.0017 to 0.0021) less than those afforded by the same prism at the ordinary temperature, as given on page 734, the usual effect of rise of temperature being to diminish the refraction.

A second prism yielding α and γ gave results for γ practically identical with those afforded by the prism the results for which have just been quoted, while for α the following values were given:

Light.	α .
Li	1.4671
C	1.4675
Na	1.4700
Tl	1.4725
F	1.4757

This example concludes the discussion of the essential details of the practical work on the determination of the refractive indices of a crystal by the prism method.

General Formula for Refractive Index.—The refractive index μ of an isotropic crystal or other transparent substance, or any one of the refractive indices of a doubly refractive crystal, may be expressed for any wave-length λ of the spectrum by the well-known general formula of Cauchy:

$$\mu = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots \dots \dots ,$$

in which A, B, and C are constants characteristic of the specific substance. In order to determine them it is necessary to determine μ for three wave-lengths, which may conveniently be red C-hydrogen light, yellow sodium light, and green thallium light; these are three of the six wave-lengths for which it has been recommended in this book that the determinations of the optical constants should always be made. The three values of μ and the three wave-lengths corresponding are then inserted in three equations of condition of the above form, as below, and these equations are subsequently solved for A, B, and C.

$$\begin{aligned} \mu_1 &= A + \frac{B}{\lambda_1^2} + \frac{C}{\lambda_1^4} + \dots \dots \\ \mu_2 &= A + \frac{B}{\lambda_2^2} + \frac{C}{\lambda_2^4} + \dots \dots \\ \mu_3 &= A + \frac{B}{\lambda_3^2} + \frac{C}{\lambda_3^4} + \dots \dots \end{aligned}$$

The solution of these equations is no light matter, and it may save other workers much valuable time to give here the values of the invariable portions of the calculation, and the simplest possible expressions for the three constants involving these invariables. The expressions were found by the method of determinants, and as well as the invariables have been tested times innumerable, and their accuracy proved by the absolute manner in which the refractive indices are reproduced by use of the formula calculated with their aid. The wave-lengths employed were practically identical with those given in the table on page 566.

TABLE OF WAVE-LENGTHS, THEIR SQUARES, AND 4TH-POWER VALUES.

Nature of Light.	λ .	λ^2 .	λ^4 .
Red Li	6705	44957025	2021134096850625
Red C, $H\alpha$, $=\lambda_1$	6562	43059844	1854150165304336
Yellow Na, $=\lambda_2$	5892	34715664	1205177326960896
Green Tl, $=\lambda_3$	5348	28601104	818023350018816
Greenish-blue F, $H\beta$	4861	23629321	558344810921041
Violet $H\gamma$	4340	18835600	354779827360000
Extreme Red A-line	7604	57820816	334315240000000

Solving the three equations in their symbolic form above given, by determinants, A works out to an expression which eventually simplifies to the following, which is a particularly convenient form :

$$A = \frac{\lambda_1^4}{\lambda_1^2 - \lambda_2^2} - \frac{\lambda_2^4}{\lambda_1^2 - \lambda_3^2} + \frac{\lambda_3^4}{\lambda_2^2 - \lambda_3^2}.$$

μ_1
 μ_2
 μ_3

Calling these three fractions k_1 , k_2 , and k_3 respectively, we have :

$$A = k_1 - k_2 + k_3.$$

The two other constants are then very simply expressed as follows :

$$B = - \{k_1(\lambda_2^2 + \lambda_3^2) - k_2(\lambda_1^2 + \lambda_3^2) + k_3(\lambda_1^2 + \lambda_2^2)\}.$$

$$C = k_1\lambda_2^2\lambda_3^2 - k_2\lambda_1^2\lambda_3^2 + k_3\lambda_1^2\lambda_2^2.$$

In these expressions the top halves of k_1 , k_2 , and k_3 are invariables, and their values are given once for all in the next table. There are also given in the table the sums and the differences of the squares of each pair of wave-lengths, and also their products. With these values taken from the table, the calculation of A, B, and C is merely one of simple arithmetic. Having obtained A, B, and C, they can be checked by calculating the refractive indices for the three wave-lengths used in obtaining them, from the formula, the squares and fourth powers of the wave-lengths being taken from the first table. The indices should be reproduced to the last unit in the fourth place of decimals absolutely, if no arithmetical blunder has been made. The reproductions may also be extended to the refractive indices of the other three wave-lengths experimentally used, and also for wave-length A at the beginning of the spectrum if desired, as was usually done by Gladstone and Dale in their well-known work. The reproduction should occur absolutely for Li-light, that for F-light to within one or two units in the fourth place of decimals, while that for violet hydrogen light near G of the spectrum may be as much out as one unit in the third place, as a further term would be required in the formula to afford the refractive index absolutely for this more distant part of the spectrum.

TABLE OF INVARIABLES.

Required for Calculation of :	1st Term.	2nd Term.	3rd Term.
A . . .	$\lambda_1^4 = 222208793$ $\lambda_1^2 - \lambda_2^2 = 1445874$	$\lambda_2^4 = 197099599$ $\lambda_1^2 - \lambda_2^2 = 8344180$	$\lambda_3^4 = 56576392$ $\lambda_2^2 - \lambda_3^2 = 6114560$
B . . .	$\lambda_2^2 + \lambda_3^2 = 63316768$	$\lambda_1^2 + \lambda_3^2 = 71660948$	$\lambda_1^2 + \lambda_2^2 = 77775508$
C . . .	$\lambda_2^2\lambda_3^2 = 992906316$	$\lambda_1^2\lambda_3^2 = 1231559076$	$\lambda_1^2\lambda_2^2 = 1494851076$

Only significant figures have been included in the table. As the refractive indices μ_1 , μ_2 , and μ_3 have four places of decimals k_1 , k_2 , and k_3 must be likewise worked out to four places of decimals, the numbers of figures in the invariables given being just adequate for this to be done accurately. The result for A will consequently have four decimal places. The invariables given for B are also such that on multiplying by k_1 , k_2 , or k_3 the result will have four places of decimals. After obtaining the final result for B the decimal places can be struck off, as B is only required to whole numbers. It is usually a six-figure quantity. It must be remembered that the sign of B is negative, so that if the value inside the large bracket works out to be negative, as it usually does, the middle term being larger than the sum of the first and third terms, the actual value of B is positive. As regards C, the invariables given have six fewer figures than correspond to their real value, the six right-hand end figures having been struck off as giving unnecessary labour, for they do not influence the result. Hence, after multiplying the product of two wave-length squares by k_1 , k_2 , or k_3 , which have the usual four decimal places, we require to add two ciphers (6-4) to the end result for C, in order to obtain its real value. It

generally has 13 figures, although sometimes only 12, which should be spaced in threes for clearness. Of the 13, the first 5 are alone of significance, and after expressing the fifth to the nearest whole number the rest can be left as ciphers. The sign of C varies.

An example will now be given to render all quite clear. The case of the β refractive indices of monoclinic ammonium zinc sulphate, $(\text{NH}_4)_2\text{Zn}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$, will be taken.

The observed refractive indices were: for red C-hydrogen light 1.4904, for yellow sodium light 1.4930, and for green thallium light 1.4957.

If it be desired that the Cauchy formula should represent the true refractive index *in vacuo* these numbers, as taken from the published table of the refractive indices of the salt,¹ should be corrected to a vacuum before use, the correction being usually +0.0004. For the refractive index of air (for light passing from a vacuum into air)

is 1.00029, and the observed refractive indices given by the formula $\mu = \frac{\sin \frac{\alpha + \delta}{2}}{\sin \frac{\alpha}{2}}$

(p. 583) require to be multiplied by this in order to obtain the refractive index for a vacuum. Now $1.4904 \times 1.00029 = 1.4908$. Hence, +0.0004 being the correction, the three refractive indices actually used were:

$$\mu_1 = 1.4908, \mu_2 = 1.4934, \mu_3 = 1.4961.$$

Finding the values of k_1 , k_2 , and k_3 with the help of the table of invariables we have:

$$k_1 = 222208793 \div \frac{1445874}{1.4908} = 22.9113$$

$$k_2 = 197099599 \div \frac{8344180}{1.4934} = 35.2759$$

$$k_3 = 56576392 \div \frac{6114560}{1.4961} = 13.8430.$$

Hence $A = k_1 - k_2 + k_3 = 22.9113 - 35.2759 + 13.8430 = 1.4784$.

Proceeding next to find B , using these values of k_1 , k_2 , and k_3 and the table of invariables we have:

$$k_1(\lambda_2^2 + \lambda_3^2) = 22.9113 \times 63316768 = 1450669466.6784 \quad (1)$$

$$k_2(\lambda_1^2 + \lambda_3^2) = 35.2759 \times 71660948 = 2527904435.5532 \quad (2)$$

$$k_3(\lambda_1^2 + \lambda_2^2) = 13.8430 \times 77775508 = 1076646357.2440 \quad (3)$$

Then $B = -\{(1) - (2) + (3)\} = 588611.6308$, or 588612.

Similarly to find C we have:

$$k_1\lambda_2^2\lambda_3^2 = 22.9113 \times 992906316 = 227487744777708 \quad (4)$$

$$k_2\lambda_1^2\lambda_3^2 = 35.2759 \times 1231559076 = 434443548090684 \quad (5)$$

$$k_3\lambda_1^2\lambda_2^2 = 13.8430 \times 1494851076 = 206932234450680 \quad (6)$$

Then $C = (4) - (5) + (6) = -23568862296$.

Adding two ciphers to this result for C , as already explained, and retaining only the first five significant figures, we have for the significant value of the constant C :

$$C = -2\ 356\ 900\ 000\ 000.$$

The general formula expressing the refractive index β of ammonium zinc sulphate for any wave-length λ is thus:

$$\beta = 1.4784 + \frac{588\ 612}{\lambda^2} - \frac{2\ 356\ 900\ 000\ 000}{\lambda^4} + \dots$$

The proof that this formula is correct is that it reproduces the original refractive

¹ *Journ. Chem. Soc.*, 1905, 87, 1144.

indices for C-light, Na-light, and Tl-light, when the squares and fourth powers of those wave-lengths, given in the first table, are inserted, and the constants B and C divided by them. In doing this division only the first four or five figures of the squares and fourth powers are significant and need be used, so that the reproducing calculations are only the work of a minute or two. The reproduced values thus actually calculated by use of the formula are given below.

Light.	Reproduced Values.	Observed Values.
Li	1.4904	1.4904
C	1.4908	1.4908
Na	1.4934	1.4934
Tl	1.4961	1.4961
F	1.4991	1.4994
G	1.5030	1.5040

The only deviations from absolute reproduction are thus observed, as expected when only three terms are used, as the blue end of the spectrum is approached, being 0.0003 for F-light and 0.0010 for violet hydrogen light near G.

It is unnecessary to calculate three separate formulæ for the three refractive indices α , β , and γ of a biaxial crystal, or for the two ω and ϵ of a uniaxial one, one calculation for β sufficing in the former case, and one for the ordinary ray ω in the latter case. It is then only necessary to decrease and increase the constant A by the amount of $\beta - \alpha$ and $\gamma - \beta$ respectively, or to alter it by $\omega - \epsilon$, for sodium light in each case, in order to reproduce the other indices practically as well as β or ω . For the difference of dispersion rarely exceeds a few units in the fourth decimal place. Thus, in the case of our example, the α -indices of ammonium zinc sulphate are reproduced when the constant 1.4784 is diminished by 0.0042, and the γ -indices when it is increased by 0.0064, as absolutely, within 0.0002, as in the case of β .

Determination of the Specific and Molecular Optical Constants.—

An investigation of the optical constants of a series of chemical salts such as an isomorphous series, or other definitely related set of compounds, is not complete unless it includes the calculation of the molecular refraction and dispersion, in which besides the refractive index the specific gravity of the crystals is taken into consideration, as well as the molecular weight of the substance. In other words, molecular refraction is the combination of the refractive index and the molecular volume. As two kinds of formulæ are in current use for the expression of these optical constants, those of Lorenz and of Gladstone and Dale, and as both have their special value and usefulness, it is advisable that the calculations should be made by means of both formulæ. It will be adequate, however, as a rule for all practical purposes to give the complete set of Lorenz calculations, and only the molecular refraction for red C-hydrogen light in the case of the Gladstone formula, as this quantity is that which has the special usefulness. The formula of Lorenz was obtained from theoretical and mathematical considerations, based on the electromagnetic theory of light, while the formula of Gladstone was a purely arbitrary one, which was found to express the experimental facts. It is singular that experience has shown that there is little to choose between these formulæ, each having their excellences

and their slight drawbacks, but as far as the purpose for which the crystallographer employs them is concerned, to exhibit the optical relationships of related molecules, the drawbacks referred to are inoperative, and so either, or better still, both formulæ may be used, confirming the truth of such relationships by the agreement of their indications.

The **Specific Refraction** according to Gladstone and Dale is the refractive index μ , minus unity, divided by the density d , that is, $\frac{\mu - 1}{d}$.

The **Molecular Refraction** is this expression multiplied by the molecular weight M , or $\frac{\mu - 1}{d}M$.

The quantity $\mu - 1$ was termed by Gladstone and Dale the "refractive energy" and $\frac{\mu - 1}{d}$ the "specific refractive energy."

The specific refraction according to Lorenz is afforded by $\frac{\mu^2 - 1}{(\mu^2 + 2)d}$ which represents, in accordance with the electromagnetic theory of light, the relation between the velocity of transmission of light and the density of the medium through which it is propagated. The molecular refraction is this quantity multiplied by the molecular weight of the substance, namely, $\frac{\mu^2 - 1}{\mu^2 + 2} \cdot \frac{M}{d}$.

The **Specific Dispersion** is the difference between the values of the specific refraction for two rays near the two ends of the spectrum, generally C red hydrogen light H_α and the violet hydrogen ray H_γ near G of the spectrum.

The **Molecular Dispersion** is the difference between the values of the molecular refraction for the same two rays.

The mean molecular refraction of the crystal is the mean of the values for all three indices α , β , and γ , taken for the same wave-length of light, in the case of a biaxial crystal, and in the case of a uniaxial crystal it is one-third of the sum of the value for the extraordinary index and twice the value for the ordinary index, that is in the two cases respectively we use $\frac{\alpha + \beta + \gamma}{3}$ and $\frac{2\omega + \epsilon}{3}$.

An example will be given to illustrate clearly the mode in which the results are set forth, taking the case of the same salt, ammonium zinc sulphate, crystallising in the well-known monoclinic series, as was used in the calculation of a general formula for the refraction. In the table the symbol n is used for the refractive index instead of μ , in accordance with the continental practice, as the memoir¹ from which these results are taken was published in both England and Germany.

¹ *Journ. Chem. Soc.*, 1905, 87, 1145; *Zeitschr. für Kryst.*, 1905, 41, 341.

REFRACTIVE INDICES AND MOLECULAR OPTICAL CONSTANTS OF
AMMONIUM ZINC SULPHATE, $(\text{NH}_4)_2\text{Zn}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$.

Refractive Indices.

Index.	Nature of Light.	Index.
α . Vibrations parallel to 2nd median line.	Li	1.4858
	C	1.4862
	Na	1.4888
	Tl	1.4914
	F	1.4947
	G	1.4992
β . Vibrations parallel to symmetry axis.	Li	1.4900
	C	1.4904
	Na	1.4930
	Tl	1.4957
	F	1.4990
	G	1.5036
γ . Vibrations parallel to 1st median line.	Li	1.4963
	C	1.4967
	Na	1.4994
	Tl	1.5021
	F	1.5056
	G	1.5102

Mean of α , β , and γ for Na light = 1.4937.

Molecular Optical Constants.

Axis of optical indicatrix	α .	β .	γ .
Specific refraction, $\frac{n^2-1}{(n^2+2)d} = n$	$\left\{ \begin{array}{l} \text{C } 0.1487 \\ \text{G } 0.1520 \end{array} \right.$	$\left\{ \begin{array}{l} 0.1497 \\ 0.1532 \end{array} \right.$	$\left\{ \begin{array}{l} 0.1514 \\ 0.1549 \end{array} \right.$
Molecular refraction, $\frac{n^2-1}{n^2+2} \cdot \frac{M}{d} = m$	$\left\{ \begin{array}{l} \text{C } 59.27 \\ \text{G } 60.62 \end{array} \right.$	$\left\{ \begin{array}{l} 59.71 \\ 61.07 \end{array} \right.$	$\left\{ \begin{array}{l} 60.36 \\ 61.74 \end{array} \right.$
Specific dispersion, $n_G - n_C$	0.0033	0.0035	0.0035
Molecular dispersion, $m_G - m_C$	1.35	1.36	1.38
Molecular refraction, $\frac{n-1}{d} M$	$\left\{ \begin{array}{l} \text{C } 100.34 \\ \text{Mean of } \alpha, \beta, \text{ and } \gamma, 101.35 \end{array} \right.$	$\left\{ \begin{array}{l} 101.21 \\ \end{array} \right.$	$\left\{ \begin{array}{l} 102.51 \\ \end{array} \right.$

The density of the salt at 20° compared with that of water at 4° , which was used in the calculations, was 1.932. The molecular weight employed was 398.72. The molecular volume $\frac{M}{d}$ was consequently 206.38.

If it should be desired to compare the mean molecular refraction of the crystal with the molecular refraction of its substance in the state of solution in water, a determination which proved very interesting in the cases of the sulphates and selenates of potassium, rubidium, caesium and ammonium, it is only necessary to determine both the refractive index and the density for two or three concentrated but not quite saturated solutions of accurately known strength. In the year 1868 Gladstone¹ advanced the generalisation that "the refraction

¹ *Proc. Roy. Soc.*, 1868, 18, 49.

equivalent of a solution is the sum of the refraction equivalents of the solvent and of the substance dissolved," and time has shown that this law is substantially correct, such slight differences as are observed, due to change of state, being sometimes on one side and sometimes on the other.

It may be expressed by the following formula, in which the letter p represents the total amount of solvent and substance, and p_1 and p_2 are the respective percentages of the two substances present.

$$p\left(\frac{\mu - 1}{d}\right) = p_1\left(\frac{\mu_1 - 1}{d_1}\right) + p_2\left(\frac{\mu_2 - 1}{d_2}\right).$$

It is convenient to consider p as unity, however, and p_1 and p_2 as the proportions of the two constituents expressed as decimals, rather than p as 100 and p_1 and p_2 as actual percentages; it is equivalent to dividing out by 100. The expression on the left-hand side of the equation is then the specific refraction of the solution. The first expression on the right is the unknown required specific refraction of the crystalline substance in the dissolved condition multiplied by p_1 the amount of it present, and the second and last expression is the specific refraction of distilled water multiplied by its proportion p_2 . The unknown desired specific refraction of the dissolved substance may be called X ; the refractive index of water for C-light as determined by the author and corrected to a vacuum is 1.3321, and its specific gravity at the standard temperature of comparison of 20°, compared with its maximum unit density at 4°, is 0.99824.

Hence, the equation for solution is as under :

$$\frac{\text{refractive index of solution} - 1}{\text{density of solution}} = \text{prop. of salt} \times X + \frac{\text{prop. water} \times 0.3321}{0.99824}.$$

The practical operations required are (1) to make up the solution by weighing out a quantity of the crystals and dissolving them in such a weighed quantity of water recently distilled as will make a nearly but not quite saturated solution; (2) to determine the density of the solution thus made, by the pycnometer method with cap described in Chapter XXXII., 10 c.c. being required for this operation; and (3) to determine the refractive index of another quantity of the solution, using a small hollow prism, with truly worked plate-glass sides as thin as is compatible with rigidity and inclined at 60°, for the purpose. The author employs a small prism of about one cubic centimetre capacity, mounted with hard optician's wax on one of the crystal-holders of the No. 1a Fuess goniometer-spectrometer. This prism is also very useful for the determination of the refractive indices of highly refractive immersion liquids.

An example will make the determination and calculation quite clear. It is taken from the latest case investigated by the author, that of **ammonium selenate**, $(\text{NH}_4)_2\text{SeO}_4$, and is one of two determinations made with solutions of different concentrations. Weak solutions must never be employed, as electrolytic dissociation may begin to enter. By exception, the common crystals of this salt are monoclinic, the rhombic form isomorphous with ammonium sulphate never being obtained in

the pure state. This has been explained in Chapter XXVI. and an illustration of a crystal given in Fig. 417 (page 433).

Preparation of solution; 8·7021 grammes of crystals of the pure salt were dissolved in 9·3513 gr. of water, which corresponds to 48·20 *per cent* of salt and 51·80 *per cent* of water.

Density of solution; the specific gravity of the solution at 20°/4° by the method shown in Fig. 472 (page 521), using one of the 10 c.c. pycnometers, was 1·3627.

Refractive index of solution; the observed refractive index for C-light was 1·4073, and the value corrected to a vacuum 1·4077. Hence $\mu - 1 = 0·4077$.

We have then the following equation, in which the above data have been inserted :

$$\frac{0·4077}{1·3627} = 0·482 X + 0·518 \times \frac{0·3321}{0·99824}$$

Working this out we obtain as the value of X, the only unknown, $X = 0·26319$, which is the specific refraction of the crystals in solution.

In order to get the molecular refraction, MX, where M is the molecular weight of the salt, we multiply the specific refraction thus found by the molecular weight 177·98, and on doing so obtain the value **46·84** for MX, the molecular refraction of ammonium selenate in the condition of solution in water.

A second determination with a somewhat weaker solution, containing 43·88 *per cent* of the salt, yielded the molecular refraction 46·74, the two results being thus in close agreement. The mean, **46·79**, was accepted as the true value.

It will be instructive in conclusion, in order to afford evidence of the general correctness of Gladstone's law—that the refraction equivalent of a solution is the sum of the refraction equivalents of the substance and solvent, and that such minute changes as accompany the change of state occur on both sides, the differences between the molecular refraction of the dissolved substance and the mean refraction of its crystals being sometimes in one direction and sometimes in the other—to append the following table of the molecular refractions of the alkali sulphates and selenates for the two conditions. It will be seen that the exceptional monoclinic crystallisation of ammonium selenate does not affect the conclusion in the slightest.

Salt.	Mean Molecular Refraction of Crystals.	Molecular Refraction in Solution.	Difference.
K ₂ SO ₄ . . .	32·04	33·21	+ 1·17
Rb ₂ SO ₄ . . .	37·55	38·21	+ 0·66
(NH ₄) ₂ SO ₄ . . .	38·75	39·43	+ 0·68
Cs ₂ SO ₄ . . .	47·48	47·27	- 0·21
K ₂ SeO ₄ . . .	38·50	39·65	+ 1·15
Rb ₂ SeO ₄ . . .	44·06	44·63	+ 0·57
(NH ₄) ₂ SeO ₄ . . .	45·92	46·79	+ 0·87
Cs ₂ SeO ₄ . . .	54·35	54·10	- 0·25

The values thus pass, along with increase of the atomic weight of the alkali metal, from a relatively very appreciable positive difference in the case of the potassium salt of each group, to a small negative difference in the case of the caesium salt, the rubidium salt standing intermediate with a small positive difference.

CHAPTER XLVI

DETERMINATION OF THE REFRACTIVE INDEX OF CRYSTALS BY THE METHOD OF TOTAL REFLECTION.

THE foundation of this method has already been discussed in Chapter XXXVI. It was shown that the limit at which total reflection occurs is so sharp that it can be made to appear as the line of demarcation between a brightly and a feebly illuminated portion of the field of an observing telescope, and thus adjusted to a spider-line. Many forms of total reflectometer have been devised, of more or less efficiency. They are roughly divisible into three types, and the best of each of these will alone be described, namely, (1) those in which a plate of the crystal is simply immersed in a highly refractive liquid, (2) those in which the crystal plate is laid against one of the faces of a prism of highly refractive glass, exclusion of air being secured by a film of a highly refractive liquid, and (3) those in which a cylinder or hemisphere of glass replaces the prism.

The critical angle ϕ corresponding to the limit of total reflection is in all cases measured by means of a divided circle, and the refractive index $\mu = \frac{1}{\sin \phi}$, as shown on page 578.

In the case of doubly refractive crystals there are, of course, two limits, and the two are each duly defined, one after the other, as the boundaries of portions distinguished by differently graded shading in the field of the telescope. A typical field afforded by a doubly refracting crystal-plate is shown in Fig. 615, with one of the limiting curves adjusted to the crossed spider-lines.

(1a) **By Kohlrausch's Total-Reflectometer.**—The simplest of the liquid total-reflectometers is that of Kohlrausch, shown in Fig. 616. A lengthy description of this instrument is unnecessary, as it is now largely superseded by the use of the suspended goniometer of the type shown in Fig. 443 (page 494) in Chapter XXX., although it has some special conveniences of its own which render it at times a very useful instrument.

The crystal, cemented to a cork plate with gum arabic, is attached to a somewhat

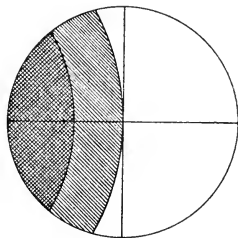


FIG. 615.—The two Limiting Curves of Total Reflection afforded by a Doubly Refractive Crystal.

complicated adjusting apparatus *a*, suspended within a detachable flat-bottomed flask *b*, containing the highly refractive liquid, provided with a plate-glass window *c* opposite the adjustable telescope. A divided circle *d*, with two verniers read by a pair of small microscopes *e*, forms the upper part of the apparatus, and is rotatable above a steel ring *f* carried by the supporting column *g*. A special adjustment *h*, manipulated by the milled head *k*, is provided for the rotation of the crystal in its own plane, which enables the limiting curves to be followed throughout the various azimuths in the plane of the optical ellipsoid parallel to which the crystal has been cut. The adjustable mount for the crystal bears a small graduated circle *l* to record the amount of rotation and identify any particular azimuth, and the cork mount of the

crystal is attached to three pins carried by the annulus *m*, pushed into the central boring of the little circle plate. The crystal plate can be adjusted parallel to the rotation axis of the main circle *d* by means of a black-glass mirror *n*, which has been adjusted once for all parallel to that axis. The telescope *o* carries a detachable lens, like a goniometer telescope, for converting it into a low-power microscope with which to view the crystal. Either α -monobromonaphthalene $C_{10}H_7Br$ or methylene

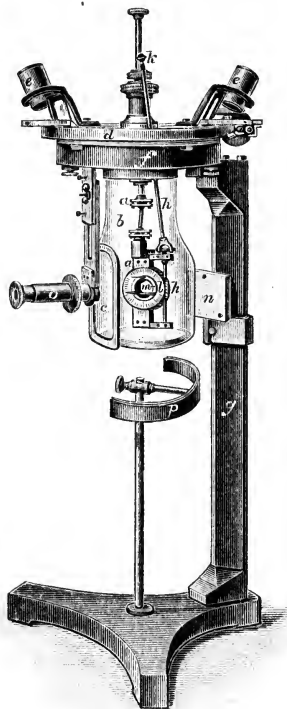


FIG. 616.--Kohlrausch's Total-Reflectometer.

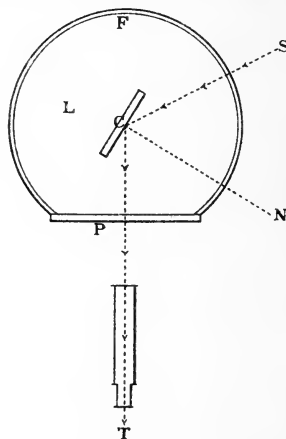


FIG. 617.

iodide CH_2I_2 is employed as the highly refractive liquid, their refractive indices at 20° for sodium light being respectively 1.6657 and 1.7421. When a liquid is required of still higher refraction, sulphur may be dissolved in methylene iodide until the refractive index is raised to 1.85. The observations should be made in a darkened room, and the cork support and any marginal faces of the crystal visible round the plate should be blackened, as the curves are not brilliantly defined with small crystals, and other reflections tend to mask them.

The principle of the method will be clear from Fig. 617, which represents a horizontal section through the apparatus and crystal-plate *C*. The direction of the normal to the plate is indicated by the dotted line *CN*, while *CT* is the direction of the telescope axis, *CS* is that of the incident light, *L* is the highly refractive liquid in the flask *F*, and *P* is the plane-parallel glass window-plate through which the

observations are made. Diffused sodium or other monochromatic light is allowed to proceed to the crystal along SC from a window, filled with oiled paper, in a blackened cardboard or metal screen, which is fitted round the flask and carried by the semi-circular metallic support *p* in Fig. 616. The monochromatic source of light, usually a sodium flame, is placed near the window, and the crystal rotated until the limiting curve or curves of total reflection are seen and adjusted in the field of the telescope. The latter is conveniently removed at first, so as more readily to locate the limiting curves with the naked eye accommodated for long distance.

(1*b*) **By the Suspended Goniometer used as Total-Reflectometer.**—With the advent of the suspended goniometer the necessity for the special total-reflectometer just described disappears, although when such a goniometer is not possessed the Kohlrausch apparatus, if available, is, of course, equally efficient. The suspended goniometer of Miers has already been described in Chapter XXX. and is illustrated in Fig. 443 (page 494), and the simple cylindrical cell with plate-glass front in which the observations are carried out is also shown in Fig. 444. All the necessary adjustments can be given to the crystal-plate, which is suspended in the highly refractive liquid in the cell from the crystal-holder at the base of the adjusting movements. Equally convenient is the author's cutting-and-grinding goniometer described in Chapter XLII. and illustrated in Fig. 583 (page 682); moreover, in this instrument the adjusting movements are more elaborate and are graduated, enabling any desired orientation of the crystal-plate to be attained with accuracy, with respect to the natural crystal faces left uninjured on the edge of the plate. The cell of liquid is placed on the permanent supporting table of the grinding apparatus, instead of a lap. The cell shown in Fig. 444 is particularly suitable for this purpose, as it carries its own adjusting tripod, for which there is ample room on the grinding table.

A later suspended goniometer devised by Miers especially for refractive index work is shown in Fig. 618, and is a model of convenience and ingenuity. It was constructed by Messrs. Troughton and Simms.

The milled head *a* is the screw for adjusting the height of the crystal-holder. The disc *b* provided with handles, and the milled head *b'*, are for the rotation of the inner cone carrying the crystal-holder, either from above or below, as may be most convenient. The circle *c* is carried by the middle axial cone, and is rotated by the milled head *c'*. A bracket *d*, rotating on the outer fixed cone, carries the telescope *e*, a counterpoise *f*, the verniers *g* for recording the rotation with respect to the circle *c*, a clamping screw *h* and a slow motion for fine adjustment of the relative positions of the circle and telescope, and a clamping screw *k* for fixing the disc *b* to the telescope. The clamping screw *l*, and adjacent fine adjusting screw to its left also marked *l*, adjust the telescope and verniers with respect to the fixed stand and the collimator. The long handles *m* are arranged for convenience of adjusting the circle with respect to the stand while sitting at work in front of the instrument. The milled heads *b'* and *c'* can be clamped together by the fixing screw *n*, which thus locks the disc *b* to the circle *c*. The collimator *o* is fixed to the pedestal column at the back of the apparatus as seen in Fig. 618. The trough for the reception of the highly refractive liquid, or the solution when the instrument is being used for the study of growing faces of a crystal in its mother liquor, may be adjusted for height by the milled head *p*, and fixed at the convenient height by the clamping screw *q*. Three levelling screws *r* are provided for the adjustment of the trough so that the plate-glass window is normal to the telescope

axis. The temperature of the liquid in the cell may be regulated by a stream of water flowing down the outside of the trough from the perforated ring pipe *s*, and carried off by a channel in the adjustable supporting table. A thermometer *t* of suitably bent shape is held by a carrier attached to the bracket *d*, and a stirring arrangement *u* is provided, worked by a motor, the pulley, band, and eccentric gear for ensuring effective stirring by means of the bent glass rod being clearly shown in front and to the left of the figure.

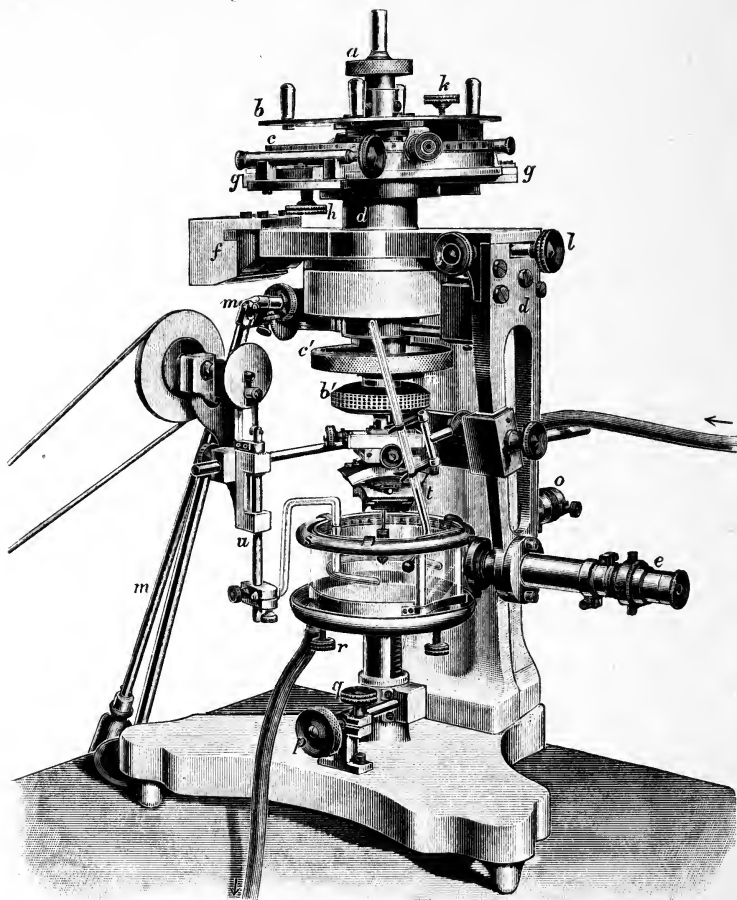


FIG. 618.—Miers Suspended Goniometer-Refractometer.

A suspended goniometer is also now constructed by Fuess. To this, or to any form of the Miers suspended goniometer or the cutting-and-grinding goniometer used as refractometer, a direct-vision spectroscope can be fitted if desired, in order to employ the method of Soret, to be described in the next section.

(1c) By Soret's Total-Reflectometer.—The method of Kohlrausch ¹

¹ *Ann. der phys.* N.F., 1882, 16, 603.

suffers under the disadvantage that the refractive index of the liquid employed in the cell changes rapidly with variation of temperature. To avoid this serious difficulty, which influences the fourth place of decimals of the refractive index, reversion is generally made to a method employed by Wollaston¹ so long ago as the year 1802, in which a solid prism of known refractive index replaces the liquid. But Soret,² who had invented a total-reflectometer on the Kohlrausch principle but with much more elaborate adjustments, in order to eliminate the refractive index of the liquid from his calculations, employed the device of making the determinations in the liquid alternately in quick succession with the crystal and with a prism of known refractive index. If n be the refractive index of the crystal and N that of the prism, and if θ be the

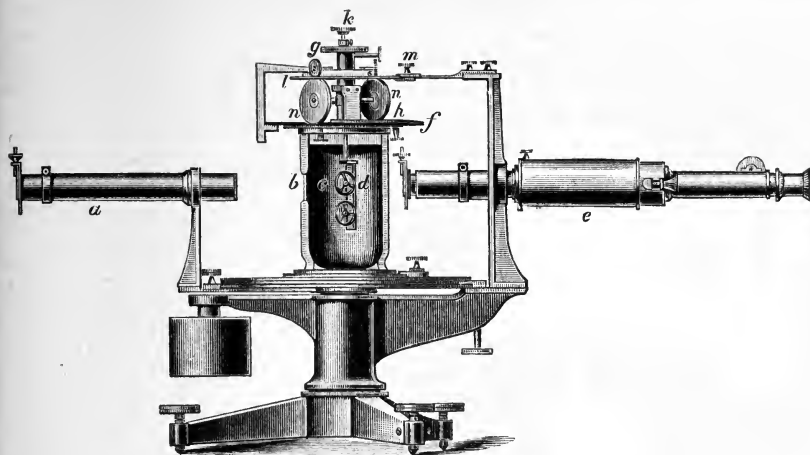


FIG. 619.—The Total-Reflectometer of Soret.

limiting angle of total reflection for the crystal and ϕ that for the prism, then

$$n = N \sin \theta / \sin \phi.$$

With this method of Soret and his identical apparatus some admirable determinations of refractive indices have been carried out by Perrot,³ particularly of a number of double sulphates of the monoclinic series $R_2M(SO_4)_2 \cdot 6H_2O$, which have since also been investigated by the author, with the aid of 60° -prisms, by the minimum deviation method. The agreement between the two sets of results is so admirable that it is obvious that this method of Soret, as perfected by Perrot, is capable of enabling very accurate work to be carried out.

The original apparatus of Soret is shown in Fig. 619, and the upper part of the

¹ *Phil. Trans.*, 1802, 92 (2), 381.

² *Archives des sciences phys. et nat.*, Geneva, 1883, 9, 1.

³ *Archives des sciences phys. et nat.*, Geneva, 1891, 26 and 669, and 1893, 3.

apparatus with the later adjusting apparatus of Perrot in Fig. 620. The source of light employed was the sun, the solar rays being directed on the apparatus by means of a heliostat; the refractive indices could thus be determined for the various Fraunhofer lines, those employed by Perrot being *a*, *B*, *C*, *D*, *b*, *F*, and *G*. The rays were condensed on the slit of the collimator *a*, and passed through a truly plane plate-glass window at *b* into the liquid cell *c*, in which the object-plate was immersed, being supported at the end of the detachable adjusting and rotating apparatus *d*. After reflection from the crystal the light rays passed out of the cylindrical glass vessel, which with its liquid acted as a cylindrical lens, and were concentrated thereby on the slit of a direct-vision spectroscope *e*. The latter is counterbalanced and adjustable to any azimuth round the axis of the circle and the liquid vessel. The circle-plate *f* carries within its central boring two axes, the outer of which, manipulated by the milled head *g*, is rigidly connected with the vernier *h*, whilst the inner axis, rotated

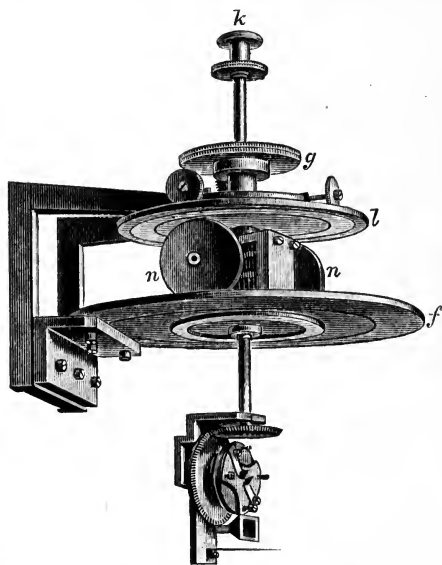


FIG. 620.—Perrot's Adjusting Apparatus for Soret Total-Reflectometer.

by the head *k*, passes down into the liquid vessel and carries the object-holder *d*. There is an ingenious device for maintaining the reflected rays from the crystal-plate always on the slit of the spectroscope as the angle of incidence is altered. The spectroscope carries with it in its rotation the circular plate *l*, to which it is attached at *m*, and this plate is kept pressed down on two rollers *n*, which rotate over the circle-plate *f* and carry with them both the head *g* and the vernier *h*, so that the latter moves through half the angle of rotation of the spectroscope.

When the angle of incidence is such that the light is all reflected from the plate the spectrum appears brilliant, but as the angle is diminished the various wave-lengths in succession find their limit of total reflection and become less intense owing to loss of light by reflection. A kind of dark curtain thus traverses the spectrum from one end

to the other as the spectroscope is moved, and the more or less sharp edge of this curtain, parallel to the Fraunhofer lines, is adjusted to such of the latter as it is desired to make determinations of refractive index for. After thus taking a series of readings with the light incident on one side, say the left, the spectroscope is rotated round until the light is similarly and symmetrically incident on the other side, the right, and a second set of readings taken. Half the difference of the two readings for any wave-length is the angle of total reflection required. A similar set of determinations is then carried out with the glass prism of known refractive index, for each of the same Fraunhofer lines, the prism having been previously adjusted with its face parallel to the object-plate. The refractive index of the crystal is then calculated with the aid of the formula given on the last page, which does not involve the variable index of the liquid. Perrot employed carbon bisulphide as the liquid. His method of work was to determine the angle of total reflection for each wave-length with both the crystal and the prism before passing on to the next wave-length, the adjusting holder being merely raised or lowered to bring the one and the other

not affect the result, while avoiding the well-known difficulty of fitting together two polished surfaces without inclusion of air. For even if in the end air be excluded without the use of a liquid film, another difficulty arises, namely, that the two surfaces adhere as if cemented, and it is very difficult to rotate the plate in its own plane (at any rate without scratching the prism face), as is required in total reflection work. Hence, a layer of highly refractive liquid is of great value, not only guaranteeing contact and lubricating the azimuth movement, but also enhancing the transparency and the sharpness of the limiting line or curve.

When the prism is so arranged that the light passing through the first face AC falls on that part of the second surface BC of the prism which is also in common with the crystal-plate, assuming the film of liquid to be infinitely thin, at an angle of incidence less than the critical angle, it is partly reflected and partly transmitted, and the crystal-plate looks darker than the other part of the prism face not covered by it. But on rotating the prism until the critical angle is reached the part touched by the plate looks brighter than the rest of the surface of the prism, owing to all the light striking it being reflected. The actual limit is, as with the liquid method just described, a sharp line or curve clearly visible in the telescope arranged for parallel rays, this being the all-important fact, first pointed out by Abbe, on which the method depends. A highly refractive glass is necessary, its index being higher than that of any crystal likely to be used, and such glass is usually, unfortunately, more or less yellow, and absorbs blue light, and those glasses with the highest refractive index attainable, about 1.96, are also very soft and easily scratched. When, therefore, the refractive index of the crystal is known not to be high, a flint glass prism of 1.64 refractive index can be much more safely employed. The liquid composing the film should, if possible, be of higher refractive index than either the crystal-plate or the prism, but at any rate of higher index than the crystal; α -monobromonaphthalene ($\mu_{Na} = 1.6657$) is a very suitable liquid, and in extremely high cases methylene iodide ($\mu_{Na} = 1.7421$).

The best form of total-reflectometer based on this principle of Wollaston is due to Liebisch, and is constructed either as a separate instrument, or as a fitting to the Fuess goniometer No. 2a in the form of an additional crystal-holder, or again as a larger fitting with more elaborate adjustments, replacing the whole crystal-adjusting apparatus, to the large Fuess goniometer-spectrometer No 1a. As the last is the most efficient it will be here described, although the smaller model works almost as well. It is shown in Fig. 623, in the later form actually used by the author.

A large vertical circle a , divided directly on its silver limb into half degrees, and reading with a pair of verniers b to single minutes, is carried in a suitable horizontal axial bearing cd . Its axle terminates on the silvered side of the circle in a large milled head e , by which the circle, its axis, and a pair of the usual goniometrical crystal-adjusting segments g , are rotated together. The bearing-support c is

carried at one end of a bevelled elongated plate *h*, which is rigidly attached below to a slider *i*, moving by rotation of the screw *j* over a fixed dovetailed piece *k* attached rigidly to the circular levelling table *l* which forms the base of the fitting, a boss *m* below the second plate of the table actually screwing to the top of the central crystal-axis *n* of the goniometer No. 1*a*. A second slider *o* moves over the long dovetailed plate *h* at the other end to that on which the circle-bearing is carried, and its sliding motion is brought about by the screw *p*. This slider carries the prism *q* and its means of adjustment, the slider itself effecting the adjustment for approach to the crystal-holder. The 60°-prism has two faces, *r* and *s*, polished plane, but the third face *t* (corresponding to AC in Figs. 621 and 622) is ground to diffuse the incident light. One of the polished faces, *s* (corresponding to BC), is arranged parallel to the circle. The prism is cemented on to a rather tightly fitting little slider *u*, by which the prism can be moved horizontally parallel to the face *s*. Besides

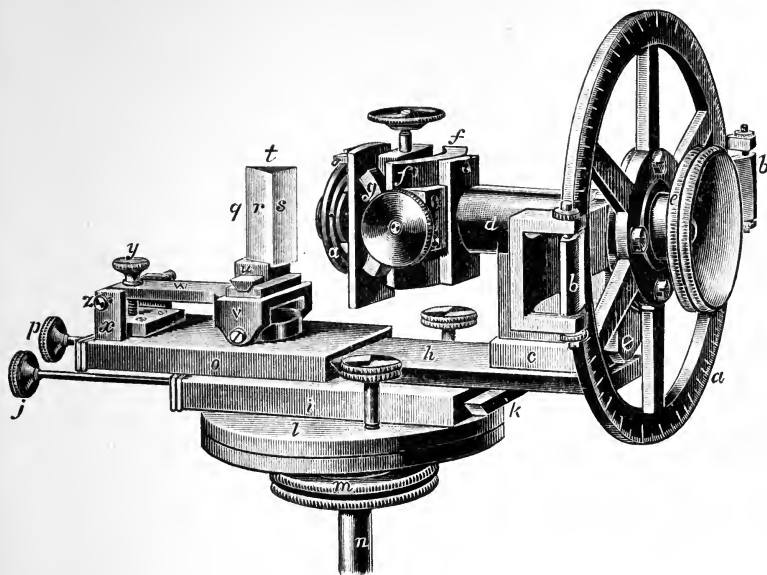


FIG. 623.—The Liebig Total-Reflectometer.

the two horizontal rectangular movements of transference, two rectangular adjusting movements are also provided, one for altitude in the vertical plane and the other for azimuth in the horizontal plane, about two axes parallel to the horizontal and vertical diameters of the circle, which enable the prism face *s* to be set absolutely parallel to the circle if not already so adjusted. They are achieved by mounting the dovetailed bed of the slider carrying the prism on a bracket *v*, which is rotatable about trunnion screws, and by carrying back from the bracket a lever arm *w*, a screw *y* through the outer end of which adjusts the tilt of the face *s* about the trunnion-axis parallel to the horizontal diameter of the circle, a strong spiral spring confined by the screw ensuring the rigidity of the adjustment. The arm is also adjustable for sideways motion, about a vertical axis parallel to the vertical diameter of the circle, by being held within another bracket *x*, in which it is adjustable by the two side-screws *z*, the trunnions of the bracket *v* being attached to a circular disc passing below into a short pin rotatable in the slider *o*. Thus every desirable adjusting movement

is given to the prism, enabling it to be brought up to the crystal in the ideal position.

The crystal-holder is of special construction. The crystal-plate is mounted with optician's wax on a little circular disc which is suspended within a pair of gimbal-rings α , the axes of rotation of which are at right angles to each other, the outer gimbal being hinged from a bracket carried at the top of the basal plate of the holder which carries the fixing peg, a spring pressing this outer gimbal somewhat outwards, so that a little pressure has to be exerted to bring the ring vertical. This pressure is just adequate to ensure good contact between the crystal-plate and the prism when the latter is brought up to the former by manipulation of the screw p . The other inner gimbal ensures that the crystal touches the prism equally all over and that the contact remains equal when the circle and the crystal-plate with it is rotated.

The conditions will be rendered clear by Fig. 624, with the aid also of Figs. 621 and 622. The rays I from the source of monochromatic light are allowed to fall on the matt-ground face of the prism t (AC in Figs. 621 and 622), by which they are diffused. They then pass in this condition to the second face s (BC in Figs. 621 and 622), which is the polished face in contact with the crystal with the intervention of the thin film of liquid. A little screen is arranged between the light source and the prism and quite close to the latter, and the screen is pierced by an aperture of such a size and so arranged

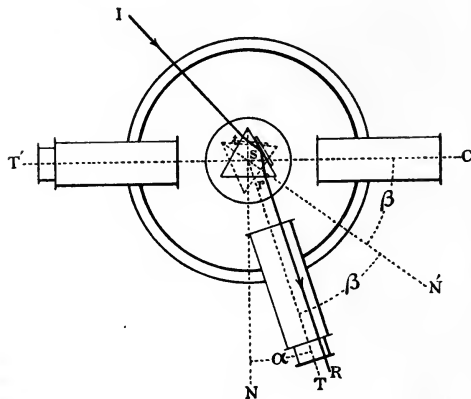


FIG. 624.—Principle of Liebisch Total-Reflectometer.

that only that part of the surface s is illuminated which is common to both crystal and prism. The rays leaving this face, therefore, are only such as are reflected from the crystal-plate; they pass thence to the third face r (AB in Figs. 621 and 622), also polished, at which they are refracted to the telescope T arranged to receive them, the line R parallel to T (T itself in Figs. 621 and 622) indicating their path diagrammatically. The angle α which the totally reflected rays make with the normal to this exit-face r , when the limiting line (or one of the two lines assuming the crystal to be doubly refracting) has been adjusted to the vertical spider-line of the goniometer, is what is actually measured, and it is determined in the following simple manner.

We require first to know the position of the normal to the exit-face r , and can find it at once by use of the glass-plate mirror, the Becker fitting shown in Fig. 13 and described on page 34, placed in front of the telescope eyepiece, in order to observe simultaneously the spider-lines and their image reflected from the face r . Or, the position of the normal may be found goniometrically as follows.

The collimator C is arranged to the right, about the position shown in Fig. 624, and the telescope T , before being arranged in front, as shown, to receive the rays, is rotated to T' on the left in order to take the direct reading of the position of the collimator, the total-reflectometer being lowered sufficiently to be out of the way while this is achieved. The telescope is then rotated back to the position T in front which is convenient for the reception of the reflected rays, the total-reflectometer being raised also into position again, and a reading taken for this convenient position of the telescope. The supplement of the difference between the two readings is obviously the angle between the collimator and telescope, an arbitrary but convenient angle which

we may term 2β , so that the half of it, the angle between the lines T and N' (TrN' in Figs. 621 and 622) is β . The telescope and collimator remaining fixed, the circle is rotated, with the total-reflectometer which it carries, until the limiting line of total reflection is adjusted to the spider-lines of the telescope at T, and a reading is taken. The circle is then further rotated until the Websky signal-slit of the collimator reflected from the exit-face r of the prism is also adjusted to the spider-lines, and a reading is taken also for this. The angle rotated through between the two positions is obviously that between the lines N and N' (in Figs. 621 and 622 it is NrN'); for N is the position of the normal to the face r when the limiting line is set to the vertical spider-line of the telescope, the angle between N and T (NrT) being the angle α which the limit of total reflection makes with this normal N; and N' is the position of the normal to the same face r when the latter has been brought to reflect the collimator signal into the telescope, that is, when the face is equally inclined to collimator and telescope as shown by the dotted prism in Fig. 624, the normal thus bisecting the angle between the two optical tubes. Now this measured angle between N and N' is clearly $\alpha + \beta$ in the case depicted in Fig. 621, and $\beta - \alpha$ in the case shown in Fig. 622. That is, $\alpha = NN' - \beta$ in the one case, and $\beta - NN'$ in the other. As β is known, the angle α of total reflection made with the normal to the exit-face is consequently at once obtained by taking the difference between this measured angle NN' and β .

Representing now the prism angle between the exit-face r (AB) and the face s (BC) against which the crystal-plate was pressed by θ , and μ being the refractive index of the glass of the prism, then, as by definition $\mu = \frac{\sin \alpha}{\sin \phi}$, the angle ϕ between the limiting ray inside the prism and the normal N to the face r is afforded by the equation $\sin \phi = \frac{\sin \alpha}{\mu}$; and the required angle of internal incidence i for total reflection at the face s is equal in the two respective cases to the sum and difference of the angle of the prism θ and the internal angle of refraction ϕ , that is, in the case represented in Fig. 621 :

$$i = \theta + \phi,$$

and in the case shown in Fig. 622 :

$$i = \theta - \phi.$$

These facts should be clear from the two figures, aided by Fig. 624 showing the disposition of the apparatus, the angles being clearly indicated in the two former figures 621 and 622.

(3) **The Pulfrich Total-Reflectometer.**—This instrument possesses a highly refractive (more highly refractive than the crystal) glass cylinder instead of a prism, as shown in section in Fig. 625, the general appearance of the instrument being represented in Fig. 626.

From the latter illustration the arrangement of the cylinder, the circle, and the elbow telescope will be obvious. If l in Fig. 625 be the limiting ray and μ represent the refractive index of the crystal-plate laid on the plane-polished normal top of the cylinder, μ_1 that of the glass of the cylinder, ϕ the true limiting angle of total reflection within the cylinder, and θ the measured angle at which the limiting ray l emerges from the cylinder, then :

$$\sin \phi = \frac{\mu}{\mu_1}, \text{ and } \mu = \sqrt{\mu_1^2 - \sin^2 \theta}.$$

If the light enter the cylinder say on the left in Fig. 625, the rays a and b will be partly refracted and transmitted through the crystal-plate

and partly reflected to *a* and *b* on the right; while rays *c* and *d*, being inclined to the normal (the vertical) at angles greater than the critical angle, will be totally reflected, the limiting line appearing between the two zones at *l*. The limiting line *l* is, however, clearest of all when the rays are not incident on the cylindrical surface at all, but on the crystal as at *e* and *f* on the top-right of Fig. 625. The incident rays are then partly refracted, *e* to *b* and *f* to *a* on the left; the rays of grazing incidence (along the direction of the basal plane of the cylinder and under surface of the crystal-plate in contact therewith) will leave the cylinder at the limit of total reflection *l*; and rays entering on the right below the line of contact of plate and cylinder, say at *d* and *c*, will be totally reflected to *d* and *c* on the left, leaving the upper part of the field

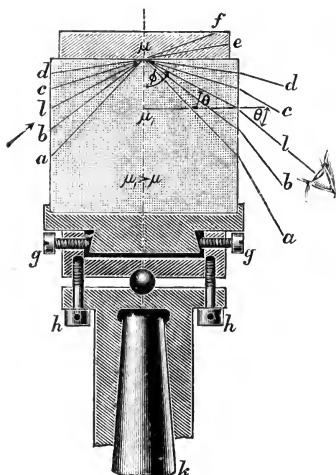


FIG. 625.—Section of Cylinder of Pulfrich Total-Reflectometer and its Adjustable Supports.

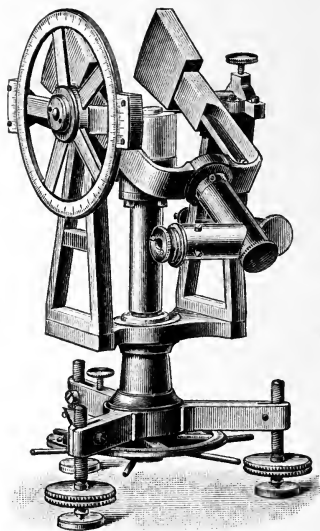


FIG. 626.—The Pulfrich Total-Reflectometer.

seen through the telescope quite dark and the limiting line at its maximum sharpness and distinctness. A drop of highly refractive liquid is employed between the plate and the cylinder as usual.

It is a great advantage either with this form of total-reflectometer or with that next to be described in which a hemisphere is employed, to work with a crystal-plate which has been ground circular, and the cylindrical edge-surface polished, as then the light can be readily utilised incident in the manner just described. This, of course, is practically impossible with small crystals of artificial chemical preparations, but when investigations of the refractive indices of minerals occurring in large crystals are concerned it is readily possible. For this reason, moreover, this form of total-reflectometer lends itself readily to class demonstration, especially if the glass cylinder be replaced by one of bromonaphthalene, enclosed in a cylindrical glass vessel, and the crystal-

plate is just immersed in it and sunlight reflected sideways against the cylindrical edge-surface of the crystal-plate.

When doubly refractive crystals are being investigated, the two curves can be followed by rotation of the cylinder and crystal round the vertical axis of the former, and measurement of the total-reflection angle for each curve at a large number of different azimuths carried out. The screws *g* and *h* serve to adjust the axis of the cylinder, so as to form precisely a continuation of the inner axis of rotation *k* of the instrument,

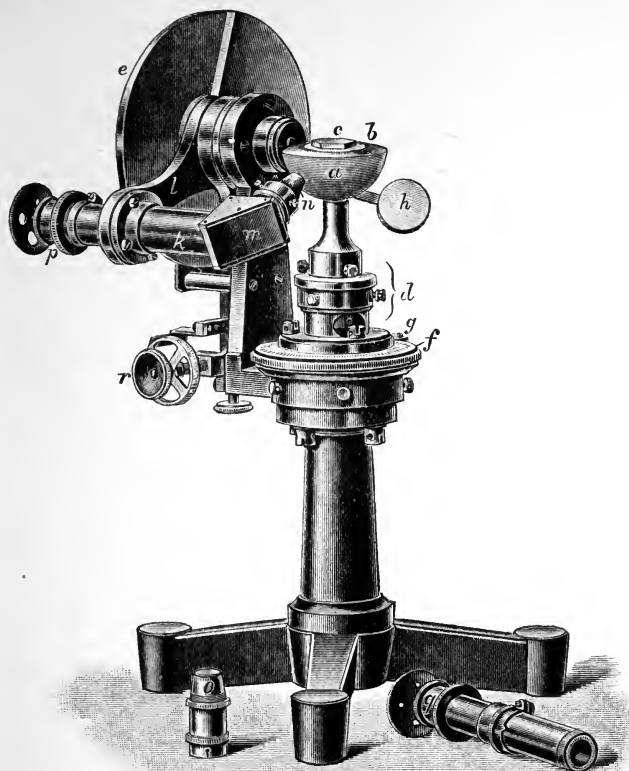


FIG. 627.—The Abbe Total-Reflectometer.

and a circle at the end of this axis below the tripod, shown in Fig. 626, enables the azimuth positions to be read off.

The Abbe Total-Reflectometer.—Bertrand was the first, in the year 1885, to suggest the employment of a hemisphere of dense glass instead of a prism or cylinder. Abbe greatly improved the experimental arrangements, and this form of apparatus has since been perfected by Pulfrich, his successor at Jena, to whose constructive genius we also owe the later form of the total-reflectometer with cylinder just described, the use of a cylinder having been first suggested to succeed the prism of Liebisch by

Abbe. The total-reflectometer with glass hemisphere in its best form as constructed by Zeiss is shown in Fig. 627.

The hemisphere *a* is of Jena flint glass of refractive index for sodium light 1.8904. The method consists in directly measuring the angle of total reflection at the plane surface *b* of the hemisphere, on which the crystal-plate *c* is laid, with a thin film of highly refractive liquid intervening of refractive index at least higher than that of the crystal and preferably higher also than that of the hemisphere, in order to avoid any chance of total reflection before the glass surface is reached. The hemisphere is adjustable by the arrangement *d*, so that its axis of rotation, normal to the plane face of the hemisphere *b*, and that of the divided circle *e*, intersect each other at exactly right angles in the centre of the sphere of which the hemisphere is half. This adjustable support for the hemisphere rests on a second smaller divided circle *f*, the fixed vernier *g* of which is so arranged that its zero mark lies in the plane of the incident light. The large vertical circle *e* is for the measurement of the angle of total reflection, and carries two verniers to right and left, so that the angle can be read on either side of the hemisphere. A mirror *h* serves to illuminate the common face of the crystal and hemisphere, either from below or from above. The telescope *k*, which rotates by an arm *l* about the axis of the circle *e*, is arranged elbow-wise, the two parts being connected by a reflecting prism *m*, and the objective fitting *n* is given in duplicate, to suit larger and smaller crystal-plates, the alternative one *o* to that shown in position lying in the foreground. The lens of this fitting forms along with the hemisphere the objective of a telescope arranged for parallel rays, and magnifying two to three times in the case of one of the alternative fittings, while the other fitting reduces the apparent size of the object to about the same extent as the first magnifies. Each corresponds to a separate eyepiece, one *p* being shown in position, and the other *q* lying in the foreground to the right of the alternative objective in Fig. 627. Another lens can be attached in front of the eyepiece to convert the telescope into a low-power microscope, in which the crystal can be viewed when the telescope is rotated over it, and the plate may thus be centred and orientated as regards azimuth with respect to any existing edge-face and the spider-lines of the eyepiece.

Sodium light is used with small crystal-plates, but if larger plates are available sunlight may be used and the refractive indices determined for the chief Fraunhofer lines. In this case a direct-vision spectroscopy eyepiece is employed instead of the ordinary eyepiece *p* or *q*. The fine adjustment of the telescope, during the placing of the limiting curve or line of total reflection to the spider-lines, is carried out by a micrometer screw *r* with divided drum.

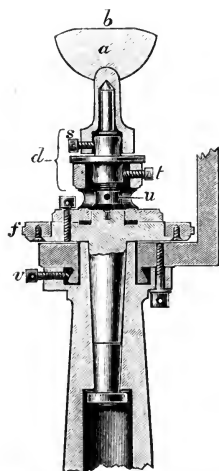


FIG. 628.—Section of the Hemisphere and its Adjustments.

As the plane top of the hemisphere, the central part *b* (the outer part of the top being slightly curved and ground matt), becomes more or less scratched with use, owing to the inevitable softness of highly refractive flint glass, a means of polishing it is given, without the necessity for removal from its position and the remaking of the very troublesome centring adjustments. A groove is cut in the cylindrical boss which rests on the circle *f*, for the reception of the band from a whirling apparatus or motor, and when the hemisphere is thus in rapid rotation it may be repolished with a polisher's tool, a disc with handle and covered with soft leather, with the aid of the finest rouge. In case it should, however, be necessary to readjust the hemisphere, the nature of the adjusting movements will be clear from the section given in Fig. 628. The screws *s* and *t* enable the axis of the hemisphere to

be adjusted exactly coincident with that of the circle. The capstan-nut u engaging with the screw thread on the lower end of the fitting which carries the hemisphere affords the means of raising or lowering the latter. Four screws, of which one is shown at v , finally enable the axis of the hemisphere to be made coincident with the goniometrical axis of the telescope.

With all these forms of total-reflectometer the limiting line or curve of total reflection is much sharper when the incidence is "grazing," as already explained in connection with the Pulfrich instrument with cylinder, and illustrated in Fig. 625. In the case of this particular form with hemisphere, in ordinary incidence from below, the common surface of the crystal and hemisphere is illuminated from below by means of the mirror, h in Fig. 627, through the hemisphere, and the limiting line found by the telescope and adjusted to the spider-lines, the angle of total reflection ϕ being that made with the normal to the plate and surface of hemisphere. The experiment should be made

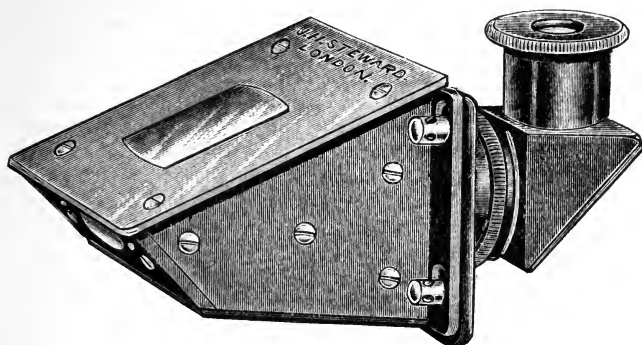


FIG. 629.—The Herbert Smith Refractometer.

from either side and the mean of the two values of ϕ taken. According to the other preferable method of grazing incidence the incident rays are caused to arrive from the mirror at a very small angle downwards from above the crystal, passing through its edge-face, which again renders it obvious how advantageous it is, if possible, to have the crystal provided with a ground normal cylindrical edge, although it is at the same time a disadvantage not to have actual crystal faces visible on the edge for azimuth reference purposes. When the rays are thus directed slightly down on the crystal the distinctness of the limit is very marked, corresponding to the refraction of rays parallel to the grazing line along the surface of the hemisphere.

The Herbert Smith Refractometer.—An ingenious little instrument, shown in Fig. 629, has been devised by Herbert Smith¹ for the rapid approximate determination of the refractive index of a crystal, a gem-stone for instance, and in which at the same time compensation for the spherical curvature of the focal plane, has been effected.

¹ *Mineralogical Magazine*, 1907, 14, 354.

This latter is also achieved in the Pulfrich total-reflectometer by the introduction of a corrective lens before the achromatic objective. The field of view of the Herbert Smith refractometer is quite aplanatic for its entire range of refractive index, 1·300 to 1·800.

A hemisphere of dense glass is employed as in the Abbe-Pulfrich instrument, and by means of a reference scale, shown separately in Fig. 629*a*, the refractive index may be read off directly to the second place of decimals, and by estimation as far as an approximation to the third. The hemisphere is a centimetre in diameter and of refractive index 1·8049. A convex lens of crown glass of 25 millimetres focal length is arranged with the surface of greater curvature nearest the hemisphere, which enables the spherical focal surface of totally reflected rays to be brought to a plane focus, so that the edges separating the light and dark fields are sharply defined throughout the whole effective range of the instrument. The plane surface of the hemisphere is arranged to be parallel with and to project slightly above the inclined surface-plate of the instrument, the angle between the normal to this plane and the axis of the tube of the instrument being 61°.

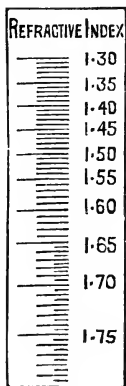


FIG. 629*a*. — Reference Scale of Refractive Indices.

The centre of the field thus corresponds to a refractive index of 1·61. The corrective lens is carried in a vertical sliding frame, adjustable by means of a screw (below, not visible in Fig. 629) for its vertical position. The reference scale, photographed on a glass plate, is carried in a short inner tube, sliding in an outer tube carried by a second vertical frame close to the first one, in near proximity to the corrective lens; it is also adjustable by means of slots and four screws provided with capstan holes, two of which are shown in Fig. 629 proceeding from the front edge of the frame. A positive eyepiece magnifying ten times slides in the outer end of the inner tube, and between its two lenses a totally reflecting prism is mounted, the tube being thus made elbow-shaped instead of straight, so that the eye may be more conveniently placed during the observations. Light is admitted to the instrument by means of a lens of 25 millimetres focus fitted in the back face of the box, which is also inclined at 61°.

The effective range of the instrument is from 1·40 to 1·80 of refractive index. Each interval on the scale corresponds to a difference of index of 0·005 in the higher indices and 0·01 in the lower. In sodium light the limiting line of total reflection separating the light and dark fields is adequately sharp to enable tenths of divisions to be approximately estimated, and this light is used in making the scale with a few substances of known refractive index, so that the scale readings are actual refractive indices.

Even in white light, owing to the violet end of the spectrum being in shadow, the yellow of the spectrum produced instead of a limiting line is sharp enough to enable determinations to the second place of decimals to be made. A sheet of white paper laid on the table is an excellent source of white light. The width of the spectrum seen affords a measure of the amount of dispersion possessed by the substance, and is in itself a valuable indication. For such spectrum width depends on the difference of the dispersion of the crystal and of the glass of the lens; so that a substance of high dispersion, similar to that of the lens, shows an almost colourless limiting line of total reflection, while a substance of low dispersion shows a broad spectrum instead of a sharp line.

Small fragments of crystals only a millimetre in diameter may be used with this instrument, and there is practically no upper limit to the size of crystal or gem-stone which can be tested with the apparatus, the surface plate being itself of some size and its plane clearing all the other parts of the instrument. As only one plane surface is required on a crystal for use with the method of total reflection, a small facet of a gem-stone serves as well as the large table facet of the stone, except that there is more difficulty in being sure of true contact and of not scratching the somewhat soft plane surface of the hemisphere.

The upper limit being 1.80, the instrument includes the ruby and sapphire and other forms of corundum within its range. It is constructed by J. H. Steward, Strand.

Hutchinson Universal Apparatus.—A simple form of Universal Apparatus has recently (January 1911) been described to the Mineralogical Society by Hutchinson, which is well adapted for the measurement and optical examination of small crystals.

The instrument is a goniometer of the suspended type, mounted on a solid base-plate P, to which a telescope A, a microscope B, and a collimator C, can be clamped in the manner shown at K. The circle D, provided with fine adjustment and fixing screw E, is carried by a rigid column S. The crystal is suspended below the usual adjusting and centring apparatus G, and the inner axis carrying the latter may be fixed at any height by the screw F. The microscope is so arranged that its axis

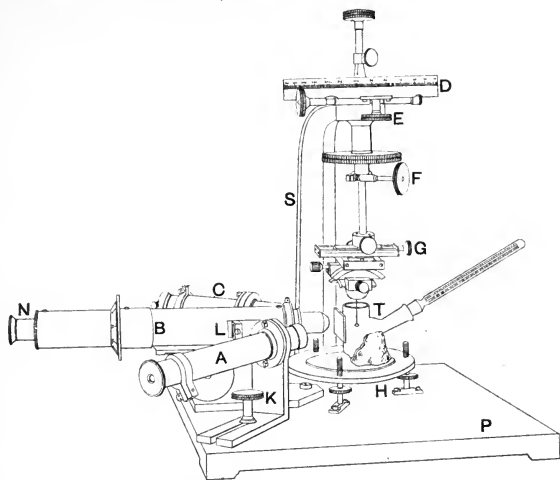


FIG. 630.—The Hutchinson Universal Apparatus.

bisects the angle between the optical axes of the collimator and telescope. A cap carrying a Nicol N, may be attached in front of its eyepiece, and it is fitted with a Bertrand lens L. The latter, when used with a $\frac{1}{2}$ -inch objective, converts the microscope into a diminishing telescope, with which the images of the collimator signal reflected by minute faces can be observed. Another fitting, not shown in the figure, carries a Nicol and a converging lens. When this is placed opposite the objective of the microscope, the optic axial angle of a biaxial crystal can be measured. Further, since any face of the crystal must be perpendicular to the microscope axis, when the image of the collimator signal reflected by the face is seen on the cross-wires of the telescope, a means is afforded of finding the angle at which an axis emerges from a face without employing auto-collimation. Similarly, the plane face of the tank T can be rapidly adjusted perpendicular to the microscope axis by the three levelling screws of the table H, and the instrument then becomes a Kohlransch total-reflectometer, with which the limits produced by very small faces can be readily distinguished.

CHAPTER XLVII

THE DETERMINATION OF THE OPTIC AXIAL ANGLE AND DISPERSION OF THE MEDIAN LINES.

THE optic axes of a biaxial crystal are, as already stated in Chapter XLI., experimentally indicated by the positions of the vertices of the two hyperbolic brushes of the interference figure afforded in convergent polarised light by a plate perpendicular to the acute bisectrix of the optic axial angle, when the two Nicols are crossed and arranged at 45° to the plane of the two optic axes. The acute bisectrix is either the maximum or the minimum axis of the optical indicatrix (or *vice versa* for the Fresnel optical ellipsoid) according as the crystal is one of positive or one of negative double refraction. The obtuse bisectrix will be the converse, either the minimum or the maximum axis of the indicatrix. As the crystal-plate is rotated the hyperbolic brushes rotate about their vertices, until when the line joining the optic axes is parallel to the plane of polarisation of either of the Nicols the brushes unite to form a rectangular cross, one arm of which passes through both axes and the other crosses this at right angles midway between the two optic axes, the rings around the latter remaining more or less the same, as described in detail in Chapter XLI., and illustrated by photographs.

If, therefore, the section-plate be adjusted to the first-mentioned position, when the optic axial brushes are opened to their fullest extent, the line joining the optic axes being inclined 45° to the planes of polarisation of the two Nicols, as shown in Figs. 558, 568, and 579 on the Plates I. and II. facing pages 670 and 674 in Chapter XLI., and the plate is rotated about the diameter at right angles to this line joining the optic axes, first one and then the other of the vertices (the vertex being fortunately the sharpest part of the whole hyperbola in each case) may be brought to the spider-lines of the eyepiece of the polariscope in turn, as shown in Fig. 631, and the angle between these two positions read off on a divided circle carried by the instrument normally to the rotating axis and concentric with it.

The optic axes thus indicated by the hyperbolic vertices are, in all ordinary cases, both the primary and the secondary optic axes, the distinction between which was made clear in Chapter XL. For the angle between a primary and secondary optic axis is usually so small

that the hyperbolic vertex overlaps both, even when the crystal section-plate is so relatively thick as to give very small rings around, and a very sharp brush-vertex at, each optic axis. Hence, for all the usual practical purposes primary and secondary optic axes are identical.

The angle thus measured by the rotation from one to the other vertex is not, however, the true optic axial angle within the crystal, generally symbolised by $2V_a$, but only the apparent angle in air between the optic axes. The angle thus seen in air is conventionally known as $2E$, and is often very large, and only in the widest-angle polariscopes and in cases of a small true angle, are both axes visible at once in the field. When the crystal is immersed in oil, monobromonaphthalene, or other highly refractive liquid the apparent angle is reduced, and both axes are more frequently capable of being brought to be visible in the field at once.

It has been shown in Chapter XL that of the two rays into which a beam of light is converted—as the effect of double refraction, on traversing a doubly refractive biaxial crystal along a direction lying in that principal section of the optical ellipsoid which has for its axes the directions corresponding to the minimum and maximum indices of refraction α and γ , and which also contains the two optic axes—one of them, vibrating perpendicularly to this plane, resembles an ordinary ray as it has the constant intermediate refractive index β in all directions; the other, vibrating in the optic axial plane, is an extraordinary ray, its refractive index varying from α to γ , and along the directions of the two optic axes it has the value β , the same as the perpendicularly vibrating ordinary ray. Hence, if we know the value of this intermediate index β of the crystal we can convert the apparent angle $2E$ of the optic axes into the real angle $2V_a$ by calculation with the aid of the formula:

$$\sin V_a = \frac{\sin E}{\beta}.$$

Moreover, when the apparent angle in air is too large for the emergence of the optic axes, but the latter are visible when the section-plate is immersed in oil or monobromonaphthalene, if we also know the refractive index μ of this immersion liquid, we can calculate the true angle $2V_a$ from the apparent angle $2H_a$ in the liquid by means of the formula:

$$\sin V_a = \frac{\mu}{\beta} \sin H_a.$$

If we prepare another section-plate perpendicular to the obtuse bisectrix of the optic axial angle, the axes will probably be invisible in air, but in the highly refractive liquid the angle will usually be sufficiently reduced to enable the axes to be seen, and the angle itself

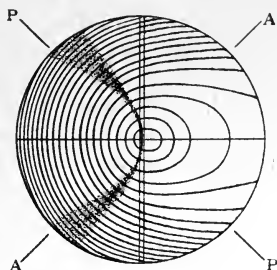


FIG. 631.—Optic Axial "Brush"
Vertex adjusted to Spider-lines
of Polariscope.

to be measured. Now, the knowledge of both the apparent acute and apparent obtuse optic axial angles, $2H_a$ and $2H_o$, thus obtained for liquid immersion enables us at once to calculate the true angle between the optic axes within the crystal. For if $2V_a$ represent this true acute angle, $2H_a$ the apparent acute angle as measured while the plate perpendicular to the first median line is immersed in the liquid, and $2H_o$ the apparent obtuse angle similarly measured while the plate perpendicular to the second median line is immersed in the same liquid, in immediate succession before the temperature has had time to change, then :

$$\tan V_a = \frac{\sin H_a}{\sin H_o}.$$

That is, the quotient of the sines of the halves of the two measured angles gives the tangent of half the true angle, the whole true angle being thus obtained by doubling the angular result of the calculation.

This method is of fairly general application, but it fails when the true angle is so small that the apparent obtuse angle, even in monobromonaphthalene or methylene iodide, is too large to be measurable, owing to the axial brushes not emerging or not being able to be brought into the field, the necessary rotation of the section-plate approaching very close to 90° on each side. In this case, however, if a liquid can be found of the same refractive index for sodium light as the mean index of the crystal, that is, the mean of all three indices α , β , and γ , for sodium light, the acute angle for this same light, as measured in this liquid with the plate perpendicular to the first median line, will be the true angle, and for other wave-lengths will be approximately the true acute angle; and if the dispersions of the crystal and liquid are similar, the true angle will be afforded for all wave-lengths.

Indeed it is not always necessary to cut a section-plate at all, if such a perfectly matched liquid be available, although it is usually much preferable to do so if possible. For the interference figure is seen almost as well through the whole crystal itself, when immersed in a parallel-sided cell of liquid of the same refractive index; the edges and faces of the crystal almost perfectly disappear if the crystal be colourless. This fact is of great use in the investigation of microscopic crystals, as we shall see later in Chapter LII.

It is possible to calculate the optic axial angle from the three refractive indices α , β , γ when the latter are known, by means of either of the formulæ :

$$\tan V_a = \frac{\gamma}{\alpha} \sqrt{\frac{\alpha^2 - \beta^2}{\beta^2 - \gamma^2}} \text{ or } \cos V_a = \sqrt{\frac{1 - \frac{1}{\beta^2}}{1 - \frac{1}{\alpha^2}}}.$$

Only an approximate result is afforded, however; for a slight error of a unit or two in the fourth place of decimals of the refractive index causes a relatively very large difference in the optic axial angle. Hence,

this method of arriving at the optic axial angle is only for use in the last resort, when no direct measurement is possible. An experimental determination is usually possible, however, and is infinitely more valuable and to be preferred.

The best form of polariscope for the measurement of $2E$, $2H_a$, and $2H_o$ is the larger optic-axial-angle goniometer constructed by Fuess. This excellent instrument is shown in Fig. 632, and also in Fig. 633 as employed with the spectroscopic monochromatic illuminator and lime-light lantern. It is also shown in Fig. 634 with the electric lantern, the arrangement of the whole apparatus being that now always adopted by the author; this illustration also shows the apparatus for heating the crystal when it is desired to repeat the measurements at higher temperatures.

The instrument is essentially a combination of the horizontal-circle goniometer and a polariscope. The circle a is the same as that of the No. 2a Fuess goniometer, and the crystal-adjusting apparatus b and centring apparatus c is also similar, but is suspended below the circle instead of rising above it, the vernier table d and covering bevel-case d' for the circle being supported by two rigid wide columns e . The crystal, mounted on its little circular glass plate f , is suspended from the crystal-holder g below the adjusting and centring movements, and the polariscope hk is arranged at the same height with its horizontal axis parallel to a diameter of the circle. It consists of a polarising tube h with convergent lens system l , in this instrument a single hemispherical lens, and in the same axial line an analysing tube k with a similar lens system l' (also a simple hemisphere of glass in this polariscope), to collect the light rays again after their passage through the crystal suspended between the two tubes and their lens systems. Each tube is held in a collar m forming part of one of the columns e , which is widened at this height and pierced for the passage of the optical tube.

The polarising tube h carries an objective lens n at its outer end, where the light enters from the white-light goniometer lamp or the monochromatic illuminator; then comes the polarising Nicol prism in the middle of the optical tube, and subsequently,

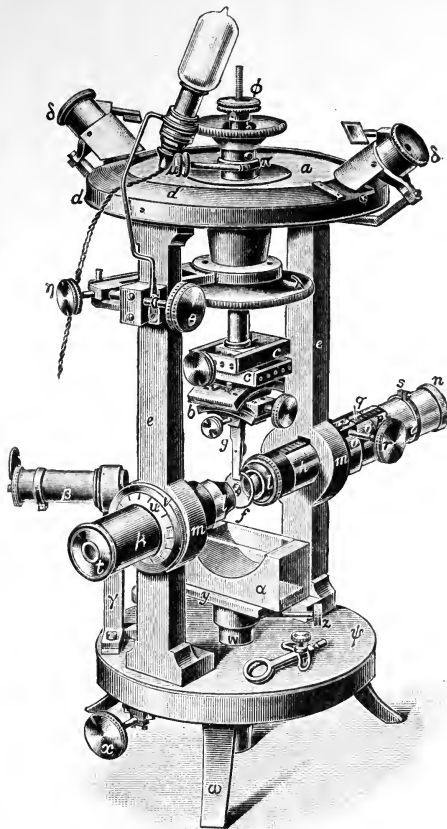


FIG. 632.—Larger Optic-Axial-Angle Goniometer.

comes the analysing tube k with a similar lens system l' (also a simple hemisphere of glass in this polariscope), to collect the light rays again after their passage through the crystal suspended between the two tubes and their lens systems. Each tube is held in a collar m forming part of one of the columns e , which is widened at this height and pierced for the passage of the optical tube.

about an inch from the inner end of the tube nearest the crystal a second lens similar to the first. Within this inner end there fits finally a short tube bearing

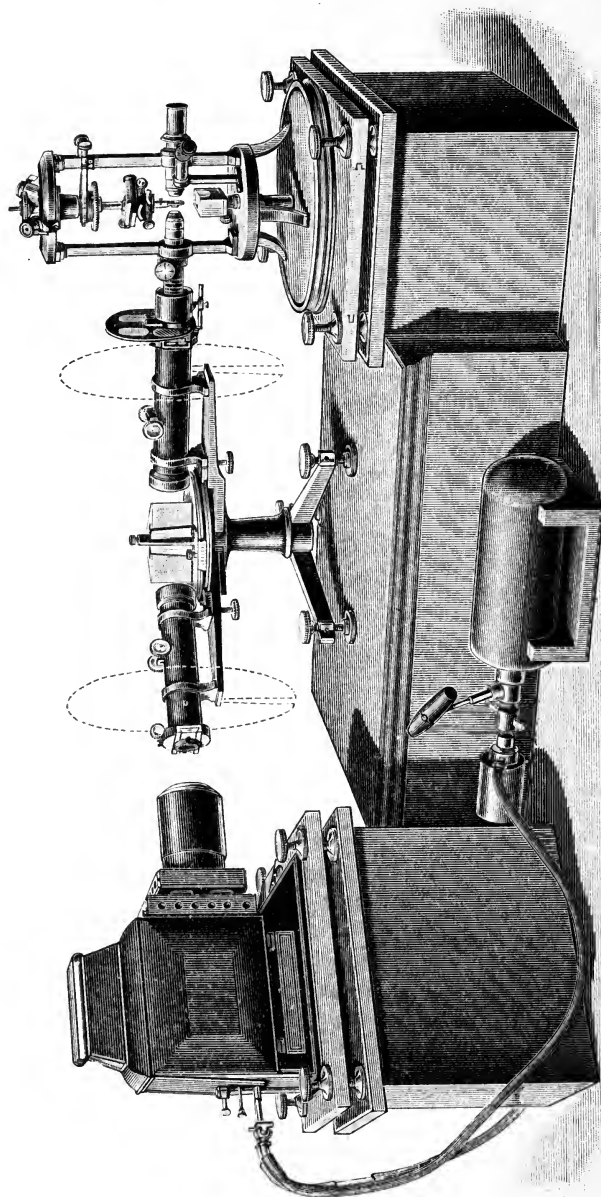


FIG. 633.—Determination of Optic Axial Angles with the Aid of the Monochromatic Illuminator and Lime-light Lantern.

at its extremity nearest the crystal the hemispherical lens *l*. The collar *m* supports this polarising tube in an outer tube *o* rigidly attached to *m*; and a second short inner

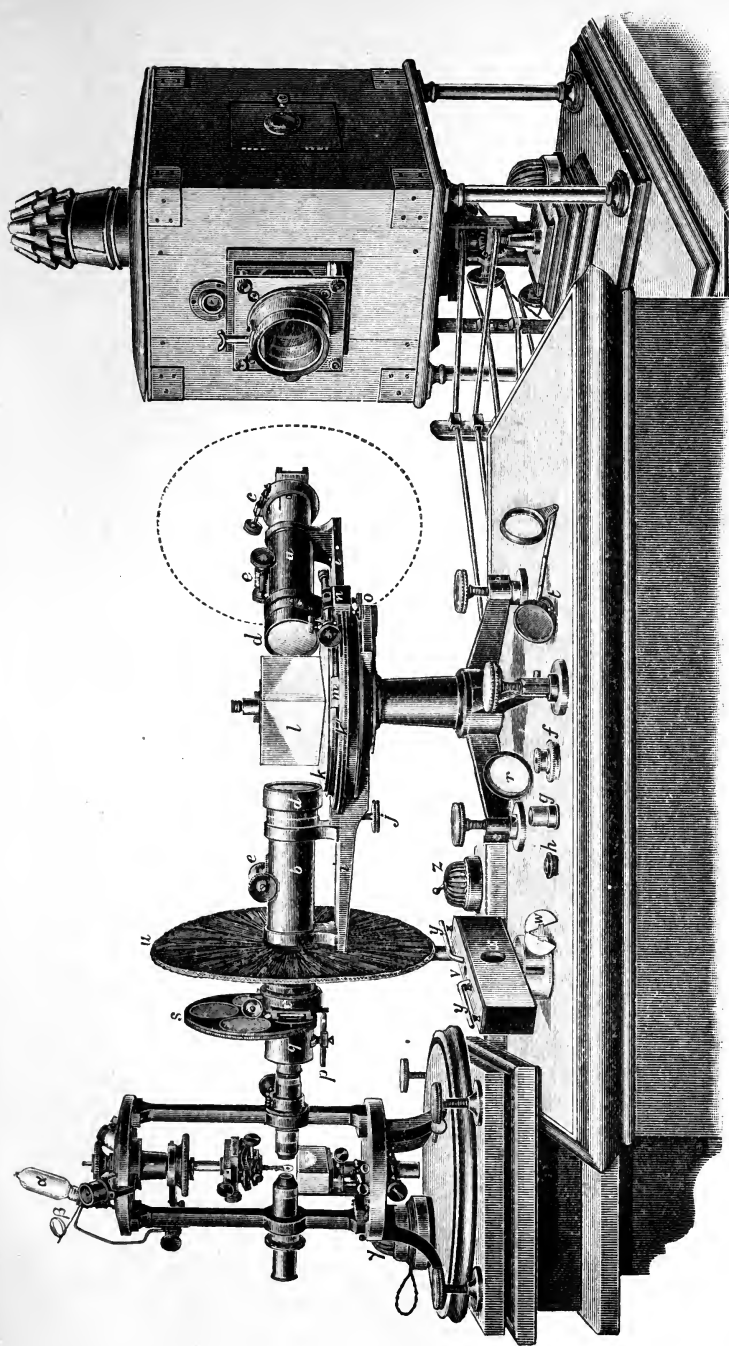


FIG. 634.—Apparatus for Measurement of Optic Axial Angles at the Ordinary and Higher Temperatures, employing the Electric Arc as Source of Light.

tube p slides within o by rack q and pinion r , by which the polarising tube h itself, fitting tightly in the rack-tube p , can be racked closer to or further away from the crystal as may be required for the production of the best-lighted field of interference rings and brushes. The rack-tube has two notches in it, at 45° apart, and the polarising tube carries a tightening collar s bearing a corresponding projection, so that the tube may be automatically arranged with the plane of polarisation of the Nicol horizontal (or vertical) or at 45° thereto.

The analysing tube k comprises the inverted duplicate hemispherical lens l' as convergent system, an adjustable glass plate bearing the engraved cross-lines—one horizontal and two vertical lines, the latter an apparent couple of millimetres apart—the eyepiece t and the analysing Nicol, the latter rotatable and provided with a beveled silver circle u moving over a flanged collar v bearing also on silver the indicator mark. The collar is grooved and the outer tube of the analysing Nicol correspondingly keyed, to prevent rotation when the Nicol is rotated.

In the centre of the basal table is a bored cylindrical column w , adjustable for height by means of a rack and pinion x , and in the bore of which may be inserted the peg of a small rectangular table y , clamped by the screw z , to support the parallel-sided cell a for the highly refractive liquid to be employed as immersion liquid during some of the determinations. A little collimator tube β is also mounted on a separate column γ , about 35° to the left of the analysing tube, bearing a cross-line signal on glass instead of a slit, for occasional use in obtaining a reflection from a crystal-plate which it is desired to arrange normally to the axis of the polariscope tubes.

The circle is read by the two microscopes δ , and is fixed by the clamping screw η , and finely adjusted by the screw θ . The rotation of the circle is accomplished by the large milled wheel λ ; it may also be clamped to the crystal axis by the tightening screw μ of the locking collar π . The crystal axis may be raised or lowered by means of the milled head ϕ at the summit of the instrument. The whole is mounted on a stout base ψ with three feet ω .

This instrument has been found most satisfactory, in the form here described, for the measurement of optic axial angles. Some later innovations in the construction of the polariscope introduced by Fuess have not, in the author's opinion, been improvements. The small illuminating spectroscopic attachment provided with it is of little use, especially in cases of optic axial angles of wide or crossed-axial-plane dispersion. But the parts just described form the best and most delicately accurate goniometrical polariscope with which the author has ever experimented, and when used with the spectroscopic monochromatic illuminator described in Chapter XLIII. and shown in position in Fig. 633 and 634 the apparatus works to perfection.

When it is desired to conduct the determinations for higher temperatures than the ordinary, the little table supporting the cell of liquid is replaced by a metallic air-bath fitting, shown at v in Fig. 634 on the plinth of the monochromatic illuminator. It is an elongated rectangular box, the top of which is divided in the middle so as to be capable of being drawn off in two halves, a half from each side, leaving the centre open for the lowering of the crystal-plate into the middle of the bath, after which the two cover-slides can be slid in their grooves (cut in the upper edges of the sides) back again so as to close up the bath. A semicircle is cut out of each half of the cover at its inner end in order that the two parts may close together without quite touching the crystal-holder, which thus passes readily through the circular hole formed by the approximation of the two halves. To ensure still better closing of the hot-air chamber a split disc, w in Fig. 634, hinged at one extremity of the diameter along which it has

been cut, and with a square hole in the centre the exact shape of the section of the crystal-holder, may be fitted round the latter so as to rest loosely on the top of the bath over the circular aperture. A window x filled with truly plane-parallel glass, is arranged at the centre of each side of the bath, opposite the optical tube in each case, the plane surface of the hemispherical lens of which is brought up to within a couple of millimetres of it during a determination. Each of these windows is fitted in a circular cap, screwed into the box side, and with just sufficient play to permit of the different expansion of the metal and glass during the heating. Two thermometers y , bent at an angle near the bulbs, may be introduced through two other holes near the central aperture, the stems being supported horizontally on two little stirrup-rests standing up from the lid in each case, while the bulb is within the bath quite close to the crystal. The author employs two miniature Bunsen burners to heat the apparatus, one near each end, the gas supply, regulated by a Stott governor and a graduated tap with lever-indicator, being delivered equally to them from a T-piece, on each side of which is a screw pinch-cock to effect any desirable difference in the minute flames in order to ensure equal temperatures in the bath on each side of the crystal, as indicated by the two thermometers.

The conduction of heat by the crystal-holder is a serious matter when the crystal-plate or its glass mount is held directly by the clip-holder, but is considerably reduced by the use of a platinum holder, constructed out of thick foil and held firmly by the ordinary clip-holder. The correction for conduction under these conditions was determined in the case of the author's instrument by suspending the bulb of a miniature thermometer in the bath in the place of the crystal. The necessary correction was found, as the mean of several closely agreeing experiments, to be as much as 7° in the neighbourhood of 100° C.

The angle taken in by the above described optic-axial-angle goniometer for accurate work is necessarily small, in order to ensure adequate refinement of measurement, and the much wider angle afforded by the polariscopical goniometer of the von Groth universal apparatus renders that instrument preferable for demonstration work. It is shown in Fig. 635 partly in section and with the optical tubes separately exhibited in section below, and in Fig. 636 in position in front of the monochromatic illuminator. It includes a much wider field, embracing both optic axes at once with their complete system of lemniscates, unless the apparent angle $2E$ between the optic axes be uncommonly large, corresponding to a true angle $2V_a$ of nearly 90° . This instrument may also be used for measuring the optic axial angle, but one must then be content with a lower degree of accuracy.

The optical tubes of this apparatus designed by von Groth and constructed by Fuess have already been partially described in connection with the stauroscope discussed in Chapter XLIV. and shown in Fig. 599 (page 708). The polarising tube i bears the two lenses j and k with the large Nicol prism l between them just as in the stauroscope. But immediately after the lens k the condensing system of four lenses a , b , c , d follows closely, each mounted separately in its own screw cap so that it can be readily detached or added, according to the amount of convergence required to produce the most suitable rings and brushes with any particular section-plate, the number and closeness of the rings depending largely on two factors, the thickness of the plate and the amount of the double refraction, the former varying inversely as the latter. This polarising tube slides freely in a short outer tube e carried rigidly by one of the two

columns f of the instrument, each of which is cast with a ring expansion g just below the centre, in which ring the outer tube is screwed. This outer tube bears two notches at 45° apart, to fit a V-projection on a tightening collar clamped round the polarising tube, so that when the plane of polarisation of the Nicol has once been determined, the tube can be inserted with this plane either vertical or obliquely at 45° .

The analysing tube t is supported in a similar manner in an outer tube h carried by the ring expansion g of the other column, and a couple of notches are cut in the outer tube at 45° apart to gear with a corresponding V-projection on a collar carried by the analysing tube. Moreover, there is a vertical vernier-plate m for a circle,

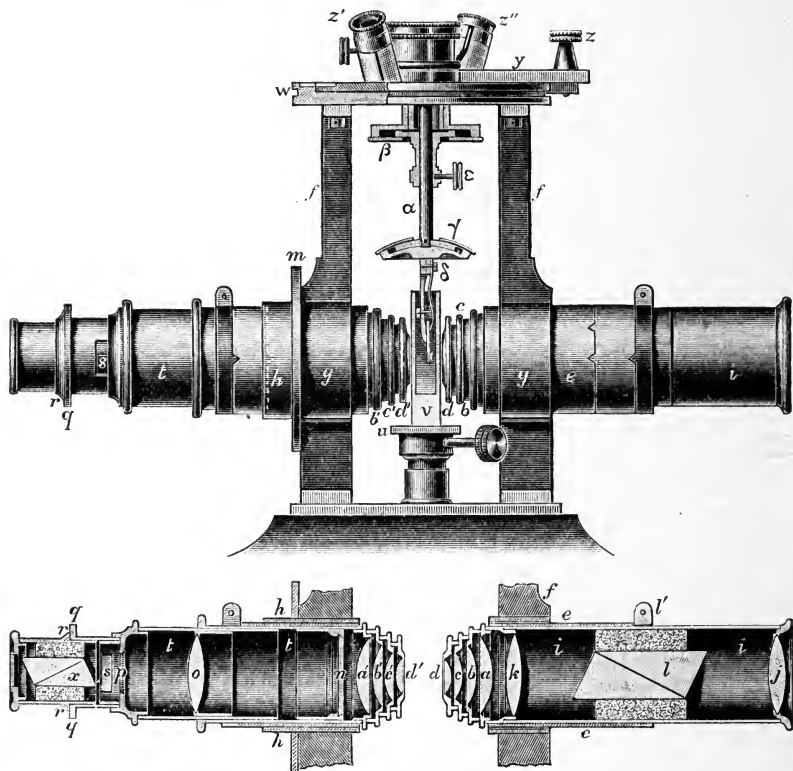


FIG. 635.—Optic-Axial-Angle Goniometer of the von Groth Universal Apparatus.

carried round the outer tube and close up to and in front of the ring of the column, in order that a cap provided with a large bevelled silver divided circle may fit over it; this special cap bears a short tube, in which the analysing Nicol prism α to be presently referred to, together with its carrying tube, may be inserted, when the instrument is to be used for another purpose to be described in Chapter XLIX., that of determining the angle of rotation of the plane of polarisation of light by a plate of an optically active crystal (see Fig. 655, page 805).

The analysing tube carries at the end nearest the polarising tube and the crystal a series of four short focus lenses a' , b' , c' , d' , similar in every respect, including focal length and curvature, to those of the polarising tube. Just beyond the fourth provision is made for the adjustable insertion of a glass plate n bearing an engraved scale, which

can be calibrated, by comparison with the interference figures of crystals of known optic axial angle, so that the value of the divisions in optic-axial-angle magnitudes shall be known. This is best removed if the instrument has to be used either for real measuring of the optic axial angle by the attached goniometer, or for the study or photography of the interference figures, as it disturbs the beauty of the interference figures. Further along the tube is the viewing lens *o*, carried in an adjustable inner tube, slotted to gear with a pin in the main tube in order that no rotation shall be possible. A diaphragm *p* follows a little distance further, where the inner tube narrows and eventually terminates in a flange *q* which gives it a broad flat end, which is silvered and engraved with an indicator mark for the circle of the analysing Nicol *x*. This latter is carried in a short tube provided with a flange bearing the bevelled silver divided circle *r*. The narrow part of the inner tube close to the thicker part has a pair of slots *s* cut in it at positions diametrically opposite to each other and

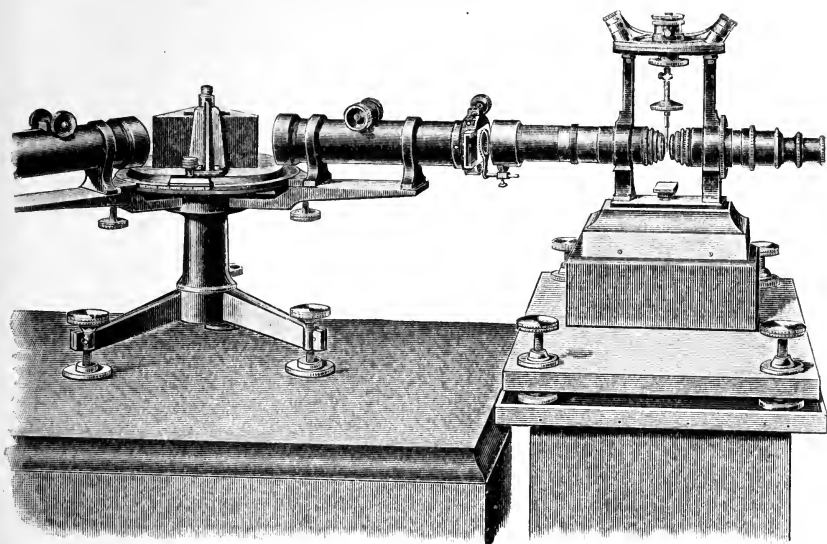


FIG. 636.—The Wide-angle von Groth Apparatus arranged with Monochromatic Illuminator for the Measurement of Optic Axial Angles or Observation of the Interference Figures.

inclined 45° to the vertical and horizontal directions, for the reception of a mica quarter-undulation plate or a quartz wedge. The slots can be closed by a rotating inner cylindrical shutter, manipulated by a pin projecting through one of the slots, when the mica or quartz is not required.

The base of the instrument carries in the centre under the crystal a little table *u*, adjustable for height by means of a rising and falling column, in order that the crystal may be immersed in a cell of highly refractive liquid *v*, such as monobromonaphthalene, for the measurement of $2H_a$ and $2H_o$.

The two columns carry at their upper terminations the goniometer circle *w*, divided on its wide silver limb directly to $20'$, and reading to single minutes with the aid of a vernier carried on a similar flat and broad silver limb moving concentrically within the circle plate and flush with it. Rotation of this vernier disc is effected by means of a lever arm *y* carrying a large milled head *z*, which is the head of a clamping screw by which the vernier and circle-plates may be clamped together with the aid of a small bracket-piece fitting a groove in the periphery of

the circle-plate. Around the central boss of the vernier plate, from which the arm radiates, a ring is loosely fitted, having two light arms branching off opposite to one another and terminating in holders for a couple of small microscopes z' and z'' , with which to read the circle at diametrically opposite positions.

Within the central boss there is capable of either independent or clamped rotation the axis a , terminating in the crystal-holder and carrying the means of adjustment of the crystal. The nature of the centring and adjusting movements will be clear from Fig. 635. They are both simple movements of a sliding character, the centring movement β being that of a flat circular disc within a circular box of large diameter, and the adjusting movement γ that of a shallow cup-shaped disc of spherical curvature within a similarly shaped box, both boxes having on one side (the lower and the upper in the two respective cases) six radial saw-cuts in three pairs close together so as to grip the discs by three strips which can be slightly bent in. The adjusting movement is carried just above the crystal-holder δ , one of the usual spring gripping type, and from its disc rises the inner axial rod α , which slides within a short wider tube carried below by the centring disc β , which is pierced in order that a long length of inner axis α may be available. A fixing screw ϵ enables the axis to be clamped at the right height for the most perfect passage of the rays from the polariser through the crystal.

A heating apparatus consisting of a rectangular bronze air-bath, similar to that of the larger optic-axial-angle goniometer, is provided for use with the instrument. As the convergent system, when all the lenses are employed, is a much more powerful one, however, than that of the polariscopical goniometer just referred to, the two optical tubes require to be brought with their convergent systems closer together, nearer to the crystal. Hence, the two circular parallel-plate glass windows are fitted in circular frames tapped with a screw thread on their periphery, to fit similar threads in the two circular apertures facing each other in the box sides, so that by means of a key provided they may be screwed more or less deeply into the box, and thus enable the two convergent systems to penetrate adequately within the box without quite touching the window.

As an example of an actual determination the measurements and results of the calculations may be quoted for the optic axial angle of monoclinic ammonium magnesium sulphate. The plane of the optic axes of the crystals of this salt is the symmetry plane, and the first median line (acute bisectrix of optic axial angle) is inclined $12^\circ 8'$ to the axis a , and the second median line (obtuse bisectrix) is inclined $4^\circ 58'$ to the vertical axis c , in front of the latter.

DETERMINATION OF OPTIC AXIAL ANGLE OF $(\text{NH}_4)_2\text{Mg}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$.

With Section-plate Perpendicular to Acute Bisectrix.		With Section-plate Perpendicular to Obtuse Bisectrix.
Apparent Angle in Air 2E.	Apparent Angle in Monobromonaphthalene 2H_a .	Apparent Angle in Monobromonaphthalene 2H_o .
Li-light $79^\circ 15'$	$45^\circ 21'$	$106^\circ 35'$
C „ $79^\circ 14'$	$45^\circ 18'$	$106^\circ 30'$
Na „ $79^\circ 10'$	$44^\circ 56'$	$106^\circ 0'$
Tl „ $78^\circ 59'$	$44^\circ 38'$	$105^\circ 28'$
F „ $78^\circ 44'$	$43^\circ 57'$	$104^\circ 35'$

Calculations.

	Calculated Value of V_a .	Calculated Value of $2V_a$.
Expression for $\tan V_a = \frac{\sin H_a}{\sin H_o}$.		
For Li-light, $\tan V_a = \frac{\sin 22^\circ 40\frac{1}{2}'}{\sin 53^\circ 17\frac{1}{2}'}$	$25^\circ 41'$	$51^\circ 22'$
C ,, $= \frac{\sin 22^\circ 39'}{\sin 53^\circ 15'}$	25 40	51 20
Na ,, $= \frac{\sin 22^\circ 28'}{\sin 53^\circ 0'}$	25 34	51 8
Tl ,, $= \frac{\sin 22^\circ 19'}{\sin 52^\circ 44'}$	25 30	51 0
F ,, $= \frac{\sin 21^\circ 58\frac{1}{2}'}{\sin 52^\circ 17\frac{1}{2}'}$	25 19	50 38

The values of $2V_a$ derived from the measurements of two other pairs of section-plates were within a few minutes of the values just given in the last column, and the final mean values for $2V_a$ derived from all three sets of determinations are set forth below :

Li-light	.	.	.	$51^\circ 20'$
C ,,	.	.	.	51 18
Na ,,	.	.	.	51 11
Tl ,,	.	.	.	51 2
F ,,	.	.	.	50 36

The values for G-light are not as a rule adequately accurately determinable in monobromonaphthalene, owing to the absorptive power of the liquid for violet light.

Determination of the Dispersion of the Median Lines. Case 1. Inclined Dispersion of both Median Lines (both lying in the symmetry plane).—The theoretical discussion of this subject has already been given in Chapter XLI, p. 676, and it only remains here to describe the practical operations involved in the determination.

It will be obvious that the first median line can only be truly perpendicular to the section-plate for one colour, and the plate is generally cut truly for sodium light, a convenient radiation not very far from the middle of the spectrum, and one for which under any circumstances the stauroscopic observations will have been made. The amount of the dispersion is usually very small, rarely exceeding a degree and generally only a few minutes, and consequently too small to introduce any appreciable error in the measurement of the optic axial angle for wave-lengths other than that of sodium light. It is so small, indeed, as a rule, as to be almost undetectable in the determination of the extinction directions in the symmetry plane by the stauroscope for red light and for blue light, and this is, therefore, by no means a method of precision for determining such a small amount of dispersion of the median lines.

In order to determine it accurately, after measuring the optic axial angle in the ordinary manner just described for the usual wave-lengths, including the limiting wave-lengths between which we wish to find the dispersion, we also measure the separate movement of each optic axis while the crystal is immersed in a liquid of the same refractive index for sodium light as the mean index of the crystal. The series of measurements in monobromonaphthalene do not serve for both purposes, because the dispersion of the axes varies with the liquid, and the true angle within the crystal is only afforded when the refractive index of the liquid is the same as that of the crystal.

The crystal axis of the optic-axial-angle goniometer is rigidly locked to the circle during the measurements, and the orientation of the section-plate is also carefully noted as regards the crystal faces, so as to ascertain in which direction the dispersion of the median line occurs. For this reason it is always advisable to leave as many faces uninjured on the edges of the section-plate as possible, to be used as reference faces during these measurements, that is, to inform us on which side, right or left, of the adjusted section-plate the crystallographic axes a and c are situated. From the actual circle readings for the positions of the two optic axes for Li-light and for F-light (the angle for G-light being rarely obtainable with adequate accuracy on account of the feebleness of the light escaping absorption) the movement of each axis can be readily found, and the mean of the two determinations (or the half of their difference) for the two axes is taken as the amount of the dispersion of the first median line.

The measurement of the optic axial angle in a liquid of the same refractive index is also of use as indicating, at any rate approximately, the magnitude of the true angle $2V_n$, and of confirming the order of dispersion of the axes, as shown by the calculated results of $2V_n$ from $2H_a$ and $2H_c$, even in the cases of rhombic crystals, whenever the variation of the optic axial angle for different wave-lengths is very small, a few minutes only for instance.

The example of monoclinic ammonium magnesium sulphate may again be taken as an illustration as regards the determination of inclined dispersion; for the optic axes of the crystals of this salt lie in the symmetry plane. The mean refractive index (mean of all three indices α , β , γ for sodium light) is 1.4744. From the list of refractive indices given on page 577 in Chapter XXXVI. it will be observed that oil of turpentine has a refractive index of 1.4725 at 20° , and at the temperature of the experiment, about 15° , the index of this liquid is 1.474. It is thus particularly suitable for the purpose. Observations in this liquid with the same section-plate as was used in the measurements in monobromonaphthalene quoted on page 776 showed that the first median line lay $17'$ nearer to the a axis for red C-hydrogen light than for green F-hydrogen light.

Case 2. Horizontal Dispersion of the First Median Line.—The plane of the optic axes in this case is perpendicular to the symmetry plane and the first median line lies in the symmetry plane. As an example of the mode of conducting the measurement of the amount of horizontal

dispersion, the determination in the case of the monoclinic form of ammonium selenate $(\text{NH}_4)_2\text{SeO}_4$ may be quoted.

The section-plate was adjusted so that the line joining the two optic axes was arranged vertically instead of horizontally, and the Nicols were set at 90° and 0° instead of at 45° and 135° . Each of the optic axes in turn was brought to the centre of the cross-lines by rotation of the back-to-front cylindrical adjusting segment of the polarising goniometer, and measurements were then made of the lateral difference of position of the axis for the two limiting wave-lengths of light, in this instance red C and greenish-blue F light; while the plate was immersed in (1) oil of cassia ($\mu = 1.5862$) and (2) oil of anis ($\mu = 1.5540$), the refractive indices of which two liquids are slightly higher and lower respectively than the mean index of ammonium selenate crystals (1.5694). Both series of determinations agreed in indicating that the two optic axes are dispersed $12'$ between C-light and F-light, which, taking into consideration the position of either optic axis between the fixed second median line (parallel to the immovable symmetry axis) and the movable first median line, as indicated by the optic axial angle ($37^\circ 19'$ for C-light and $38^\circ 44'$ for F-light), corresponds to a dispersion of the first median line of $13'$.

The problem is to find the angle θ in the spherical triangle in Fig. 637, in which B_{aC} and B_{aF} are the two positions of the movable acute bisectrix for C- and F-light respectively, B_o is the immovable obtuse bisectrix, C the position of one of the optic axes (that lying in this quadrant) for C-light and F the same for F-light. The actual $12'$ angle measured was the line CF at right angles to the base B_oCB_{aC} ,

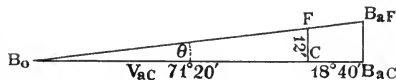


FIG. 637.

that is, the displacement of the optic axis parallel to the horizontal cross-wire, which is parallel to $B_{aC} B_{aF}$, the angle required between the positions of the first median line for the two wave-lengths. Now this latter required angle is the angle θ , and from the data now acquired it can be calculated at once. For FCB_o is a spherical triangle with a right angle at C, the side CB_o is the complement of V_a for C-light which is $18^\circ 40'$, and the short side CF is the measured $12'$. Hence, constructing a Napierian diagram (Fig.

638), we obtain from it the relation :

$$\sin 71^\circ 20' = \tan 12' \cot \theta; \text{ and } \cot \theta = \sin 71^\circ 20' \cot 12'$$

$$\text{Log. } \sin 71^\circ 20' = 1.97653$$

$$\text{Log. } \cot 12' = 2.45709$$

$$\text{Log. } \cot \theta = 2.43362 \quad \theta = 13'$$

Thus as a rule this calculation is not absolutely necessary, for the

difference of one minute between 12' and 13' is less than the possible error in determining the amount of horizontal displacement of the optic axis experimentally. Hence, when the angle is small the displacement of the first median line may be taken as being sensibly afforded by the displacement of either of the optic axes. After some experience with optic axial angles and dispersions of different magnitudes it is, however, quite possible to draw up a scale of these minute corrections. Until this information has been accumulated, however, it is safest to make the simple calculation in every case.

Case 3. Crossed Dispersion of the Second Median Line.—The plane of the optic axes is again perpendicular to the symmetry plane, but the movable bisectrix lying in the symmetry plane is now the second median line instead of the first. The measurement of the displacement is carried out just as for case 2, except that it is the plate perpendicular

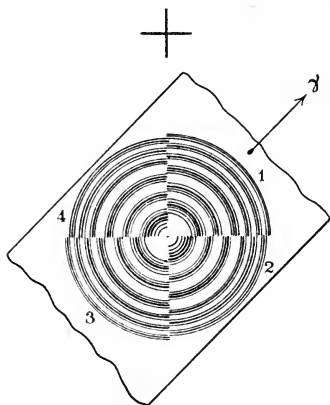


FIG. 639.—Effect of Quarter-wave Mica Plate on the Interference Figure of a Positive Uniaxial Crystal.

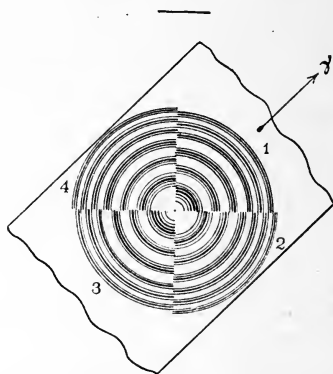


FIG. 640.—Effect of Quarter-wave Mica Plate on the Interference Figure of a Negative Uniaxial Crystal.

to the second median line which is employed, instead of the one perpendicular to the first median line.

Use of the Quarter-Wave Mica Plate and Quartz Wedge for the Determination of the Sign of the Double Refraction.—When the quarter-wave mica plate referred to in Chapter XLI., in the form of the 6 by 1 centimetre strip described on page 662, is inserted diagonally at 45° to the planes of the crossed polariser and analyser, during the examination of the section-plate of a uniaxial crystal perpendicular to the axis in convergent polarised light, the mica being inserted between the crystal-plate and the analyser, the figure is altered in opposite ways in alternate quadrants. The $\frac{1}{4}\lambda$ effect of the mica is added to that of the crystal in one pair of quadrants, and subtracted from it in the alternative opposite pair. Supposing the elongated mica plate to be arranged as shown in Fig. 639, passing across quadrants 1 and 3, and that the crystal is of positive double refraction, the portions of the

rings in these quadrants are expanded, while if the crystal be negative they are contracted, as shown in Fig. 640. The dark cross, with arms parallel to the planes of the Nicols, vertical and horizontal, also disappears. The innermost ring of each contracted quadrant becomes a dark spot, and a useful rule to remember is, that if the two spots lie on the diagonal line along which the mica strip is inserted, as in Fig. 640, the crystal is negative; whereas, when the two spots are on each side of that diagonal line, on a line at right angles to it (parallel to the short edges of the rectangular mica plate and thus perpendicular to the γ axis of the indicatrix which joins the apparent positions of the two optic axes of the mica), as represented in Fig. 639, the crystal is positive.

An analogous effect is produced with a biaxial crystal-plate, as shown in Figs. 641 and 642. The crystal must be arranged with the line

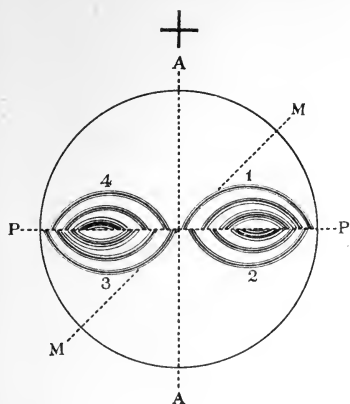


FIG. 641.—Effect of Quarter-wave Mica Plate on the Interference Figure of a Positive Biaxial Crystal.

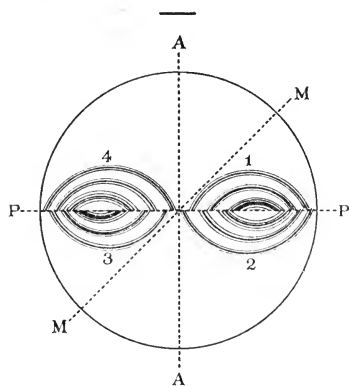


FIG. 642.—Effect of Quarter-wave Mica Plate on the Interference Figure of a Negative Biaxial Crystal.

joining the pair of optic axes parallel to one of the planes of the crossed Nicols, and on inserting the mica plate diagonally as before the following effects are observed. If the crystal be positive the two spots produced from the two innermost rings (the innermost of each of the two systems) will be respectively above the horizontal line on the left (fourth quadrant) and below (second quadrant) on the right; while if the crystal be negative, the spots will be conversely arranged, below on the left (third quadrant) and above on the right (first quadrant).

When two quarter-wave plates are employed, one before and one after the crystal-plate, the dark cross of either a uniaxial or a biaxial figure will disappear, and the rings appear alone and undisplaced, that is as continuous rings; in the case of a uniaxial crystal they are truly circular, exactly resembling Newton's rings or those afforded by a concave film of gypsum (selenite) or a concave plate of quartz. On

rotation of the analyser the rings will expand or contract, according to the direction in which the analyser is turned, very much like those of an optically active crystal such as quartz.

The sign of the double refraction of a biaxial crystal may also be determined by use of the quartz wedge also described in Chapter XLI. (page 663). Quartz being positive (uniaxial), it subtracts from the effect of a negative crystal, making it appear as if thinner, while it adds to the apparent thickness of a positive crystal. The crystal-plate is arranged at 45° to the crossed Nicols, and the quartz plate is inserted

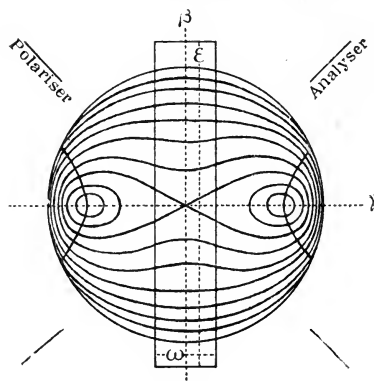


FIG. 643.

between the crystal-plate and the analyser, first along one and then along the other of the 45° -diagonals, that is, parallel to the direction of the line joining the optic axes and perpendicularly thereto. The latter case is shown for negative (biaxial) mica in Fig. 643. In this case the direction of the maximum γ axis of the optical indicatrix of the mica crystal coincides with that of the minor axis ω of the quartz, and thus the two effects are subtractive, so that the result is as if a thinner plate were being examined, and the rings will consequently expand. As

the quartz wedge is pushed further and further in so as to increase the thickness of quartz in action, the rings will successively expand, and pass through the loop stage to the ellipse-like variety of lemniscate, one after the other.

If the wedge be inserted parallel to the line joining the optic axes, the maximum axis γ of the crystal in the case of mica, this axis and the major axis ϵ of the wedge being in conjunction the effects are additive, and the appearance is that corresponding to a thickening of the section-plate, that is, a closing up of the rings.

The converse phenomena are observed when the crystal is a positive biaxial one, the rings expanding when the quartz wedge is inserted parallel to the line joining the optic axes, and contracting when the insertion occurs at right angles thereto.

The effect of the quartz wedge is most strikingly shown with section-plates too thick to show the optic axial rings. If the crystal be negative and not too extremely thick, the rings will make their appearance when the wedge is inserted perpendicularly to the line joining the optic axes, or if it be positive, when it is inserted parallel to the optic axial line.

In concluding this chapter it may prove convenient to give a short table of a few easily procurable substances exhibiting graduated values of the optic axial angle.

TABLE OF SOME USEFUL OPTIC AXIAL ANGLES FOR NA-LIGHT.

	2E.	2V _A .	Observer.
Nitre (potassium nitrate) . . .	10° 51'	7° 12'	Schrauf
Cerussite (lead carbonate) . . .	17 8	8 14	"
Sanidine felspar	19 45	12 50	Des Cloizeaux
Aragonite (rhombic calcium carbonate)	30 52	18 11	Kirchhoff
Borax (sodium borate)*	59 23	39 36	Tschernak
Potassium magnesium selenate (with 6H ₂ O)	61 7	39 38	Tutton
Barytes (barium sulphate) . . .	63 12	37 28	Heusser
Potassium copper sulphate (6H ₂ O)	72 7	46 32	Tutton
" magnesium sulphate (6H ₂ O)	72 47	47 54	"
Ammonium magnesium sulphate (6H ₂ O)	79 11	51 11	"
Ammonium sulphate	84 6	52 12	"
Ammonium magnesium selenate (6H ₂ O)	88 4	54 47	"
Topaz	100 40	56 39	von Koksharov
Potassium sulphate	111 0	67 20	Tutton
Potassium zinc sulphate (6H ₂ O) .	112 29	68 14	"
" nickel sulphate " . . .	130 57	75 16	"
Ammonium zinc sulphate " . .	144 41	79 0	"
" " selenate " . . .	170 0	82 7	"

CHAPTER XLVIII

CROSSED-AXIAL-PLANE DISPERSION OF THE OPTIC AXES.

It has been made clear in the last chapter that the positions of the two optic axes of a biaxial crystal are in general different for different wave-lengths of light and at different temperatures, and mention was made of extreme cases in which the axes are separated at a considerable angle along one plane of the optical ellipsoid for red light or a particular temperature, and along another principal plane, necessarily at right angles to the first, of the ellipsoid for blue light or a second specific temperature. These instances are not so rare as they were formerly thought to be, and the author has met with and studied in detail no less than seven such cases in the course of his investigations, and from the experimental evidence has evolved the general law governing the phenomenon.¹ It was shown that the essential conditions for crossed-axial-plane dispersion are :

“(1) The simultaneous occurrence of extremely small double refraction (nearness of α and γ indices of refraction) and close approximation of the intermediate index β either to the α or to the γ index. The latter condition is necessary for the possibility of crossing, and the former for wide separation of the optic axes in the two planes for the two ends of the spectrum, or for two different temperatures. (2) Change of wave-length of the light employed at the ordinary temperature, or change of temperature while using light of the same wave-length, or both kinds of change simultaneously operating, must so act as to bring about equality, at a particular temperature for each wave-length, of two of the three refractive indices, namely, of the intermediate index β and of that one of the other two which is already nearest to equality with it.”

It is the first condition which renders the crystal so sensitive to even minute physical change, such as is brought about as the result of the usual difference shown by the three indices in their spectral dispersion, and the slightly different effect of change of temperature along each of the axial directions of the optical ellipsoid, both of which causes are adequate to upset the balance. For when all three refractive indices are so close together at the start, a change only affecting the fourth decimal place in the refractive index may suffice to reverse the

¹ “Allgemeine Erklärung des Phänomens der Dispersion in gekreuzten Axenebenen,” *Zeitschr. für Kryst.*, 1907, 42, 554 ; “The Optical Constants of Gypsum at Different Temperatures,” *Proc. Roy. Soc., A*, 1908, 81, 40.

relative positions of two of the indices, and thus for some intermediate wave-length or temperature to effect the temporary equality of those two indices, with production of uniaxial conditions, and display of the rectangular cross and circular rings by a section-plate perpendicular to what under ordinary conditions is the first median line, one of the three axes of the optical ellipsoid of general form, but which under the temporary conditions referred to becomes the axis of a rotation ellipsoid.

The crystals which exhibit crossed-axial-plane dispersion are more or less divisible into two main classes, which are respectively especially sensitive to change of temperature and to change of wave-length. Of the first class gypsum (selenite), the monoclinic hydrated sulphate of lime $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$, is the most striking example, while brookite, the rhombic form of dioxide of titanium, TiO_2 , and the so-called "triple" tartrate of potassium, sodium, and ammonium, the crystalline product obtained as an isomorphous mixture from mixed solutions of the double tartrates of potassium and sodium, and ammonium and sodium, known as the Seignette salts, $\text{C}_4\text{H}_4\text{O}_6\text{KNa} \cdot 4\text{H}_2\text{O}$ and $\text{C}_4\text{H}_4\text{O}_6(\text{NH}_4)\text{Na} \cdot 4\text{H}_2\text{O}$, and also the picrates of iron and manganese, are clear cases of the second class. But the two types merge into one another on the border line, and six new cases met with by the author belong to this intermediate type sensitive to variations of both temperature and wave-length.

They are rhombic rubidium sulphate, Rb_2SO_4 , and caesium selenate, Cs_2SeO_4 , monoclinic ammonium selenate, $(\text{NH}_4)_2\text{SeO}_4$, monoclinic caesium magnesium sulphate and selenate, $\text{Cs}_2\text{Mg}(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$ and $\text{Cs}_2\text{Mg}(\text{SeO}_4)_2 \cdot 6\text{H}_2\text{O}$, and the monoclinic form of an organic substance ethyl triphenylpyrrholone, the triclinic methyl analogue of which (both being dimorphous and exhibiting a triclinic variety) has already been referred to on pages 85 and 283. Caesium selenate is a particularly interesting case, for within the limits of the ordinary temperature and 250°C . each of the three axes of the optical ellipsoid in turn becomes the first median line. The substance saccharine, $\text{C}_6\text{H}_{10}\text{O}_5$, is also one of these border-line cases remarkably sensitive to both kinds of physical change.

The following table expresses concisely the facts as regards the three definite substances chiefly sensitive to change of wave-length; the value of the apparent optic axial angle in air 2E is shown for Li, Na, and Tl-light, together with the direction of the plane of the optic axes.

SUBSTANCES WITH OPTIC AXIAL ANGLES VARYING ENORMOUSLY
WITH CHANGE OF WAVE-LENGTH.

Light.	Brookite.		Picrate of Iron.		Picrate of Manganese.	
	2E.	Plane of Optic Axes.	2E.	Plane of Optic Axes.	2E.	Plane of Optic Axes.
Li	$58^\circ 0'$	(001)	$50^\circ 16'$	(100)	$41^\circ 53'$	(100)
Na	$38 10$	(001)	$24 48$	(100)	$15 30$	(010)
Tl	$21 40$	(010)	$46 54$	(010)	$57 13$	(010)

In the case of brookite and iron picrate the uniaxial figure is exhibited for a wave-length between yellow Na-light and green Tl-light, in the former case in the neighbourhood of wave-length 0·00055, whereas in the case of manganese picrate the critical wave-length is between lithium and sodium light. A mixed solution containing specific proportions of the two picrates has been found to yield crystals showing the uniaxial cross and circular rings for sodium light.

Extremely interesting observations have been made by Brugnatelli¹ concerning saccharine, $C_6H_{10}O_5$, an optically active substance crystallising in the bisphenoidal (hemihedral) class of the rhombic system, like the triple tartrate of potassium, sodium, and ammonium, and the two double tartrates admixed in the latter. Saccharine is the lactone or anhydride of saccharic acid, $C_6H_{12}O_6$, and is obtained by the action of lime on a solution of invert sugar; it crystallises from a hot saturated solution in excellent large crystals of prismatic habit, which melt at 160° , and yield a dextro-rotatory solution. The crystals cleave readily along {010}, and the first median line is perpendicular to this cleavage face. The temperature at which the uniaxial figure is produced varies considerably in different crystals; while the average temperature observed by Brugnatelli was 15° , actual temperatures were 12° , 17° , and 26° for sodium light. The more perfect the crystal, however, the nearer to 15° was the temperature at which the uniaxial rings and cross were formed. A cleavage plate two millimetres thick from a magnificent clear crystal gave it at $15\cdot8^\circ$ for sodium light. For all wave-lengths between D and violet the optic axes lie in the plane {100}, whilst for light between D and the red end of the spectrum they lie in {001}. The temperatures at which the uniaxial figure was produced for the various wave-lengths with the plate just alluded to is shown in the following table:

Light corresponding to Spectrum Line.	Temperature for Production of Uniaxial Figure.
B	$6\cdot6^\circ$
C	$9\cdot0$
D	$15\cdot8$
Ca β	$21\cdot3$
Tl	$23\cdot8$
E	$26\cdot0$
b	$27\cdot4$
F	$34\cdot7$
Sr δ	$39\cdot5$
G	$48\cdot0$

The results are also expressed in the accompanying curves in Fig. 644, of which the abscissæ are wave-lengths and the ordinates values of $2E$. Separate curves are given for the temperatures 16° , 21° , and 26° . The values of $2E$ in the plane {100} are considered as positive, and those in {001} as negative. It is clearly obvious from this

¹ *Zeitschr. für Kryst.*, 1898, 29, 54.

diagram that the curves are very steep in the neighbourhood of the critical zero angle of the optic axes, when the uniaxial figure is produced; that is to say, near the critical position of identity the optic axes move extraordinarily rapidly with change of wave-length. Also, the curves on the two sides of zero are unsymmetrical.

Pockels¹ has shown that the conditions for such a substance the optic axial angle of which is very sensitive to change of wave-length may be expressed by a formula. He starts with the well-known formula connecting $2E$, the apparent angle in air, with the refractive indices a , β , γ , namely:

$$\sin E = \gamma \sqrt{\frac{\beta^2 - a^2}{\gamma^2 - a^2}} = \sqrt{\frac{\gamma}{\gamma^2 - a^2}} \sqrt{\beta^2 - a^2},$$

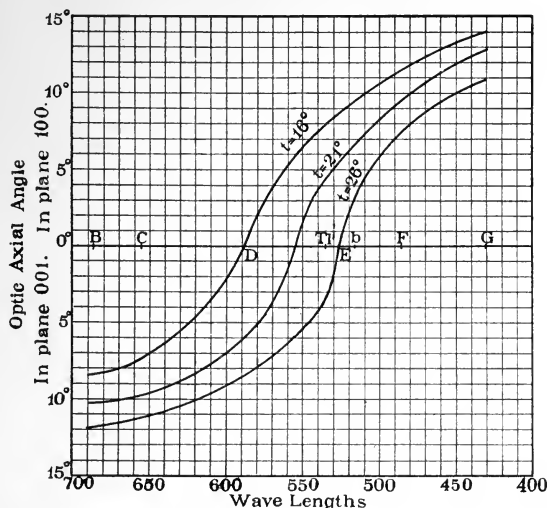


FIG. 644.—Optic-Axial-Angle Curves of Saccharine for the Temperatures 16°, 21°, and 26°.

and replaces the refractive indices by their Cauchy dispersion formulæ of the type $A + \frac{B}{\lambda^2}$, thus obtaining the following differential equation as the expression of the change of the apparent optic axial angle with the wave-length λ :

$$\frac{dE}{d\lambda} \text{ or } \frac{d \sin E}{d\lambda} = -\frac{\gamma^2}{\gamma^2 - a^2} (B_2 - B_1) \cdot \frac{1}{\sin E} \cdot \frac{1}{\lambda^3}.$$

From this formula it is apparent that when $E = 0$ the differential quotient $\frac{dE}{d\lambda}$ becomes infinite, and that in the neighbourhood of zero E must change very rapidly with λ ; also that the two parts of the curve

¹ *Lehrbuch der Krystalloptik*, p. 71.

for the two different sides of zero must be unsymmetrical, as is evident from the diagrams, because for equal values of E the change becomes the greater the smaller the wave-length.

A somewhat similar expression has been derived by S. Kreutz,¹ where g and h are the constants of the Cauchy formula; it runs as follows:

$$\frac{d \tan V}{d\lambda} = - \frac{g(\frac{h}{g} - \frac{h'}{g'})}{\tan V \lambda^3}$$

Both these formulæ of Pockels and of Kreutz agree in indicating that when a small optic axial angle becomes smaller it does so more rapidly than a larger angle, the existing dispersion becoming greater; indeed, it becomes a maximum for E or $V = 0$. Moreover, the order of the dispersion comes into play, that is, whether it be $\rho > \nu$ or $\nu > \rho$. The shorter the wave-length, the more rapid the change, so that the optic axial angle diminishes more rapidly for blue light than for red. Hence, when the dispersion is $\rho > \nu$ and the angle is diminishing, both these influences (the approach to the zero angle and the energetic action of a short wave-length) co-operate to produce large change; but if $\nu > \rho$, the effect is differential. The formulæ also show that large dispersion of the axes is to be expected when the double refraction is weak.

Thus the rules found by the author in the year 1894, purely from experiment, to apply in the case of rubidium sulphate, and which have since been amply confirmed by four other new examples among the sulphates and selenates of the alkalis and their double salts, as well as in the case of gypsum, have since been absolutely verified from the mathematical point of view by Pockels and Kreutz. We may now, therefore, inquire more in detail into some of the cases of sensitiveness to both change of wave-length and alteration of temperature, which have been investigated experimentally by the author, and which include the historic case of gypsum.

Rubidium sulphate is the central member of the series of three isomorphous rhombic sulphates of potassium, rubidium, and caesium (K_2SO_4 , Rb_2SO_4 , Cs_2SO_4). This group affords one of the clearest instances of "eutropic" or family-group isomorphism, which series are governed by the general law that the crystallographic properties are functions of the atomic weights of the interchangeable elements (here potassium, rubidium, and caesium) of the same family group which give rise to the series (see page 484). Now the double refraction is already low in the first member of the series, potassium sulphate, and this particular property obeys the law just referred to in the sense of diminution, so that the extreme refractive indices α and γ converge towards equality as the atoms of potassium are replaced by the heavier ones of rubidium and the still heavier atoms of caesium. This is shown graphically by the curves in Fig. 645, which represent the reciprocals of the three refractive indices,

¹ *Ber. der kais. Akad. Wien*, 1908, 117, 9.

the axes of the Fresnel ellipsoid, for the three salts, the lengths of these ellipsoidal axes being taken as abscissæ and the atomic weights of the alkali metals as ordinates. The dotted curves express the total change in the axial lengths of the ellipsoid, while the continuous curves represent the relative values of the three axes for each salt, when the b -value is taken as unity. So rapid is the convergence that the a and c curves (these letters representing the crystallographic axes, identical in direction in these rhombic salts with the axes of the optical ellipsoid) intersect before the rubidium salt is reached, and the b and c curves do so almost exactly at the rubidium salt, so that the double refraction, represented in the

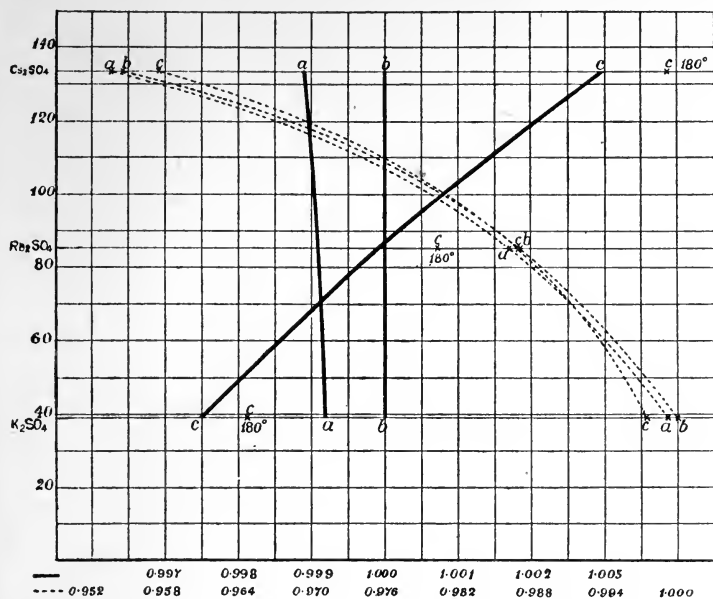


FIG. 645.—Curves of Axial Values of the Fresnel Ellipsoid for the Sulphates of Potassium, Rubidium, and Cesium.

case of potassium sulphate by the distance between b and c , is reduced in rubidium sulphate to the smaller amount indicated by the distance between the a and b curves. As, however, besides very small double refraction we have also the condition of nearness of two of the refractive indices, those corresponding to the directions of the crystallographic axes b and c , the conditions are ideal for the exhibition of large dispersion of the optic axes even at the ordinary temperature, and for the possibility of crossing of the optic axial planes at a somewhat elevated temperature. On heating the section-plate perpendicular to the first median line, rise of temperature is observed fortunately to act in the right direction, in the way of further reduction of the optic axial angle, so that at 40° the uniaxial figure is produced for red lithium light, and at succeeding

higher temperatures for the other wave-lengths in their order, until it is finally formed at 65° for G-violet light.

Cæsium selenate, Cs_2SeO_4 , the third member of the analogous group belonging to the same rhombic series of eutropically isomorphous selenates, behaves similarly, and the curves for this group exhibiting the fact are shown in Fig. 646. In this case, for the sake of variety, the relative values of the axes of the optical indicatrix are portrayed, which represent the refractive indices directly. The reduction of the double refraction according to the law in this group only results in equality of two of the indices being reached about the position of the cæsium salt, the c -value (γ index) coming to equality with the a -value (β index) just before arrival at cæsium selenate, the two curves crossing, and the

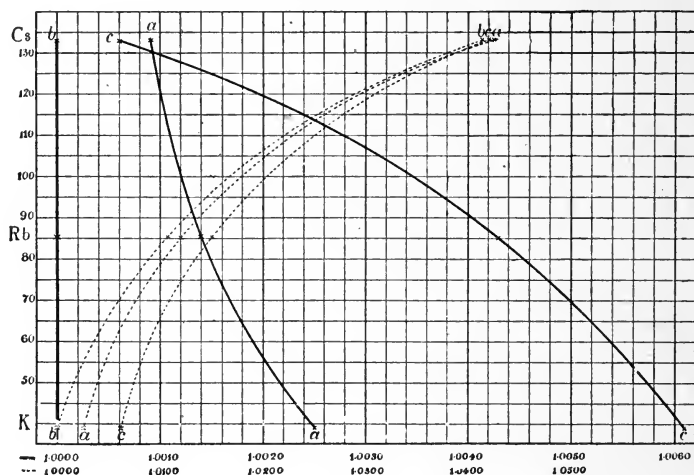


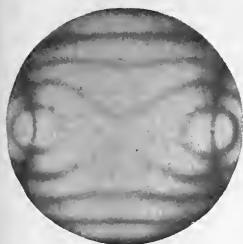
FIG. 646.—Curves of Axial Values of the Optical Indicatrix for the Selenates of Potassium, Rubidium, and Cæsium.

c -value being brought inside the a and b values as the now intermediate index β , while the a -value corresponds henceforth to γ .

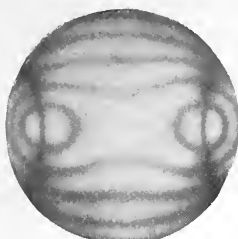
The double refraction, represented by the distance between a and b , is now enormously less than it was in the potassium salt, represented by the distances between b and c for that salt, and so one of the conditions for large dispersion of the optic axes is brought about.

The first effect of heat on the section-plate perpendicular to the first median line, the axis b , brings the c -curve still further inside the a -curve, so that it passes the intermediate position between a and b , the optic axial angle increasing to 90° , and the optic axial brushes subsequently passing out of the field of view; the c -curve then approaches the b -curve, with diminution of the optic axial angle about a new acute bisectrix, the axis a , which had formerly been the second median line, the double refraction having also changed from negative to

SERIES 1. ORDINARY TEMPERATURE.



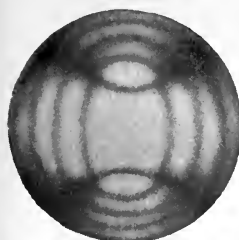
For Li-light.



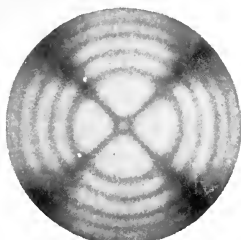
For Na-light.



For Ti-light.



For F-light.



For Wave-length 466.

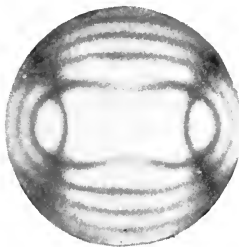


For G-light.

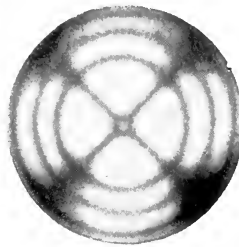
SERIES 2. TEMPERATURE 78°.



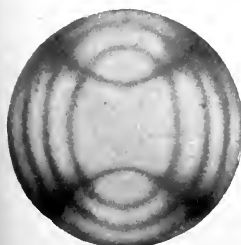
For Li-light.



For C-light.



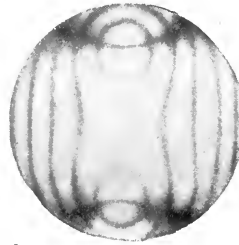
For Na-light.



For Ti-light.



For F-light.



For Wave-length 466.

FIG. 647.—INTERFERENCE FIGURES OF CÆSIUM MAGNESIUM SELENATE.



positive; this occurs so rapidly, moreover, that if a second section-plate perpendicular to this new acute bisectrix a , be adjusted on the polariscopical goniometer and similarly heated, it is observed that at 92° to 98° for the different wave-lengths of light from red to blue the c -curve actually intersects the b -curve, the uniaxial cross and rings being produced as this occurs for these different wave-lengths in succession. Continuing the heating,* the optic axes then again bifurcate, but along a new diameter at right angles to the old one, corresponding to the crossing of the optic axial planes, and they separate more and more until they disappear out of the field of the polariscope. If now a plate be ground perpendicular to a third bisectrix, the axis c , and heated to 150° , the optic axial rings become visible on opposite sides of the extreme edge of the field, and approach each other about this new bisectrix, until at 250° , when the heating has to cease for obvious reasons, the angle has become less than 90° and another change of sign of double refraction has occurred, from positive back again to negative, as it is at the ordinary temperature, and the angle is still becoming less moment by moment. Thus this unique substance has for its first median line within the comparatively short range of 250° of temperature every one of the three axes of the optical ellipsoid in turn.

The double sulphate and double selenate of caesium and magnesium afford two cases when the uniaxial figure is produced at the ordinary temperature, for a wave-length in the blue, 0.000450 and 0.000466 millimetre in the two respective cases. The interference figures of caesium magnesium selenate, for six wave-lengths at the ordinary temperature and at 78° , are illustrated in the photographic reproductions given in Fig. 647, Plate III. It will be clear that the optic axes are well separated in the horizontal plane at the ordinary temperature for red lithium light, and that they approach each other more and more as the wave-length is diminished, through Na-light, Tl-light, and F-light, until for the critical wave-length 0.000466 the uniaxial cross and circular rings are formed. Subsequently, as the violet is approached, the optic axes separate again, more and more, but along the vertical diameter, as illustrated by the figure for G-light. On heating the section-plate the uniaxial figure travels through the spectrum towards the red for successive higher temperatures, and the other six figures represent the conditions at 78° , for which the crossing of the axial planes occurs for sodium light, the uniaxial figure being produced; the optic axes are then separated in the horizontal plane for wave-lengths on the red side, and in the vertical plane for wave-lengths on the blue side. By the time the temperature of 97° is reached, the optic axes for all wave-lengths are separated in the vertical plane, at a small angle for red, and at greater and greater angles towards the blue.

The illustrations are actual photographs of the figures observed. The section-plate requires to be half a centimetre thick in order to exhibit such sharp figures, an emphatic illustration of the necessary condition of low double refraction for the production of crossed-axial-plane dispersion.

The Case of Gypsum. The Mitscherlich Experiment and Apparatus for its Projection.—It has been shown in the last chapter that at the

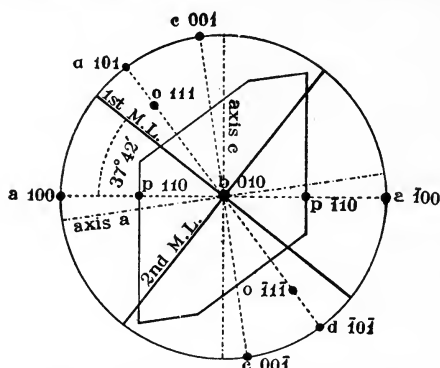


FIG. 648.—Positions of the Bisectrices of the Acute and Obtuse Optic Axial Angles of Gypsum.

ordinary temperature the optic axes of gypsum, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$, lie in the symmetry plane of the monoclinic crystals, the inclined dispersion being strongly marked. The first median line lies in the obtuse angle of the axes a and c , and is inclined $37^\circ 42'$ to the normal to $a = (100)$ and $46^\circ 40'$ to the axis a ; also $52^\circ 18'$ to the axis c and $43^\circ 20'$ to the normal to $c = (001)$, as clearly shown in Fig. 648.

The true optic axial angle $2V_a$ at 10° , and the apparent angle in air $2E$ at 11.5° , are given for seven wave-lengths in the following table :

VALUES OF TRUE AND APPARENT OPTIC AXIAL ANGLES OF GYPSUM.

	$2V_a$ at 10° .	$2E$ at 11.5° .
For Li-light	$60^\circ 27'$	$99^\circ 16'$
" C "	$60 31$	$99 27$
" Na "	$61 1$	$100 36$
For wave-length 0.000573	$61 4$	$100 43$
For Tl-light	$60 51$	$100 34$
" F "	$60 34$	$99 58$
" G "	$59 48$	$98 24$

One of the most remarkable things about gypsum is that the optic axial angle does not go on steadily increasing or diminishing with the regular progression of wave-length, but that when a certain wave-length in the greenish yellow is reached, 0.000573 millimetre, the progress is arrested and the angle begins to alter in the opposite sense, in other words, the optic axial angle has a maximum for wave-length 0.000573.

On heating the section-plate the optic axes move towards one another and the centre of the field, and at temperatures varying in different crystals from 105° to 115° coalesce in the centre to produce the uniaxial cross and circles. The following table shows the exact corrected temperatures at which this occurred in the same crystal of gypsum as gave the above values for $2V_a$ and $2E$.

For G-light cross and rings produced at 102.5°				
"	F	"	"	104.5
"	Tl	"	"	105.1
Last wave-length (0.000573) for which				
uniaxial figure is produced				105.5
For Na-light cross and rings produced at 105.2°				
"	C	"	"	104.7
"	Li	"	"	104.5

After the production of the cross for the particular wave-length of the illuminating light the axes separate along that diameter of the field which is perpendicular to the original direction of separation, the plane of the axes now changing to one at right angles to the symmetry plane. Between the ordinary temperature and 95° also, as shown in the last chapter, the first median line itself moves 5½° in the symmetry plane, towards the axis *c*. These phenomena of gypsum were demonstrated by Mitscherlich to the Berlin Academy in the year 1826, and their exhibition has since been referred to as the "Mitscherlich experiment." The usual mode of performing this experiment with the lantern polariscope has hitherto been to place a somewhat large section-plate of gypsum, cut perpendicularly to the acute bisectrix of the optic axial angle, in a metal frame having a projecting part which can be heated by a spirit lamp or small Bunsen flame. But the author has demonstrated to the Royal Society,¹ and also to the British Association at the 1909 meeting at Winnipeg,² a much more elegant mode of performing this beautiful experiment, without any extraneous heating whatsoever. The improved form of lantern polariscope was employed which, as regards its arrangement for parallel light projection, has already been described in Chapter XXXIX. and illustrated in Fig. 515, page 609. Its arrangement for projections in convergent light, including the Mitscherlich experiment, is shown in Fig. 649. The convergent lens-system is such as concentrates the light rays so perfectly in the centre of the crystal, on a spot not exceeding a couple of millimetres in diameter, that the heat rays simultaneously thus also concentrated are quite adequate to effect the crossing of the optic axial plane. This is true, moreover, even although the greater proportion of the heat rays are removed by the water cell, which is always essential in order to protect the balsam of the very valuable large Nicol prisms from softening. There is no necessity for the crystal to be much larger than this focal spot, so the author employs a plate, any one of several prepared by Messrs. Steeg and Reuter of Homburg, which is only 6 millimetres square, but 2 millimetres thick in order to ensure a sharp interference figure (the double refraction being so low), mounted in a miniature blackened thin sheet-brass or better platinum-foil carrier-frame with circular aperture 3 millimetres in diameter on each side, through which the light enters from the convergent system of lenses and leaves to enter the similar collecting system. The little frame itself, only 7 by 6 by 3 millimetres, has a

¹ *Proc. Roy. Soc., A*, 1908, *81*, 41.

² *Brit. Assoc. Reports*, Winnipeg Meeting 1909, First Evening Discourse.

lip-like continuation on one side, which serves for gripping by the crystal-holder ; it is held directly by a hard-wood holder, to prevent loss of heat

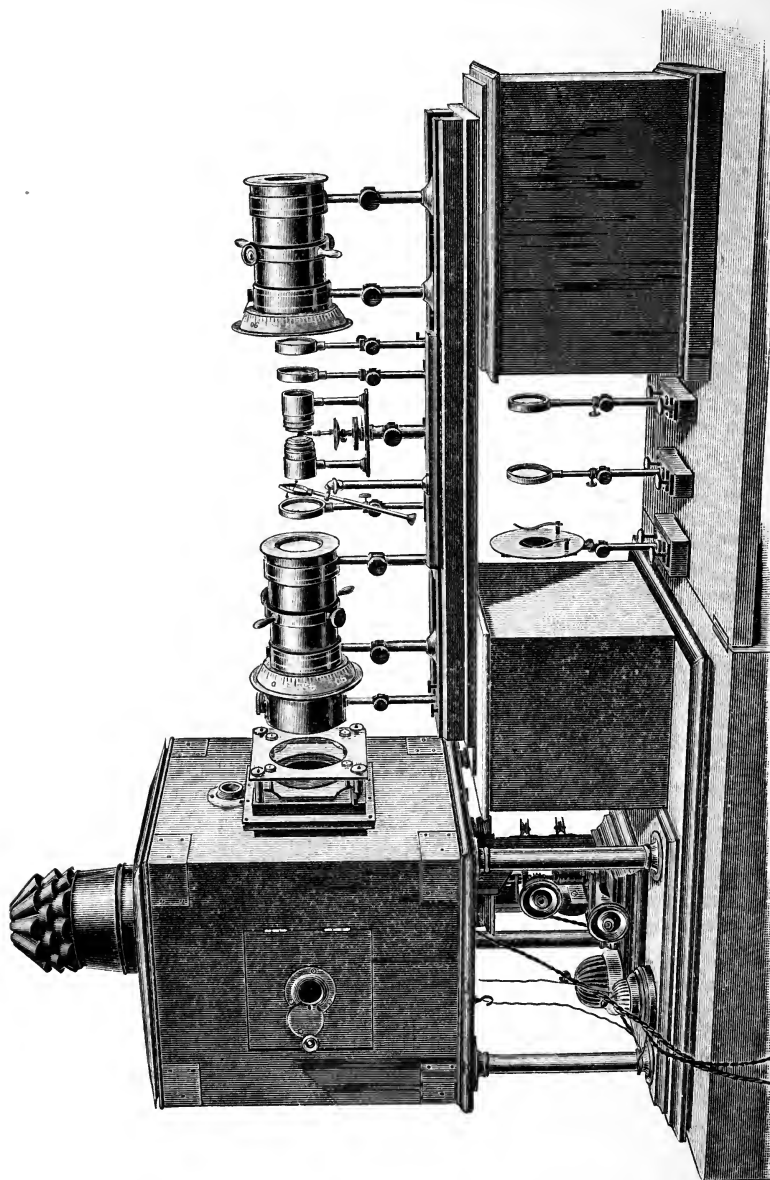


FIG. 640.—Projection Polariscopes arranged for the Mitscherlich Experiment and other Projections in Convergent Polarised Light.

by conduction, and which in turn is gripped by the ordinary metallic pincette of the crystal-adjusting apparatus.

The disposition of the apparatus for the performance of the Mitscherlich experiment, is that shown in Fig. 649. The electric lantern is provided with a $4\frac{1}{2}$ -inch double condenser, and an adjusting table for the electric arc are furnished with three rectangular adjusting movements, which are particularly useful, indeed, imperative in this experiment; the electric lamp is a Brockie-Pell self-feeding one, but an improved Oliver lamp does equally well. The water-cell of 2 inches thickness may be supported in front of the lantern-slide frame, as in Fig. 515, or, as now shown in Fig. 649, separately on an adjustable holder with semicircular stirrup, as the first of the fittings on the hard mahogany guiding bed on which the various accessories are mounted. After trying both this and the lathe-bed method of mounting the parts of the apparatus the author now prefers the hard mahogany bed fitted with rigid and very true rabbets, which latter can be cut away for about 4 inches to the right of the centre near the analyser in order to afford a place for the removal or interchange of the various supporting stands carrying lenses or other fittings without having to remove the large Nicols. All the bases of the supports (seen better in the case of the parallel-light fittings also shown, in the recess underneath, in Fig. 649), whether as narrow as $1\frac{1}{2}$ inch or broader, are partially cut away underneath and packed with smooth felt to facilitate sliding, and are also fitted with a couple of fixing screws, passing through the little fixed transverse dovetailed brass guiding bed for the bevelled slider which directly carries the column, and which enables the transverse adjustment to be carried out.

After the water-cell comes the large polarising Nicol, already fully described in Chapter XXXIX., with its convexo-concave parallelising lens at the end nearest the water-cell, the light being so converged upon this lens by adjustment of the distance of the electric arc from the condenser of the lantern that the rays leaving the lens are rendered strictly parallel, and pass as such through the polariser. Besides the main plinth guiding bed, and fixed along the middle part of its front rabbet, there is a subsidiary metallic dovetailed bed, in which moves a bevelled slider carrying a columnar tube with inner rod adjustable for length, and terminating in a ball-and-socket universal joint, which carries at its summit a very convenient clip-holder for the cork or other mounts of section-plates of crystals or other objects which it is desired to introduce into the path of the rays. In the Mitscherlich experiment illustrated in Fig. 649 it is not used, being rotated away in front from the path of the light rays.

The convergent light arrangements after the polariser are as follows. First comes an independently mounted 5- or 6-inch plano-convex lens (two alternative ones being provided) to commence the convergence of the light rays. Then follows the optic-axial-angle goniometer, with its two exactly similar but inverted systems each of four plano-convex lenses of very short focus, of which only three are usually employed, the outer smallest one in each case being only attached when extremely strong convergence is required. The largest of the battery of lenses, all the individuals of which are separately mounted and readily detachable by unscrewing the mounts, in each case is screwed into a short tube, which slides in an outer one supported by a column rising from an elongated metallic plate. This base-plate bearing the two columns is pierced at the centre by, and rigidly attached to and carried by, a stout tubular axis, adjustable for height as an inner tube within a strong outer column rising from a transverse slider in a dovetailed metallic bed, screwed on to a mahogany slider in the main guiding bed similar to the others but much broader, having more to carry. The two convergent systems are thus supported by the two columns rising from the plate in such a manner that each can be independently approached to or drawn away from the central space between them, where the crystal-plate is held at the head of an adjusting and centring arrangement of the kind provided with the von Groth universal apparatus (Fig. 635, page 774). The central boss of this adjustable crystal-holder is not mounted directly at the head of the tubular sliding column, but to a second inner sliding and rotating short tube, which carries a radial pointer forming

an indicator of the rotation of the crystal-axis about a silver divided circle fixed above the elongated cross-plate and concentrically around the tubular support. The central axis of the crystal-adjusting apparatus passes freely down through both these supporting tubes and the still narrower tube of the adjusting apparatus, and is adjustable for height so as to bring the crystal absolutely to the common focus of the convergent systems. Transverse and longitudinal adjustments, and a universal circular motion are also afforded by the centring and adjusting apparatus so that the crystal is adjustable in all directions, and the whole arrangement is an absolutely symmetrical one.

After this very convenient form of projection optic-axial-angle goniometer comes a doubly convex field lens on its adjustable pedestal, and this is closely followed by an achromatic lens combination likewise independently mounted, which acts as a most efficient projecting lens for the focussing of the interference figures in convergent polarised light on the screen. Finally comes the large analysing Nicol. As now described this arrangement is of general application for the projection of the rings and brushes of crystals, and for any other purposes for which convergent polarised light is required.

To perform the Mitscherlich experiment, the little crystal in its miniature platinum-foil frame is supported in the wooden holder, which grips the lip of the frame, and the wooden holder is held in turn by the ordinary pincette of the adjusting apparatus, and the crystal adjusted to the centre of the space between the convergent systems in such a manner that the clear aperture of the crystal is at the approximate common focus of the latter. The planes of the Nicols are arranged at 45° left and right respectively of the vertical zero line, being thus crossed for the production of the dark field. The electric arc is then switched on, preferably by a switch on the lantern itself, and if the lenses and the crystal are correctly adjusted the focal spot of light will be seen to be impinged right on the centre of the 3 millimetres of clear aperture of the crystal, which is thus brilliantly illuminated. On regarding the screen, if the crystal-plate has been arranged correctly as regards its orientation, which will have been attended to in mounting in the platinum frame, the two systems of optic axial rings will be observed almost at once to appear at the right and left margins of the field, and to be visibly moving towards the centre. A little centring of the light will now generally be necessary by means of the large milled heads of the three rectangular movements of the adjusting table of the electric lamp, in order to ensure even, well-centred, and brilliant illumination of the field. The axial brushes themselves soon become visible, and march rapidly with the rings towards the centre of the field, where brilliant colours begin to break out, constantly changing and giving place to others, as first looped spectrum-lemniscates and subsequently ellipse-like ones appear round the brushes. The last complete ring round each of the two brushes finally opens out, the two rings becoming first a loop lemniscate, then an apparent ellipse enveloping both brush-vertices, and lastly a true circle, surrounded by five or six others, all in spectrum colours, and the brushes themselves coalesce to form a black uniaxial St. Andrew's cross.

In order to effect the changes steadily and without hesitation the light must be kept well centred, so that the focal spot is concentrated always on the crystal. The cross and circular rings soon break up

again into hyperbolæ and lemniscates, at first of the ellipse-like character but afterwards forming loops and then complete rings round the two separated axes; the vertices of the hyperbolæ are now, however, separated along the vertical diameter of the field, and when they are well asunder the experiment should be stopped, as gypsum loses its water of crystallisation slightly above 120° . It is only necessary to blow a gentle stream of air across the crystal, however, or to introduce a glass plate a little obliquely somewhere in the optical train between the lantern and the crystal, to cause the figure to recede again to the crossing point and beyond it, owing to the reduction of the temperature of the crystal by the cool air or the deflection of the focussed rays. Or an opaque screen may be intermittently introduced into their path before reaching the crystal, to effect the same purpose.

Experimental Confirmation of Conditions for Crossed-Axial-Plane Dispersion.—In order to confirm the truth of the conditions for crossed-axial-plane dispersion laid down at the beginning of this chapter, the author has determined the refractive indices of all the substances referred to which exhibit the property, at not only the ordinary but the higher temperatures, and the phenomena observed have in all cases been both highly interesting and exactly in accordance with the conditions stated. In every instance when the 60° -prism affording the two indices of refraction which are supposed to become identical—at the temperature for which the section-plate perpendicular to the first median line shows the uniaxial figure—was heated to that temperature, the two large and brilliantly illuminated coloured images of the Websky signal-slit in the monochromatic light employed were seen steadily to approach one another, until at the exact temperature corresponding to the production of the cross and circular rings for that same wave-length of light which was illuminating the slit, the two images appeared identical; after this, when the temperature was still increased, or the wave-length was varied in the right direction by rotation of the prism circle of the monochromatic illuminator, they separated again on the other side of each other. When apparently identical, they were seen to be still two images, but overlapping; for no apparent extinction occurred when the Nicol prism carried in front of the eyepiece was rotated, one image waxing equally as the other waned (modified slightly by the fact that the 90° image, corresponding to light vibrating parallel to the refracting edge of the prism, is always slightly the weaker, owing to such light as is polarised and lost by reflection vibrating parallel also to the edge). The moment they were just separated again one was extinguished by the Nicol placed at its 0° -position, while the other was extinguished when the Nicol was arranged at its 90° -position.

In the case of gypsum, in order to avoid any error due to slight differences between different crystals, two section-plates (B and E in Fig. 650) perpendicular to the first and second median lines respectively, with which measurements of optic axial angle were made, and a complementary pair of 60° -prisms (C and F in the figure) affording all three refractive indices between them, were cut and polished from a single

large and highly perfect crystal, one complete set of optical constants being thus derived from one and the same crystal. The plan on which the crystal was cut will be clear from Fig. 650, the crystal lying on a clino-pinakoid face $b = \{010\}$, parallel to the symmetry plane. The prism C afforded the indices β and γ , while F gave α and β . It was with this latter prism that the observation of the approach to equality of

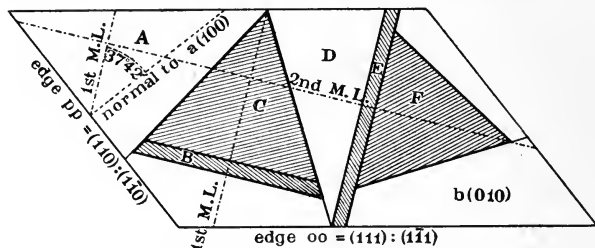


FIG. 650.—Plan on which Complementary Pairs of Plates and Prisms were cut out of the same Crystal of Gypsum, in order to afford all the Optical Constants.

two of the indices, in the case of gypsum α and β , with rise of temperature, was made, and their subsequent attainment of identity and then their passage past each other, followed, all of which occurred precisely in accordance with expectation derived from the optic axial angle phenomena afforded by the plate B perpendicular to the first median line.

Thus the conditions specified in the opening of this chapter as necessary for the production of crossed-axial-plane dispersion of the optic axes have been absolutely verified experimentally.

CHAPTER XLIX

ROTATION OF THE PLANE OF POLARISATION BY CRYSTALS.

THE phenomenon of optical activity, or rotation of the plane of polarisation of light, is not confined to the well-known liquids which display it, such as solutions of sugar and of many organic substances, but is likewise exhibited by many crystals belonging to the eleven enantiomorphous classes possessing lower degrees of symmetry (no plane of symmetry) than the full symmetry of their system, as explained in Chapter X. (pages 138 and 139). Indeed the phenomenon itself was first observed in rock crystal, quartz, dioxide of silicon, SiO_2 , so long ago as the year 1811, by Arago. Owing to the nature of the inner structure, which is often of a screw or spiral character as in the case of quartz, the two rays propagating themselves through the crystal, instead of vibrating in definite planes, are circularly polarised as in the quarter-wave mica plate. On reaching the second surface of the crystal plate, one is retarded behind the other by reason of the difference in velocity, and possibly also because of difference in length of path, one moving more obliquely than the other. Two such circularly polarised rays, on attaining the second surface, will thus in general be in different phases. On emerging into the air each of these two circular motions, right-handed and left-handed, may be represented by a pair of forces, one radial and the other tangential. The two tangential forces are in opposite directions and destroy each other, but the radial forces are each directed towards the centre of the original circular movements, and so compound on leaving the crystal into a rectilinear vibration which is rotated from the original direction by an amount (half the difference of phase on emergence) depending on the retardation of one of the original circular motions behind the other. The amount of rotation is a minimum for rays corresponding to the red end of the spectrum, and a maximum for the violet end. Biot, who investigated the relationship between the colour of the light and the angle of rotation, found that it is approximately inversely proportional to the square of the wave-length.

Hence, not only is the direction of vibration of the light rotated by such crystals, but the rays are also dispersed by them; for the differently coloured rays, after emergence, will be vibrating in planes differently inclined to the original plane of polarisation of the polarising

Nicol prism, and the analysing Nicol will require to be rotated to different extents for the different colours, in order to restore the dark field. Or, if colour be produced, the complementary colour to that extinguished will be displayed at each position of the analyser. For a given thickness of quartz, for instance, cut perpendicularly to the axis, along which the property is exhibited at its maximum, the order of the colours is inverted when the direction of rotation of the analyser is reversed.

The following general formula has been shown by Boltzmann to express more accurately than Biot's approximation the relationship between ρ , the angle of rotation for a plate one millimetre thick, and λ , the wave-length in air of the light employed :

$$\rho = \frac{A}{\lambda^2} + \frac{B}{\lambda^4} + \dots$$

The amount of rotation also depends directly on the thickness of the crystal-plate, a plate twice as thick as another affording double the angle of rotation for light of the same wave-length, and the interference colours afforded under crossed Nicols in white light become less brilliant as the thickness increases beyond that corresponding to colours of the first two or three orders of Newton, passing into white of the higher orders. It takes a much greater thickness to bring about this result, however, than in the case of interference owing to double refraction. The amount of rotation also obviously depends on the orientation of the ray within the crystal, becoming less as the orientation departs from that for the maximum effect, which in the case of quartz and other uniaxial substances is the direction of the optic axis.

The interesting fact was pointed out in Chapter XXIII., and illustrated in Figs. 344 and 345 (page 357), that right- and left-handed crystals of quartz are distinguished by the appearance on the right and left respectively of the subsidiary faces of the forms $s = \{41\bar{2}\}$ and $x = \{4\bar{1}2\}$, the symmetry of quartz being that of the enantiomorphous trapezohedral class (class 18) of the trigonal system.

Fresnel's Prism.—That the above explanation of optical rotation is correct is confirmed by the result of an interesting experiment carried

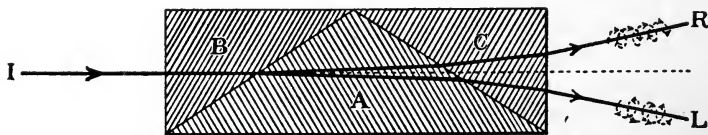


FIG. 651.—Fresnel's Compound Quartz Prism, and Isolation by its Means of two oppositely Circularly Polarised Rays.

out by Fresnel, who constructed a compound prism of the two varieties of quartz, of the kind shown in Fig. 651, by which he was actually enabled to separate the two oppositely rotating rays. It was composed of an

obtuse-angled prism of right-handed quartz A to which were cemented, one on each side, the two halves B and C of another prism of left-handed quartz, the whole forming a rectangular block. All were cut and arranged so that the rays of light traversed them along the direction of the optic axis, indicated by the dotted line. The normally incident ray I becomes divided into two oppositely circularly vibrating ones in the semi-prism B, which on reaching the first surface of the prism A are refracted in opposite ways, as the more slowly circularly vibrating ray in B becomes the faster one in A, the divergence being further accentuated on reaching the second surface of A, where a further reversal occurs. The two rays which emerge, R and L, are thus vibrating in circular orbits in opposite directions, and afford, on looking through the prism, two images of an illuminated spot or slit of light or other object employed as source of light, which show no variation in brightness on rotation of an analysing Nicol held between the prism and the eye, the test for circular polarisation. Moreover, when a quarter-wave mica plate is introduced one of the images disappears, namely, that one which was formed by light vibrating in a circular orbit opposite to that permitted by the mica, thus proving by their diverse behaviour to this test the opposite nature of the circular polarisation of the two rays.

Von Lang has also since shown that a single 60° -prism of quartz, cut so that the bisecting plane is perpendicular to the optic axis, affords in the same way, at minimum deviation two oppositely circularly vibrating rays, about half a minute apart. The specific rotation ρ , according to Pockels, is related to the velocities of the two circularly polarised rays O_r and O_l in the following manner :

$$\rho = \frac{\pi}{\lambda} \left(\frac{1}{O_r} - \frac{1}{O_l} \right).$$

Interference Colours displayed by Plates of Optically Active Crystals, especially Quartz.—Calcite and quartz, as we have seen in Chapter XXXVIII, are both uniaxial crystals, and a plate of calcite cut perpendicularly to the optic axis shows neither double refraction nor polarisation effect in parallel polarised light, remaining quite dark under crossed Nicols. On the other hand a plate of quartz normal to the optic axis, even one seven or eight millimetres thick, shows bright colour in the dark field, truly even all over when the plate is furnished with a pair of surfaces which are truly plane and parallel. Moreover, the colour changes and passes through all the tints of the spectrum as the Nicol analyser is rotated, and never becomes either colourless or dark. When the light is homogeneous, monochromatic, extinction is produced for a specific position of the analyser, corresponding to the amount of rotation of the plane of polarisation by a plate of that thickness for the wave-length employed. The angle of rotation from the crossed position becomes greater as the violet end of the spectrum is approached, and the direction is right or left of the zero position of the crossed Nicols (at which more or less light is

transmitted) according as the plate belongs to a right- or a left-handed crystal. Plates of quartz cut obliquely to the optic axis also exhibit rotation, but the phenomena are complicated by the introduction of double refraction, and cannot be studied alone. Eventually when the obliquity becomes considerable, the circular polarisation, already become elliptical on leaving the direction of the optic axis, passes into two ordinary plane vibrations, and thus a plate parallel to the optic axis shows no rotation, but only ordinary double refraction.

The colour exhibited in white light by such a plate of an optically active substance has already been shown to be more or less complementary to that extinguished. The colour varies, as will be clear from the preceding statement, with the position of the analyser with respect to the polariser, the variation being greatest when the colour extinguished is that of greatest intensity in the spectrum, namely, yellow in the neighbourhood of wave-length 0.000550. In the case of quartz

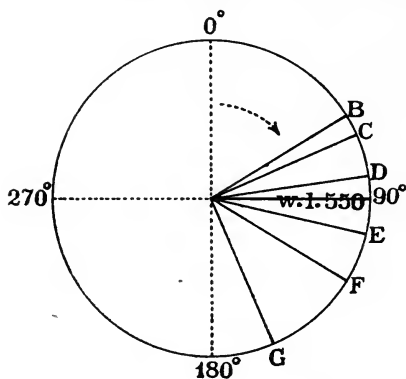


FIG. 652.—Angles of Rotation for Right-handed Quartz Plate 3.75 mm. thick.

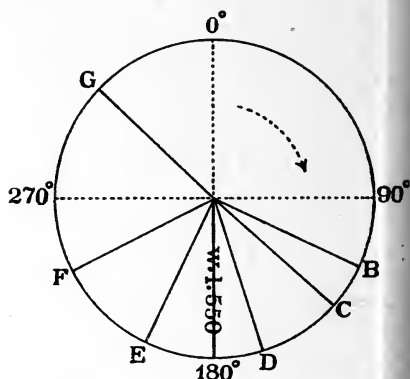


FIG. 653.—Angles of Rotation for Right-handed Quartz Plate 7.5 mm. thick.

under crossed Nicols this occurs for a plate perpendicular to the axis of 7.5 mm. thickness, and with parallel Nicols for one 3.75 mm. thick. For a plate of 7.5 mm. rotates this yellow ray for 180° , while one of 3.75 mm. thickness rotates it for 90° . This is graphically illustrated in the two diagrams Figs. 652 and 653, which give with strong lines the rotation of rays of wave-lengths corresponding to those of the Fraunhofer lines B, C, D, E, F, and G, and to wave-length 0.000550, by right-handed plates of these two thicknesses. The interference colour actually observed when wave-length 0.000550 is extinguished is the transition violet between the first and second orders of spectra in the cases of both these plates. That is, a plate of quartz 3.75 mm. thick gives the transition tint when the Nicols are parallel, while one of 7.5 mm. thickness gives it when the Nicols are crossed, a rotation of 180° having the same effect as no rotation at all, as it merely brings about the crossing again. A slight rotation of the analyser from the position for the violet transition tint, to the right (clockwise) or left

(anti-clockwise), according as the crystal is right-handed or left-handed, causes the colour to change to red (first order). On the other hand, a rotation of the analyser contrary to the rotatory character of the plate causes the violet transition tint to change to blue or green (second order).

The colours afforded by plates thicker than 7.5 mm. are less brilliant; for when the angle of rotation for red becomes a multiple of 180° , other colours than that may also be rotated for still higher multiples of 180° , and be simultaneously extinguished by the analyser. A plate 10 mm. thick rotates the bright red near C for 180° and a plate of 20 mm. for 360° , two semicircles. A 20-mm. plate, however, also rotates the yellowish green for three semicircles, and the indigo blue for four semicircles, so all three colours are extinguished under crossed Nicols, and white of the higher orders is largely admixed with the colour shown. More colours still are simultaneously extinguished with yet greater thicknesses of plate, and eventually pure white light is afforded. On the other hand, the colour becomes again enfeebled when the thickness is much less than 3.75 mm. A plate 1 mm. thick gives only feeble tints, one of 0.5 mm. feebler still, and one of only 0.1 mm. only gives rotations for red and violet of 1.7 and 4.4 degrees, and under crossed Nicols shows practically no light.

The Biquartz.—A most valuable application of the rule stated at the top of this page is found in the biquartz, a double plate of 3.75 or 7.5 mm. thickness, composed of one semicircular plate of right-handed quartz and another of left-handed quartz, both cut truly perpendicular to the axis; the two normally ground and polished plane bases of the semicircles are cemented together with balsam so as to show only a faint line when the plate is regarded normally, as shown in Fig. 654, the different-handed nature of the rotation of the two halves being indicated in the figure by the shading. Such a composite plate gives the transition violet equally throughout when the Nicols are parallel (for a 3.75-mm. plate) or crossed (for a 7.5-mm. plate), and a slight rotation of the analyser causes one side to become red of the first order and the other blue or green of the second order spectrum. In sodium light such a biquartz appears equally brightly yellow in the two halves, when the analyser is parallel or crossed in the two cases, while a very slight rotation of the analyser is sufficient to effect a clearly perceptible difference of intensity between the two halves.

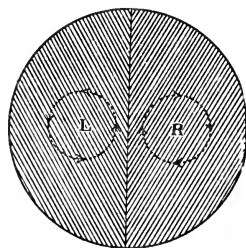


FIG. 654.—The Biquartz.

A biquartz thus affords a means of rendering more exact the adjustment of the analysing Nicol to the crossing position with the polariser, and also for detecting rotation of the plane of polarisation by any other plate or a cell of liquid introduced between it and the polariser, after the latter and the analyser have been set for the transition tint. For obviously any disturbance of the plane of polarisation

will affect the two sides of the biquartz oppositely. The biquartz is consequently much used in saccharimeters, polariscopes for the determination of the optical rotation of liquids, and may be effectively used also for the practical determination of the angle of rotation of an optically active crystal-plate.

Sometimes it is possible to obtain a natural biquartz, cut from a natural twin crystal of right- and left-handed rotatory power and crystallographic development. The author possesses two such natural biquartzes of 3.75 mm. thickness, in which the violet colour produced in both halves when the Nicols are parallel is exquisitely even and beautifully delicate, while the line of junction of the two individual crystals is invisible, except for a very thin white line in one case, and a mere trace of such in the other, the plane of composition of the twin being perpendicular to the plate. On the slightest rotation of the analyser the two halves pass respectively into brilliant red and blue. With crossed Nicols an even yellow is shown, with a thin black line of demarcation between the two semicircles; the yellow passes rapidly on rotation of the analyser into orange on one side and green on the other.

When the plane of the plate is not at right angles to the plane of composition of the twin, but inclined, the effect with parallel Nicols is to produce a broad white band extending for the width of the inclined overlapping of the two halves, and under crossed Nicols a black band, the dark field being produced along this strip where the two opposite rotations neutralise each other. It is similar to the dark band produced in the centre of the field by a rectangular block composed of two superposed wedges of oppositely active quartz, the so-called "biprism" of quartz, described on page 815.

Determination of the Rotatory Power of Optically Active Crystals.

—The direct determination of the angle of rotation of the plane of polarisation by a crystal plate is conveniently carried out with the aid of a special fitting to the polariscope of the von Groth Universal Apparatus already referred to in Chapter XLVII., and illustrated in Fig. 635 (page 774), in connection with its use in optic-axial-angle determinations. The apparatus as arranged for the determination of optical rotation is shown in Fig. 655, together with the monochromatic illuminator of Chapter XLIII. used, however, in this case as a spectroscope for the analysis of the coloured light transmitted by the biquartz, and localisation of the extinguished wave-length, as will be fully described in a later section of this chapter (page 808).

The system of converging lenses *a*, *b*, *c*, *d* (Fig. 635) is removed from the polarising tube *i*, so as to leave it arranged as for parallel light instead of for convergent light. Further, the whole analysing tube *t* is removed (it is shown standing up on the spectroscope base-board in Fig. 655), and replaced by a special rotatable cap *e*, fitting over the circle plate *m* (Fig. 635) which carries an engraved vernier, and which has been permanently attached to the circular widening *g* of the front column *b* (*f* in Fig. 635) for the express purpose of this measurement. The cap is provided with a bevelled silver divided circle, for use with the vernier engraved on *m*. The front of the cap carries

a short tube θ for the reception of the Nicol analyser α , and the whole cap is then rotated until the dark field is produced with the large polarising Nicol in the polarising tube i . It is sometimes an advantage to place in front of the analysing Nicol the small-aperture cap with its little lens, shown at κ leaning up against the tube t in Fig. 655. The crystal-plate is suspended from the crystal-holder δ , or is held in a special pincette of platinum foil held in turn by the ordinary one δ , and it requires to be first adjusted, perpendicular to the optical axis of the polarising and analysing tubes, with the aid of the adjusting and centring movements γ and β .

This adjustment has to be effected absolutely, as otherwise the rays will traverse a greater thickness of the crystal than is represented by its measured thickness. To enable it to be accurately achieved, the cap and the analysing Nicol are temporarily withdrawn, and replaced by the ordinary analysing tube t ; but instead of the analysing Nicol, which is left attached to the removed cap, there is inserted a so-called Gauss's mirror, a short open tube carrying a plate of truly plane-parallel glass, rotatable on an axis diametral to the tube, the rotation being effected by a little bent lever the handle

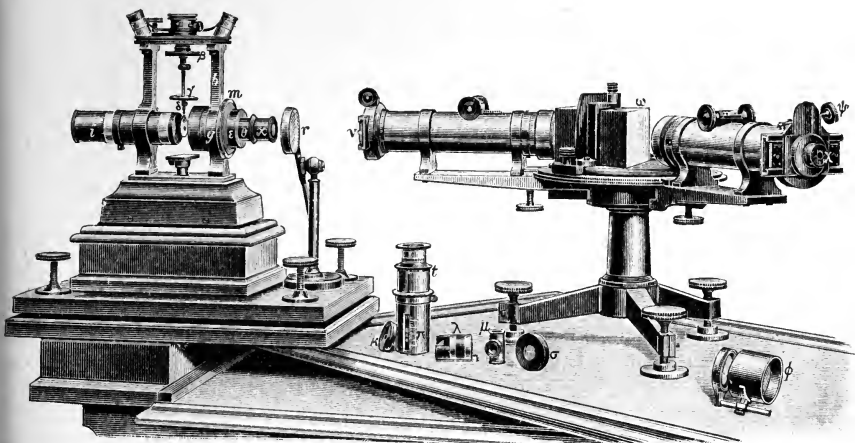


FIG. 655.—Determination of Angle of Rotation of Plane of Polarisation by an Optically Active Crystal.

of which projects outside in front. It is shown at λ in front of the apparatus, and separated from t for the sake of clearness, in Fig. 655. Or the Becker fitting described on p. 34 and also shown in Fig. 13, and in front at μ in Fig. 655, may be adapted instead. The rays from a goniometer or table lamp placed to one side of the instrument are admitted through a slot in the tube t , and are reflected from the glass-plate mirror normally to the crystal, and thence back again, passing straight through the glass plate to the observer's eye. In order to serve as a signal, the glass plate bearing an engraved micrometer scale and also a vertical diametral line at right angles to the horizontal one of the scale, which is given by Fuess with the Universal Apparatus for use in approximate estimations of optic axial angles, should be placed in position at π in the tube t , if not already there; the reflected image of the cross-lines and scale is then made, by adjustment of the crystal-plate (reflecting the image) exactly normal to the axis of the tube t , to coincide with the actual cross-lines and scale themselves, as seen directly.

After this important adjustment has been carried out, the tube t is removed and the cap and Nicol reinstated. The determination of rotation can then be carried out

either (1) without or (2) with the aid of a biquartz. The former simpler case will first be considered.

(1) The Nicols being accurately crossed for production of maximum darkness, the crystal-plate will appear coloured in white light, and bright in monochromatic light with the colour corresponding to the wave-length employed. Using either sodium light or its equivalent furnished by the monochromatic illuminator as one of the wave-lengths for which a determination is to be made, in the manner shown in Fig. 636 (p. 775), it will be found to be necessary to rotate the analysing Nicol, together with the large divided circle-cap which records the amount of the rotation much more finely than the circle of the Nicol itself, for some degrees on either one side or the other in order to restore the dark field. By approaching this position from the original crossed position and taking a reading, and then going too far and returning to the position of extinction and taking a second reading, and afterwards taking the mean of the two readings, or better still the mean of several such pairs of readings, a close approximation to the extinction position will be obtained. The angle of rotation required is the difference between this mean reading for extinction and the reading for the crossed position of the Nicols.

(2) The determination of the extinction position can be more delicately achieved with the aid of a biquartz of either 7.5 or 3.75 millimetres thickness. The biquartz is inserted in another cap, which acts also as a mount and fits over the further end of the tube projecting through the collar *g*, nearest the crystal. Its fine diametral line of cementation if it be an artificial biquartz (such a biquartz is shown in front at σ), or line of composition if a natural twin (a natural biquartz being in position in the cap), is arranged vertically (except when a spectroscope is to be used to analyse the colours, when it is arranged horizontally, as will be described in the next section), and this line serves excellently for focussing purposes when the small-lens cap supplied with the apparatus is used in front of the Nicol in order to restrict the eye-opening and enlarge the field. The junction-line may also be utilised as the line of reference instead of the micrometer scale and cross-lines if desired or convenient. The crystal-plate should be raised out of the field while the biquartz is adjusted to afford absolutely identical transition tints in the two halves, the Nicols being parallel if the biquartz be of 3.75 mm. thickness, or crossed if it be a 7.5-mm. plate. Then on lowering the crystal into position again, after confirming that the normal adjustment of the plate has not been impaired, by using the Gauss or Becker fitting and the dividing line of the two halves of the biquartz itself as reference line, these two halves will be observed to have taken up different tints, due to the rotation of the plane of polarisation by the crystal.

The analysing Nicol is then rotated through the angle of this rotation, to the position at which the transition tint is equally restored in the two halves of the biquartz. The transition violet should be approached from the two different sides several times, equal in number, and the mean of the readings taken as the true position; the difference of this from the reading for the original transition tint when no crystal was in the field is the angle ρ of rotation required.

This value, however, will only be the mean value of the rotation for the middle of the spectrum, and if the dispersion of the rotation is considerable for different wave-lengths, it can only be regarded as an approximate value. Hence, in truly accurate work monochromatic light is essential, and the observations should at least be made for sodium light, equality of intensity of illumination of the two halves of the biquartz being accepted as indicative of the true position of the rotated plane of polarisation. To render the observations complete, however, they should be made for a series of wave-lengths, and when the spectroscopic monochromatic illuminator described in Chapter XLIII. is available (arranged as in Fig. 636) the same six wave-lengths as have been recommended in all the other optical work may conveniently be employed, namely, those corresponding to L-light, C-light, Na-light, Tl-light, F-light, and violet hydrogen light near G of the spectrum.

There are many other devices which may be employed instead of a biquartz, to enhance the sensibility of the determination of the rotation angle. A Laurent half-shadow plate may be used as in the saccharimeter of that name, in which case monochromatic light is essential, sodium light giving very sharp results. The Laurent plate is a plate of glass of which one half is covered with a half-wave mica, selenite (gypsum), or quartz plate, so that the circular field of the polariscope is divided by the diametral line of junction, as in the case of the biquartz. The two halves appear equally bright when the analyser is arranged parallel to the plane of polarisation of the light reaching it, but the least divergence from parallelism introduces a considerable difference of intensity. In the event of the Laurent plate having one-half constructed of quartz, the latter is cut parallel to the axis, and of such thickness that there is a retardation between the ordinary and extraordinary rays of exactly one-half-wave.

In many saccharimeters now, moreover, the sensitive-enhancing plate is a triple one composed of three vertical strips; the central strip shows one intensity for a slight deviation from the rotation position, and the two flanking strips on each side, being similarly orientated, exhibit the other intensity to an equal degree, that is, exhibit equally another different intensity. It appears easier to discriminate the difference of intensity when the observer has a central strip of one intensity and two equally different ones on each side to deal with, the required position when all three are equally intense being very sharply determinable.

Another device for enhancing the sensitiveness of the determination is to replace the Nicol analyser by a Jellett compound calcite prism. This little-known but very valuable prism was described by Jellett to the British Association at their meeting in the year 1860.¹ It enables the plane of polarisation of light to be determined with great precision. It consists of a long prism of calcspar, which is first reduced to the form of a rectangularly terminated prism by grinding off its ends, and is then sliced lengthwise by a plane nearly but not quite perpendicular to the principal plane. The two parts are joined in reversed positions, and a diaphragm with a circular opening is placed at each end. The light which passes through both diaphragms produces a circular field divided by a diametral line into two parts, in which the planes of polarisation are slightly inclined to one another. When light which has been previously plane polarised is transmitted, it will be extinguished in the two parts of the field for positions which are fairly close together, and the light will become of uniform intensity in a position midway between the two. When, therefore, the double prism is arranged to give this equality of tint on placing it in the path of the light rays leaving the polarising Nicol, and the optically active crystal-plate is then introduced, the balance of intensity in the two halves will be upset; the Jellett double prism is then rotated to follow the rotation of the plate, and arrested when the two halves are again equally illuminated. The difference of the circle readings corresponding to

¹ *British Assoc. Reports*, 1860, vol. 2, 13.

these two positions will then obviously afford the required angle of rotation ρ .

Although each of these forms of sensitiveness-enhancer has its special merits, as above recorded, the use of a perfectly constructed biquartz is, however, on the whole perhaps the most satisfactory of all devices for enhancing the sensitiveness of this important determination of optical rotation, and any slight error of setting of the biquartz and the crystal-plate to exact normality to the polariscopic axis may be corrected by taking a second set of readings with the analyser rotated 180° from the position which it occupied in the first series.

Spectroscopic Analytical Method of Determining the Rotation Angle.—The most accurate method of all, however, is to analyse spectroscopically the light proceeding from the arrangement of apparatus with a biquartz above recommended. By means of a lens of suitable focal length, r in Fig. 655, a real image of the biquartz is thrown on the slit v of the spectroscope, so as to produce one above another two spectra in the field of view of the telescope of the latter, one from each half of the biquartz, which should obviously be arranged for this determination with its diametral cementation-line horizontal. Each spectrum will then be found to exhibit a dark band corresponding to the colour extinguished by the analysing Nicol. On rotation of the latter the two bands will move through the spectra in opposite directions, and the object is to arrange the Nicol so that they are exactly one above the other in the same straight line. When this is achieved it will be clear that light of the same wave-length has been extinguished in the two halves of the field, and that the plane of vibration of the Nicol analyser is parallel to the plane of vibration of the light leaving the crystal-plate. The biquartz may, however, be eliminated as an alternative method, when a band or bands will be seen in the spectrum corresponding to the radiations extinguished by the crystal-plate itself.

If sunlight be used and the solar lines clearly focussed, the position of the analyser, and therefore the angle of rotation of the crystal, can be read off for the coincidence of the dark band with each of the principal Fraunhofer lines, such coincidence being brought about by rotation of the analysing Nicol to the necessary extent. The band effects its passage through the spectrum from red towards violet as the analyser is rotated more and more from the original crossed position. If the dispersion of the angle of rotation exceeds 180° for the two ends of the spectrum, but is less than 360° , two bands will appear in the spectrum, and an additional band for every 180° of dispersion. In fact the spectral band or bands never disappear, but simply move along the spectrum in one direction or another, according to the direction of rotation of the analyser and the character of the rotation, right- or left-handed, of the substance under investigation.

The apparatus recommended in the last section serves, when differently disposed, admirably for this spectroscopic method. For the monochromatic illuminator described in Chapter XLIII., employed to supply the polariscope with the necessary monochromatic light, is also

an excellent spectroscope. It is only needful to replace the fitting containing the ground-glass diffusing screen in front of the exit slit, shown as thus removed at ϕ in Fig. 655, by one of the observing eyepieces χ , in order to convert it at once into an ordinary spectroscope, just, in fact, as it is used during the calibration of the instrument for the passage through the exit slit of monochromatic light corresponding to specific wave-lengths. The exit slit merely requires to be opened wide, with the aid of the adjusting screw ψ , in order that the whole field of the eyepiece may be filled with the spectrum. The large dispersion of the single prism ω of colourless special-flint glass, which has the merit of transmitting a very perfect violet end although so highly dispersive, is particularly suitable for this analytical purpose. The disposition of the whole apparatus is shown in Fig. 655, and it is of perfectly general application, whatever may be the nature of the optically active substance which it is desired to investigate. Sun-light is, of course, used, reflected into the polarising tube i from an adjustable mirror (such as t in Fig. 634), which receives its light from a heliostat if one be available; if not, the mirror merely requires periodic further adjustment as the sun gets off the instrument.

Determinations of the Rotatory Power of Quartz. — Accurate determinations of the rotation of quartz have been made by Broch,¹ by von Lang,² and by Soret and Sarasin.³ Broch¹ was the first to use the spectroscopic method, and devised it for the determination of the rotatory power of quartz. His apparatus consisted only of (1) a polariscope, the quartz plate cut perpendicular to the axis being placed in a movable diaphragm between the two Nicols, (2) a slit at some distance from the polariser, and (3) a refracting prism and telescope after the analyser. The Nicol analyser was first arranged exactly at 90° to the polariser, for perfect production of the dark field, in the absence of the quartz plate; then, after introducing the latter precisely perpendicularly to the axis of the Nicols, exact coincidence of one of the dark interference bands with one of the Fraunhofer lines was brought to occur by rotation of the analyser. Broch used many plates of quartz of very diverse thicknesses, and both right- and left-handed. He was able to determine the angle of rotation for any specific Fraunhofer line to about ten minutes of arc.

Von Lang² subsequently endeavoured to render the two operations, just specified as concerned in a determination, of equal facility and value, by the use of a double-prism or block of the nature of a biquartz, and production of two spectra one above the other from the two halves; the method is that illustrated in Fig. 655, except that no substance plate other than the biquartz was employed, this double-block of quartz serving both as substance plate and biquartz, the material being quartz in both cases and duplication being thus unnecessary. By rotation of the Nicol analyser the interference bands in the two spectra were made to move in opposite directions, until for a certain position of the Nicol they stood identically over each other. The analyser was then rotated first to the right and then to the left, until first in one spectrum and then in the other the nearest interference band coincided with a certain Fraunhofer line. When ψ_1 and ψ_2 are the corresponding azimuths of the Nicol, the angle of rotation ρ for this line is afforded by the formula :

$$\rho = m90^\circ \pm \frac{1}{2}(\psi_1 - \psi_2),$$

¹ *Ann. chim. phys.*, 1852, 34, 119.

² *Wien. Akad. Ber.*, 1876, 74, 209.

³ *Arch. Soc. phys. et nat. Genève*, 1882, 8, 5, 97, and 201.

where m is a whole number easily obtained from the thickness of the plate and preliminary determinations.

The + sign is to be employed when the interference band lies at the first adjustment on the red side of the Fraunhofer line, and the - sign when the band lies on the blue side. The angle of rotation is in this manner afforded by the difference of two similar observations.

Von Lang found the usual 3.75-millimetres biquartz too thin for his purpose, the bands afforded being too broad. He used, therefore, a double block of right- and left-handed quartz 33.4 millimetres in length, in the direction of the optic axis, along which the light travelled. The exact length was arranged so that the sodium D-line should be practically opposite two coincident bands in the two spectra at the first adjustment of the Nicol, so as to render $\psi_1 - \psi_2$ very small. Such a thick biquartz has to be arranged very near to the slit of the spectroscope, so that the plane of separation of the two halves may bisect the slit. The polarising Nicol was consequently the only one in front of the spectroscope, and was made in von Lang's apparatus the rotating one carrying the divided circle. The second Nicol was fixed in the telescope tube. In order to determine the influence of temperature on the angle of rotation von Lang also enclosed his biquartz in a heating apparatus furnished with plate-glass windows through which the light passed. The maximum

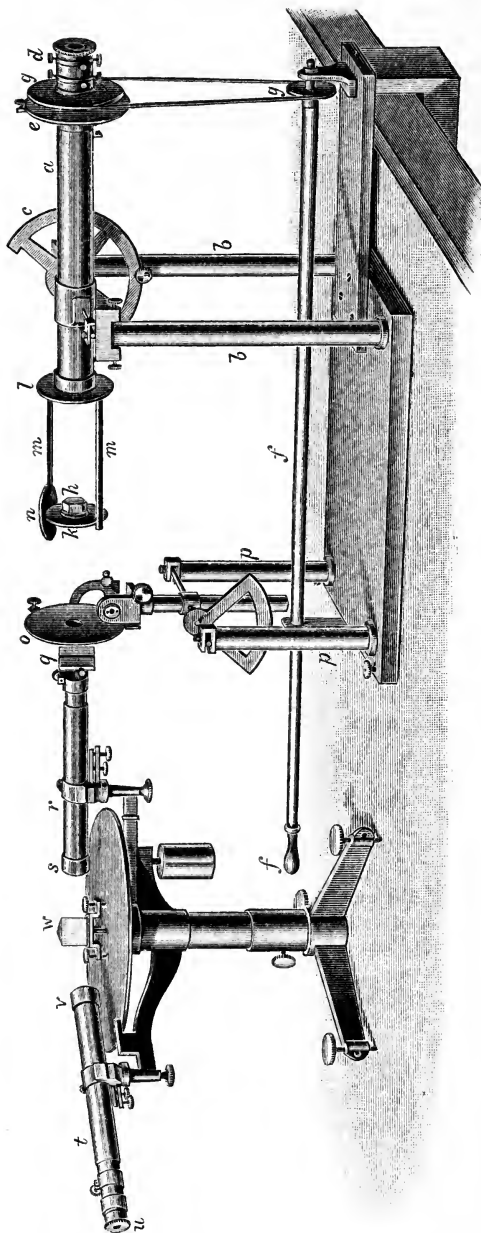


FIG. 656. — Apparatus employed by Soret and Sarasin for the Determination of the Optical Rotation of Quartz.

error in von Lang's determinations did not reach the one-hundredth of a degree. He found that the temperature change is represented by the following formula :

$$\rho = \rho_0(1 + 0.000149t),$$

where ρ and ρ_0 are the angles of rotation at t° and 0° .

Sohncke afterwards showed that this two-term formula is only correct for the mean effect between 20° and 100° , and that the rotation is not constant, but alters with the temperature in accordance with the three-term formula :

$$\rho = \rho_0(1 + 0.000\ 099\ 9t + 0.000\ 000\ 318t^2).$$

Soret and Sarasin¹ employed the method of Broch with greater refinements, and used no biquartz, merely a much longer prism of simple right- or left-handed quartz parallel to the axis. They used the apparatus shown in Fig. 656, in which a is a brass tube supported by trunnions on a pair of columns b , and adjustable for altitude by means of a divided circle c . At the right end the tube carries a polarising Nicol d provided with a divided circle e , and manipulated by the observer at the spectroscope by means of a shaft f and pulley gear g . The quartz block h is supported in a diaphragm k , and a similar diaphragm l carrying rectangularly a couple of rods m and a plate n afford a simple means of effecting the necessary adjustment of the quartz. The analysing Nicol is carried by the rotatable divided circle o , which was large enough to read to minutes with the aid of a vernier; it was mounted independently in a manner which provided it with elaborate adjustments, on the two columns p . It was arranged fairly close to the slit q of the spectroscope, of which latter r is the collimator with its objective s , and t the telescope, with its eyepiece u and objective v . The 60° dispersing prism w was of Iceland spar in certain of the experiments, cut so that the rays at minimum deviation travelled along the optic axis with only single refraction; in other determinations it was constructed of white flint glass. The glass Steinheil achromatic lenses in many of the experiments were replaced by a complete quartz train, which enabled the observations to be continued into the ultra-violet region of the spectrum. Sunlight was reflected into the apparatus by a heliostat, a quartz lens of 1.4 metres focal length being used to produce an image of the sun on the slit of the spectroscope. Two quartz blocks were found most satisfactory among a large number used; one was of 30 millimetres thickness, and of left-handed quartz, and the other was of right-handed quartz and no less than 60 millimetres thick. Both were single crystals, no biquartz being employed by Soret and Sarasin, and the ends were truly plane and parallel to each other, and perpendicular to the optic axis, which direction it was most carefully arranged should be traversed by the rays during the determinations.

In some further measurements with monochromatic light Soret and Sarasin (*loc. cit.* p. 201) employed the method of the Laurent saccharimeter, the essential point of which has already been shown to be that precision in the angular measurement of the position of the analysing Nicol is attained by the use of a half-shadow plate, which is a plate of glass half covered by a half-wave plate of quartz, gypsum, or mica. The plane of polarisation, with or without the plate of rotatory substance under investigation (quartz in the experiments under consideration), is determined by turning the analyser until the two halves of the Laurent plate show equality of illumination. With sodium light, and using quartz for the half-wave portion of the composite plate, this method of Laurent proved likewise very satisfactory in the hands of Soret and Sarasin.

The angles of rotation ρ for a plate of quartz one millimetre thick, at the temperature of 20° C., for light of the wave-lengths of the principal Fraunhofer lines, as derived from these three series of measurements by Broch, von Lang, and Soret and Sarasin, are given in the following table :

¹ *Arch. Soc. phys. et nat. Genève*, 1882, 8, 5, 97, and 201.

ROTATION ANGLES OF QUARTZ 1 mm. THICK AT 20°.

Solar Line.	Broch.	Von Lang.	Soret and Sarasin.
A	12° 39' = 12·65°
B	15° 18' = 15·30°	...	15 45 = 15·75
C	17 14 = 17·24	17° 18' = 17·30°	17 19 = 17·31
D	21 40 = 21·67	21 44 = 21·73	21 41 = 21·69 D ₁
E	27 28 = 27·46	...	21 44 = 21·73 D ₂
F	32 30 = 32·50	32 43 = 32·72	27 32 = 27·54
G	42 12 = 42·20	...	32 46 = 32·76
H	42 35 = 42·59
			51 11 = 51·19

Further Colour Phenomena of Quartz due to its Optical Activity, and Discrimination of the Two Varieties.—Right- and left-handed quartz are readily distinguished by the order of succession of the colours on rotating the Nicol analyser. A right-handed crystal becomes first red, then orange, yellow, green, blue, and violet, as the analyser is rotated in the direction of the hands of a watch; whereas a left-handed crystal gives this order on rotating the analyser anti-clockwise. The two varieties are also distinguished by the manner of movement of the circular rings, afforded in convergent polarised light by a plate perpendicular to the axis, when the analyser is rotated. The usual black cross of a uniaxial figure is absent from the centre of the figure afforded by a plate of quartz perpendicular to the axis of the thickness required to produce several rings, although the arms of the cross appear towards the margin of the field; this is owing to the rotatory polarisation, indeed, the disappearance of the cross is the more complete the thicker the section-plate. Moreover, the centre is never black, but coloured just as the

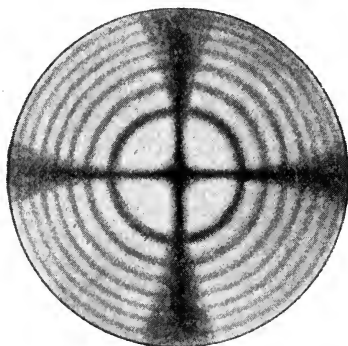


FIG. 657.—Interference Figure afforded by Quartz Plate 1 mm. thick in strongly Convergent Polarised Light.

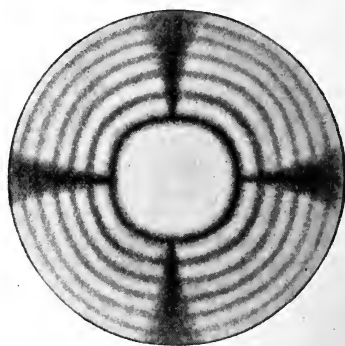


FIG. 658.—Interference Figure afforded by Quartz Plate 3.75 mm. thick in moderately Convergent Polarised Light.

whole plate would be in parallel light. The succession of colours at the centre, on rotating the analyser, will thus be the same as that just

given for parallel light. A plate one millimetre thick, however, which has already been shown to afford only weak colours in parallel light, and only exhibits a very few widely separated rings in the usual field, and using the same convergent system of lenses as that convenient for the thick plate, does show the black cross. A photograph of such a figure, but using a more strongly convergent beam, is reproduced in Fig. 657. A right-handed or left-handed quartz plate, of 3.75 mm. thickness, affords an interference figure such as that reproduced from an actual photograph in Fig. 658.

On rotating the analyser from the crossed position, in the direction of the hands of a watch, the rings of such a quartz figure as Fig. 658 expand when the crystal plate is right-handed, and contract if the section has been cut from a left-handed crystal, as they change to produce the figure for parallel Nicols, the central ring also becoming nearly square. When such an equally thick pair of right- and left-handed plates are superposed, a double spiral figure is produced, the well-known "Airy's Spirals," photographically reproduced in Fig. 659 as afforded by the same two plates one of which was used for Fig. 658.

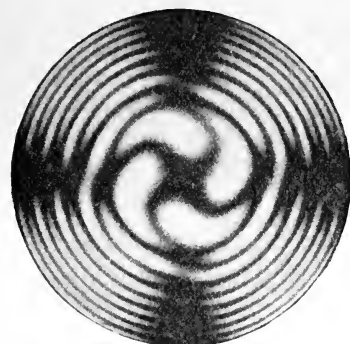


FIG. 659.—Airy's Spirals obtained by Superposition of a Right-handed and a Left-handed Plate of Quartz 3.75 mm. thick in moderately Convergent Polarised Light.

It is of considerable theoretic interest that Reusch has succeeded in artificially reproducing these figures for right- and left-handed quartz,

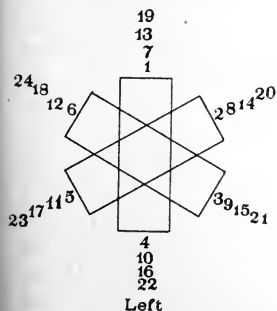


FIG. 660.

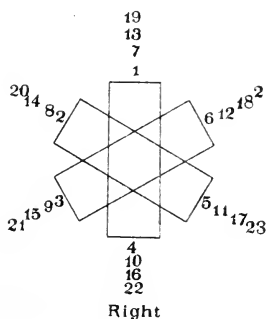


FIG. 661.

Reusch's Simulated Left- and Right-handed Quartzes composed of spirally arranged Mica Films.

crossed Nicols. In each of Reusch's preparations, which may now be obtained from Messrs. Steeg and Reuter of Homburg, 24 lamellæ are laid over each other at angles of 60° , thus making four complete screw turns. The arrangements are indicated in Figs. 660 and 661, the numbers indicating the order of the laying down of the strips.

by arranging an adequate number of strips of mica films over each other in a clockwise and anti-clockwise manner respectively; and that a pair of these imitation quartzes of mica, of opposite rotations, also reproduce Airy's spirals when placed over each other under

Regarding them from above, as laid on the stage of the polariscope, which is the opposite way in which the polarised light from the polarising Nicol meets them, the right-handed preparation has the strips arranged in anti-clockwise order, or clockwise to the rays of incident light; while the left-handed preparation has the strips arranged clockwise, or anti-clockwise to the incident light rays. All the lamellæ are taken from the same cleavage sheet of mica of equal thickness throughout. One-eighth-wave thickness of mica-film is the best for the purpose. The direction of the line joining the optic axis is diametral in all the 24 strips, that is, parallel to the longer edges of the strips. Hence, this direction is rotated four times during the passage of light through all the 24 films.

These two mica preparations, artificial quartzes, are obviously the mirror images of each other, and form a splendid proof of the mirror-image or "enantiomorphous" similarity of the two varieties of quartz.

Striking Screen Demonstration of Right- or Left-Handed Character of Quartz.—A very beautiful screen experiment, due to S. P. Thompson, for exhibiting the right- and left-handed rotations of the two varieties of quartz in a striking manner is as follows. A circular plate is prepared, composed of 24 sectors of half-wave mica, all cut from the same half-wave film and in such a manner that the direction of the line joining the two optic axes of the mica bisects each sector; this direction is consequently radially arranged when the composite plate is complete. The 24 sectors are carefully cemented closely side by side radially, with their points at the common centre, with hard balsam in benzene, on a circular glass plate of the usual $1\frac{7}{8}$ -inch diameter for projection purposes in parallel light. When the balsam has set, some days subsequently, a second glass plate is likewise cemented over them, so as to enclose the sectors permanently in balsam between the two protective glass plates. When the whole is hard the composite plate is mounted in the usual rectangular mahogany frame with rabbeted 2-inch circular aperture, in which it is maintained

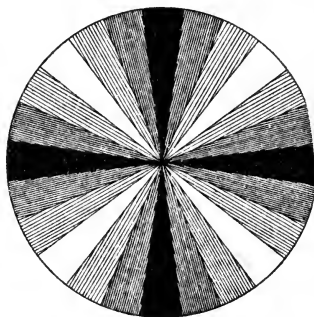


FIG. 662.

by a brass spring-ring; it should be arranged so that the bisecting line of one pair of sectors is vertical when of course that of another pair will be horizontal. On placing it as a polarising object on the stage of the projection polariscope, exactly as shown in Fig. 515 on p. 609, and under crossed Nicols, it affords, on actuating the arc of the lantern, the appearance on the screen which is shown in Fig. 662, the shading lines indicating the direction in each sector of the line joining the optic axes. The vertical and horizontal sectors

are at extinction, and appear as a black cross, while the sectors occupying the diagonal 45° -positions are brilliantly white; the intermediate sectors show regularly varying tones of grey, the whole being a very beautiful

object of exquisite shades of French grey, passing in each quadrant into jet black at the vertical and horizontal positions and into white at the bisecting 45° -positions.

When now a right-handed plate of quartz 1 mm. thick cut perpendicularly to the axis is introduced normally into the beam, that is, parallel to the mica plate, preferably behind the latter, the black cross rotates one sector to the right, and a little colour of delicate shades is introduced into the whole figure. When, on the other hand, a left-handed one-millimetre plate is introduced, the black cross rotates one sector to the left. The experiment is very striking if the two quartz plates are mounted independently in the same carrier frame, so that they can be rapidly inserted in succession or alternated; the rotation of the cross is then particularly marked, first to one side and then back through two sectors to the other, according to the right- or left-handed character of the plate introduced. A greater thickness of quartz than one millimetre is a disadvantage, as the black cross begins itself to colour, and to lose its distinctive and striking character.

The Four-Wedge Biquartz.—A form of biquartz which also affords a striking method of demonstrating optical rotation on the screen, and which is also used sometimes in determinations of rotation, when the spectroscopic analytical method is not adopted, consists of a compound rectangular block made up of four equal wedges, composed of the two varieties of quartz, alternately arranged as shown in Fig. 663. The wedges marked r are of right-handed quartz, and those marked l of left-handed. Each rectangular half, composed of two wedges of opposite rotation, affords in the dark field of the crossed Nicols a black band in the central line, where the thickness of the two wedges is the same and the rotations in opposite directions exactly neutralise each other. On each side of this central band, separated by a narrow interval illuminated with white light, are a series of spectrum-coloured bands at regular intervals, corresponding to the orders of Newton, due to the regular increments of thickness and consequently preponderating amount of rotation of one wedge over the other. On cementing the two blocks together as one block like a biquartz, inverted as regards the front and back positions of the right- and left-handed wedges, as shown in Fig. 663, and again placing between crossed Nicols, the two sets of bands are observed to be exactly in line with each other, and the black central line of one to form a continuation of that of the other, when the analyser is exactly crossed to the polariser. But on the slightest rotation of the analyser, or on the introduction into the optical system of a plate of a rotating substance, the bands move in opposite directions, and the angle through which the analyser has to be rotated in the latter case in order

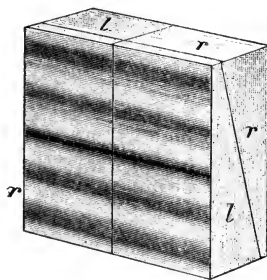


FIG. 663.—Four-wedge Biquartz, and Effect produced by it under Crossed Nicols.

to restore the two halves of the black band to the same straight line, is the measure of the angle of rotation of the crystal-plate.

The Biot Quartz Plate.—A sensitive quartz plate of 3·75 millimetres thickness, cut perpendicularly to the axis, is also often used in crystal microscopy, instead of the sensitive gypsum plate giving an even red of the first order referred to in Chapter XLI. page 656, and is known as a “Biot quartz plate.” It is generally inserted over the objective as a slider in a slot of the optical tube of the microscope, between the polariser and analyser. For as it changes so rapidly into red of the first order or blue of the second order, when the doubly refractive effect of another thin crystal is superposed, it is most useful for determining the sign of the double refraction of microscopic crystals, by the sense in which the colour changes as the stage is rotated towards the 45° position of the planes of vibration of the small crystal, with respect to those of the crossed Nicols, the uniform tint of the quartz being interrupted by the higher or lower order colour of the crystal.

Discrimination between the Double Refraction Interference Colours of Quartz and Colours due to Dispersion of Optical Rotation.—It was stated in Chapter XLI. that quartz, when examined in very thin plates under the microscope, is one of the most brilliantly polarising of crystals, owing to interference due to double refraction. For this reason it is readily recognised in rock sections, the usual thickness of which is about 0·04 millimetre. A quartz section-plate of approximately this thickness affords brilliant first or second order double refraction colours under crossed Nicols, when the plate is not perpendicular to the optic axis or nearly so (when extinction occurs under crossed Nicols), and when the vibration directions are in the neighbourhood of 45° to the planes of the Nicols. As the thickness increases the colours pale off as described in Chapter XLI., more and more white light becoming admixed until after the seventh order of spectrum colour has been passed we have first the “white of the higher orders,” and eventually pure white light. But when the thickness approaches a millimetre the rotation colours may begin to be prominent, if the direction of the section-plate is not too far from normal to the optic axis, due to the other phenomenon of optical activity, and for a plate of 3·75 mm., as we have seen, these colours are brilliant, and particularly so when the plate is actually normal to the axis. This is even more so when the double thickness of 7·5 mm. is reached. Hence, we have the two distinct sets of colour phenomena to discriminate between in the case of quartz, both brilliant, that due to double refraction in very thin plates, and which attains its maximum in a plate parallel to the axis; and that due to optical activity in thick plates, which is most intense when the plate is perpendicular to the axis. It is this latter attribute of quartz, of polarisation in thick plates, which renders it so magnificent a subject for screen experiments with the lantern polariscope, large and thick plates of quartz being exceedingly valuable for this purpose.

Other Optically Active Crystalline Substances.—Quartz is relatively powerful as an optically active substance, and in order to form a true

perspective of the subject it is necessary to compare its rotatory power with that of other active crystalline substances. Another very prominent substance also endowed with the property is sodium chlorate, NaClO_3 , and in this case it is only developed to a typically low degree. In the following table the angles of rotation for a plate of sodium chlorate one millimetre thick, as determined by Guye, are given for the same wave-lengths, those of the prominent Fraunhofer lines, as those for quartz already tabulated on page 812.

ANGLES OF ROTATION ρ FOR SODIUM CHLORATE.

Fraunhofer line.	B	C	D	E	F	G	H
Rotation angle .	$2^\circ 16'$	$2^\circ 30'$	$3^\circ 8'$	$3^\circ 56'$	$4^\circ 40'$	$6^\circ 0'$	$7^\circ 10'$

The two oppositely optically active varieties of both quartz and sodium chlorate exhibit equal amounts of the two opposite rotations, so that the angles ρ are identical for the two varieties, but the signs are different.

The reason is that in the crystals of one variety, the right-handed, the clockwise circular movement or vibration is the more rapid, whilst in the other modification, the left-handed, the anti-clockwise orbital movement is the quicker. With equal thickness of plate, however, the amounts of rotation exhibited by the two varieties are exactly equal, corresponding to their exact mirror-image symmetry. The more important substances the crystals of which rotate the plane of polarisation of light are arranged below according to their symmetry, and the amounts of their rotation for yellow sodium light (except in the case of cinnabar, for which the only available determination is for red light), for a plate 1 mm. thick, are appended.

CUBIC SYSTEM.

Tetrahedral-pentagonal-dodecahedral class (cubic-tetartohedral), Class 28.

	ρ_{Na}	Observer.
Sodium chlorate, NaClO_3	$3^\circ 8'$	Guye
„ bromate, NaBrO_3	$2 10$	Traube
„ sulphantimoniate, $\text{Na}_3\text{SbS}_4 \cdot 9\text{H}_2\text{O}$ (Schlippe's salt)	$2 40$	Marbach
„ uranyl acetate, $\text{NaUO}_2(\text{C}_2\text{H}_3\text{O}_2)_3$	$1 29$	Traube

TETRAGONAL SYSTEM.

Trapezohedral class (trapezohedral-hemihedral), Class 11.

Hydrogen Zinc malate, $\text{Zn}(\text{C}_4\text{H}_5\text{O}_5)_2 \cdot 2\text{H}_2\text{O}$	$3^\circ 1'$	Traube
Strychnine sulphate $(\text{C}_{21}\text{H}_{22}\text{N}_2\text{O}_2)_2 \cdot \text{H}_2\text{SO}_4 \cdot 6\text{H}_2\text{O}$	$13 15$	„
Ethylene diamine sulphate, $\text{C}_2\text{H}_4(\text{NH}_2)_2 \cdot \text{H}_2\text{SO}_4$	$15 30$	von Lang
Guanidine carbonate $(\text{CN}_3\text{H}_5)_2 \cdot \text{H}_2\text{CO}_3$	$14 34$	Bodewig
Diacetyl phenolphthalein, $\text{C}_{20}\text{H}_{12}\text{O}_4(\text{C}_2\text{H}_3\text{O})_2$	$19 42$	„

HEXAGONAL SYSTEM.

Hexagonal-pyramidal class (hemimorphic-hemihedral), Class 23.

Potassium lithium sulphate, KLiSO_4	$3^\circ 26'$	Traube
		3 G

TRIGONAL SYSTEM.

Trapezohedral class (trapezohedral-tetartohedral), Class 18.

Quartz, SiO_2 (see special table)	21° 43'	Soret and Sarasin
Rubidium dextro tartrate, $\text{Rb}_2\text{C}_4\text{H}_4\text{O}_6$	10 12	Traube
Cesium dextro-tartrate, $\text{Cs}_2\text{C}_4\text{H}_4\text{O}_6$	19 0	"
Potassium dithionate, $\text{K}_2\text{S}_2\text{O}_6$	8 23	Pape
Lead dithionite, $\text{PbS}_2\text{O}_6 \cdot 4\text{H}_2\text{O}$	5 32	"
Calcium dithionate, $\text{CaS}_2\text{O}_6 \cdot 4\text{H}_2\text{O}$	2 6	"
Strontium dithionate, $\text{SrS}_2\text{O}_6 \cdot 4\text{H}_2\text{O}$	3 23	"
Benzil, $\text{C}_6\text{H}_5 \cdot \text{CO} \cdot \text{CO} \cdot \text{C}_6\text{H}_5$	25 0	Des Cloiseaux
Cinnabar, HgS , most powerfully rotatory substance known, 20 times as great as quartz	325° for } red light }	"

Pyramidal class (hemimorphic-tetartohedral), Class 16.

Sodium periodate, $\text{NaIO}_4 \cdot 3\text{H}_2\text{O}$	23° 18'	von Groth
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In the cases of the cubic and therefore isotropic crystals of the first four substances mentioned in the list, the rotation is the same in all directions, the angle only varying with the wave-length of the light and the thickness of the plate. The latter may be cut out of the crystal in any direction. The crystals of the remaining substances in the list, all of which are optically uniaxial, only show the pure phenomenon of rotation of the plane of polarisation in the direction of the optic axis, the plate for investigation being in all cases cut perpendicularly to the optic and principal crystallographic axis. For only along this direction is there no double refraction to complicate the phenomenon, the crystal behaving as if isotropic along this direction of the axis. Hence, all the measurements of rotation given in the list refer to the axial direction.

The moment the inclination of the plate to the optic axis commences to vary from 90° the influence of double refraction begins to be manifest. A plate of quartz cut parallel to the axis, in fact, shows with light of ordinary intensity no sign of rotatory power, the interference phenomena being apparently entirely those due to double refraction. Airy, Gouy, and Wiener have shown that as the inclination to the axis alters from 90° the circular orbits of the vibrations become elliptical, and finally, as parallelism to the axis is attained, these become converted into truly linear vibrations. Voigt has shown, however, that it is impossible absolutely to extinguish the light when very intense, with a plate parallel to the axis. But speaking in general terms, and as regards readily observable phenomena, it has been clearly demonstrated experimentally by Airy that elliptic vibrations are really produced by sections inclined at angles other than 90° to the principal axis, and he actually determined the precise dimensions of the axes of the ellipse, and the complete form of the vibrations. Jamin added to this information knowledge concerning the relative

rétardation of the two series of elliptic vibrations behind one another, and Hecht further confirmed the complete agreement of observation with theory. Later Croullebois and MacConnel also determined the difference of path traversed by the elliptic vibrations. Croullebois went so far, indeed, as to render the two elliptically vibrating rays separately visible, by means of a double-prism of a similar character to that of Fresnel shown in Fig. 651, only composed of two simple prisms of 82° angle, of right- and left-handed quartz respectively, and cemented together by their hypotenuse faces, and with the further difference that the rectangular entrance and exit faces inclined 80° to the optic axes instead of being normal thereto. Later work by Brunhes, Beaulard, and Voigt, and a masterly analysis of the whole subject by Pockels, have resulted in the perfect establishment of these facts, in exact accordance with the theory of the superposition of the effects of double refraction of the ordinary linear-vibration kind on the phenomena of circular polarisation, as the direction of the optic axis of a uniaxial crystal is departed from.

The question of possible OPTICAL ROTATION on the part of BIAxIAL CRYSTALS is a very difficult one, as there is no direction of absolute single refraction, along which the phenomena can be observed uninfluenced by double refraction effects. The two optic axes have been shown in Chapter XL to be only the optic binormal approximations to true directions of single refraction for a specific wave-length. The case of cane sugar, $C_{12}H_{22}O_{11}$, belonging to the sphenoidal class 4 of the monoclinic system, has, however, been investigated by Pocklington,¹ and the dark centres of the hyperbolic brushes marking the two optic axes found to exhibit rotation, only becoming black on rotation of the analyser from the crossed position, one to the right, that perpendicular to the cleavage direction, and the other to the left, corresponding for a plate 1 millimetre thick and for sodium light, to $\rho = +2^\circ 12'$ and $\rho = -6^\circ 24'$ respectively. As the optic axes lie in the symmetry plane in the case of crystals of cane sugar, it was to be expected that their rotations would be unequal.

Seignette salt, potassium sodium dextro-tartrate, $KNaC_4H_4O_6 \cdot 4H_2O$, crystallising in the bisphenoidal class of the rhombic system, was also investigated by Pocklington, and for sodium light a plate 1 mm. thick gave $\rho = -1^\circ 12'$ for both optic axes, the two being equal in accordance with rhombic symmetry.

Dufet² has also investigated three other salts also belonging to the bisphenoidal class of the rhombic system, namely, Epsom salts (magnesium sulphate), $MgSO_4 \cdot 7H_2O$, which gave a rotation of $\rho = +2^\circ 36'$ for a plate 1 mm. thick perpendicular to an optic axis; sodium dihydrogen phosphate, $NaH_2PO_4 \cdot 2H_2O$, which afforded a left rotation of $\rho = -4^\circ 27'$; and ammonium Seignette salt, $Na(NH_4)C_4H_4O_6 \cdot 4H_2O$, which gave a dextro rotation of $\rho = +1^\circ 33'$, although its solution is laevo-rotatory. Dufet also investigated two monoclinic substances belonging to the

¹ *Phil. Mag.*, 1901, 2, 368.

² *Journ. de Phys.*, 1904, 3, 757; *Bull. Soc. Fran. de Min.*, 1904, 27, 156.

same hemimorphic (sphenoidal) class as cane sugar, namely, the sugar known as rhamnose, and tartaric acid (the ordinary dextro variety), $C_4H_6O_6$. In the case of the latter, the plane of the optic axes is perpendicular to the symmetry plane, and so the rotations along the two optic axes are equal; they amount to $\rho = +8^\circ 33'$ for Li-light, $11^\circ 24'$ for Na-light, and $14^\circ 14'$ for Tl-light. In the case of rhamnose the optic axes lie in the symmetry plane, so that the rotations for the two are different, namely, $\rho = +12^\circ 54'$ and $\rho = +50^\circ 24'$. As this is a substance of extremely feeble double refraction the optical rotation is very marked.

From these facts it will be clear that the phenomenon of rotation of the plane of polarisation of light extends throughout the eleven enantiomorphous classes of crystal symmetry, in accordance with the rules which have been laid down in Chapter X., page 139.

CHAPTER L

THE COLOUR OF CRYSTALS, ABSORPTION OF LIGHT, AND PLEOCHROISM.

MANY crystals are coloured, and the colour may be due either to minute enclosures of foreign matter, or to the intrinsic nature of the substance itself. Thus the sapphire and ruby are composed of colourless oxide of aluminium, Al_2O_3 , equally with corundum, and chemical analysis is only just able to detect the traces of ferric oxide or chromic oxide, to which the colour is due which renders the ruby and sapphire so valuable as gem-stones. When a ruby is examined in plane-polarised light, under the polarising microscope to be described in the next chapter for instance, with only the polarising Nicol in position, the colour appears deep red in one direction, that in which the ordinary ray ω is transmitted, while if examined in a direction permitting only the extraordinary ray ϵ to escape the colour appears to be a violet red. Likewise the sapphire appears to change colour according to the ray extinguished or transmitted, being deep blue when the ordinary ray is transmitted and greenish-blue when the extraordinary ray is alone allowed to escape. These gems are, therefore, said to be "dichroic," and their dichroism is owing to admixture with a minute amount of foreign oxides, modified by the structure of the crystals, which renders them doubly refractive.

On the other hand many crystalline substances are coloured in their own right, due to absorption of light of certain wave-lengths in its passage through the crystal; and if they are birefringent the colour is different for the light corresponding to the two rays travelling through the crystal with different velocities and vibrating parallel to different directions of the crystal, the difference being made particularly manifest when one of them is extinguished by a Nicol prism. Tourmaline is an extreme case not requiring a Nicol, where one ray is completely absorbed by the crystal itself, permitting only the other which vibrates parallel to the axis of the uniaxial crystal to escape, as fully explained in Chapter XXXVI. (page 575). It frequently happens, however, that the crystals of a coloured birefringent substance show different colours in different directions, owing to the different selective absorption of light, from the gamut of wave-lengths presented by the visible spectrum, which occurs for the two rays vibrating perpendicularly

to each of the different directions of transmission examined, leaving the light actually transmitted tinted complementarily to the colour chiefly absorbed.

Pleochroism or Dichroism, as it is variously called, may be quite well investigated under the microscope, using a single Nicol. Or it may be examined in a manner which places both colours side by side in the same

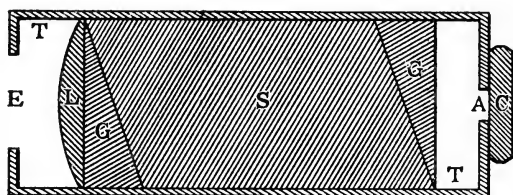


Fig. 664.—Section through the Dichroscope.

spar *S* fills the greater part of the optical tube *T*, which has a large round aperture at the eye end *E* and a small square aperture *A* at the other end, against which the crystal *C* is to be held. The oblique ends of the rhomb are squared with prisms of glass *G*, and immediately in front of the near one is placed a plano-convex lens *L*, calculated to focus the square aperture when the rhomb is removed. When the latter, however, is in position two images of the aperture are afforded, and the length of rhomb and size of square aperture are so chosen as to just completely separate them. The conditions are shown in Fig. 665, the shaded square marked ω , occupying a central position in the field, being that due to the ordinary rays, while the other, labelled ϵ , represents the image due to the extraordinary rays. When the crystal plate is introduced behind the rhomb, with the extinction directions parallel to the longer and shorter diagonals of the rhomb, ω shows a colour due to rays which have vibrated in the crystal and in the rhomb in the direction parallel to the longer diagonal of the rhomb, and perpendicular to the principal section containing the optic axis, while ϵ exhibits the colours imparted to the rays which have vibrated in the crystal and rhomb parallel to the shorter diagonal of the calcite, which lies in the principal section containing the optic axis. The colour of each square is the residual effect left after light of certain wave-lengths has been absorbed within the crystal from the original white light. On rotation of the crystal-plate in its own plane and about the axis of the dichroscope, the tints approach similarity, until at the 45° -positions they are the same; further rotation again invokes a difference, which becomes a maximum when 90° of rotation have been effected, the colours of the squares being now inter-

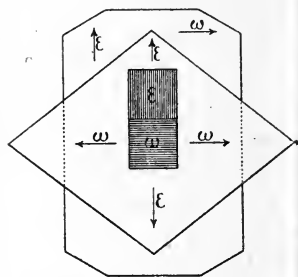


Fig. 665.—The Conditions obtaining in the Dichroscope when the Vibration Planes of the Calcite Rhomb and the Crystal are parallel.

changed compared with those displayed at first for the 0° -position. The maximum difference of colour is thus always shown when the vibration directions of the crystal and the rhomb are parallel. It is an advantage to see in this manner side by side the two colours produced at their maximum difference, for very small degrees of dichroism can then be detected which would not perhaps be recognised with the polarising microscope.

Absorption of Light by Crystals.—The determination of the absorption coefficient, and the quantitative proof of its dependence on the direction of transmission or vibration of the light rays, the qualitative proof of which is afforded by the dichroscope, requires an apparatus which combines a spectroscope with a photometer for comparing the intensities of the two light patches. The two beams are separated by double refraction as in the dichroscope, and are composed of light polarised at right angles; they are also arranged to be of equal intensity. The introduction of the absorbent crystal-plate then disturbs the equality of intensity, which is, however, restored again by rotation, for the necessary amount, of a Nicol polariser or analyser. There are two types of apparatus, of which the spectrophotometer of Glan and the microphotometer of Königsberger are characteristic examples which will be now described.

Spectrophotometer of Glan.—Any efficient spectrometer will serve as the basis of this instrument, and the one shown in Fig. 666 is the Fuess No. 2*a* goniometer, suitably modified for the special object in view. In this, and the succeeding sectional figure 667, *c* is the collimator, *p* the 60° -prism, and *t* the telescope.

In front of the objective of the collimator *c* a double-image prism *r* is mounted. It may be either of the Rochon type, which is in some ways preferable, and which gives an extraordinary image rotating about the ordinary one, or of the Wollaston type which gives double the separation of the two sets of rays, and furnishes two images rotating about a common centre, as fully described in Chapter XXXIX., page 611. This double-image prism *r* is followed by a Nicol prism *n* provided with a large rotating circle *c'*. The slit *s* of the collimator, shown enlarged in the inset of Fig. 667 in order the more clearly to elucidate these essential details, is divided into an upper and a lower half by an opaque bar, and thus appears as two rectangular strips, A and B in Fig. 668, when the double-image prism is removed. Each half is doubled, however, by the double-image prism when the latter is in position, the two images of each pair being rectangularly polarised and arranged to be above each other, as indicated in Fig. 668 at *A_e*, *A_o*, and *B_e*, *B_o*, the ordinary ray image *A_o* of A being brought next to the extraordinary image *B_e* of B. These four images are broadened out by the dispersion of the 60° -prism into four spectra almost in contact one above the other, as seen through the telescope *t*. The two outer ones are screened off by an iris diaphragm, so that only the two middle ones, corresponding to *A_o* and *B_e*, are visible. A second slit *s'*, devised by Vierordt, is placed at the common focal plane of the objective and eyepiece of the telescope, and this slit is adjustable for its position in the focal plane like a micrometer spider line, and also for its opening. This enables the observer to select any part of the spectrum exclusively for examination at a time. The whole apparatus as constructed by Fuess is shown in Fig. 666, and a section through the optical tubes and prism, with an enlarged inset of the first slit, in Fig. 667.

In using the instrument the zero is first found by observing that position of the Nicol for which that one of the two spectra is extinguished, the vibrations of which are parallel to the slit. The Nicol is then rotated for an angle α to that position for which the two spectra are of equal intensity, this angle being usually in the neighbourhood

of 45° . In order to determine the absorption of a crystal-plate, the latter is placed in front of one of the halves of the slit; it will then be found necessary, in order to restore the balance of intensity of the spectra, to rotate the Nicol for an angle β (counting from the zero). When the plate is one of a singly refractive crystal it may be arranged at any azimuth to the slit; but when a birefringent crystal is under investigation, it must be arranged so that one of its extinction directions is parallel to the slit, and the observations must subsequently be repeated with the other extinction direction similarly adjusted.

If A_A and A_B are the coefficients of reduction of intensity by passage of the two sets of rays through the apparatus, coefficients depending on the apparatus itself and

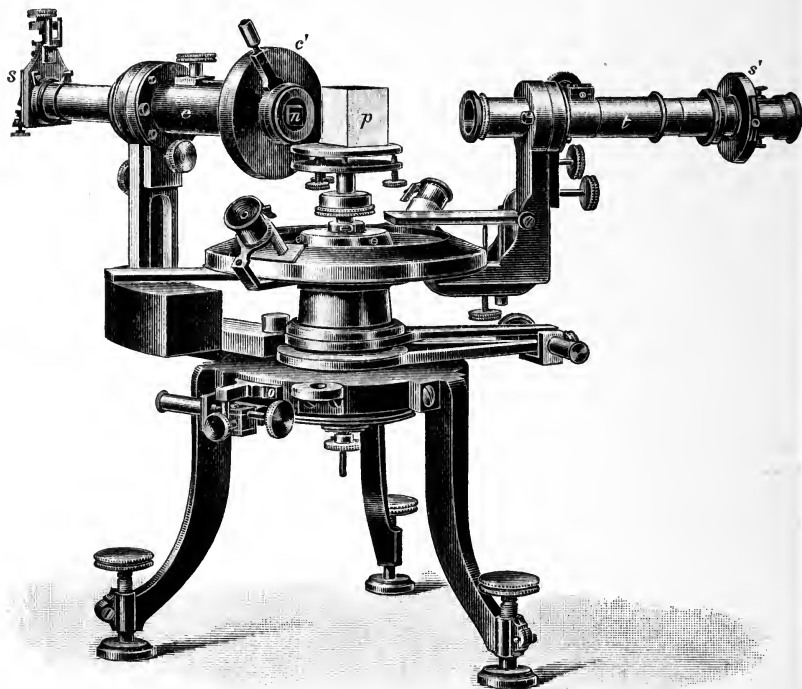


FIG. 666.—The Glan Spectrophotometer.

determined experimentally, and if I_A and I_B are the two actually observed intensities, then :

$$A_A \sin^2 \alpha = A_B \cos^2 \alpha, \text{ or } \frac{A_B}{A_A} = \tan^2 \alpha,$$

and

$$I_A A_A \sin^2 \beta = I_B A_B \cos^2 \beta,$$

when the Vierordt slit is used to compare equal portions of the two spectra. Hence,

$$\frac{I_A}{I_B} = \frac{A_B}{A_A} \cot^2 \beta, \text{ or } \frac{I_A}{I_B} = \tan^2 \alpha \cot^2 \beta.$$

This latter equation thus at once gives the ratio of the two intensities, from the data of the two observed angles α and β .

Königsberger's Microphotometer, shown in Fig. 669, is essentially a low-power polarising microscope in which the polarising Nicol prism is replaced by a double-image prism of calcite a , or a simple rhomb of adequate thickness of the mineral to afford the required separation of the images of two apertures employed.

Behind the double-image prism or rhomb is a diaphragm pierced by the two rectangular apertures. Four images of the latter are thus produced, and are so disposed, by suitable adjustment of the size of aperture and its distance from the double-image prism, that the extraordinary image of one aperture is partially superposed over the ordinary image of the other opening, the other two outer images being screened off by an iris diaphragm in the optical tube. The latter has an objective b of $3\frac{1}{2}$ inches (9 c.m.) focus, an eyepiece c and a fixed Nicol d . It is mounted in an adjustable manner which permits of its arrangement truly perpendicular to the calcite double-image prism. Either before or behind the objective comes a double quartz plate e , showing Savart's bands in feebly convergent light. A quartz plate cut parallel to the

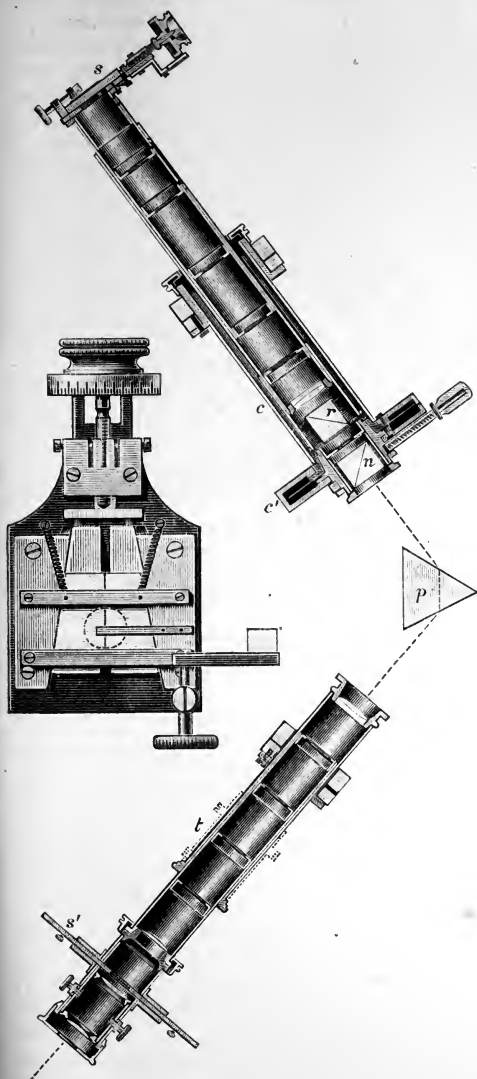


FIG. 667.—Section of Glan Spectrophotometer, and Enlarged View of the Slit.

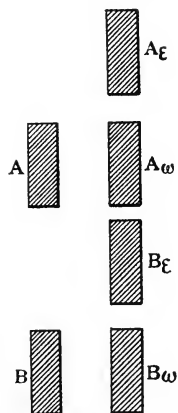


FIG. 668.

axis shows in convergent polarised light a series of hyperbolic curves, which have been illustrated in Fig. 580 (page 680) and which become narrower and sharper the thicker the plate, but beyond a certain thickness the hyperbolæ become fainter, and finally disappear for a very thick plate. The Nicols may be either crossed or parallel, the

dark hyperbolic bands merely exchanging places with the bright interspaces. Two such plates superposed with their axes arranged perpendicularly afford coloured hyperbolæ. When a plate of quartz is cut, however, at 45° to the axis the curves become approximately straight lines, and two such plates arranged with their principal section planes perpendicular to each other give straight lines, coloured in white light, but dark in monochromatic light, which bisect the angle between the principal sections. These are known as "Savart's bands," and have been illustrated in Fig. 582 (page 680). Such a compound quartz-plate acts as a very delicate test for the presence of polarisation, however feeble. It is quite adequate, in fact, to detect the polarisation of the light of the sky. The two plates of which the compound plate is composed should be the two halves of one and the same plate originally prepared, in order to secure identical thickness of the two crossed plates. The sensibility of Savart's bands increases, moreover, as the plane of polarisation of the incident light, that of a polarising Nicol for instance

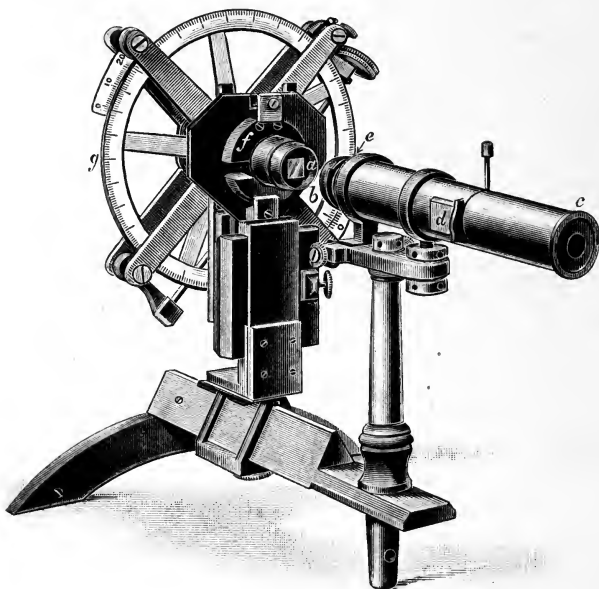


FIG. 669.—Königsberger's Microphotometer.

in front of which the compound plate is held, approaches the bisector of the angle between the principal sections of the two plates, that is, the direction of the bands themselves.

The Savart's bands disappear when the two superposed images of Königsberger's microphotometer, composed of ordinary and extraordinary rays respectively, vibrating perpendicularly to each other, are of absolutely equal intensity. Hence, to begin with, the light from the two orifices is equalised so that the bands are invisible. The moment one aperture is covered with the crystal-plate under investigation, which deranges the equality, the bands reappear. The absorption by the crystal is determined by the amount of weakening required by the light proceeding from the other aperture, in order to cause the bands to disappear again, and which is effected by means of a rotating Nicol prism *f*, provided with a large divided circle *g*, which is operated as in Glan's photometer. Monochromatic light is required for the observations, which is furnished very satisfactorily by the spectroscopic monochromatic illuminator described in Chapter XLIII.

CHAPTER LI

THE CRYSTALLOGRAPHIC MICROSCOPE AND ITS ACCESSORIES.

THE polarising microscope is one of the most useful and important instruments of the crystallographer's laboratory. For although all the optical constants are measured eventually by the instruments described in preceding chapters, constructed specifically for each particular purpose, provided adequately large crystals can be obtained for a complete investigation, still the microscope is a sort of *multum in parvo* of them all, most convenient for preliminary work, and the last resort in all cases where crystals cannot be procured of the necessary size (minimum, that of a pin's head) for individual measurement of each constant on its own specific measuring instrument. Moreover, the crystallographic microscope is invaluable in all work of a petrological character, such as the study of the natural crystal-sections revealed in the thin slices of rocks specially prepared for examination under the microscope, for the study of the minute crystals obtained in micro-chemical analysis, or for that of excessively small crystals alone obtainable by any process, for rapid work on the optical characters of larger crystals for identification purposes, and lastly, in carrying out a preliminary examination of the more apparent optical properties of a new crystalline substance. Fortunately, this country has for many years excelled in the construction of microscopes and their objectives and other accessories, and the late Mr. James Swift, made for his firm of James Swift and Son a world-wide reputation for petrological microscopes, a just reputation which is now being added to by his son as regards the construction of microscopes suitable for all varieties of crystal work. Also, the continental firms of Fuess of Berlin, Voigt and Hochgesang, now taken over by Steeg and Reuter of Homburg, Zeiss of Jena, and Nachet of Paris, and the Société Gènevoise, are all constructing admirable instruments for this specific purpose.

The best crystallographic microscope which has come within the author's own experience is the large pattern one constructed by Messrs. Swift and Son, designed in its earlier form by Mr. Allan B. Dick, and known as the "large Dick microscope." With the subsequent additions which have been made to this instrument on the recommendations of Dr. H. A. Miers, Professor Bowman, Mr. G. W. Grabham, and the author, this instrument is in every way admirable for the

crystallographer's purposes. It is shown in Fig. 670 in its completest form for ordinary work, with the measuring stage, which includes both a rotating divided circle of great accuracy and two rectangular measuring

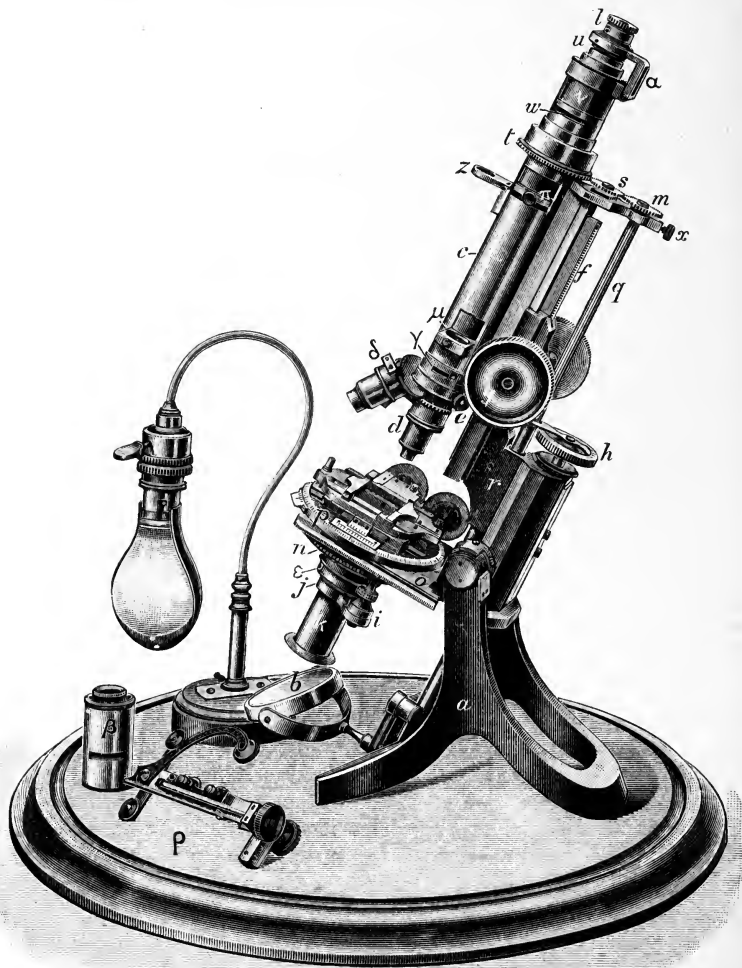


FIG. 670.—The Dick Crystallographic Microscope constructed by Swift, fitted with Measuring Stage.

movements; in Fig. 671 as fitted for work with the Miers goniometrical stage; and in Fig. 672 as arranged horizontally with the latter for optical-angle measurements in a highly refractive liquid.

In Fig. 673 it is also shown upright in its simplest form, with the fixed rectangular stage more clearly revealed, and Swift's mechanical

provided with two rotatory axial motions and two movements of transference, for adjusting the illuminating light reflected by the mirror so as to pass exactly along the axis of the body-tube *c*. The objective *d* is provided with a centring adjustment above the standard screw attachment, manipulated by two screws *e*. The coarse adjustment by rack *f* and pinion *g*, and the fine adjustment manipulated by the divided milled head *h*, act with great smoothness and precision, without any backlash, owing in the one case to obliquity of the rack teeth, and in the case of the fine adjustment to a strong

pressure-maintaining spring. This fine adjustment depends on a differential screw, with which in the instrument possessed by the author the greatest possible care was taken in the turning, and which, with the aid of the divisions on the top of the milled head, reads to the thousandth of a millimetre of axial movement of the microscope. It is of special use in determining thicknesses by the refraction method. The coarse adjustment is also graduated, a silver scale, shown in Fig. 673, divided directly into half millimetres being carried by the supporting bracket, and a silver vernier carried by the body-tube enables it to be read to a fiftieth of a millimetre.

A principal feature, introduced by Mr. Dick, and since adopted on most modern instruments, is that the polarising and analysing Nicol prisms k and l may be rotated together from the upper end of the microscope by the toothed wheel m , or from below

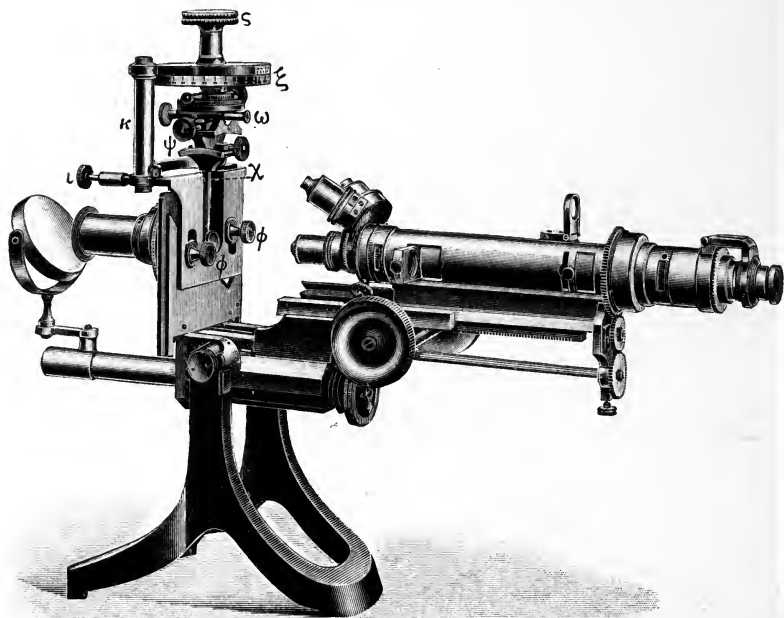


FIG. 672.—The Measurement of Optic Axial Angles with the Dick Microscope and Miers Stage Goniometer.

the stage by means of the large milled disc n , the fundamental stage o being fixed, and of rectangular shape. Hence, instead of rotating the crystal with its stage, which is not always convenient, the crossed Nicols may be rotated simultaneously, in which case the rotating stage p in Fig. 670 is not required. The amount of the rotation is registered by a silver divided circle carried by the large milled disc n , seen in Fig. 673, a vernier also on silver being carried alongside it by the fixed stage, at the bottom of a recess cut and scooped out of the latter for the purpose of revealing the graduations of this circle, and for convenience of manipulation of the crystal-holder of the goniometer stage as shown in Fig. 672. The gear for effecting this simultaneous rotation of the Nicols is very simple. The toothed wheel m , which is smoothly made so that it may also act as a milled head for rotation by the hand, is carried at the upper end of a steel fluted shaft q , which passes without contact through a boring in the body

bracket r ; a similar toothed wheel is also carried at the lower end of the shaft under the fixed stage. With the aid of an intermediate smaller toothed wheel s in each case, the upper and lower wheels each gear with a larger one t , which is really only an annulus, the centre being perforated in both upper and lower wheels with the full aperture of the optical tube and stage opening respectively. For the upper one is concentric with the optical tube at its upper end, and rotates about the tube and its axis in a plane perpendicular to the latter, carrying with it all the upper fittings, including the eyepiece u , its outer tube v , which has a slot w for quarter-wave mica or other insertions, and the analysing Nicol prism l . The Nicols can be fixed at any desired position by means of the little clamping screw x . The lower large wheel under the stage carries with it, below, the large milled disc n which bears the silver divided circle on its upper surface, the divisions being single degrees; the vernier carried by the fixed stage enables the readings to be carried out to minutes. The circle may be conveniently read by a lens z hinged, at a convenient height for focussing, to the front of the body tube. Both wheel and circle plate are perforated with a wide central aperture to permit of the passage of the light rays, and of the insertion of a sub-stage condenser when required. To this annular disc n the polarising Nicol prism k is fitted, a short column i proceeding downwards from near the margin of the disc and carrying at right angles at its lower expanded extremity a rotatable arm terminating in a ring j , in which is fixed the outer tube-case of the polarising Nicol and in which the latter and its inner mount-tube are rotatable. The pivoting of the arm about the column is so arranged, by means of a stop-pin, that the Nicol is placed exactly in position in the optic axis of the microscope when the arm is at one end of its possible swing, and is conveniently out of the way when at the other end of its swing.

The polarising Nicol is larger than usual on the author's instrument, having a clear minimum aperture of $\frac{7}{8}$ ths of an inch. It has a bevelled graduated circle and an indicator on the outer tube; a spring claw also falls into a shallow notch with a click when exact crossing or parallelism of the Nicols is attained. The analysing Nicol l is also carried by a double elbow arm a , hinged so that the Nicol can be rotated backwards out of the way when not required, or when it is desired to insert a different eyepiece. The Nicol itself is provided with a bevelled divided circle and indicator, and is separately rotatable, a notch and spring click like those of the polariser being also provided to indicate the crossed and parallel positions even when the light is shaded off. Hence, the two Nicols can either be rotated separately as

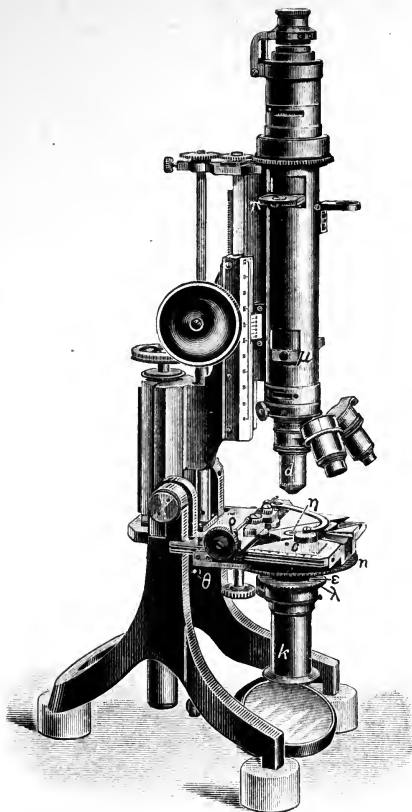


FIG. 673.—The Dick Microscope with Swift Mechanical Stage

in the ordinary polarising microscopes, or together, mutually crossed or parallel, or in any desired position with respect to each other's planes of polarisation, by means either of the milled disc *n* or the toothed wheel *m*. As the eyepiece rotates with the analyser, the cross-wires (spider-lines) carried by it also simultaneously rotate. The eyepieces are of the form shown at *β* resting on the baseboard in Fig. 670, and a projecting pin near the upper end of each eyepiece tube fits into a notch cut in the rotating tube in such a position that the cross-wires are parallel to the planes of polarisation and vibration of the Nicols. Each eyepiece also has a slot corresponding to that in the rotating tube, for the insertion of the quarter-wave mica plate or quartz wedge, the slot in the tube

being covered by an outer shutter-tube when the latter are not required. A similar pair of slots with covering shutter-tube are provided at *γ* just above the centring apparatus.

A nose-piece *δ* for three objectives is provided. The following three are the most useful objectives for crystal work, and should be left on the nose-piece as a rule, namely, a $1\frac{1}{2}$ -inch objective, for rock-section work or the preliminary examination of ordinary goniometrically measurable crystals; a 1-inch objective for parallel light work with crystals just too small for goniometrical investigation in the ordinary way; and a $\frac{1}{2}$ -inch objective for convergent light work. It is, of course, very convenient to possess a greater range of objectives than these essential ones, and those for which the author has found most use in addition to the

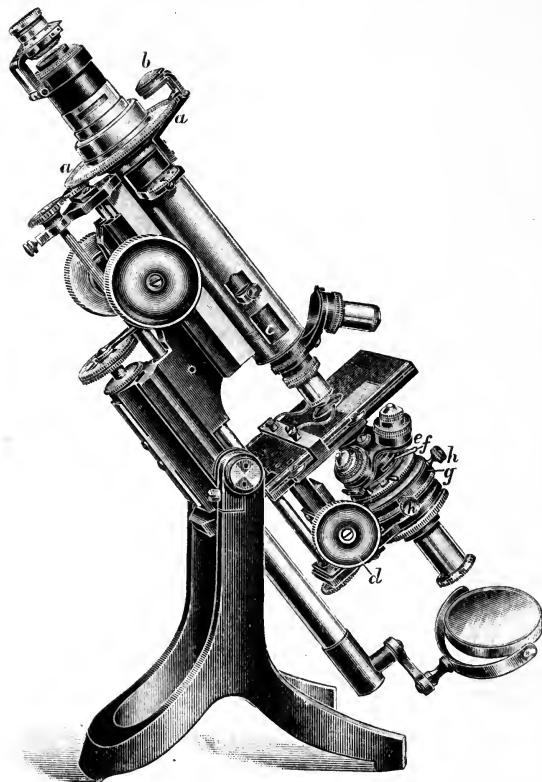


FIG. 674.—The Dick Microscope with Graham Sub-stage Fittings.

three mentioned are a 3-inch, 2-inch, $\frac{1}{2}$ -inch, $\frac{1}{8}$ -inch, and $\frac{1}{16}$ -inch, all dry lenses, and a $\frac{1}{2}$ oil-immersion lens.

A specially good $\frac{1}{4}$ -inch objective, shown in position in Figs. 672 and 673, marked *α* in the latter, is supplied by Messrs. Swift for use with convergent light, acting as the collecting lenses of the system. The converging lenses consist of a sub-stage condenser *ε* and an additional lens *η* in Fig. 673, to complete the convergence when necessary, as it usually is; this lens *η* is carried in the central line but nearer one end of a bevelled slider let into a corresponding dovetailed groove cut diagonally in the top of the rectangular fixed stage *o*. This slider can be pushed along so that the lens *η* is brought into or out of position by means of the little knob-handle *θ* belonging to an arm attached to the slider underneath the stage; an aperture 21 millimetres in

diameter, the same as that of the stage aperture itself, is also made in the slider, and this open aperture is centrally placed when no additional lens is required, and is always in position when the microscope is being used for parallel light observations. The sub-stage condenser is altogether removable when desired, screwing directly into the fixed stage, the central aperture of which is tapped with a corresponding screw thread to that carried at the top of the condenser tube. The achromatic condensing lens system is carried in a short inner tube sliding within the latter; a pin projecting from the inner tube fits in a vertical groove in the outer tube to prevent rotation of the lenses, and by means of a third outermost freely rotatable tube—carrying a large milled flange λ for convenience of effecting the rotation, and the inner bore of which has a spiral groove, into which the pin of the inner lens tube also falls—the condensing lens system can be raised or lowered nearer to or further away from the stage. Its normal position for use along with the additional stage-lens η is almost at the upper extremity of its path, quite close to the lens η . This vertical adjustment of the condenser is also very convenient when the latter is being used for ordinary microscopic sub-stage condenser work. An iris diaphragm, particularly useful in the latter operations, is also fitted at the lower end of the fitting, its opening being varied by means of a small handle projecting from and moving in a slot. The polarising Nicol k comes into position immediately below this diaphragm. The large size of this Nicol enables a particularly well-lighted field of interference bands to be obtained, with a crystal plate perpendicular to the first median line on the stage, excellent for photographic purposes.

The complementary collecting lens system consists of the $\frac{1}{4}$ -inch objective d and a plano-convex Bertrand lens mounted in a slider μ in the body-tube c ; the slider has two apertures, each of 17 millimetres diameter, one filled permanently by the Bertrand lens, and the other either left empty or filled with a removable Biot quartz plate, 3.75 millimetres thick and cut perpendicularly to the axis. This is the correct thickness and orientation to give the violet transition tint between the first and second order spectra, for use as described on p. 816 of Chapter XLIX. The slider is inserted in a short inner tube sliding up or down within the body-tube c , which latter is cut away here for the width of the slider and for rather more than the same height. This enables the Bertrand lens to be adjusted, by sliding up or down the body-tube, so as to obtain the clearest definition of the interference figure. A second Bertrand lens is also provided in another slider π higher up the body-tube c , and this one has fitted over it a little circular diaphragm pierced with six apertures of different diameters, the largest being 3.5 millimetres and the smallest 0.7 mm. The slider is fitted in an adjustable tube sliding in the body-tube, which is cut out for the width of the slider, as in the case of the lower one; the slider has also a second open aperture, which is in position when the Bertrand lens in the other aperture is not required.

The lower Bertrand lens is that usually employed for the production of interference figures. Together with the $\frac{1}{4}$ -inch objective, and with both the sub-stage condenser and the additional lens on the stage slider in position, it affords a magnificent interference figure with a crystal-plate cut perpendicularly to the acute bisectrix of the optic axial angle of a biaxial crystal or normally to the optic axis of a uniaxial one. The upper Bertrand lens gives a much smaller but very clear interference figure; it is intended for use with a small crystal, say a particular one in a rock section.

The course of the light rays in the microscope thus used as a "conoscope," or convergent light polariscope, is essentially as follows. The more or less parallel rays from the mirror at the foot of the microscope are rendered convergent by the sub-stage condenser, and still more so by the hemispherical or very short focus lens of the stage-slide. They then pass as a highly convergent cone through the crystal-plate, the apex of the cone being in the centre of the crystal when the adjustment is ideal, to the $\frac{1}{4}$ -inch objective, which collects them and brings about the formation of the interference image, which is about 5 millimetres in diameter, just above. This primary real image

is then reproduced by the Bertrand lens as a secondary real image just below the eyepiece, and this is magnified by the eyepiece as a large virtual image, of which the apparent plane is just below that of the secondary real image.

Stage Fittings.—Three important stage fittings are provided, which are readily attached when required, the two ordinary 'spring-clips for holding microscope slides on the fixed rectangular stage being easily removable, as they are simply carried by pegs fitting into two cylindrical holes in the stage.

The first is a simple **Mechanical Stage** for rapidly moving an ordinary 3-by-1-inch microscope object-slide about on the stage, so as to bring the desired object quickly into the centre of the field. It is shown at ρ lying on the mahogany baseboard in Fig. 670, and in position in Fig. 673. It consists of a narrow brass bar which slides over the fixed stage by means of two roller-wheels on a T-piece carried at one end of the bar, and which roll in a V-shaped groove in one side-edge of the stage, close contact and motion being both obtained by means of a short fluted cylindrical rod terminating in a milled head, resembling a screw in appearance but with flutings instead of a helix, which is carried at the other end of the bar so that the flutings may revolve, with sufficient friction to effect the movement of the whole mechanical stage, in corresponding closely parallel grooves in the other side of the fixed stage; the fluted rod is pivoted on a short arm pressed inwards by a spring, thus maintaining close contact of flutings and grooves. Rotation of the milled head causes the mechanical stage with its micro-slip to traverse the fixed stage in a direction parallel to the sides of the latter. Movement of the micro-slip in the direction rectangular to this, parallel to its own length, is effected by three rollers, one of which is carried freely at the end of a curved arm, and is maintained pressed against the micro-slip by a strong straight spring screwed at one end to the bar and pressing against the arm end, which is so shaped that a little force has to be expended in order to open out the bent arm sufficiently to insert the micro-slip in position, between the free roller at the end of the arm and the two other rollers, the force of the spring retaining it then in that position. The other two rollers are carried by the bar itself, which is here bridged over them sufficiently to give them free play when rotated; the rotation is effected by means of a second milled head at the end of a steel shaft, on which, opposite the two roller axles, are two portions of the thread of an endless screw, which gear with worm wheels carried on the axles. Rotation of this shaft by its milled head thus effects movement of the micro-slip in the direction of its length, while rotation of the other milled head with fluted axle moves the slip perpendicularly to its length.

The second is a **Measuring Stage**, a rotating stage with divided circle, and two divided rectangular measuring movements. It is shown lying on the baseboard at σ in Fig. 671, and in position in Fig. 670. This stage is intended for use when the ordinary manner of rotating the crystal is desired to be employed, instead of the simultaneous rotation of the Nicols and cross-wires, and also for the accurate measurement of lengths in two rectangular directions. The lower circle carries two projecting screw-pegs which fit into two holes in the rectangular fixed stage of the microscope, and when the whole fitting is laid on the latter with the two pegs in their holes it can be secured, and the lower circle thus rendered a fixed one, by screwing two milled-headed nuts from below on to the threads of the pegs. The rotatable circle p has a silver bevelled limb divided directly into degrees, and a silver vernier carried by the fixed circle enables the readings to be made to 5 minutes of arc. The two rectangular movements are effected by rotation of the large milled heads r and v , and each movement bears a silver scale divided directly into half millimetres, and reading with the aid of a silver vernier to the fiftieth of a millimetre, 0.02 mm. The upper movement is furnished at the top with a sliding object-slip holder, provided with tension springs for secure gripping of the slip. The whole of the movements are constructed to work very smoothly without backlash, and afford all that is required for either circular or rectilinear

measurement, the combination enabling the latter to be carried out in any azimuth. When it is desired to use convergent light with this stage, which is but rarely, as its chief uses are with parallel light, a duplicate convergent system, equal to the sub-stage condenser and the stage lens together, is provided. It is mounted in a second but much longer inner tube sliding in the sub-stage fitting just like the ordinary sub-stage condenser, having a guiding pin gearing with the slot of the fitting. The top lens can be brought by rotation of the milled flange λ (Fig. 673) flush with the surface of the measuring stage, or adjusted at any position below this.

The third is a **Stage Goniometer**, which is shown in position in Fig. 671, and as used with a cell of immersion liquid, and the microscope consequently arranged horizontally, in Fig. 672. It is a small Miers goniometer specially adapted by him

to the microscope stage, to which it is screwed by two milled-headed screws ϕ passing through wide slots in the stage-plate of the goniometer, broad washers being provided to cover the slots; a considerable amount of adjustment is thus permitted in the fixing. The stage-plate is cut away largely at its central part, in order to permit of the introduction of a circular cell for the highly refractive liquid, and also to afford free play to a long needle-shaped crystal-holder χ , carried axially at the end of the adjusting and centring movements ψ and ω of the goniometer. The fixed stage of the microscope, as already mentioned, is also cut away and scooped out in a conical manner to give full room for the play of the crystal-holder. The short rectangular elbow bracket of the stage-plate is attached to the

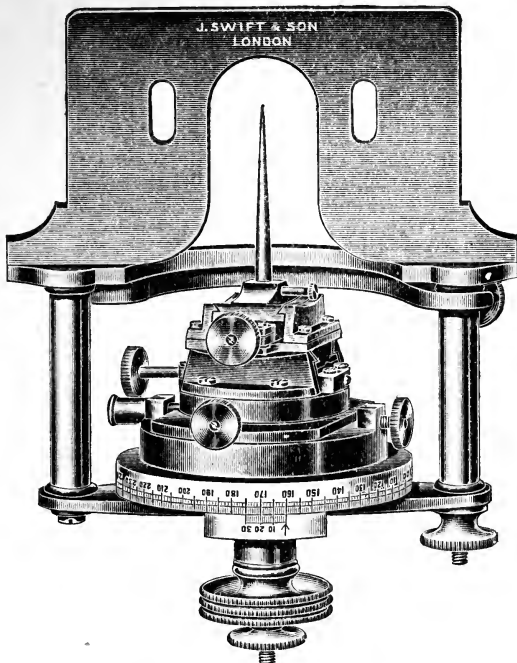


FIG. 675.—The Stage Goniometer.

goniometer by two columns κ , about one of which, hidden in the figures, it is pivoted, while the other column is attached in a slot in the bracket in such a manner that its position in the slot may be varied by a piston screw with milled head ι , a strong spring, curved so as to be well out of the way of the adjusting movements, maintaining the column and piston in rigid contact. This enables the whole goniometer to be moved, until the crystal-holder (the needle, to the end of which the crystal is attached by goniometer wax) is quite clear of the fixed stage of the microscope and its conical depression, in all positions during rotation of the goniometer circle ξ by means of the large milled head s . The circle is divided directly into half degrees, and reads with the aid of the vernier to single minutes.

The stage goniometer shown in Figs. 671 and 672 was constructed by Messrs. Troughton and Simms. In Fig. 675 is shown a similar stage goniometer constructed by Messrs. Swift, with modifications on that of Dr. Miers introduced by Prof. Bowman,

largely for the purpose of strengthening the parts and obtaining greater freedom of movement. Its construction will be quite clear from the figure.

Some further additions and modifications have been introduced into the Dick crystallographic microscope by Mr. G. W. Grabham, and the instrument embodying them, as supplied by Messrs. Swift, is shown in Fig. 674. In the first place the divided circle rotating with the Nicols, and divided to read with the vernier to $5'$, is placed at *a*, just below the eyepiece fittings, and is read by a lens *b*. In the second place an alternative analysing Nicol is fitted at *c* in the body-tube between the objective nose-piece and the Bertrand lens; it may be pushed in or pulled out of position with the aid of a little handle-rod on the other side of the instrument and therefore not shown in the figure. This is for use in the older way still preferable in many kinds of crystal work, and with which the rotating stage shown in Fig. 670 is so useful. The third feature is a focussing adjustment, by rack and pinion *d*, to the sub-stage, and the provision of a triple revolving-piece *e* carrying a battery of three different condensers, any condenser having the Royal Microscopical Society's standard screw being able to be attached to one of the three apertures of the piece just as any objective with the standard screw may be fitted to the nose-piece of the Dick microscope. An iris diaphragm *f* is provided as in the ordinary Dick instrument already described. A cell *g* for the reception of any one of a variety of stops for oblique and dark ground illumination is also provided, on a swinging arm, the mode of mounting also permitting of the rotation by the handles *h* of the cell and the stop it may contain. This arrangement is of great use in observing the shadow effects specially investigated by Becke, and which are indicative of the relative refractive indices of adjacent crystals in a rock section. A selenite or mica plate may also be fitted in this cell.

The three condensers found most useful are the Swift oil immersion of numerical aperture 1.40, the Swift dry condenser of unit numerical aperture, and a low-power lens for producing weak convergence. When parallel light is desired the latter lens is removed, and as the other two condensers give most of the necessary variety of angular field required this may conveniently generally be the case, so that the observer can change from parallel light to either degree of convergence instantly.

The instrument is otherwise like the ordinary Dick model already described.

Accessory Fittings to the Polarising Microscope.—There are a number of very useful and important accessories, which may be attached to either form of the Dick microscope. Their description will now be given.

The Ramsden Micrometer Eyepiece.—This is a positive eyepiece, composed of two plano-convex lenses with the convex surfaces inside facing each other, as shown in the section given above the general view of the micrometer in Fig. 676. The focus is outside the combination, fairly close to the second (larger) lens, the closeness depending on the power of magnification, which of course again depends on the curvature of the convex surfaces. The combination thus behaves as a single lens,

but is better than the latter as the spherical and chromatic aberrations are largely corrected. It focusses not only the real image formed by the objective at the top of the microscope body-tube, but at the same time the spider-lines of a micrometer with drum-headed screw, arranged in the plane of the image afforded by the objective. The whole arrangement is shown in Fig. 676.

The lens combination is carried in its own inner tube, which is adjustable, by sliding in an outer one attached to the top of the micrometer box, so as to focus both the spider-lines and the "comb" or scale of saw-like teeth and notches in a metallic plate, which forms a horizontal chord across the circular field (the plate cutting off the lower segment) somewhat below the centre, as shown in Fig. 708 (p. 892) in Chapter LIII. Every fifth notch is deepened, and from the central one, marked by an expansion of the notch at the bottom into a circular aperture, the fixed spider-line is stretched vertically across the field. A second movable spider-line, parallel to the first, is carried by a sliding frame, the movement of which is effected by the very fine micrometer screw of $\frac{1}{100}$ inch pitch, the drum and milled head of which are seen on the

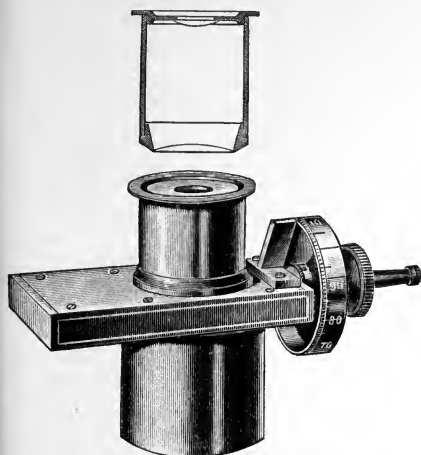


FIG. 676.—The Ramsden Micrometer Eyepiece.

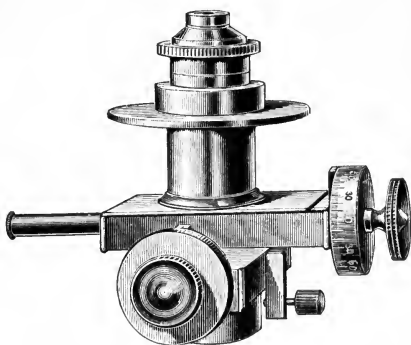


FIG. 677.—The Wright Double Micrometer.

right of the micrometer box in Fig. 676. The silvered drum is divided into 100 parts, every tenth being numbered and every fifth elongated. An indicating mark also on silver is carried by a projection from the box. When the spider-lines are coincident (one really just clearing the other in front but appearing coincident as seen through the eyepiece) the reading is 0° , and 100 drum divisions, one complete revolution, correspond to one notch of the comb, and about a fiftieth of a field diameter. Hence, the micrometer reads to the $\frac{1}{10000}$ of an inch. The wider lower tube below the micrometer box fits into the body-tube of the microscope.

The Wright Double Micrometer.—A micrometer on Ramsden's principle, but in which there is a second movable spider-line arranged at right angles to the ordinary one, the movement being measured in a precisely similar manner by means of a second screw with divided drum-head, is recommended by F. E. Wright for use in connection with Becke's methods of investigating crystals under the microscope, and is shown in Fig. 677. Its mode of employment will be fully discussed in the next chapter.

The Goniometer Eyepiece.—This is an eyepiece with rotatable spider-line and corresponding circle, shown in Fig. 678. The divided silver circle reads to 2° directly, and is fixed to a broad flange attached to the top of the lower tube of the eyepiece,

and with the silver vernier carried on the rotatory bevelled plate attached to and moving with the upper narrower eye-tube reads to half degrees. In the focal plane of the eye-lens, seen in section in its adjustable tube above the general representation of the eyepiece in Fig. 678, and in the same diametral line with the zero division of the vernier, a spider-line is stretched, which thus revolves with the vernier, while a second spider-line is permanently fixed diametrically from the 0° to the 180° graduation of the fixed circle, also practically in the focal plane, room being just left between the two lines for the one to rotate without touching the other. The lenses are arranged on the ordinary Huyghenian principle to be described in the next paragraph, as a negative eyepiece, and the focal plane of the spider-lines is that of the front, or eye-lens. This goniometer eyepiece thus enables the angle to be directly measured between any pair of crystal faces (on a tabular crystal) which are perpendicular to the plate and the microscope stage, thus forming the boundary faces of the plate and

appearing completely foreshortened as its edges. The fixed spider-line is laid parallel to, or better coincident with, one of the edges, and the movable spider-line is adjusted to coincide with the other. The crystal-plate must also be brought so that the corner formed by the intersection of the two edges is coincident with the intersection of the two spider-lines. The angle between the two faces is then at once given by the reading of the circle of the goniometer eyepiece.

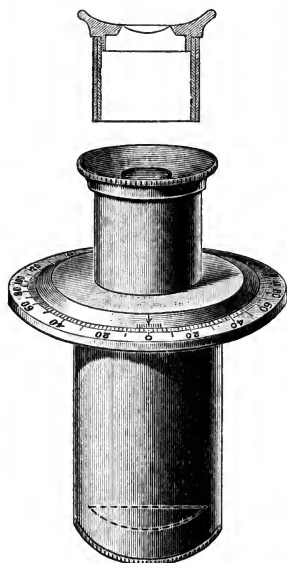


FIG. 678.—The Goniometer Eyepiece.

The Huyghenian Eyepiece provided for ordinary work with the microscope is shown in Fig. 679, and its arrangement of two plano-convex lenses will be clear from the figure, in which they and the diaphragm between them are represented by dotted lines. The two lenses both have their plane side facing the observer, contrary to the method of the positive eyepiece. The image is formed between the two lenses instead of outside

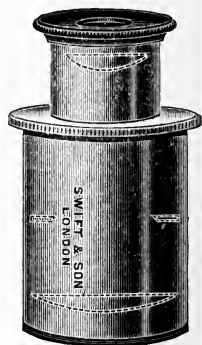


FIG. 679.—The Huyghenian Eyepiece.

both, and the whole arrangement is called a negative eyepiece. The larger lens further from the eye is known as the field lens, as it produces an image of the microscope field between itself and the eye-lens, which the latter then observes and enlarges. A diaphragm carrying the crossed spider-lines is arranged at the focal plane. The Huyghenian eyepiece has the great advantage of giving a wide field. The actual form given to it in the Dick microscope is shown at β (the eyepiece on the baseboard) in Fig. 670. The goniometer ocular just described is a special application of it, and another will be described in the next paragraph.

The Bertrand Quartz-Plate Eyepiece.—This is a Huyghenian negative eyepiece with a special compound quartz plate in its focal plane between the two lenses. Its appearance is just like that of the ordinary eyepiece, and while the latter is in position at the top of the body-tube in Fig. 670 the Bertrand eyepiece is the one marked β lying on the baseboard. The Bertrand eyepiece is employed for stauroscopic purposes, instead of a Calderon double-plate of calcite, that is, for the enhancement of the sharpness and accuracy of determinations of extinction. In the place of the crossed spider-line diaphragm

of the ordinary Huyghenian eyepiece a composite plate of right- and left-handed quartz, known as a "Bertrand quartz plate," is fitted. The arrangement of the quartz is shown in Fig. 680. The right and left quartzes occupy alternate quadrants, so that opposite quadrants are of like kind; these opposite quadrants are cut from the same plate of right- or left-handed quartz of 3.75 millimetres thickness, which has been cut and polished perpendicular to the optic axis. The rectangular edge-strips of the four quadrants are carefully polished and cemented together so as to form a truly plane-parallel plate absolutely perpendicular to the axis. The lines of junction are just visible, appearing like a pair of cross-wires, the purposes of which they admirably serve. When the Nicols are parallel, therefore, the transition violet tint afforded by a plate of quartz 3.75 mm. thick is produced, the whole field being evenly lighted and coloured, crossed by the rectangular fine black lines of division of the quadrants. The

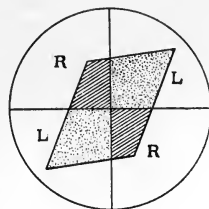


FIG. 680.—Effect of Introduction of Crystal on Even Field of Bertrand Quartz Plate.

least rotation of the Nicol, however, in either direction, or the introduction of a crystal-plate into the optical train, on the stage of the microscope, upsets the colour balance, unless the extinction directions of the crystal happen to be parallel to the vibration planes of the Nicols, and introduces red in one pair of opposite quadrants and blue in the other pair. Or if the Nicols are crossed, an even yellow tint is shown, which on the slightest rotation of the analyser or the introduction of a crystal plate, obliquely as regards its extinction directions compared with the planes of the crossed Nicols, passes into brilliant colours corresponding to opposite directions in the spectrum, a bright orange being produced in one set of quadrants and a deep bluish green in the other. The usual mode of use is to have the Nicols crossed, to place the corner of one of the quadrants, that is, the centre of the plate, to a corner of the crystal formed by the intersection of two edges, and one of the lines to one of the edges, to be used as the edge of reference; and then to rotate either the Nicols and eyepiece together in the manner of the Dick microscope, the crystal being fixed on the stage, or to rotate the crystal, the Nicols and eyepiece being fixed, until the even tint is again restored. The angle between the two positions, as measured in the two respective cases on the circle *n* of Fig. 673 or *a* of Fig. 674, or on the rotating stage circle *p* of Fig. 670, will be the desired extinction angle of the crystal-plate.

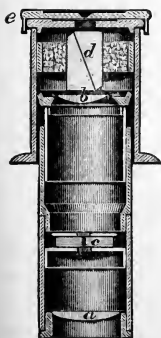


FIG. 681.—Calderon Double-Calcite-Plate Eyepiece.

This eyepiece is also useful with high powers for detecting double refraction and determining its sign in minute crystal fragments.

The Calderon Double-Calcite-Plate Eyepiece.—A special eyepiece fitted with a Calderon stauroscopic double plate of calcite similar to that employed in the von Groth stauroscope shown in Fig. 599 on p. 708, may, however, be employed equally well in stauroscopic determinations of extinction with the microscope. Such an eyepiece is shown in Fig. 681, and it includes a special analysing Nicol, which is not essential, however, the hinged ordinary analysing Nicol of the Dick microscope serving equally well, provided the eye-aperture is similarly restricted by a cap with small hole. The eyepiece is of the negative Huyghenian type, *a* and *b* being the two plano-convex lenses. The Calderon double-plate of calcite *c* is of similar construction to that of the stauroscope described in Chapter XLIV., and is enclosed between two diaphragms with somewhat small apertures, which afford the convenient amount of field and the upper one of which is focussed by the adjustable eye-lens *b*; the line of demarcation between

the two halves of the calcite plate forms the vertical diameter of the circular field. The analysing Nicol d is rotatable in the usual manner and carries a divided bevelled circle, rotating over a vernier-flange at the top of an outer tube not shown in the figure, which fits directly in the upper end of the optical tube of the microscope. Above the Nicol is fitted the cap e with very small circular aperture, which is absolutely necessary in order that the observations of extinction may be made for the precise direction of the microscope axis.

The Projection Eyepiece.—The large microscope of the Dick pattern now described, with an unusually large polarising Nicol such as has been fitted to the author's instrument, and an especially large analysing Nicol which is given with the instrument to fit directly over the eyepiece and replace the one at l in Fig. 670 (which latter can be thrown back on its hinge out of the way), serves admirably for projection purposes, and particularly for the photography of interference figures round the optic axes of crystals in convergent light. But for this object a special "projection eyepiece," one of a pair magnifying respectively three and six diameters, is advisable. This eyepiece is specially corrected to give a flat field on the screen or photographic plate. Such a projection eyepiece is shown in Fig. 682. The eye-lens, in this case the projecting lens, is a specially calculated achromatic combination of two lenses, which serves its purpose admirably. The draw tube containing it is furnished with a cap adapted for attachment to it of the large Nicol analyser; and it is provided with a spiral adjustment for the amount of its withdrawal

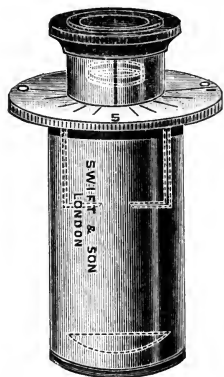


FIG. 682.
Projection Eyepiece.

out of the main tube, for accurately focussing a diaphragm in the latter on the screen or sensitive plate; the flat surface of the annulus closing the main tube is graduated to afford, with the aid of an indicator mark on the cap, a record of the proper adjustment for a given screen distance.

With the aid of these projecting eyepieces fitted to the Dick microscope here described, the interference figures in convergent polarised light given in this book have been directly photographed, some in white light and others in monochromatic light produced by the spectroscopic monochromatic illuminator described in Chapter XLIII., employing the electric arc as source of light; a Wratten and Wainwright K yellow screen was sometimes used in the case of the white-light figures, held in the free aperture of the slider π (Fig. 670), and either Wratten and Wainwright panchromatic or improved drop-shutter plates or Sanger-Shepherd B plates, and a Watson premier half-plate camera, were employed.

The Universal Stage of von Fedorow for the Investigation of Microscopic Crystals.—The methods of von Fedorow,¹ to which further reference will be made in the next chapter, depend on the use of an ingenious little stage apparatus, which is to the Miers stage goniometer what the three-circle goniometer is to the ordinary single-circle goniometer. It permits not only the usual rotation of the microscope stage, but also affords rotation about four other axes. In its latest form,

¹ *Zeitschrift für Kryst.*, 1893, 21, 574; 1894, 22, 229; 1896, 25, 113 and 351; 1896, 26, 225; 1897, 27, 337; 1898, 29, 604.

including two additional circular arcs due to F. E. Wright,¹ it is shown in Fig. 683, no less than four divided circles being provided.

The whole apparatus is carried by a stage-plate a with wide central aperture, which can be clamped to the fixed stage of the microscope by the screws b . From this stage-plate rise vertical brackets c , for the support in suitable bearings of the main horizontal goniometer-axis. Rotation of the milled head d to the left brings about rotation of this axis and of the vertical goniometer-circle V_1 carried directly and rigidly by it, and of the whole of the other circles carried indirectly about this horizontal goniometer-axis, which latter may be conveniently distinguished as axis 1 and is lettered correspondingly A_1 . The screw e serves to fix this axis at any position of the circle V_1 , as indicated by the vernier f . The second circle H_1 , which may be used as a horizontal ordinary rotating stage-circle if desired, is carried by the goniometer-axis A_1 , which thus forms a diameter; but the circle itself is rotatable, by means of its milled periphery, in its own plane, its position being indicated by a vernier g , and fixation in any position being possible with the aid of the clamping screw h ; the axis A_2 of this second circle H_1 is consequently at right angles to the goniometer-axis A_1 , and may

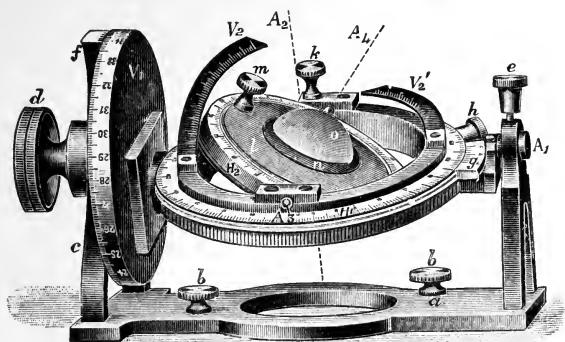


FIG. 683.—The Universal Microscope Stage of von Fedorow.

thus be called axis 2. The plane of this ring-circle H_1 obviously rotates with V_1 , while H_1 also rotates in its own plane. Within this in gimbal manner a second ring H_2 is swung, on an axle A_3 , which may be termed axis 3, lying in the plane of H_1 , and capable of fixation by the clamping screw k . Its altitude with respect to H_1 can be measured by a graduated arc V_2 or a similar one V'_2 on the other side, both being hinged so that the one not required can be turned down out of the way. Within this circle H_2 and in its plane a glass stage l is capable of rotation, the position being indicated by another vernier just hidden underneath out of sight, and fixation secured when desired by the screw m . The axis of rotation A_4 of this innermost disc may be called axis 4. In a metallic annulus n in the centre of this glass stage there fits a circular glass plate with truly plane and parallel polished surfaces, on which the crystal under investigation is simply laid when it is to be examined in parallel polarised light. When convergent light is to be used, for the measurement of the optic axial angle, a glass hemisphere o is laid over the minute tabular crystal or larger crystal section-plate as shown in Fig. 683, with a drop of a highly refractive liquid, cedar oil or glycerine being very suitable, to form a uniting film between the glass plate and hemisphere, and in which the crystal is immersed. A similar hemisphere is fitted

¹ *American Journal of Science*, 1907, 24, 343.

with its liquid film also beneath the glass plate, the two hemispheres thus together forming an optic-axial-angle apparatus on the principle of the method of Adams, which, while generally now discarded for the ordinary measurement of good-sized crystals, has found its real utility in microscopic optic-axial-angle goniometry. Two brass annuli, at the ends of pivoted or hinged arms and fitted with locking screws, serve to secure the two hemispheres in position.

Klein's Universal Stage Rotation Apparatus.—This is another very useful little stage apparatus, especially for the investigation of a small crystal in a cell of liquid the refractive index of which is approximately the same as the mean of the three indices of the crystal for sodium light. It is shown to the left in Fig. 684, as constructed by the firm of Steeg and Reuter. It enables measured rotations to be effected about three rectangular axes, the movements being of a similar nature to those of the ordinary goniometer, with the addition that the two circular adjusting segments movable in planes at right angles are

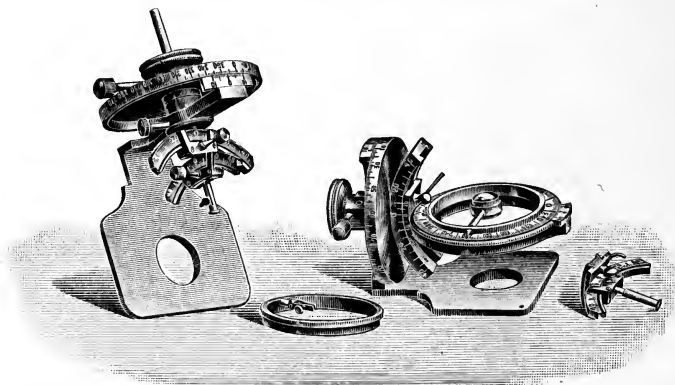


FIG. 684.—Universal Stage Apparatus of Klein.

graduated, as they are, in fact, in the case of the cutting-and-grinding goniometer described in Chapter XLII. It can be used equally for the determination of the extinction angles in a zone of faces, in the search for the optic axes, and for the measurement of the approximately true angle $2V_a$ between the optic axes, in an immersion liquid of the same refractive index as the crystal. An additional holder, shown more to the right in the figure, carries, instead of the crystal directly, a couple of Adams hemispheres, between which the crystal is enclosed with a drop of the liquid, the liquid cell being discarded when this special fitting is used for the measurement of the optic axial angle.

The Compensator of Babinet.—For the determination of the difference between the two refractive indices corresponding to the vibration (extinction) directions of a crystal-plate, that is, for the determination of the amount of its double refraction, a special fitting essentially involving a double-wedge plate of quartz and known as a Babinet compensator, is employed. It is placed under (behind) the Nicol prism

analyser in any form of measuring polariscope, but is most conveniently used as a special positive eyepiece, carrying the quartz-plate compensator at its focal plane, on the crystallographic microscope. The analysing Nicol prism is then carried above the eyepiece.

The compensator of Babinet consists of two slender quartz wedges, shown very much thickened, diagrammatically, in Fig. 685, and in Fig. 686, above in general appearance and below in section, as constructed especially satisfactorily as an eyepiece for the microscope by the Société pour la Construction des Instruments de Physique à Genève.

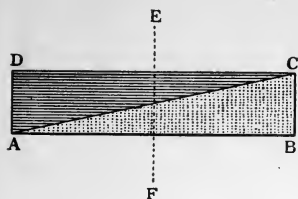


FIG. 685.—Babinet Compensator.

The wedge ABC is cut so that the optic axis of the quartz crystal is parallel to the edge of the wedge, which is perpendicular to the page in the figure, as indicated by dots in the shading, representing lines seen end on. The other wedge ACD is made of equal angle and so that the axis is perpendicular to the edge of the wedge and

parallel to the face CD. Two such wedges laid together form a parallel plate, which under crossed Nicols in parallel light when arranged at 45° to the vibration directions of the Nicols shows a black band in the centre parallel to the edge of the wedge, and

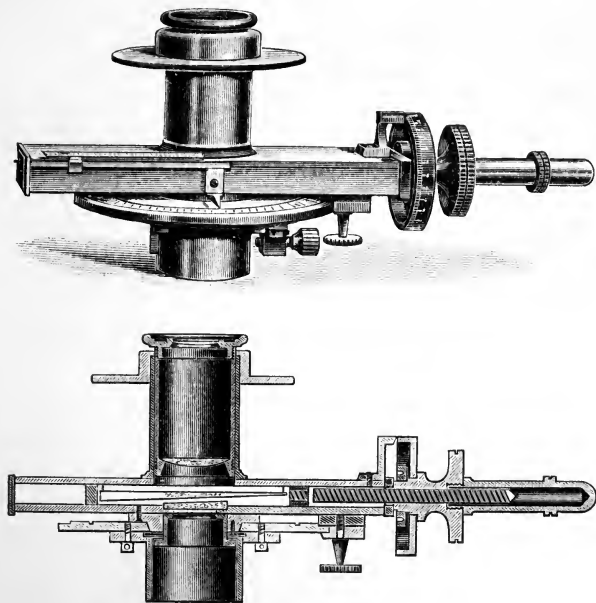


FIG. 686.—Babinet Compensator Eyepiece.

spectrum bands on each side, separated by white interspaces and showing the usual sequence of Newton's orders. This effect in white light is very much the same as that shown in Fig. 663 (p. 815) in Chapter XLIX., as produced by the four-wedges biquartz, the only difference being that the bands are there produced by thicker wedges and by utilising the optical rotational property of quartz, whereas here it is the double

refraction which is utilised. In monochromatic light Babinet's compensator shows equally spaced dark bands with luminous interspaces. When the Nicols are parallel and white light is used, a white band occupies the centre, and the colours of the other bands are complementary to those exhibited when the Nicols are crossed. In monochromatic light dark bands occupy the places where with crossed Nicols the bright interspace-bands occur.

The wedges are not cemented together, in the actual compensator, but are separated by a sufficient air layer to permit of the movement of one of them parallel to its base and in the direction of its length, by means of a micrometer screw furnished with a divided drum, the number of whole turns of which is also recorded by a scale on the rectangular micrometer box; in another form of the compensator, supplied to the author by Fuess for use with the large No. 1a spectrometer-goniometer, a divided numbering wheel, which turns one division for one whole revolution of the screw, replaces the scale. One of the quartz wedges is fixed, while the other, the upper one in Fig. 686, is mounted on a frame moved by the screw, so that the wedge which it carries is caused to move one way or the other over the fixed wedge. Just below the quartz a pair of parallel spider-lines are arranged, at the focus of the positive eyepiece, and the drum can be so set that its zero corresponds to the placing of the black band symmetrically between the two spider-lines, as shown in Fig. 708 (p. 892) in Chapter LIII., in the case of other interference bands. When the zero of the drum is thus arranged to correspond to the adjustment of the black band, the indicator of the scale or wheel should also be opposite a division mark. The whole compensator is rotatable about the axis of the microscope; it can readily, therefore, be placed so that the directions of vibration of the quartz are at 45° to those of the Nicols.

When thus arranged at 45° with the crossed Nicols the dark band seen in white light, in the centre of the parallel coloured bands, corresponds to the trace of the plane of equal thickness, EF in Fig. 685, of the two wedges, where compensation is complete for all rays and the dark field of the Nicols prevails unaffected as the net result. For plane-polarised light falling on either face of the composite quartz plate at 45° is separated into two rays vibrating respectively parallel and perpendicularly to the optic axis, that is, to the horizontal rectangular edges of the plate. On entering the second wedge the directions of vibration are retained but the velocities are interchanged, the ray which travels the faster in the first wedge travelling the slower in the second. Hence, if the thicknesses are equal there is no relative retardation of one ray behind the other, the sum of the two velocities for each ray being the same. As we recede from the centre, however, or as we move the line of equal thickness (as we do in the compensator), differences of phase will more and more manifest themselves, and wherever the difference amounts to a whole wave-length from the equality conditions of the central line there will be another dark band in homogeneous light, or a spectrum band in white light owing to the distances from the centre for the extinction of light of particular wave-lengths being different. When the Nicols are parallel the black bands in monochromatic light occupy the points corresponding to $\lambda/2$, $3\lambda/2$, $5\lambda/2$ and so on. That is, with Nicols crossed black bands occur at intervals corresponding to even multiples of the half wave-length, and with Nicols parallel to odd multiples of the half wave-length.

The Fuess Microscope.—The Dick method of simultaneously rotating the Nicols has now been adopted by Fuess in the more recently constructed and truly excellent models supplied by the firm. The fine instrument shown in Fig. 687 was constructed by Fuess for J. Hirschwald.¹

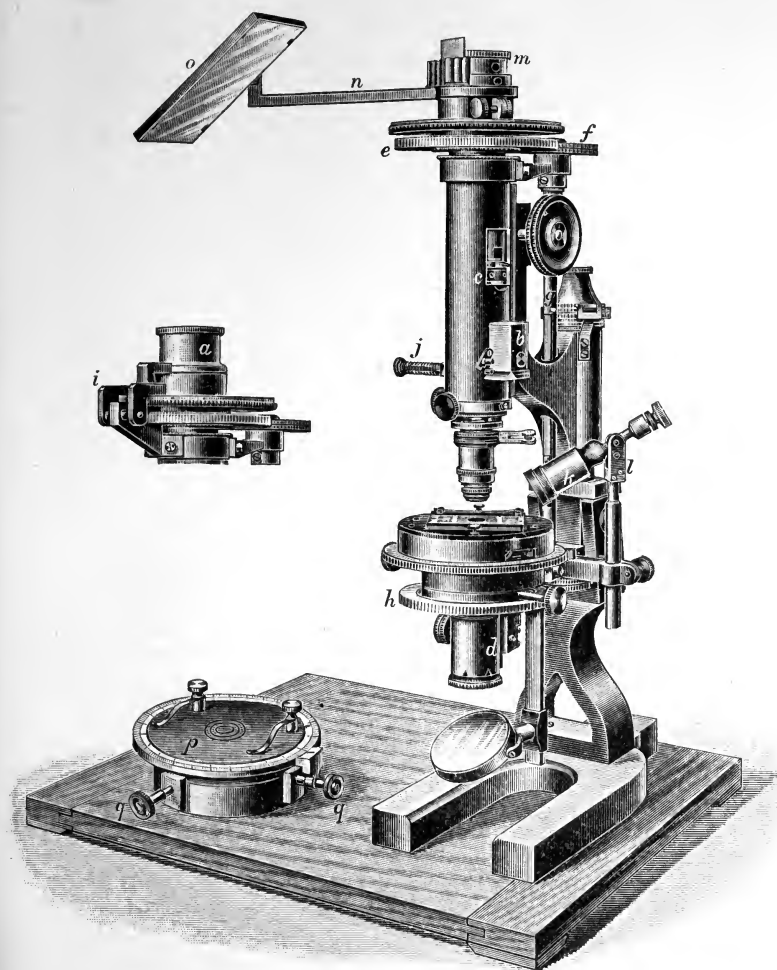


FIG. 687.—The Fuess Crystallographic Microscope, with latest Improvements by Hirschwald, and the Abbe Camera Lucida and Becke Drawing Table in Position.

It has the novelty of a second inner body-tube, which is rotatable and carries with it the eyepiece *a* (which is shown separately to the left in Fig. 687, as the main figure shows the Abbe camera lucida in position covering the eyepiece, as described in the

¹ *Centralblatt für Mineral.*, 1904, 20, 626.

next section), the analysing Nicol *b* which is mounted in the old place just above the objective, in order to avoid restricting the field, and the Bertrand lens *c*; the polarising Nicol *d* also rotates simultaneously with it, the rotation being effected either by the large toothed wheel *e* which rotates a smaller one *f* attached to the shaft *g*, or by the similar large wheel *h* at the lower end of the shaft and which carries the polarising Nicol. Sometimes it is inconvenient to rotate the eyepiece, and in this case it can be locked to the outer body-tube by the arrangement shown at *i*. When unpolarised light is desired, the analysing Nicol *b* can be pushed out of the two body-tubes by means of a spring handle *j*. Also, the polariser may be kept stationary, while the analyser and eyepiece with its cross-wires rotate, or the polariser and the eyepiece may both be maintained stationary while the analyser alone rotates. The eyepieces employed on this instrument afford a much larger field than the older ones supplied by Fuess, so that equal results may be obtained with lower-power objectives, and with very little loss of definition. The Bertrand lens can, of course, be pushed out of both tubes when convergent light is not required. A further novelty on this microscope is an electric glow-lamp mounted above and to one side of the stage in a small tube *k* closed by a condensing lens, in order to effect the illumination of opaque crystals from above; it is held in such a manner by the fitting *l* as to be readily directed in the most favourable manner for attaining the illumination desired.

The Becke Drawing Table and Abbe Camera Lucida.—For the microscopic investigation of crystals by the methods which have been described and recommended by F. Becke¹ a camera lucida and drawing table are required, and in the form in which these further accessories are constructed by Zeiss they are also shown in Fig. 687, as combined with the Fuess crystallographic microscope just described. The method of Becke consists essentially in preparing a graphical representation on a stereographic projection of the optical phenomena observed with a crystal-plate, as will be more fully described in the next chapter. By means of an Abbe camera lucida the interference figure in convergent polarised light is rendered simultaneously visible with the drawing paper laid on a small rotatable drawing table, arranged on the large baseboard on which the microscope is mounted; this drawing table can be accurately centred by means of three screws placed at intervals of 120°, or two screws at 90°. The rotating plate of the drawing table has a series of concentric circles engraved on it, in order to facilitate rapid centring and to obtain identity of the axis of rotation with the centre of the image of the interference figure.

The camera lucida *m* is fitted over the eyepiece *a*, and is connected by means of a supporting arm *n* with an adjustable mirror *o*, which reflects the light rays from the drawing table *p* to the camera lucida, the phenomena visible to the eye looking down the microscope being thus as it were projected on the image of the drawing paper, and the pencil point held on the latter will also be in focus at the same time. An accurate draw-

¹ Tschermak's *Min. petr. Mittheil.*, 1894, 14, 563; 1896, 16, 180; 1905, 24, 35.

ing of the appearance of the interference figure can then be made. Such records enable geometrical constructions to be made and graphical methods of calculation to be employed, which result in approximate determinations of the optic axial angle and other optical constants being arrived at in cases which would otherwise be hopeless.

The Abbe camera lucida has the advantage of focussing both the drawing and the pencil equally, even under considerable magnification. The optical part consists of the "Abbe cube," composed of two equal totally reflecting rectangular and 45° glass prisms, cemented together and placed at the eye position above the eyepiece. The cemented surface of the upper prism is silvered, except within a central circle of two millimetres diameter, through which the rays from the field of the microscope emerge to the eye looking through. At the same time the drawing table is reflected in the silvered mirror, together with the pencil in the observer's hand when drawing. The brightness of the image can be modified by a smoked glass, a small rack for three such smoked glass plates being provided. The accuracy of the method depends on the faithfulness of the drawing, the precision with which the drawing is centred by the screws q , and the care with which measurements are subsequently made on the finished drawing.

Spectroscope for Use with the Microscope.—An efficient yet very simple direct-vision spectroscope is furnished with the Dick microscope by Messrs. Swift, for the study of the absorption bands of crystals under the microscope. It consists of two parts, shown in Figs. 688 and 689. The first part (Fig. 688) is the direct-vision spectroscope itself, which is attached over the eyepiece, the large analysing Nicol prism referred to on page 840 being fitted over it if desired. The second part (Fig. 689) consists of an adjustable slit, which fits as a slider in the

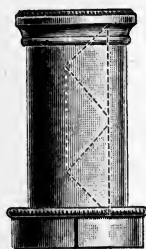


FIG. 688.—Direct-vision Spectroscopic Attachment.

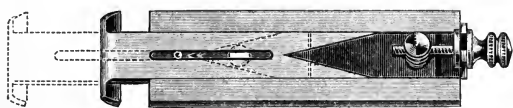


FIG. 689.—Adjustable Slit of Spectroscopic Attachment.

slot (w in Fig. 670) cut in the eyepiece tube. The construction will be obvious from the figure. Even crystals affording but relatively faint bands can readily be studied as regards their light absorption with the aid of this spectroscopic eyepiece fitting.

Microscope for Higher Temperature Work and for the Study of Liquid Crystals.—This chapter on the crystallographic microscope may well close with a brief description of an instrument specially adapted for microscopic work on crystals at higher temperatures. It is shown in Fig. 690, and is constructed by Messrs. Steeg and Reuter of Homburg; it is particularly suitable for the study of the remarkable liquids which afford the phenomena of "liquid crystals," which have been so ably

studied by Lehmann, and an account of which will be given in Chapter LV.

The whole instrument and its heating and cooling arrangements are designed to effect changes of temperature as rapidly as possible consistent with safety. Every precaution is also taken to protect the optical and finely constructed parts from the heat and from condensing

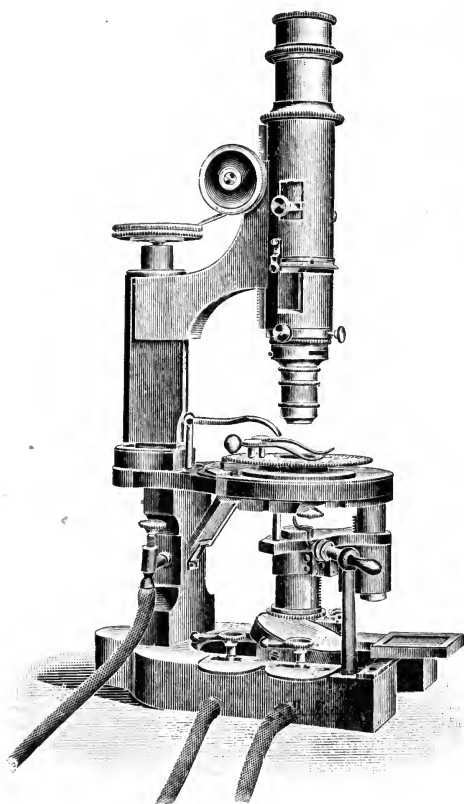


FIG. 690.—Steeg and Reuter Microscope for the Study of Crystals at Higher Temperatures.

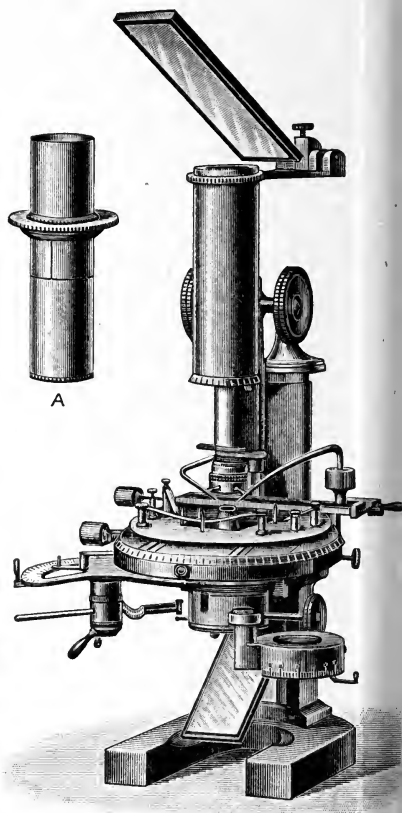


FIG. 691.—Zeiss Microscope for Higher Temperature Work.

vapours. Hence, it is possible with the instrument to continue the study of the phenomena up to a low red heat. There are two object stages, the lower completely covered one being divided into single degrees, and rotatable. The upper one may be rotated with or independently of the lower, and is removable to facilitate thorough cleansing after the operations. There are provided two blow-pipe gas jets, two cooling air jets, one for slow and the other for rapid cooling, together with the necessary gas and blow-pipe connections

provided with finely adjustable taps. Between the polarising prism and the stage a capillary-tube ring of miniature gas jets can be inserted, and the rise of temperature may be effected either with great slowness or very rapidly, as desired. The usual fittings, such as an analysing Nicol, Bertrand lens, convergent light condenser, all readily detachable, together with a coarse and a fine adjustment, are all given on this useful microscope, with which most of the experiments described by Lehmann may be repeated.

A very efficient microscope for higher temperature work, especially suitable not only for the study but for the projection of the phenomena of liquid crystals, has recently been perfected by Zeiss. Its latest form is represented in Fig. 691, as arranged for projection experiments, the eyepiece shown separately to the left at A being replaced by an adjustable mirror for projecting the image of the microscope field on the screen. A detailed description of the instrument will be given in Chapter LV. in connection with the discussion of the work and demonstration experiments of Lehmann.

Projections with the Microscope.—The Dick microscope, when supplied with larger polarising and analysing Nicols than usual and a set of projecting eyepieces, as in the case of the author's instrument, may be employed for screen projections of ordinary temperature crystallographic phenomena, when a disc not exceeding three feet in diameter is desired. It is also eminently suitable for photographic projection purposes, as already mentioned.

But screen projections on a larger scale with the polarising microscope, which have ever been difficult on account of the large amount of light lost by polarisation, may be very efficiently carried out with the form of projection polarising microscope shown in Fig. 692. It is very simple, yet the author, who has spent much time in investigating all the possible methods of screen projection of optical phenomena, and in personally testing all the apparatus yet introduced for the purpose, finds it more efficient than any other. A 6 to 8 feet disc may readily be obtained brilliantly illuminated, of crystals in polarised light, employing a Newton $1\frac{1}{2}$ -inch projection objective; moreover, almost equally good results are obtained with the $\frac{1}{16}$ -th inch higher-power Newton projection objective. No eyepiece, but an adjustable sub-stage condenser also supplied by Messrs. Newton is used, to concentrate the parallel beam received from the polarising Nicol. The latter, with its parallelising concavo-convex lens at the lantern end, the lantern itself with its condenser and water-cell in front, are precisely as already described in Chapter XXXIX. for projections in polarised light, but the second large analysing Nicol may be quite well replaced by a smaller Nicol, such as is shown at the extreme right in Fig. 692, in position after the objective; for the beam is very narrow as it leaves the objective. As a matter of fact there is practically nothing to choose between the screen effects produced with the large or the small analyser, so that whichever is most convenient at the moment can be used. The microscope fitting is simply arranged as an additional accessory on one

of the adjustable stands sliding in the guiding-bed basal plinth or lathe-bed of the projection apparatus, the particular stand having a longer base than the others (except those of the Nicols), in order to secure rigidity. The stout carrying rod rising from within the hollow supporting column, in which it is adjustable for height by a fixing screw, terminates above in a horizontal tube with flanged head, the two forming a T-piece. The cylindrical bracket-column of the microscope fits within

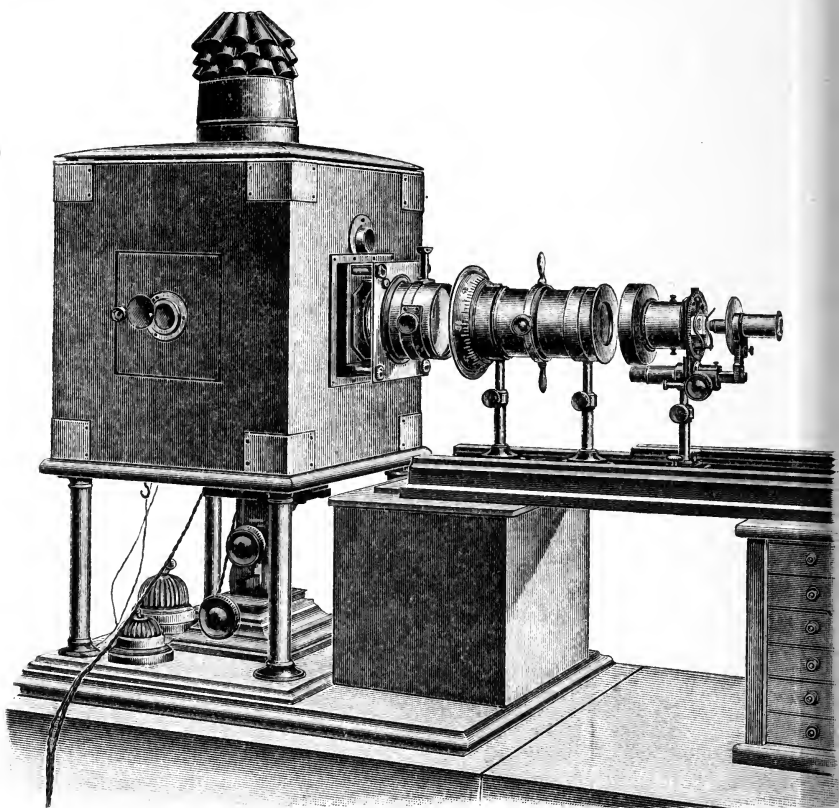


FIG. 692.—Arrangement for Screen Projection with the Polarising Microscope.

this tube, and is secured to the flange by two milled-headed screws. The objective and analysing Nicol are carried by a second bracket, the column of which is of triangular section fitting and moving by rack and pinion in a similarly shaped boring in the fixed bracket-column just referred to, which carries the stage and sub-stage condenser. The objective may thus be racked to the proper focus in the usual manner, a circular metal plate being provided immediately after it to ward off from the screen any stray reflections from the illuminated object or parts of the apparatus. The fitting of the small Nicol is hinged so

as to enable it to be rotated out of the way when not required. The stage is rotatable and graduated, and the sub-stage condenser in its mount is adjustable in an outer tube to the most efficient position, found by experiment to give the best results on the screen. With the ready means afforded by the adjusting table of the lantern lamp for centring the electric arc, and for adjusting its distance from the condenser, every desirable adjustment of the illumination is provided for.

The above microscope fitting may be also attached directly to the front of the water-cell when it is not desired to employ polarised light, but for crystallographic purposes this latter is usually desirable and necessary, so that the arrangement shown in Fig. 692 is the most convenient one. It has the further advantage, besides being simple and involving few lenses with the corresponding few occasions for loss of light by reflection, of keeping remarkably cool, being sufficiently far removed from the lantern, and being unenclosed in restricting and unventilated tubes.

Provision is also made for use of the fitting vertically upright, for experiments involving the direct projection of crystallisations from solutions. The stout columnar supporting rod (shown in Fig. 692) which passes down into the hollow column of the stand is replaced for vertical work by a longer column, and the tube and flange for the reception and fixation of the microscope bracket-column form a vertical rigid continuation of the column itself. The fitting is thus raised also sufficiently high to enable an adjustable mirror to be fitted to the column at the height of the polarising Nicol. The parallel beam of light is reflected by the mirror vertically upwards through the sub-stage condenser and microscope, and the light rays issuing upwards from the latter are projected to the screen by means of an adjustable totally reflecting rectangular prism, which is suspended by a bracket just over the polariser. Some loss of light is always inevitable in vertical reflected projection, but the results are still adequate as regards illumination with a 6-feet disc, and very brilliant with a 5-feet disc.

The Zeiss heating microscope shown in Fig. 691 may very conveniently be used with the electric lantern disposed as in Fig. 692, for the projection of liquid crystals or other experiments with crystals at higher temperatures. The water-cell is left in position on the lantern front, but the large Nicol and the author's projection microscope are removed and the Zeiss microscope substituted, raised on a rigid stand to the right height above the plinth bed. The concavo-convex parallelising lens is detached from the end of the large Nicol, and separately carried on one of the adjustable stands, in order that a parallel beam of light from the lantern may be directed to the mirror at the foot of the microscope. The light is thence reflected upwards through the microscope to the mirror at the top of the latter which is rotated and inclined to the right position which enables it to direct the rays to the screen.

CHAPTER LII

OBSERVATIONS AND MEASUREMENTS WITH THE CRYSTALLOGRAPHIC MICROSCOPE.

A GREAT deal of attention has lately been given to the determination of the characters of minute crystals, with the view of rendering the results more than merely approximate and more comparable with those obtained with crystals of larger size on the first-class measuring instruments described in the foregoing pages. The work of F. Becke, von Fedorow, and Klein stands out prominently in this domain, all of whom have devised microscope stage instruments, and the two former elaborated original methods of attack, which enable the optical constants of minute crystals or crystal fragments to be determined with a remarkable degree of accuracy, considering the difficulty of the undertaking. The stage goniometer of Miers and the other useful fittings which, as well as the stage accessories just referred to, have been described in the last chapter, together place at the disposal of the crystallographic investigator methods of far greater accuracy than formerly, and no investigation need be abandoned because of the minuteness of the crystals available.

Measurement of Crystal Angles with the Microscope.—The Miers stage goniometer may be employed for the measurement of the angles of microscopic crystals by the "illuminated face" method or "Schimmer-methode" of Brögger. The rays from a small collimating lens, with a bright spot or slit as source of light at its focus, are directed on the miniature crystal adjusted on the wax at the needle end of the axis of the stage goniometer, and the zone of faces to be measured is adjusted till the maximum brilliancy of illumination of each face is observed on rotation of the crystal axis of the goniometer. The angular position of each face is then determined by rotating the circle with the crystal till the face becomes just bright. It is then rotated further until the illumination of the face just disappears. Two or three further approaches are then made to these two positions, from the two respective sides. The mean of an equal number of readings for these positions on the two sides of maximum illumination will then give a very fair approximate value for the position of the face, when the latter, although minute, is good.

Another interesting and more accurate method of measuring crystal angles under the microscope has been devised by von Fedorow, for use with his universal stage (Fig. 683, page 841), but which is equally applicable for use with the Miers stage goniometer. A black cross is marked on the under surface of the microscope objective to be employed, and light is reflected down the microscope axis from a lamp placed to one side, by means of a glass plate, of the character of the Gauss ocular or the Becker fitting described on page 34 and illustrated in Fig. 13. When the crystal is adjusted, on a special crystal-holder which replaces the central glass hemisphere of the stage, with a face perpendicular to the microscope axis, and is clearly focussed, the cross is invisible; but on lowering the microscope so as to halve the distance between the crystal face and the objective—which is accurately possible with the aid of the scale (shown in Fig. 673) on the guide of the slider of the body-tube worked by the coarse adjustment, or if a higher power is being used by means of the graduated head-wheel of the fine adjustment—the cross becomes visible by reflection from the face, and can be used as a signal and placed to the cross-wires. The crystal is then rotated until the next face of the zone adjusted affords a similar reflection, and this also is placed to the cross-wires, the angle of rotation being measured on the goniometer vertical circle V_1 in Fig. 683, representing the universal stage, or the circle of the Miers stage goniometer, in the usual manner. The von Fedorow universal stage may be regarded as a three-circle goniometer applied to the microscope, while the Miers stage goniometer is the ordinary single-circle goniometer similarly adapted for microscopic measurement. Owing to the difficulty of re-setting a minute crystal on the needle-holder of the latter, and the obvious advantage of reducing the number of such settings, required for the measurement of a number of zones, the universal stage has here a considerable advantage, however.

The goniometer eyepiece shown in Fig. 678 may also be used to measure the crystal angles in any zone of faces on a tabular crystal so small as to be best investigated under the microscope, when the faces in question are perpendicular to the tabular plane. The method has been adequately described on page 838 of the last chapter.

The Determination of Refractive Indices with the Microscope.—The mean refractive index of a birefracting crystal, or the absolute refractive index of a cubic crystal, can be determined under the microscope by the method of the Duc de Chaulnes. This depends on the fact that the focus of a microscope alters when a transparent plate with truly plane and parallel surfaces is introduced between the objective and the object focussed. The amount of the change d is connected with the thickness t of the plate, and with the refractive index μ of the substance of which the plate is composed, by the formula :

$$\mu = \frac{t}{t - d}.$$

Let $a b c d$ in Fig. 693 represent such a plate, and F be the focus

of the microscope when the plate is absent. When the plate intervenes the rays from the focus F appear to come from the point F' , and the microscope has to be withdrawn by the amount FF' . The case before us is where F lies on or in contact with the lower surface of the plate. The apparent thickness t' of the plate, where $t' = t - d$, is the quantity experimentally determined, by focussing first the upper surface—utilising surface polishing marks or dust particles, or better still marks silvered on the plate, as objects to be focussed—and then the lower surface as seen through the plate; the real thickness t of the plate is also measured either before or afterwards with the aid of a thickness measurer such as that described on page 899 and illustrated in Fig. 710, or else with the microscope itself used as a thickness measurer. In the latter case the ordinary microscope stage is covered by a thick slab of truly parallel glass, bearing a scratch or other defining mark such as a little engraved ring in the centre, as reference mark to be focussed. The plate of the

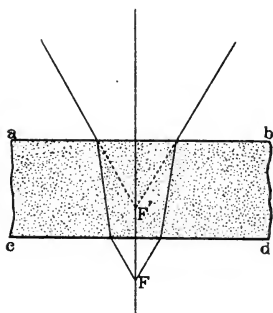


FIG. 693.—Principle of de Chaulnes.

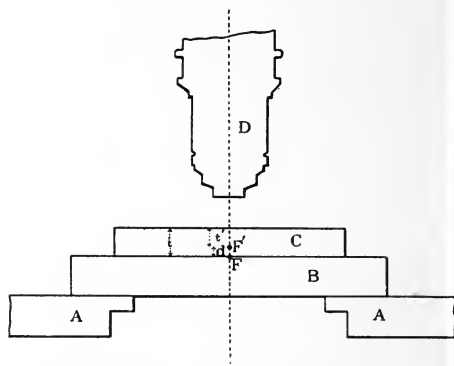


FIG. 694.—Conditions in the Microscopic Determination of Refractive Index.

crystal is then laid on this glass slab, taking care to exclude air by sliding the crystal-plate on, and its upper surface is then focussed by its polishing scratches or natural markings. The fine adjustment is employed in moving the microscope between the two focussed positions, and the difference of the readings of the divided head and the total number of its whole revolutions afford the required thickness. When the crystal-plate is thicker than the extent of the fine adjustment movement the scale of the coarse adjustment can also be requisitioned, but usually this will be unnecessary. Now it can be proved that $t = \mu t'$. Hence, $\mu = t/t'$.

In the actual determination, when accurate results and not a mere approximate value are desired, a fairly high power requires to be used, not lower than a $\frac{2}{3}$ rd-inch; and the plate must not be too thick to permit of the lower surface being focussed by such a power. The excellent fine adjustment of the Dick microscope, reading to the thousandth of a millimetre, is particularly convenient and suitable for

the measurement under these circumstances. The conditions are clearly exhibited in Fig. 694. A reading is first taken for the focus of the reference mark F on the basal slab B, laid on the ordinary stage A; the crystal-plate C is then laid in position and the mark refocussed by raising the microscope and its objective D by the necessary amount FF' , that is d , and a second reading is taken for this position; lastly the upper surface of the crystal-plate is focussed and a third reading taken. The difference between the first and third readings gives t , the real thickness of the crystal-plate, while that of the second and third gives t' , the apparent thickness of the plate, as will be evident from Fig. 694. Then the desired refractive index of the crystal μ is at once obtained by dividing t by t' . The amount of raising, the value of $d = t - t'$, increases not only with the thickness of the plate but with the refractive index.

When the crystal is a doubly refractive one, it will obviously only be the mean refractive index which will be afforded by this determination. Sorby¹ in the year 1877 endeavoured to render it possible to determine the two different refractive indices of the two transmitted rays by use of a glass micrometer or crossed grating, consisting of fine lines ruled $\frac{1}{100}$ th of an inch apart in two directions at right angles to each other. For the two sets of lines focus at different positions, the theory for which was worked out by Stokes. An excellent account of Sorby's work brought up to date has appeared in a recent number of the *Mineralogical Magazine*,² the manuscript of which was prepared for the press by Sorby himself during his last illness. The lattice-plate was placed some little distance underneath the stage, with an iris diaphragm just above it and an achromatic condenser provided with a small stop, immediately above, which focussed the lines at about the level of the stage. Below the grating a rotatable Nicol prism polariser could be thrown in or out of position, and an analysing Nicol similarly over the eyepiece. A cap pierced by a small hole was fitted over the objective in order to reduce the residual spherical aberration by stopping down to an aperture of 13° , and also a second cap was arranged to act as a semicircular stop, in order to cut off half the area of the objective field in any desired direction, the aperture of the front-lens being circular. A scale and vernier attached to the stand and body-tube of the microscope completed the essentials. The reduced image of the lattice and of the aperture of the iris diaphragm, the latter adjusted to the most suitable small opening, could thus be readily focussed by the condenser on the under or upper surface of the plate of the doubly refracting crystal, or anywhere in its interior. According to the orientation of the section-plate with respect to the original crystal the appearances of such a grating are characteristically different.

For instance, when a plate of calcite cut perpendicularly to the axis is laid on the microscope stage and the grating examined through it, there are two positions, one exactly above the other, at which the

¹ *Mineralog. Mag.*, 1877, 1, 97.

² *Ibid.*, 1909, 15, 189.

image of both sets of lines of the lattice-grating is focussed by the microscope, rotation of the plate in its own plane making no difference.

A cleavage plate of calcite, however, affords two images separated along the direction of the principal section of the plate, the one corresponding to the ordinary ray being in the centre of the field, and affording the ordinary index $\omega = 1.658$. On rotating the plate in its own plane, this latter image remains central and undistorted, while the other corresponding to the extraordinary ray moves with the principal section and is elongated and coloured at the edges. When the principal section is parallel to either set of grating lines that set is alone visible, and the microscope tube has to be raised or lowered to focus the two images in such rectangular positions. The extraordinary image is thus bifocal, and is thereby distinguished from the ordinary image which is unifocal, both sets of lines being simultaneously visible in the light of the ordinary ray at one and the same focus. The Nicol analyser can conveniently be used to discriminate between the ordinary and extraordinary images. The extraordinary image of which the lines are parallel to the axis is well defined and indicates an index of refraction of the value 1.412, while the image composed of lines perpendicular to the axis is blurred, the lines appearing as coloured bands in white light, but in red light they are sharp and afford the refractive index 1.578. The true extraordinary index is intermediate between the two, namely, 1.486. Hence the extraordinary image is truly bifocal.

A plate of calcite parallel to the axis shows three images lying over each other, which are best separated by use of the Nicol prism analyser over the eyepiece. When the vibration direction of the Nicol is perpendicular to the axis of the crystal the ordinary image only is seen, and both sets of lines are equally focussed at once, and afford the ordinary index $\omega = 1.658$. When the Nicol is arranged with its vibration direction parallel to the axis only the extraordinary images are seen, one set of lines at one focus and the other at another, the image of the diaphragm being elongated in perpendicular directions for the two positions. When the lines are seen parallel to the axis the true extraordinary index $\epsilon = 1.486$ is afforded. The index for the perpendicular direction is no less than 1.868, which agrees with the calculations of Stokes. The order of the three foci below one another, on lowering the microscope tube is as follows:—(1) the extraordinary image elongated perpendicularly to the axis and with only the lines perpendicular to the axis visible, (2) the ordinary image, (3) the extraordinary image elongated parallel to the axis and with only the lines also parallel to the axis showing.

The section-plates of calcite employed by Sorby in these interesting experiments varied from 0.3 to 0.4 inch in thickness. The objective found most convenient was a $\frac{2}{3}$ -inch, with a No. 2 eyepiece. The use of the semicircular stop over the objective front-lens has the effect of causing the ordinary ray to be well polarised in a plane perpendicular to the edge of the semicircular stop, and the extraordinary ray in the opposite sense, thus facilitating their identification.

It is remarkable that although quartz has such a weak double refraction yet it can be clearly recognised by Sorby's method. In a section $\frac{1}{8}$ th of an inch thick, cut perpendicularly to the axis, it is just possible to distinguish two foci, and in one cut parallel to the axis to see that there is both a unifocal and a bifocal image.

In the case of a plate of a biaxial crystal—rhombic aragonite and barytes being good examples investigated by Sorby, section-plates about $\frac{2}{3}$ ths of an inch thick having been used—cut perpendicularly to one of the principal axes of the optical ellipsoid, two bifocal images are always observed, comparable to the two extraordinary images seen in the case of calcite or other strongly doubly refractive uniaxial crystal. In the case of calcite cut perpendicularly to the axis neither the ordinary nor the extraordinary image is polarised, unless the semi-circular stop be used in front of the objective; but in the case of a biaxial crystal cut perpendicularly to an axis of the ellipsoid the phenomena resemble more closely those seen in a plate of calcite cut parallel to the axis, where the extraordinary ray is clearly bifocal, which it is not in the case of the section perpendicular to the axis. Also with a biaxial crystal both the rays have this bifocal extraordinary character. The phenomena with biaxial crystals are thus very complicated, there being usually four different apparent indices of refraction. From the total results with several plates cut perpendicularly to at least two axes of the ellipsoid Sorby was able, however, to calculate the three true indices, α , β , and γ from these apparent indices, with very fair accuracy to the third place of decimals.

The following laws were eventually arrived at by Sorby as to the general relations between the images observed and double refraction.

1. Crystals having no double refraction have no bifocal image.
2. Crystals having one axis of double refraction have one bifocal image.
3. Crystals having two axes of double refraction have two bifocal images.
4. Other circumstances being the same, the distance between the foci varies directly as the intensity of the double refraction, and directly as the thickness of the specimen.

The only exceptions are in the cases of very strongly dichroic substances like tourmaline, in which one ray is absorbed; tourmaline transmits only the extraordinary bifocal image, the ordinary ray being invisible through very dichroic green specimens, although slightly visible in less dichroic specimens.

There is no difficulty in carrying out Sorby's method in the cases of unifocal images, such as either the single one of a cubic crystal, or the ordinary image of a non-dichroic uniaxial crystal. For the apparent index of refraction is here also the true index, whatever be the orientation of the plate.

In the case of the single extraordinary bifocal image of a uniaxial crystal the true extraordinary index μ' can be conveniently determined, when the section is either parallel or perpendicular to the axis, from the

equation $\mu' = \sqrt{A\mu}$, in which A is the apparent index, the ordinary index μ being first determined and the two images being distinguished by use of the semicircular stop over the objective. When the section is parallel to the axis, the apparent index a for lines parallel to the axis and perpendicular to the plane of polarisation is actually the true index μ' of the extraordinary ray; it may also be calculated from the other apparent index b by means of the equation $\mu' = \frac{\mu^2}{b}$. The value of

μ' is thus obtained in duplicate from two different measurements, and when the results are concordant the mean may be accepted as the truth.

When the crystal has two bifocal images, polarised in opposite planes, as in the cases of biaxial crystals, giving the four apparent indices a, b, c, d , the three true indices of refraction μ, μ', μ'' may be either directly observed or calculated, when the section lies in the plane of any two of the three axes of the crystal. The apparent indices a, b, c, d then become respectively $\frac{\mu''^2}{\mu}, \mu', \mu, \frac{\mu^2}{\mu'}$. The apparent indices b and c

for lines perpendicular to the plane of polarisation are thus the true indices μ' and μ , while μ'' may be determined from the values of a and d by means of the equations $\mu'' = \sqrt{a\mu'}$ and $\mu'' = \sqrt{d\mu}$.

In the cases of determinations with crystals covered with thin glass cemented by Canada balsam, allowance is made for the combined layer of glass and balsam by deducting 0.53 of its value from both t and d ; for the index of refraction of the combined layer is 1.53, and the focal length is therefore increased 0.53 for every 1.00 in the apparent thickness of the glass as measured by looking through it, the upper surface being focussed with the aid of dust particles, and the lower one by means of the markings or scratches on the surface of the crystal. The value of t must, of course, be reckoned from the apparent upper surface of the crystal and not from the top of the cover-glass.

Monochromatic light is essential to accuracy with the method; Sorby used a red glass, which gave results for the middle of the red end of the spectrum.

Section-plates need not be cut if there are natural crystals available with two opposite sufficiently flat faces; and it is only necessary that in some one part the crystal should be adequately transparent over a surface which need not exceed $\frac{1}{16}$ th of an inch square. Perfect freedom from impurities is by no means essential, many good observations being possible with specimens loaded with fluid cavities and minute crystal or granular enclosures. But it is of prime importance that the crystal should not have a laminar or fibrous structure, giving rise to optically discontinuous planes, as they produce distorted images by internal reflection. Irregularities of surface may be almost completely overcome by enclosing the crystal in a liquid of approximately the same refractive index between the thick supporting glass and the cover-glass. But it is usually advantageous to polish the faces, if results of high accuracy are desired. The crystal or section-plate employed should have a thickness

somewhere between the limits of $\frac{1}{10}$ th and $\frac{1}{2}$ inch, and if the orientation of the section be not known, it can often be ascertained by the direction of elongation of the extraordinary images of the iris diaphragm, reduced to a small opening in order to act as a spot of light. Indeed Sorby recommends studying the images of this spot before passing on to observe those of the grating, and gives in his 1907 memoir a series of photographic reproductions of their appearance with various crystals.

The practical use of Sorby's method, however, is not very great; for crystals suitable for use by it are usually procurable sufficiently large for the measurement of the refractive indices by the method of the 60° -prism or that of total reflection. It is highly instructive and interesting, however, and may at any moment prove useful as a last resort. For further details of the interesting phenomena observed by Sorby with numerous minerals the posthumously published 1907 memoir (*loc. cit.*) should be consulted.

Direct Measurement by Babinet's Compensator of Retardation and of Strength of Double Refraction in a Crystal Plate.—It is often required to determine the difference of the two refractive indices μ_1 and μ_2 , that is, $\mu_2 - \mu_1$ where $\mu_1 < \mu_2$, for any particular section-plate that may happen to be available. In the cases of plates orientated parallel or perpendicular to the optic axis, or to an axis of the optical ellipsoid, such a determination is of further importance as affording the amount of the double refraction for the whole crystal, the difference between the two, or two extremes of the three, refractive indices of the substance. The most satisfactory mode of carrying out the determination is by means of the compensator of Babinet, described in the last chapter, and illustrated in Figs. 685 and 686, page 843.

The interference colour exhibited by a crystal-plate of thickness l is the same as that of an air film the double thickness $2h$ of which is equal to the product of l and the difference of the two refractive indices, μ_1 and μ_2 , corresponding to the two rays propagated in the direction normal to the plate. That is:

$$2h = l (\mu_2 - \mu_1).$$

When the plate is one of a uniaxial crystal parallel to the axis, the two rays propagated along the normal to the plate vibrate respectively parallel to and perpendicular to the axis, and correspond to ϵ and ω , so that $\mu_2 - \mu_1$ becomes the important constant $\epsilon - \omega$ or $\omega - \epsilon$, the maximum amount of double refraction for the whole crystal, according as the latter is positive or negative. Also, when the plate is one of a biaxial crystal cut parallel to the plane of the optic axes, one of the rays transmitted along the normal (the direction of the intermediate axis of the indicatrix β) will vibrate parallel to the α axis of the optical indicatrix and the other parallel to the γ axis, the former direction α being that of the acute bisectrix of the optic axial angle (first median line) when the crystal is negative, and that of the obtuse bisectrix (second median line) when the crystal is positive. Hence, the two vibration directions of such a plate correspond to the maximum and minimum refractive indices of the whole

crystal, and thus such a plate affords $\gamma - a$ (μ_2 here = γ and $\mu_1 = a$), the maximum amount of double refraction of the crystalline substance.

The actual determination of the difference of the two refractive indices, and of the actual amount of retardation R of one of the rays transmitted by the crystal-plate behind the other, is carried out with the aid of Babinet's compensator as follows. The compensator is arranged with its quartz wedges at 45° to the crossed Nicols, and with the movable wedge arranged at its zero position, that is, so that when white light is used the black central band is arranged altogether and symmetrically between the pair of parallel spider-lines. The doubly refracting crystal-plate under investigation is then introduced on the microscope stage, so that its extinction directions (the traces of its vibration planes) are also at 45° to the crossed Nicols. It will then be observed that the bands have been displaced, so that the black band is no longer between the two spider-lines, but may be several bands away. Monochromatic light of wave-length λ , say sodium light, wave-length 0.000589, is next employed, and the amount of displacement d , the movement of the movable wedge required to bring the black band back to its original zero position between the two spider-lines, is measured in scale divisions and (for the fractional part of a division) drum readings. The difference of path of the two rays or retardation R of one ray behind the other, brought about by the passage through the crystal-plate is then given in millimetres by the expression $R = Cd$, in which the only unknown quantity is C , the constant of the compensator, the value of the scale divisions in bands of known retardation.

The retardation R at any point is the difference of the two opposing retardations, r_1 and r_2 , and these depend on the thicknesses t_1 and t_2 of the two wedges at the point, and on the known difference of the refractive indices of quartz ϵ and ω . In fact, $R = r_1 - r_2 = (\epsilon - \omega)(t_2 - t_1)$, for it is

the same as is given by a plate of thickness $T = t_2 - t_1$. It will be clear from Fig. 695 that by the sliding of the wedge the total thickness is increased by ΔT , entirely due to extra material of the movable wedge, that is, $t_2 - t_1 = \Delta T$, as the amounts were equal at the zero position; also

that the displacement $d = \Delta T / \tan a$, where a is the angle of the wedge. This ΔT is the compensation thickness required to neutralise the effect of the crystal-plate of thickness t and indices μ_1 and μ_2 . Hence,

$$\Delta T (\epsilon - \omega) = t (\mu_2 - \mu_1).$$

The Determination of the Constant of the Compensator C is readily carried out by first taking the reading in white light for the adjustment of the black band to the centre of the pair of parallel spider-lines, the separation of which is adjusted to be only just slightly greater than the width of a band, and then illuminating with sodium or any other

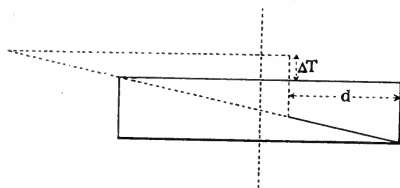


FIG. 695.

convenient monochromatic light of known wave-length; each other band, on both sides of the centre, is then brought in turn into the central position between the two spider-lines, and the reading taken for each such adjustment of a band. What we really do is to determine the amount of displacement d required, for the particular light used, to bring about a difference of path equal to one wave-length of that light, for instance 0.000589 mm. when Na-light is employed. For it will be remembered that the centre of the first band on either side of the central one corresponds to one wave-length difference, λ , from the centre of that central band, as explained on page 844 of Chapter LI., and that each successive band is due to a further retardation of a wave-length. We have, therefore, the data given in the second column of the following table for crossed Nicols. A similar set of readings, corresponding to the data given in the third column of the table, can also be taken for the bands produced in the intermediate position when the Nicols are parallel.

TABLE OF RETARDATIONS FOR SODIUM LIGHT.

No. of Band from Centre.	Retardation R corresponding to Production of Band.	
	Nicols crossed.	Nicols parallel.
1	$\lambda = 0.0005893 \text{ mm.}$	$\lambda/2 = 0.0002946 \text{ mm.}$
2	$2\lambda = 0.0011786$	$3\lambda/2 = 0.0008839$
3	$3\lambda = 0.0017679$	$5\lambda/2 = 0.0014732$
4	$4\lambda = 0.0023572$	$7\lambda/2 = 0.0020625$
5	$5\lambda = 0.0029465$	$9\lambda/2 = 0.0026518$

The constant C is thus $C = R/d$, where R is the actual retardation in millimetres for any band, and d the displacement of the band from the central position in drum units. C may thus be determined ten times over, for the five bands on each side of the central black one. The interval in scale and drum readings between every two consecutive bands for the same wave-length should be the same, but when it is not, a curve should be constructed expressing the values. The value C of a drum unit in actual retardation is thus obtained. If it be desired to make determinations with other wave-lengths, the constant C can either be determined directly for these wave-lengths, or calculated from the knowledge of the dispersion of quartz.

If now, on introducing a doubly refractive crystal-plate, the displacement of the central band be equal to, say 3.5, bands, the *Retardation* R will be :

$$R = Cd,$$

where d is the displacement corresponding in scale divisions and drum readings to the 3.5 bands, and C is the retardation value of a drum unit as above determined for the wave-length of the light employed.

Finally, the *Double Refraction*, the difference of the two refractive indices is :

$$\mu_2 - \mu_1 = R/t,$$

where t is the thickness of the crystal-plate.

For the determination of the relative retardation of minerals in thin sections, a fixed double quartz wedge has been devised by Dr. J. W. Evans (*Min. Mag.*, 1905, 14, 91) and is constructed by Messrs. R. and J. Beck. It consists of two quartz wedges of identical angle mounted side by side close together, one with its length parallel to the axis as usual, and the other with its breadth parallel to the axis. When placed obliquely between crossed or parallel Nicols they show the same bands continued across both, which are unaffected when a crystal-plate is also introduced at its extinction position. But when the plate is rotated 45° a black band indicating the position of compensation is shown by one wedge, and enhanced retardation colours are displayed by the other. The colour opposite the black band indicates double the retardation due to the crystal alone, thus affording a delicate determination of the latter.

Determination of Extinction Angles under the Microscope.—It will be unnecessary to repeat here the principles underlying the determination of the extinction directions in a crystal-plate, as they have been so thoroughly dealt with in Chapter XLI. on the theoretical considerations involved, and in Chapter XLIV. on the practical determination of extinction angles by the stauroscope. It will be sufficient to remind the reader that the operations can be carried out in a simpler, somewhat more limited and, therefore, correspondingly less accurate method, with the aid of the microscope arranged for parallel light; that is, the microscope is used without either sub-stage condenser and convergent system or the Bertrand lens, the only essentials besides the objective and eyepiece being the polarising and analysing Nicols and, if desired, one of the stauroscopic sensitiveness enhancers, such as the Bertrand quadruple-quartz-plate eyepiece described in the last chapter, page 838, and illustrated in Fig. 680, or the Calderon double-plate of calcite. The mode of carrying out a determination of extinction with the aid of the Bertrand eyepiece of the Dick microscope has been fully set forth at the conclusion of the description of the latter on page 839. If it be desired to employ a Calderon double-plate instead, it is preferably introduced also in a special form of eyepiece, which is shown in Fig. 681 in the last chapter as constructed very efficiently by Fuess, and which may quite conveniently be adapted to the Dick microscope as well as to the Fuess microscope itself. The eyepiece is of the negative Huyghenian form, the diametral cementation line of the two halves of the Calderon plate being clearly focussed by the eye-lens. The analysing Nicol fits above the latter, and is supplied with a covering cap with very small eye aperture, as in the case of the Nicol analyser of the stauroscope. A diaphragm with a small aperture is also placed just over the double-plate in order to restrict the field and the light to the rays passing nearest to the axis of the microscope, the fine line of cementation being then sharply defined. The latter is arranged to correspond to the zero of the circle of the Nicol, as indicated by a mark on the eyepiece tube, and when this Nicol is crossed to the polarising Nicol the two halves of the plate show equal intensity of half-shadow, that is appear almost dark to an equal extent. The crystal under observation is then arranged to be bisected by this line of cementation, just as in the stauroscope, and the stage carrying it is rotated, in order to determine the extinction position, until the two halves of the double-plate are again of equal slight intensity

of illumination (half-shadow) by the monochromatic light employed, as they were before the introduction of the plate. The cementation line can quite conveniently be used as a cross-wire for the purpose of adjusting an edge of the crystal outline to it, the edge representing the foreshortened face of reference, perpendicular to the plate, with respect to which the extinction is being determined. The difference between the reading of the rotating stage circle for this position and that for the adjustment of the crystal to afford two equally illuminated halves is the extinction angle required. It should be determined several times, approaching the adjustment from both directions, and if possible with the crystal inverted for a second series of readings. The mean should then afford a close approximation to the truth.

Determination of the Crystal System.—A preliminary idea of the system of symmetry to which a crystal belongs can frequently be obtained with the aid of the microscope, especially if a von Fedorow universal stage be available. A cubic crystal is isotropic in all directions, and unless in a condition of strain does not polarise in colour or afford any double refraction phenomena at all.

Uniaxial crystals appear isotropic along the one direction of the principal axis, which is the optic axis of no double refraction as well as the axis of tetragonal, hexagonal or trigonal symmetry. Although they are doubly refractive in all other directions they can usually be distinguished from biaxial crystals at once by the fact that every section of a uniaxial crystal contains one of the infinite number of equal diameters ω of the circular section of the indicatrix, perpendicular to the principal (the optic) axis. This seems impossible at first sight, but on reflection it will be seen that any section whatsoever of an ellipsoid of revolution may be moved in imagination parallel to itself, which obviously does not affect the optical phenomena which it displays, until it passes through the centre of the ellipsoid, if it does not already do so. The minimum or maximum axis of the elliptical section of the indicatrix, made by the plate, according as the crystal is positive or negative, will then in all cases be a diameter of the circular section of the optical indicatrix; in other words, it will always be one of the infinite number of directions in the circular section the vibrations along which correspond to the refractive index ω . Therefore, one of the two directions of extinction in a uniaxial crystal-plate, cut anyhow, is always an equatorial diametral direction corresponding to the refractive index ω , and the extinction direction at right angles to it will correspond to a direction in a principal section of the indicatrix ellipsoid of revolution, that is, to a direction in a section containing the optic axis, somewhere between ω and ϵ , the latter refractive index corresponding to the direction of the optic axis. A special case is when the section-plate happens to be parallel to the circular section itself, when every direction in it corresponds to ω and the plate remains dark on revolution under crossed Nicols, every direction being then one of extinction.

On examining one of the two rectangular directions of extinction in the general uniaxial case, and setting it parallel to one of the horizontal

axes of rotation of the universal stage, it will be found that under crossed Nicols such a general section will remain dark on rotation if the direction selected has been that of ω . But if the wrong one of the two has been chosen the section-plate will show the usual interference colours in parallel polarised light, unless the section-plate should happen to be one parallel to the optic axis, such as a plate or tabular crystal parallel to a pair of faces of the primary hexagonal or tetragonal prism zone, the discovery of which fact would of itself be of prime importance and simplify the work considerably.

When a tabular crystal perpendicular to the optic axis is available, discovered in parallel light by remaining dark on rotation in its own plane, and through which in convergent light under crossed Nicols the interference figure of circular rings and black rectangular cross is visible, the proof of the uniaxial character of the crystal will be complete. With the aid of the universal stage of von Fedorow, or that of Klein, and even with the Miers stage goniometer, it is often possible to see the uniaxial interference figure through crystals not tabular when the crystal is immersed in the small cell containing a highly refractive liquid—the liquid chosen being one in which the crystal is most nearly invisible and therefore of like refractive index—and when the crystal is rotated until the optic axis is brought parallel to the axis of the microscope.

It is usually fairly easy to discriminate between a tetragonal and a hexagonal crystal, for the former shows square or regular octagonal outlines sooner or later, when numerous crystals of the substance are examined, while the latter exhibits hexagonal 120° or 60° outlines, perpendicular to the optic axial direction. To discriminate between the hexagonal and trigonal systems is not so easy, unless obviously definite rhombohedra are characteristic. Further study of the nature of the end faces, those other than the hexagonal prismatic forms which are common to the hexagonal and trigonal systems, is requisite, differences of surface of alternate faces of apparent hexagonal pyramids being particularly helpful. Angular measurements will probably require to be made before certainty is arrived at, and these may often be carried out with good microscopic crystals by the methods described earlier in this chapter.

A biaxial crystal, in general, does not remain dark when rotated about either extinction direction, and when it actually happens that the crystal under examination follows this rule its biaxial character is at once discovered. For if it were uniaxial it would remain dark when rotated about one of the two extinction directions. Only section-plates parallel to the primary zones, or natural plates, tabular crystals, parallel to the actual faces of these zones, do in the case of biaxial crystals remain dark on rotation about the extinction directions. The discovery of two directions of isotropism, the two optic axes, also affords immediate proof of biaxial character, and this discovery can often be made with the universal stage, especially when the crystal is immersed in a liquid of the same refractive index. For when such a direction is brought parallel to the axis of the microscope the crystal will remain dark when rotated under crossed Nicols in parallel light, and will show the rings

and single brush of a biaxial crystal in convergent light. When rotated to the intermediate direction, that of a median line, it will show the full biaxial interference figure, with the optic axes separated at either their acute or obtuse angle, unless the latter happens to be too large to be visible even in the highly refractive liquid; if this should be so, however, the perpendicular intermediate direction will afford the acute-angle figure. Indeed, the recognition of the biaxial interference figure in convergent polarised light is at once conclusive evidence of biaxial character, that is, of rhombic, monoclinic, or triclinic symmetry. It frequently happens that even a cursory examination of a number of crystals of a biaxial substance will reveal the figure, if not symmetrical to the centre, as it probably would be in the event of the crystal being rhombic, still not displaced from the centre too much for recognition of its nature.

When the crystal is rhombic the two rectangular extinction directions will as a rule either be parallel to two rectangular edges of the crystal as seen under the microscope, or symmetrical to two parallel pairs of edges. Also the biaxial interference figure will probably show itself, if not immediately, sooner or later on examining a number of crystals, symmetrical to the centre when a tabular crystal parallel to an axial plane is laid on the simple microscope stage. A monoclinic crystal will also in some cases show extinction directions parallel to rectangular faces, those of the orthozone, but such a crystal-plate will not be likely to afford an interference figure symmetrical to the centre of the field, although it may exhibit one symmetrical to a diameter, the trace of the symmetry plane. Other plates than those of faces in the orthozone will show extinctions oblique to the faces. Also, when the interference figure is closely regarded, discrimination between the rhombic and monoclinic systems can usually be effected by observing the nature of the dispersion shown by the figure in white light. When the crystal is rhombic the distribution of the colours will always be perfectly symmetrical, that is both to the centre and to the two rectangular diameters parallel to the black cross formed when the plate is arranged parallel to the planes of vibration of the crossed Nicols. If monoclinic, it will show either horizontal, inclined, or crossed dispersion, as described at the conclusion of Chapter XLI. If the crystal be obviously biaxial, but exhibits none of the regular phenomena just referred to as typical of rhombic and monoclinic crystals, it can only be preliminarily considered as triclinic. It is much more likely to be really triclinic if it appears to be not very far from showing monoclinic phenomena, for instance, when the plane of the optic axes shows itself as only a very few degrees, yet clearly so, from being parallel to a crystallographic facial plane, than if the differences are greater. For we have always to reckon with the possibility that the crystal before us may belong to a class of the rhombic or monoclinic system of lower than full systematic symmetry. It could not be of higher symmetry, that is uniaxial, because even such low-symmetry classes of the uniaxial systems will always show the uniaxial interference figure sooner or later, whereas a biaxial crystal of lower than full systematic symmetry will yield a biaxial figure. Such difficult cases of apparently triclinic crystals

absolutely necessitate angular interfacial measurements. Moreover, we have always to remember that such crystals of low-class symmetry may belong to one of the eleven enantiomorphous classes, which may be expected to display optical activity. Hence, we have always to be on our guard that polarisation phenomena displayed are the regular ones due to double refraction, and not those due to optical rotation. The latter can always be recognised, and the angle of rotation measured in monochromatic light, quite conveniently under the microscope, provided the property be developed sufficiently strongly, which is rare, to exhibit it in plates or crystals as small as are generally investigated under the microscope. This disturbing cause is thus not so serious in microscopic work on account of the small size (thickness) of the crystals usually alone operated on.

Lastly, as regards the preliminary determination of the crystal system under the microscope, such observations should, unless no other observations are possible, only be regarded as truly preliminary, to be followed by definite angular measurements on a good goniometer, and by precise determinations of the optical constants on the accurate measuring instruments described in previous chapters. Such preliminary work with the microscope frequently very considerably facilitates the systematic investigation just referred to, as the observer knows his way about the crystals, and is familiar with their prominent characteristics. In cases where such further work is impossible, on account of inability to obtain or grow crystals of adequate size, this microscopic investigation is invaluable.

In the above statement it has been assumed that single individual crystals have been available. Simple twinning is immediately recognised under the microscope, as the two halves of a twin polarise differently. Even repeated twinning is likewise at once detected, the various strips polarising so that adjacent individuals always show different colours on rotation of the stage or the Nicols, at some position or other during the rotation, four times repeated during a complete revolution, the Nicols being crossed. But, as has been pointed out in Chapter XXVI. on twinning, microscopic laminated twinning often apparently enhances the symmetry, and may be so fine, the alternations so numerous, as to be ultra-microscopic with the low powers used. Such cases require most careful and special investigation, including that with high-power objectives; indeed, it will be obvious that the microscope becomes here the ultimate instrument of research, and not merely one for preliminary examination. But it is largely because of such possibilities that an open mind should be maintained as long as possible as to the results of a microscopic investigation, and every effort be made to supplement them by regular large-scale measurements.

Discrimination of the Class of Symmetry by means of Etched Figures.—One of the most important uses of the microscope is the examination of etched figures on the faces of crystals, not merely on microscopic ones but even on crystals of some size, for the purpose of definitely settling the particular one of the 32 classes of symmetry to which a substance conforms, the latter being known from the earlier part of an investigation to belong to a particular system.

Instead of invoking the ordinary rapid solvent action, of a solvent poured in quantity on the material of a crystal, the solvent is applied in these experiments in relatively very small quantity compared with the bulk of the crystal substance. For instance, to study the etched figures of a crystal soluble in water, the minute quantity of water may be applied by breathing on the crystal faces or slightly steaming them, or gently touching them once or twice with a damp cloth. The process of solution then begins at isolated spots instead of all over the surface, and occurs with different rapidity in different directions radiating from each of such spots, beginning first with the direction of least cohesion and, in fact, indicating the order of cohesion in the different directions, and therefore of the symmetry, by the character of the figure produced. Such etched figures are generally of the character of depressions or pits of specific shape, and their configuration is in accordance with the class-symmetry of the crystal, and forms perhaps the best of all indications of the detailed character of such symmetry. For the resistance to etching in different directions is obviously a direct outcome of the structure. The shape of all the pits on any particular face will be similar, and also on all faces of the same form, that is, on faces which are absolutely equivalent as regards the symmetry. Water, dilute acids, alcohol, ether, benzene, carbon-bisulphide and other of the usual organic solvents, are the principal solvents employed, and the essence of effective treatment is to apply them in the minimum possible quantity, for instance, by merely touching with a cloth moistened with the solvent. Mineral crystals are often found with faces naturally etched by water containing carbonic acid or other acid gases such as hydrochloric acid in solution, or by gaseous hydrofluoric acid, or even by water at high pressure and temperature or by superheated steam.

Perhaps one of the most instructive cases that could be cited of the use of etched figures to discriminate between two classes and even two systems, is one to which particular attention has been called by Miers, namely, the effect of hydrochloric acid on apatite and calcite. Apatite has been shown in Chapters XXI. and XXII. to belong to the bipyramidal class 25 of the hexagonal system, in which the elements of symmetry are the essential hexagonal axis and the equatorial plane of symmetry, the six other symmetry planes and six digonal axes of the holohedral hexagonal class 27 being absent. The etched figures afforded by a hexagonal prism of apatite with hydrochloric acid are shown in Fig. 696, and the adjacent Fig. 697 illustrates the etched figures given under similar conditions by a hexagonal prism of calcite, which has been shown in Chapters XXIII. and XXIV. to belong to the scalenohedral class 21 of the trigonal system. The essential trigonal axis of symmetry and three symmetry planes intersecting in it, together with three digonal axes lying in the equatorial plane midway between the symmetry planes, are the elements of symmetry present in calcite and its class. There is no equatorial plane of symmetry, as in the holohedral trigonal class 22, and the digonal axes differ from those of that class by not lying in the symmetry planes.

The first difference in the etched figures on the two crystals that strikes us is that in apatite they are the same on all the six faces of the

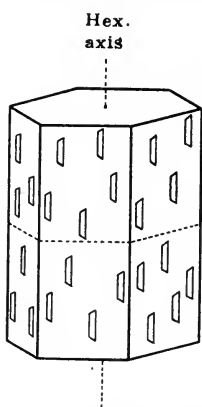


FIG. 696.—Etched Figures afforded by Apatite.

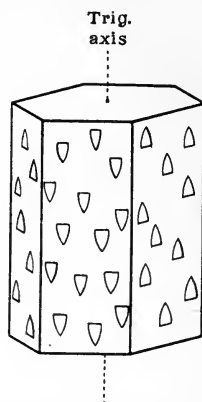


FIG. 697.—Etched Figures afforded by Calcite.

hexagonal prism, indicating truly hexagonal symmetry; while on calcite alternate faces bear differently directed figures, although the shape is the same, indicating that all the six faces belong to the same form, the pointed end, however, being upwards in one alternate set of three faces and downwards in the other, indicating trigonal and not hexagonal symmetry. It must be remembered that the hexagonal prism is a form common to both the hexagonal and trigonal systems. Moreover, each of the calcite etched figures is symmetrical to a vertical plane of symmetry, but not to an equatorial plane; and obviously the digonal axes do not emerge through the middle of the faces but through the edges, indicating class 21 and not the holohedral trigonal class 22. Again, the etched figures of apatite are symmetrical at both ends to the equatorial plane of symmetry, but not to vertical planes of symmetry, which are thus absent, indicating class 25 and not the holohedral hexagonal class 27.

An excellent set of drawings of etched figures of all the 32 classes of crystals will be found on pages 280 and 281 of Miers' *Mineralogy*. Miers has further shown that the sides of etched figures are minute facets, vicinal faces of very high indices not perpendicular to any of the planes or axes of symmetry; he has made a profound study of the character of these vicinal faces, and an account of the highly interesting results has already been given in Chapter XXX.

Determinations of Optic Axial Angles by the Microscope.—It will have been clear from Chapter XLI. that the interference figure in convergent polarised light is composed of curves or lines of two kinds. One of these consists of curves, lemniscates, commonly known as "rings," which are dark and luminous alternately when the polariscope is illuminated with monochromatic light. They become coloured when white light

is substituted, and their form is independent of the position of the axes of the section-plate relatively to those of the crossed Nicols. They do not become deformed on rotation of the section-plate in its own plane, and their form has no connection with the direction of the original plane of polarisation. These curves of the first kind are exclusively due to interference, being "Isochromatic Curves of Equal Retardation." The curves of the second kind, commonly known as "brushes," remain dark when white light is used, the Nicols being crossed, but are colourless and bright when the Nicols are parallel. They alter in the case of biaxial crystals when the section-plate is rotated in its own plane, their form being connected, in the general case, with the extinction directions of the plate, and, in the special very important case of a plate perpendicular to the first median line, with the direction of the axes of the optical ellipsoid lying in the plane of the section-plate, relatively to the plane of polarisation of the rays received from the polarising Nicol. They are generically termed "isogyres."

Curves of Equal Retardation.—Assuming the Nicols crossed, the interference is positive, causing a dark curve of equal retardation, when the retardation of one of the two interfering rays behind the other is equal to an odd number of half wave-lengths, that is, to $(2n + 1)\lambda/2$. The interference is negative, producing a bright curve of equal retardation, when the delay is equal to an even number of half wave-lengths, that is, to $2n\lambda/2$ or $n\lambda$. We have to add to these retardations within the crystal, however, the half wave-length change of phase introduced by the action of the analyser, so that the total retardation observed due to both causes is really the reverse, namely $2n\lambda/2$ or $n\lambda$ for a dark curve and $(2n + 1)\lambda/2$ for a bright one, odd and even half wave-lengths corresponding to reversed phenomena compared with the usual effect of interference of light waves, as explained in Chapter XLI. pages 666 and 670. When the Nicols are parallel the converse is true of the conditions for the two cases. These curves of equal retardation are given by the intersection of surfaces of equal retardation $\lambda/2$ (dark), $2\lambda/2$ (bright), $3\lambda/2$ (dark), $4\lambda/2$ (bright), and so on, by a plane parallel to the crystal section-plate.

Isogyres.—The intensity of the light transmitted by the analyser obviously depends not only on the retardation but on the orientation of the planes of vibration in the crystal with respect to the Nicols; that is, on the directions of the axes of the elliptic section of the ellipsoid by the plate, the extinction directions, in the general case, and on the directions of the axes of the optical ellipsoid of which the crystal-plate is a principal section, in the special case of a section-plate perpendicular to the first median line prepared for the measurement of the optic axial angle. Such rays, therefore, as emerge vibrating parallel to the polariser, are stopped by the analyser when the latter is crossed to the polariser, and this extinction gives rise to the isogyres. They are clearly independent of the wave-length of the light employed. But their form is dependent in general on that of the optical ellipsoid, on the orientation of the section-plate with respect to the latter, and of the positions of the axes of the ellipsoid with reference to the vibration directions of the Nicols.

Curves of Equal Velocity and Skiodromes.—It has been necessary thus to summarise these facts here, as a *résumé* and somewhat of a further development of those given in Chapter XLI, inasmuch as some extremely interesting and important work in improving microscopic methods of attack has recently been carried out by F. Becke of Vienna and von Fedorow of St. Petersburg, in which both curves of equal retardation and isogyres have their own important places. The form of the isogyres has been deduced by Becke¹ from the consideration of curves of equal velocity, to which he has applied the name velocity ellipses or “isotaques.” A series of models are shown in Fig. 698, constructed by Krantz of Bonn, which represent these curves on a sphere, for a typical set of uniaxial and biaxial crystals, the actual models which the figure represents being in the Science Museum at South Kensington.

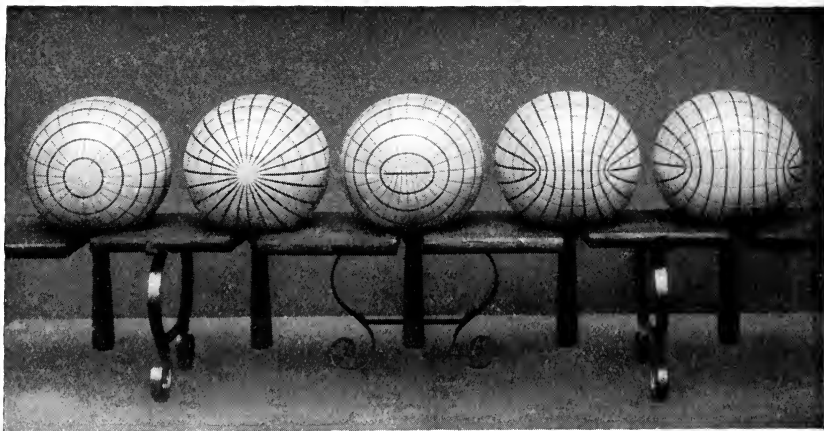


FIG. 698.—Spherical Models exhibiting Becke's Curves of Equal Velocity.

The two first are of uniaxial crystals, and the three others of biaxial crystals having optic axial angles respectively of 30° , 60° , and 90° . They give a general representation of the directions of vibration of the light waves in the crystal, for these directions at any selected point are the tangents to the two curves (drawn like lines of latitude and longitude on the sphere) which intersect at that point.

These curves of equal velocity, the isotaques, may also be projected in a stereographic projection, like the zones of crystal faces in our ordinary crystallographic stereographic projection. Their orthogonal projections have, however, been preferred by Becke, and used by him for the deduction of the shape of the isogyres, and such projections have been termed by him “skiodromes.” While of great theoretical interest, however, skiodromes are not of much practical service, as the forms of the isogyres are so readily determined by actual

¹ *Min. und petrogr. Mittheil.*, 1905, 24, 1 and 34.

observation, even if only by a microscopic method, and provided at least one of the biaxial pair be visible in the field of view. Such experimental determination is obviously far preferable, and the microscopic methods of Becke himself have rendered such practical work so much more valuable and trustworthy that it may almost always now be relied on to furnish more or less adequate information. Nevertheless, the Krantz models of the curves of equal velocity are of great value in affording a graphic and concise review of the whole scheme of light vibration in the two great optical types of doubly refractive crystals.

For all practical purposes the isochromatic curves of equal retardation, currently known as the "rings round the optic axes" of crystals in convergent polarised light, and the isogyres, better known as the "dark cross" and the "hyperbolic brushes," are alone of prime and essential importance.

Given a tabular crystal perpendicular to the first median line, and as accessory apparatus to the microscope either the Miers stage goniometer, the von Fedorow universal stage, or that of Klein, the observation of the interference figure, the localisation of the optic axes, and the measurement of their angle of separation, can be carried out without any serious difficulty. Also even when tabular crystals are not available and sections are too difficult to prepare, if a liquid of the same refractive index as the mean of the crystal indices can be found by trial from among a number of available liquids without action on the substance, such as those the refractive indices of which are tabulated on p. 577, the crystal apparently becoming invisible in such a liquid when the indices are matched, the optic axial angle can be measured with a whole crystal of other than tabular habit. But in more difficult cases, such as that of a crystal in a rock section, or one of a group not readily detachable, or a crystal too small for such manipulation, the optic axial angle can still be determined by the use of convergent light by the method of Becke, or by use of parallel light by the method of von Fedorow.

Direct Determination of Optic Axial Angle by the Miers Stage Goniometer.—The Miers stage goniometer is very useful in all simpler cases, such as for measuring the optic axial angle of a miniature crystal of tabular form perpendicular to the acute bisectrix. The crystal is mounted with wax or other cement on the end of the holder-needle, so that the plane of the optic axes is perpendicular to the goniometer axis. The cross-spider-lines of the eyepiece are to be arranged parallel and perpendicular respectively to the goniometer axis, and the directions of vibration of the Nicols at 45° to these directions. The convergent system and Bertrand lens being in position, the objective is brought down by the coarse adjustment as close to the section-plate as possible without impeding the free rotation of the needle and crystal. Even if this be not close enough, employing the $\frac{1}{4}$ -inch objective so suitable for this work and both sub-stage condenser and stage lens as convergent system, to permit of a full field of interference curves being obtained, it will be adequate to show the central part of the field, where each optic axial hyperbolic brush is to be brought in turn to the cross-wires

at the centre. When the crystal has been adjusted for symmetry of the figure to the centre, the rotation is effected first on one side and then on the other until the vertex of each brush in turn is delicately adjusted to the spider-line (that one which is parallel to the tangent to the brush at the vertex of the hyperbola), when a reading of the goniometer circle is taken. (See Fig. 562, page 672.) The difference of the readings for the two brushes is $2E$, the apparent optic axial angle in air.

If the mean refractive index β has already been determined, the true angle $2V_a$ can at once be calculated by means of the formula :

$$\sin V_a = \frac{\sin E}{\beta}.$$

But when a second plate perpendicular to the obtuse bisectrix is available, measurements with both plates can be made in oil or a still more highly refractive liquid such as monobromonaphthalene, and the true angle calculated from the apparent acute and obtuse angles in the liquid, $2H_a$ and $2H_o$, just as described in Chapter XLVII., with the aid of the formula :

$$\tan V_a = \frac{\sin H_a}{\sin H_o}.$$

The circular cell with truly parallel-surfaced plate-glass sides is used to contain the liquid ; it fits readily in the cut-away recess in the goniometer stage-plate, and when pushed as far as it will go in this recess, resting on the semicircular end, it is centrally placed. There is an opening in the cylindrical wall of the cell adequately large to admit the crystal, and the cell is arranged so that this opening is at the top when the microscope is arranged horizontally for this immersion work, as shown in Fig. 672. The goniometer is so constructed that when the cell is thus placed in position the crystal and needle-end will be completely immersed in the liquid, when the cell is three-quarters filled. Two such cells of different sizes are shown lying on the base-board in Fig. 671 ; the circular metallic basal annulus of each of these cells has the same radius as the semicircular end of the recess in the stage-plate.

Even if there be only the one plate available, perpendicular to the acute bisectrix, the true angle can be found, as accurately as microscopic observation can ever go, by using as the immersion liquid one which has a refractive index for sodium light equal to the mean of the three refractive indices of the crystal, the observed angle being then the true angle $2V_a$ for sodium light and approximately so for other wave-lengths. If the refractive indices have not been determined it is sufficient for approximate work to choose a liquid in which the crystal becomes invisible, for the refractive indices are then approximately alike.

As the true optic axial angle $2V_a$, and even the apparent angle in air $2E$, as well as the nature of the interference figure itself, are so highly characteristic of the crystals of a substance, this determination and the observations connected with it are of great importance as

furnishing data for a valuable description of the substance, and when no crystals large enough for the preparation of section-plates for use with the optic-axial-angle goniometer described in Chapter XLVII. are to be had (a crystal as large as a pin's head will often suffice, when the cutting-and-grinding goniometer described in Chapter XLII. is available) the crystallographic microscope is the only resort, and proves of the utmost value. For results almost as accurate as those obtained with larger crystals and the specific measuring instruments referred to in Chapter XLVII. can be attained by its careful use with microscopic crystals. For the simpler cases the Miers goniometer employed as just described is excellent, but for the more difficult cases recourse must be had to the universal stage, the various methods of using which will now be proceeded to.

Convergent light methods are available whenever at least one optic axis appears within the field of vision. The method requiring the use of the Becke drawing table shown in Fig. 687 (page 845) in the last chapter is generally applicable, and accurate to within five degrees. The method involving the use of the double-screw micrometer described by F. E. Wright,¹ in an excellent memoir on this subject of the new methods of microscopic investigation, and illustrated in Fig. 677 (page 837) in the last chapter, is also of general application and both more accurate and simpler. It is an up-to-date improvement on the older well-known procedure of Mallard, and utilises Wulff's method of projection. The results with both methods require to be plotted on a stereographic projection, and the optic axial angle is then measured graphically on the projection. Hutchinson's stereographic net of particularly convenient size, shown in Fig. 39, page 91, is extremely useful for the purpose; or, if preferred, the net of either von Fedorow or one of the other authors referred to in Chapter VI. may be used.

For parallel light work the method of von Fedorow is the most satisfactory, provided the position of one optic axis can be determined directly. The situation of the second optic axis is then determined by the method of extinction curves.

The most accurate results by both parallel and convergent light methods are with sections through which both optic axes are visible, while if neither is to be seen none of them is really satisfactory, and the method just described of immersion in a liquid of the same refractive index is the only hopeful one.

Indirect Method of Mallard as improved by Wright.—As the Bertrand lens affords an image, the secondary one, of the interference figure in convergent polarised light, which can be focussed by a Ramsden micrometer positive eyepiece, a micrometer spider-line can be simultaneously focussed. It is only necessary, therefore, to employ a Ramsden micrometer such as that shown in Fig. 676, in order to be able to get a linear measurement of the distance between the two vertices of the hyperbolic brushes, when both are visible in the field of view. An engraved micrometer such as is given with the Groth universal apparatus,

¹ *Amer. Journ. of Science*, 1907, 24, 317.

and likewise brought into the focus of the positive eyepiece, serves the same purpose somewhat more roughly.

The improvement introduced by Wright¹ consists in using a double-screw micrometer. It is an ordinary Ramsden micrometer with the addition of a second micrometer at right angles to the first, each micrometer having its own screw with divided drum, and its own movable spider-line following absolutely the movement of the screw, as recorded by the drum divisions and a scale of notches or "comb," every fifth notch deepened, to record the number of complete drum and screw revolutions. Its construction will be clear from Fig. 677, page 837. The two movements are adjusted so that the two movable spider-lines are exactly perpendicular to each other. Each movement can be accurately read to 0.005 mm. The Ramsden positive eyepiece is fitted above with a cap pierced by a small aperture, the function of which is to act as a small stop to minimise any possible errors of parallax. The two coordinates of any point on the dark axial bar (isogyre) of an optic axis in the field of view may thus be taken directly, when once the angular calibration value for the movement of each of the two spider-lines has been determined, in the manner which will presently be explained. The angular values may then at once be transferred to a stereographic net and the rest of the procedure of Becke's method followed (described in the next section), which is essentially based on Mallard's principle and original suggestion, for the determination of the optic axial angle.

Mallard² represented the distance of the axis of the microscope from one of the brush-vertices by D , which is thus the lineal representative of E , half the apparent optic axial angle in air. He then showed that

$$D = K \sin E,$$

where K is a constant (the Mallard constant) for the particular optical lens system employed. The lens system must satisfy the well-known Abbe condition of aplanatism:

$$\frac{\sin u}{\sin u'} = \frac{n'A}{n} = \text{a constant},$$

in which u and u' are the angles of convergence of the rays in the two parts of the convergent systems of lenses, n and n' their indices of refraction, and A the magnification. All the microscopes by the best makers, supplied with condensing system, aplanatic objective such as a Swift $\frac{1}{4}$ -inch or a Fuess No. 9 of equivalent focal length 2.7 mm. and numerical aperture 0.97, Bertrand lens, and Ramsden eyepiece, fulfil these conditions. Moreover, Mallard showed that D is always proportional to the sine of the apparent half-angle E of the optic axes in air, as will be obvious from his formula. The constant K is determined experimentally by means of a few section-plates perpendicular to the

¹ *Amer. Journ. of Science*, 1907, 24, 336.

² *Bull. Soc. Min. Fr.*, 1882, 5, 77.

first median lines of crystals of well-ascertained magnitude of $2E$, as varied as possible. We then have :

$$K = \frac{D}{\sin E}.$$

The simplest method is to measure with each of these chosen section-plates the distance between the vertices of the two dark hyperbolic brushes, as if measuring an optic axial angle but in scale and drum divisions instead of angular measure, this distance being $2D$. That is, the movable spider-line undergoing calibration is adjusted first to the summit of one hyperbola and then, after noting the reading of the scale and drum, it is traversed by rotation of the milled head of the drum until the other hyperbolic summit is similarly adjusted, when another reading is taken. The difference of the two readings is $2D$. As we know the actual angular value $2E$ of the optic axial angle in air thus linearly measured we can at once make up the equation for K above given and find K . Doing this for the several section-plates chosen, we obtain several more or less concordant values for K , the mean of which may be taken as the trustworthy value of the constant once for all. We are then in a position to employ Mallard's equation to find the E , and therefore $2E$, of a crystal of unknown optic axial angle, by use of the double micrometer to measure linearly the distance between the vertices of the two hyperbolæ when the latter are both visible in the field of the microscope, for $\sin E = D/K$.

When the section-plate is not quite perpendicular to the bisectrix of the angle between the optic axes, the spider-line is placed over one axial vertex, the section rotated 180° , and the spider-line traversed to the new position of the same vertex ; half the difference between the two positions is a value of D corresponding to the real distance of the optic axis from the centre of the field. A similar operation is then performed for the other hyperbolic vertex, and a value of D obtained for this corresponding to the distance of this second vertex from the centre. By using the formula $\sin u = D/K$ to calculate u in each case, two values of u are obtained corresponding to the real angles made by each optic axis with the normal to the plate, the sum of the two being $2E$.

Method of Becke, by use of the Drawing Table.—When the plate is not nearly perpendicular to the acute bisectrix of the optic axial angle the method in its simple form is inapplicable. But Becke (references given at foot of page 846) has admirably generalised it by the introduction of his drawing table and the use in connection with it of the stereographic net. The drawing table, however, is not absolutely indispensable when the microscope is fitted with the Dick arrangement, described so fully in the last chapter, for the simultaneous rotation of both Nicols, which was first arranged for Mr. Allan Dick by Swift and has now also been adopted by Fuess and other first-class makers of crystallographic microscopes. Becke's method depends essentially on the fact that the isogyres, the dark hyperbolic brushes, pivot round the poles of the optic axes when the section-plate is rotated

in its own plane on the stage of the microscope. The position of one optic axis, which should be visible in the field for the method to be applicable in its more accurate form, is determined by the intersection of the isogyres observed about this axis for different positions of the stage, some definite angular distance apart, such as 30° or 45° . For this purpose he projects the interference figure by means of the Abbe camera lucida (shown in Fig. 687 in the last chapter) on his rotatable drawing table (also shown in Fig. 687), which can be accurately centred by three screws arranged at intervals of 120° , or by two screws arranged at 90° . That is to say, simultaneous images of the interference figure and the drawing on the drawing table are arranged to be obtained on looking through the camera lucida. The rotating plate of the drawing table has a series of concentric circles engraved on it to facilitate rapid centring to identity of axis of rotation with that of the centre of the image of the interference figure. The drawing table is graduated in degrees, and may if desired, be supported on an arm attached to the microscope stand instead of on the baseboard, and is recommended to be so used by Wright.

Having fixed its position so that its axis of revolution coincides after reflection in the camera lucida with the optical axis of the microscope and so that it is also at the proper distance from the eye for clear vision, and having determined its constant K corresponding to the K of the microscope in the formula $D = K \sin E$ (several methods of determining which are given by Becke, analogous to that used for the Ramsden or Wright micrometer just described), any interference figure can be traced and the angular position—the polar angle ρ and longitude angle ϕ —of any point in the projection can be determined from the drawing and plotted subsequently on a stereographic projection net, such as that of Hutchinson given in Fig. 39. Thus the angular distance between any two points, for instance the two optic axes if both are present at once in the field, can be graphically found. As stated in the last section, however, this particular simple problem can be solved more readily by the use of Wright's double micrometer, the drawing table being unnecessary. The value of the method comes in when only one optic axis is present in the field of view.

When only one optic axis is visible, the method of Becke depends on the fact that the dark axial bar or hyperbolic brush, the isogyre, exhibits a curvature definitely related to the magnitude of the optic axial angle, and also on the well-known Biot-Fresnel rule explained on page 648 in Chapter XLI. This important rule, which Biot¹ demonstrated experimentally and Fresnel² explained theoretically, is that the directions of extinction in any section of a biaxial crystal are parallel to the traces, on that section, of the planes bisecting the angles between the two planes containing the normal to the section and the optic axes. The normal to the plate being the common line of intersection of all four of these planes, and the two bisecting planes being at right angles to each other, their traces, the extinction directions of the plate, are

¹ *Mém. de l'Acad. de l'Inst. de France*, 1820, 3, 228.

² *Pogg. Ann.*, 1831, 23, 542.

also at right angles to one another. They (the extinction directions) are the directions of possible light vibration in the plate, those of the two rays into which by virtue of the property of double refraction the light is divided on entering the crystal-plate. They are also the principal axes of the elliptic section of the optical indicatrix made by the plate. This law of Biot and Fresnel is a natural extension to biaxial crystals of the law of Malus¹ governing the case of a uniaxial crystal, that the lines of extinction in any given section are parallel and perpendicular respectively to the trace of the plane containing the normal to the section and the optic axis.

This fundamentally important Biot-Fresnel law can be illustrated by a very simple model, of which the essentials are (1) a flat board to represent the crystal plate, (2) a rod standing upright from the centre to represent the normal, (3) two other rods to represent the optic axes, each hinged at one end with a ball-and-socket joint to the centre of the baseboard, from which they thus spring in common with the fixed rod, but are able to be fixed at any required angle to each other, the optic axial angle, and to the fixed normal rod, and (4) a sheet of metal with one essential edge truly straight. On laying the metal sheet against the normal rod and one of the optic axial rods, with the straight edge resting as a ruler on the baseboard, the trace of the plane on the baseboard, or preferably on a card cut to the shape of the crystal plate resting on the baseboard, can be drawn by ruling. Similarly the trace of the plane of the normal and the other optic axis can be drawn. On then bisecting the two angles, acute and obtuse, formed by these two lines thus drawn, the two rectangular extinction-vibration directions are obtained, the traces of the two bisecting planes. When the card representing the crystal section has been accurately cut, the relations of the extinction directions to the faces bounding the section are thus at once seen.

The axial bar is first drawn when in a position parallel to the horizontal cross-wire, that is, when the extinction directions of the section are parallel to the vibration directions of the Nicols, as shown at A_1C in Fig. 699, which is due to Wright. The graduation of the circle of the drawing table for this position should also be made the same as that of the rotating stage of the microscope. This line A_1C is the trace of the optic axial plane. The microscope stage or the Nicols and the drawing table are then revolved either 30° or 45° in the same direction, and the axial bar is drawn in the new position A_1P ; A_1 is common to both, as it is the position of the optic axis about which the brush is rotated, that is, it is the pole of the optic axis. The drawings are then repeated after revolution of the microscope stage (or the Nicols) or drawing table alone through 180° , giving A'_1C' and A'_1P' , in order to determine the exact centre of the field O . For the centre is the middle

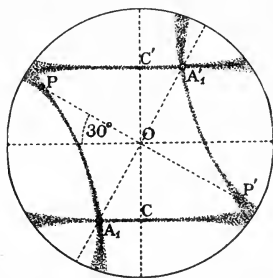


FIG. 699.

¹ *Mém. prés. à l'Inst. Sc. math. et phys.*, 1811, 2, 303.

light is chiefly used by von Fedorow,¹ with the crystal mounted on the universal stage, which is fixed on the ordinary stage of the microscope in the manner described in the last chapter.

The essence of the method is that when an optic axis is brought parallel to the axis of the microscope in parallel plane-polarised light the section-plate remains uniformly dark when revolved about that axis. The universal stage enables the optic axis to be thus brought parallel to the microscope axis with great ease and consequently to be found directly by experiment, when the optic axis in question emerges somewhere in the field of view. The positions of both optic axes are found in this way in parallel light if possible, but if the optic axes do not emerge they are found by means of determinations of extinction directions in definite zones of faces; the results are plotted on a stereographic net, with the aid of a second such net on tracing paper. It is then possible to determine graphically both the optic axial angle and the orientation of the section-plate with respect to the optical ellipsoid, although neither optic axis may be visible through the plate. The true angles in the crystal are found before plotting on the net, by use of the mean refractive index β of the crystals, which von Fedorow also gives a method of finding, together with the approximate double refraction. The Biot-Fresnel rule is again much used as a base of operations, namely, that the planes of vibration of light waves propagated in any given direction bisect the angles between the two planes each containing one of the two optic axes and the given direction. Von Fedorow recommends the use of circular glass mounts 2 centimetres in diameter for the crystals or mineral sections, instead of the ordinary microscope 3-by-1-inch slips. The universal stage enables a crystal section to be examined under the microscope in any position, and to be revolved about any axis, thus affording control over all possible orientations.

There are four cases distinguished by von Fedorow, for which he recommends certain procedure as most convenient: (1) when both optic axes are visible, and inclined between 15° and 55° to the normal to the section; (2) when one optic axis only is visible, and inclined less than 20° to the normal to the section, the second optic axis being unable to be brought into view by any revolution of the stage; (3) the same as (2) but the inclination of the optic axis is more than 20° to the normal; (4) both optic axes are outside the field of view of the microscope. The methods recommended for treating these four cases are briefly as follows.

(1) The positions of the two optic axes are determined by bringing each one, by means of the two circles V_1 and H_1 of the universal stage as shown in Fig. 683 (p. 841) into the vertical position, when the crystal will remain dark under crossed Nicols during the whole revolution of the microscope stage or when the Nicols are simultaneously rotated. Also in feebly convergent light the optic axis can actually be seen, when so adjusted, in the centre of the field. The position of each is then confirmed by means of extinction angles along definite directions, and by plotting the results on a

¹ References given on page 840.

stereographic projection; the "optical curves" thus obtained will all intersect in the optic axis in question.

To obtain such an optical curve the crossed Nicols are first placed in some definite position, then H_1 is turned through 5° at a time and for each position the angle of inclination about V_1 is found for which the crystal plate is at maximum darkness. The observed angles are then reduced to true directions in the crystal by use of the mean refractive index β of the crystal, and those directions in the crystal are found for which the extinction is 0° for a given position of the Nicols. The curve uniting these directions is the "optical curve" of von Fedorow. Other analogous curves are then obtained and plotted for other definite positions of the crossed Nicols, when they will all be found to intersect in the optic axis. In the case (1) before us, as the position of the latter has been found approximately already, only one setting of H_1 at 5° or 10° on each side of the approximately known position is necessary. Convenient positions of the Nicols are at 0° , 45° , 15° , and 30° from the axis of V_1 .

When both axes are present in the field of view their positions are determined each separately as above, and plotted in stereographic projection; then the optic axial angle $2V_a$ is measured graphically, with the aid of the formula:

$$2V_a = \cos v \cdot \cos v' + \sin v \cdot \sin v' \cdot \cos (h - h'),$$

in which v and v' are the readings on the vertical circle V_1 of the universal stage for the two respective optic axes, and h and h' the corresponding readings on the circle of the rotating microscope stage.

(2) The section-plate being nearly perpendicular to an optic axis (within 20°), the universal stage is arranged with the two gimbal circles H_1 and H_2 horizontal. H_2 is then turned slightly about its axis of rotation A_3 , and also inclined about V_2 by rotation about axis A_2 , until darkness is produced and the optic axis coincides exactly with the axis of the microscope. To find the other optic axis, the vertical goniometer circle V_1 and the gimbal circle H_1 are then rotated until darkness is again attained, the plane of the optic axes being thus found, and also the apparent position of the axis β of the optical ellipsoid. The circle H_2 is then brought back to 0° with respect to circle V_2 , and H_1 is rotated until the visible optic axis (A_1 in Fig. 701) comes to lie in the plane (OCD in Fig. 701) which is perpendicular to the axis A_1 of V_1 , and the extinction curve determined, the intersection of which with the plane of the optic axes in the projection fixes the position of the second optic axis.

To determine this extinction curve the extinction angle EOD in Fig. 701 is first found, when the stage is in the horizontal position, and also subsequently when it is at such an inclination about V_1 that the extinction angle is 45° . This can be recognised by placing the Nicols in the 45° position and then revolving about V_1 until darkness ensues. By thus ascertaining the angle of revolution necessary to attain the required 45° extinction angle, the great circle CA_2M (the plane in which the unknown optic axis must lie when the extinction angle is 45°) is fixed with reference to the horizontal diameter. The position of A_2 , the second optic axis, on CA_2M is at the point where it is intersected by a radius OB drawn at an angle DOB equal to twice the extinction angle DOE.

The results can be confirmed by determining the positions of the axes α and γ of the optical indicatrix both from the projection and directly by observation. They may also be checked by determining the extinction curve for revolution about the circle V_2 .

Owing to the effects of elliptical polarisation in producing indistinctness of extinction phenomena when the section-plate is steeply inclined, the above simple method does not always work perfectly, in which case another more general one is available. After having placed the known optic axis A_1 in the plane (NO in Fig. 702) normal to the axis of the circle V_1 , as in the method just described, the extinction angles are measured for different inclinations of the stage about the axis of V_1 (MOE

in Fig. 702 being that for the horizontal position), and reducing these angles as usual to real angles within the crystal by means of its average refractive index; the reduced angles are then plotted on a stereographic net, as in Fig. 702. The radii which make an angle with the vertical diameter NOM equal to twice the extinction angles (OA_2 in Fig. 702 is such if EOA_2 be drawn equal to EOM) will be planes containing the second optic axis A_2 , the exact location of which can be found by noting for two given radii, such as OA_2 and OA'_2 (MOE' being a new extinction angle and $E'OA'_2$ its equal, drawn in by construction), the small circle the arc $A_2A'_2$ of which intercepted is equal to the angle of revolution of the stage. The determinations should be repeated for different stage inclinations, such as those equivalent to 0° , 10° , 20° , 30° , 40° , and 45° within the crystal, on both sides of the normal to the section-plate. This method is perfectly general, and the result can be checked by drawing the great circle CF, which marks the position which the plane OA'_2 would assume were the section-plate turned back to its original position. In practice the position of A_2 is determined for a number of different angles of revolution about the goniometer axis of the circle V_1 and the mean taken as the true situation of A_2 .

(3) When one optic axis is visible, inclined between 20° and 55° within the crystal

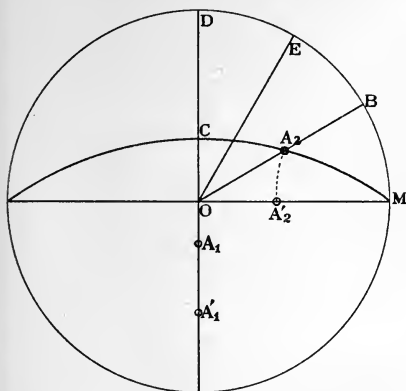


FIG. 701.

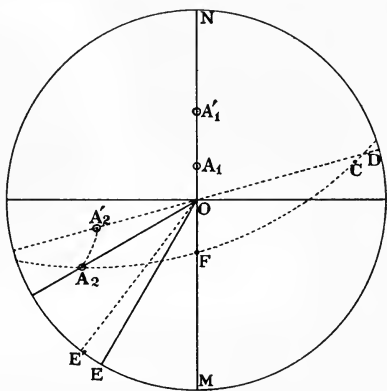


FIG. 702.

to the normal to the section-plate, the second optic axis being out of sight, the visible optic axis is first determined by optical curves as before, and the second optic axis by means of extinction curves about the axes of rotation both of the main goniometer circle V_1 and of the arc V_2 . The results may be confirmed by determinations of the positions of the three axes of the optical ellipsoid α , β , and γ .

(4) When both optic axes are invisible their location is accomplished entirely by means of extinction curves, and the accuracy is of a low order, as a small error in the determination of an extinction angle affects the optic axial angle very considerably.

The above examples will have afforded some idea of the very valuable work of von Fedorow, in devising methods of determining the optical constants with a single miniature section-plate of a crystal orientated anyhow, or a minute crystal of tabular form. In these methods, together with the method of Becke, and the simplification of the latter by the introduction of the principle of simultaneous rotation of the Nicols in crystallographic microscopes, which we owe to the ingenuity of Mr. Allan Dick, as well as the modification of the Becke

method introduced by Wright, dependent on the use of a double micrometer, crystallographers have now the means of investigating with considerable prospect of success crystals of minute size, possibly only thin miniature plates, which at first sight look most unpromising. The united labours of these workers have conferred an altogether higher value on the crystallographic microscope as an instrument of original investigation, altogether apart from its magnificent and unique value in the study of the natural crystals in thin sections of rocks, to which crystals these methods are equally applicable.

Wright has made an interesting calculation of the probable frequency with which a crystal showing at least one optic axis may be encountered in a rock-section, and finds it to be 4 out of 10 in the case of uniaxial

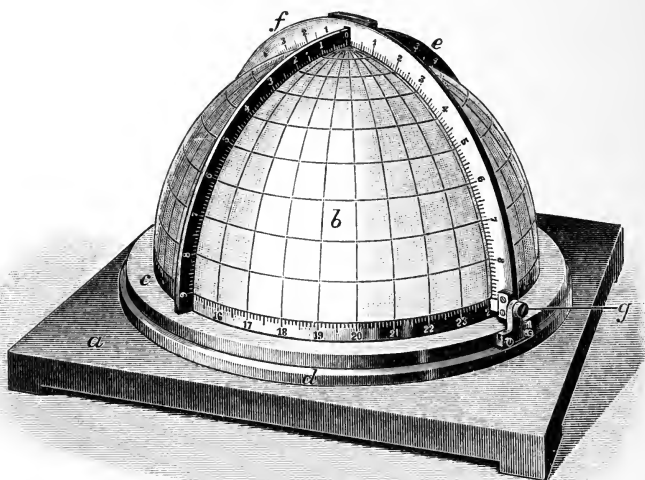


FIG. 703.—Nikitin's Hemisphere for the Graphical Solution of Universal Stage Problems.

crystals, and as high as 8 out of 10 in the case of biaxial crystals. As in all these cases results are certain to follow the careful application of these improved methods, it will be obvious that suitable individual crystal-sections for the measurement of the optic axial angle and other optical constants are likely to be found for such an investigation in every rock-section exhibiting clear crystals.

Nikitin's Hemisphere for recording Universal Stage Results.—A very useful apparatus, which is shown in Fig. 703, has been devised by W. Nikitin¹ in von Fedorow's laboratory in St. Petersburg, for the purpose of graphically recording in pencil on the surface of a porcelain hemisphere the results obtained with the universal stage. A massive wooden base *a* carries the hemisphere *b*, the equatorial plane of which is circumscribed by a fixed metallic annulus divided directly into degrees. Vertical

¹ *Zeitschrift für Krystallographie*, 1910, 47, 379.

great circles like meridians of longitude, and small circles resembling parallels of latitude, are clearly painted on the hemisphere at every 10° . The hemisphere is directly mounted on the circular plate *c* fixed to the base-board, and around it there revolves an outer annulus *d*, which carries with it two semicircular divided metallic limbs *e* and *f*; the former *e* is fixed in the vertical plane, so as always to follow a meridian of longitude, while the latter *f* is attached by hinges *g*, so that it can be inclined on either side, to reproduce the effect of the rotation of the inner gimbal of the universal stage, the inclination to the horizontal plane being measured by the limb *e*. This tilting limb *f* enables not only the angle between any two points on the sphere to be measured directly, but also any section of the sphere by any diametral plane to be traced on the porcelain surface by the pencil. As all symmetry planes of the optical ellipsoid are diametral, their traces can be at once drawn on the sphere. The positions of the optic axes can also be laid down on the sphere from the readings of the universal stage, and the optic axial angle can consequently be immediately read off between the two points thus marked with the aid of the hinged limb. The apparatus is also generally useful for the graphical solution of any stereographical problem which may arise.

CHAPTER LIII.

THE DETERMINATION OF THE THERMAL EXPANSION OF CRYSTALS.

THE thermal expansion of a solid body is expressed by its "linear coefficient of expansion," which is symbolised by a , and is the amount of extension of the body which unit length of it undergoes in the direction of that length on raising the temperature of the body through one degree centigrade. The expression is not itself a constant, but consists of two constants a and b of which twice the latter is multiplied by the number of degrees of temperature. That is, the linear coefficient of expansion a for any temperature t is :

$$a = a + 2bt.$$

The coefficient thus varies regularly with the temperature, the increment per degree of the latter being $2b$. Thus, for example, the coefficient is not the same for 50° - 51° or 100° - 101° as it is for 0° - 1° , but as a rule considerably different and greater in each case; occasionally it happens, however, that the value $2b$ is a decrement, the sign of $2b$ being negative. It is regular at any rate, so long as the chemical composition of the substance remains unchanged, and if the body be crystalline, for the same direction in the crystal; also, in the cases of polymorphous substances, so long as the symmetry remains the same, that is, so long as the particular crystalline modification persists. The expression for a above given is also the mean linear coefficient between any two temperatures of which the mean is t . The mean coefficient between 0° and t° is $a + bt$.

Variation of Linear Expansion with the Symmetry.—A crystal belonging to the cubic system expands equally in all directions, like non-crystalline substances such as glass, or heterogeneously crystallised substances such as metals. Hence a cubic or non-crystalline substance has only one expression for its linear expansion. A sphere cut out of such an isotropic crystal remains a sphere on being heated or cooled, its radius as a rule increasing with rise of temperature. The interfacial angles of cubic crystals remain unaltered by variation of temperature.

The cubical expansion of such a cubic crystal or other isotropic substance is $3a$. For when the expression $a + 2bt$ is cubed all terms beyond the second are found to be too small to affect the significant figures of the coefficient.

A crystal belonging to one of the optically uniaxial systems of symmetry, the tetragonal, hexagonal, and trigonal, expands differently in different directions, but there is this regularity in the differences, that the maxima and minima of expansion occur along the respective directions of the optic axis and any direction perpendicular thereto, along which latter infinity of directions lying in the equatorial plane the expansion is equal. Thus two expressions for the linear expansion, one for the direction of the axis and another for directions perpendicular thereto, are required in the case of a uniaxial crystal. Hence, a sphere cut out of a uniaxial crystal becomes an ellipsoid of revolution on heating or cooling, the axis of which is coincident in direction with the optic axis. All the faces of the crystal which are parallel (together with their edges of intersection) to the optic and thermal axis, or perpendicular thereto (the basal planes), remain unaffected in their angles of mutual intersection; but the inclinations of faces oblique to the axis are altered. When the axis is the direction of maximum expansion these pyramidal or domal faces become steeper with rise of temperature, but when the axis is the direction of minimum thermal dilatation the form becomes a flatter one. The mode of measuring such angles at temperatures up to 120° was dealt with fully in Chapter XXVIII. It was the study of such changes for the first time by Mitscherlich, in the case of calcite, that brought to light both the fact that such interfacial angles of crystals are dependent on the temperature, and indirectly the laws of the thermal dilatation of crystals. Mitscherlich caused the first really delicate goniometer, reading to ten seconds of arc, to be constructed by the then celebrated mechanician Pistor, for this very purpose, and with its aid showed that the rhombohedral angle of each of several fine rhombs of calcspar from Eskifjörður in Iceland was $105^{\circ} 4'$ at 10° , but became reduced to $104^{\circ} 56'$ at 110° C.

The cubical expansion of a uniaxial crystal is obtained by adding to the constants a and $2b$ for the axial direction twice the values of those constants for the equatorial direction. That is, if a represent the coefficient for the former direction and a' that for the latter, then the cubical expansion is $\alpha + 2\alpha'$.

A crystal belonging to the optically biaxial systems, the rhombic, monoclinic, or triclinic, expands unequally in different directions, and there are two directions perpendicular to each other along which the dilatation is respectively a maximum and a minimum, while the direction at right angles to both, the normal to their plane, exhibits an intermediate amount of expansion. These three rectangular directions form the axes of a thermal ellipsoid of general triaxial character, analogous to the optical ellipsoid. A sphere cut from such a crystal becomes converted into an ellipsoid of this form. In the case of a biaxial crystal, therefore, three expressions for the linear expansion are required, one for each of these three axial directions. The three axes are only coincident with the crystallographic ones and those of the optical ellipsoid in the case of crystals belonging to the rhombic system, although even here the lengths

in general have no relations. In the monoclinic system only one thermal axis is identical in direction with a crystallographic axis, the symmetry axis b , which is likewise an axis of the optical ellipsoid. The other two thermal axes lie in the symmetry plane, and although similarly rectangularly inclined have no necessary (although often a fortuitous) connection with the directions in that plane occupied by the optical axes. In the triclinic system the three rectangular thermal axes may have any orientation, consistent with the retention of their mutual perpendicularity. Consequently, only in the rhombic system do the pinakoidal forms all remain angularly unaltered at 90° .

The cubical expansion of a crystal the thermal expansion of which is thus represented by a triaxial ellipsoid is the sum of the three linear coefficients of expansion along the three axes. For when the expression for the expansion of the solid, the product of the three linear expansions along the three rectangular axes of the thermal ellipsoid, is examined, it is found that only two terms in it affect the fourth significant figure in the coefficient of expansion, namely, the sums of the 3 values of a and of b respectively.

The Practical Determination of Thermal Dilatation.—The determination of the coefficient of expansion of so small a body as a crystal usually is obviously demands a very much more sensitive experimental method than is suitable for substances obtainable in greater mass. Such a delicate and refined method is that devised by Fizeau, which utilises the production of interference fringes and employs them as a natural scale of half wave-lengths of light. It depends essentially on measuring the difference of expansion of the crystal and of a small tripod, composed of three screws of fine pitch, screwing through a little table on which the crystal is supported. This is effected by observing the alteration of the thickness of the film of air between the upper plane surface of the crystal and the lower plane surface of a plano-convex lens supported over it by the three screw-legs of the tripod, on the tops of which the lens rests. Interference fringes are generated between the light reflected from the two surfaces, and when the thickness of the air-film separating the two alters, owing to the difference of expansion of the two substances forming the crystal and the tripod, the fringes also move, and for every fringe that passes a reference spot an alteration of thickness of the air-film at the spot has occurred equal to half the wave-length of the light employed. For the light derived from the second reflecting surface has to traverse the air-film twice, on its incidence and after its reflection, so that each alteration of thickness of the film equal to half a wave-length is equivalent to a whole wave-length of retardation of this light behind the light derived from the first reflecting surface, thus corresponding to repetition of the same phenomenon of light or darkness, the phases being similar. Hence, if an interference fringe were over the reference spot before the alteration of thickness, the next fringe would have replaced it over the spot at the completion of the alteration equal to a half wave-length. Whether a fringe (total extinction of light) or a bright interval (of no interference)

is over the spot depends on the total retardation, that is, that due to traversing the film plus the constant half-wave-length change of phase due to one reflection from a denser medium (glass) into a lighter (the air-film) medium, as explained in Chapter XLI. (p. 643) in connection with Newton's rings. Hence whole wave-lengths (even numbers of half wave-lengths) of retardation due to the film correspond to the dark fringes and odd numbers of half wave-lengths to bright intervals. Fizeau employed sodium light, as the fringes are only produced in monochromatic light, and they are the sharper, and visible for greater thicknesses of the air-film, the more perfect the monochromatism.

Now, as is well known, sodium light corresponds to two lines of the spectrum, D_1 of wave-length 0.0005896 and D_2 of wave-length 0.0005890. The light waves from these two sources consequently periodically interfere, such so-called "secondary interference" causing intermittent disappearance of the interference bands as the thickness of the air-film regularly increases. Hence, although sodium light served Fizeau's purpose, care being taken to avoid the positions for maximum secondary interference, it is not an ideal light to use for interference work. Moreover, the original Fizeau apparatus does not lend itself to micrometric measurement, nor does it fulfil one of the conditions for complete interference of the rays of light, namely, normal incidence of the rays on the reflecting surfaces. Both these defects are corrected in an improved form of apparatus devised by Abbe and employed by Pulfrich,¹ but even this new form suffered from the fact that the telescope was arranged over the hot air-bath. This, and a few other inconveniences, have been corrected and eliminated in a later form of apparatus of more general application described by the author,² and employed in the determination of the thermal expansion of the crystals of the alkali sulphates, of the metals nickel, cobalt, and aluminium, and of the alloy of platinum with 10 per cent of iridium, of which alloy the interference tripod is constructed in the author's apparatus, Fizeau having used platinum and the same alloy, and Abbe and Pulfrich steel or quartz. The apparatus has also been employed for the important determination of the thermal expansion of the fine porcelain of which air or other gas thermometers are constructed, and also of that of a variety of glasses. The experience gained with these determinations has enabled the dilatometer to be brought to such a state of perfection as to render it not only of general convenience and application, but of particular value for the determination of the thermal expansion of crystals. It is constructed by Messrs. Troughton and Simms.

The Interference Dilatometer.—The dilatometer is shown in Fig. 704. It consists of two separate parts, mounted to the right and to the left on independent pedestals, the left one carrying the interference apparatus in a suspended manner, so that it can be immersed in a cylindrical double air-bath of copper, lined outside with asbestos. Sections of the optical trains of the two parts are given in Figs. 705

¹ *Zeitschr. für Instrumentenkunde*, 1893, 365.

² *Phil. Trans.*, A, 1898, 191, 313; 1899, 192, 455; *Proc. Phys. Soc.*, 1902, 18, 631.

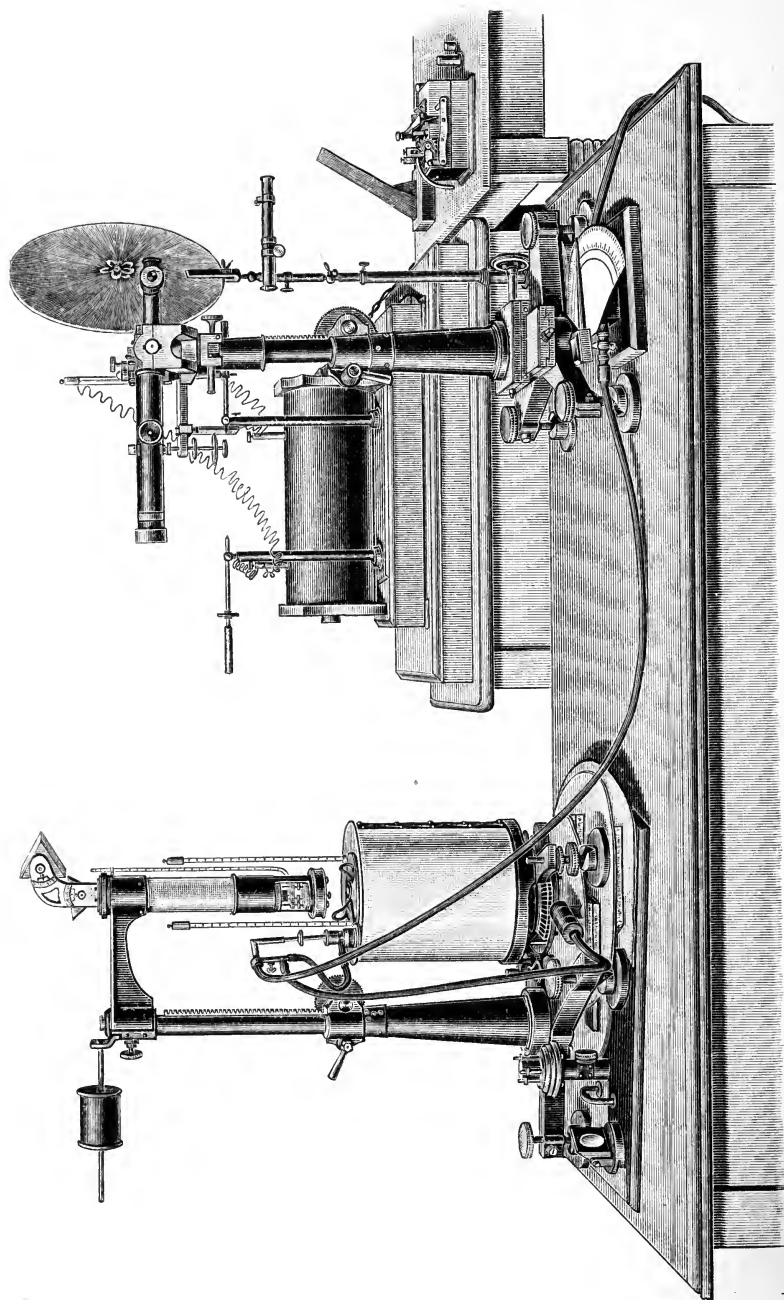


FIG. 704.—The Interference Dilatometer.

and 706, and the lettering in the following description refers to these sectional drawings.

The right pedestal carries the observing telescope, *a* in Fig. 705, which is of auto-collimating character, in order to ensure the normal incidence above referred to. It is shown in Fig. 707 in the manner in which the observer regards it from the end of the slate observing table. At the common focus of the objective *b* and eyepiece *m* or *n* the optical tube *a* is constricted by a diaphragm *g* with a fixed maximum aperture, which can be lessened to any extent by an iris diaphragm *f*; the right half of this circular aperture, as the observer stands at the right end of the

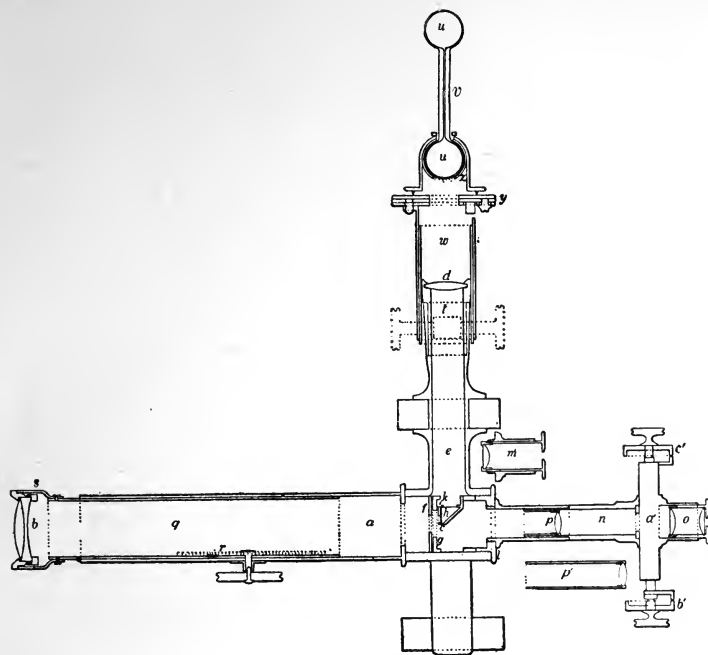


FIG. 705.—Section of Auto-collimating Telescope and Geissler Tube Attachment.

long slate table looking through the telescope, is closed by a small totally reflecting prism *h*, which receives the rays from the source of light, which is an ordinary goniometer lamp when white light is used in preliminary adjustments, and a hydrogen or cadmium Geissler tube *w* when observing in monochromatic light. The rays are concentrated on the little prism by a suitable lens *d* carried in an inner draw-tube *t* at the end of a short elbow tube *e* attached perpendicularly to the telescope opposite the reflecting prism *h*, and the rays are reflected by the latter towards the objective *b*, the focus of which is adjustable by rack-and-pinion movement of the draw tube *g*. The side tube *e* carries the Geissler tube fitting as a readily detachable accessory, a tube *w* carrying it slipping over the collimator *e* and being adjustable by rack and pinion, which move an outer tube actually supporting the fitting *z*, in order to attain the right focus of the capillary *v*, the brilliantly illuminated section of which is the source of light; sideways adjustment of the end-on capillary is secured by two centring

movements y for the tubular holder z of the supported wide limb of the H-shaped Geissler tube.

The telescope is adjustable for height by the stout rack and pinion of the pedestal, and for azimuth and altitude by fine motions; it is also adjustable in two rectangular directions by a pair of movements at the base of the pedestal. The aperture of the prism face looking towards the objective is restricted by the insertion of any one of a series of rectangular "stops," an oblong aperture in a metallic strip k provided with

handle, which latter enables the strip to be pushed down into a slot cut for it in the optical tube, the strip nearly touching the prism face. The aperture of the stop usually employed is 3 by 2 millimetres, and it is cut out of the strip very close to one of the long edges of the latter, that one which when in position occupies the vertical diametral line of the whole iris aperture of the optical tube. This brings the little rectangular aperture, which may be regarded as the direct source of monochromatic light for the production of the interference bands, quite close to the axis of the telescope, thus practically securing normal incidence; for after returning from the interference apparatus the rays are arranged to arrive at the iris aperture just in the open half, so as to be visible to the

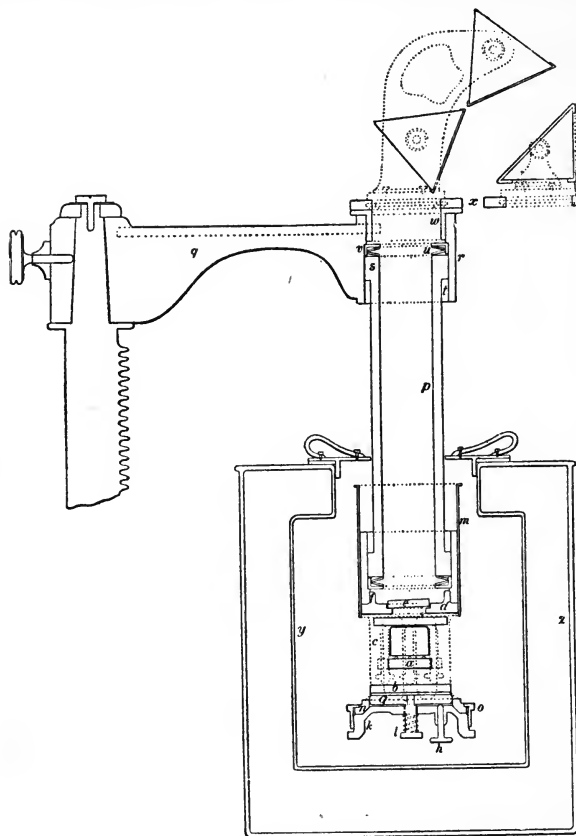


FIG. 706.—Section of Suspended Tube, Interference Apparatus, and Air-bath.

observer looking through the eyepiece, and within a couple of millimetres of the origin-stop itself.

Two eyepieces are provided, one a single-lens plano-convex eyepiece m magnifying four diameters, and the other n a positive two-lens micrometer eyepiece carrying two vertical spider-lines, one moved horizontally parallel to itself by the milled-headed drum b' on one side of the micrometer box, while the drum c' on the other side moves both simultaneously. Each drum is divided into a hundred parts, and the function of this micrometer is to enable any interference band to be brought parallel to and to lie totally between the pair of vertical spider-lines, and so that both the position of

this band and its distance from its adjacent fellows on either side may be accurately determined, as well as the width of a band, that is, the interval between two bands, in drum divisions; the fraction of any band which may have partially passed the reference centre is at once afforded by these data. The central line between the two spider-lines is not the direct, but only the secondary reference position, the primary one being the centre of a minute silver ring carried at the centre of one of the two interference surfaces, in order that no error due to movement of the telescope may influence the results, the reference centre and the bands having the ideal common focus. The pair of spider-lines are set symmetrically, one on each side of the reference centre of the ring, in order to determine the position of the latter, in a manner which will be clear from Fig. 708, which shows a typical field of interference bands as seen in the micrometer eyepiece of the telescope.

The attachment tube n of the micrometer eyepiece is somewhat long, an additional lens p , converting the telescope into a very low power microscope, being accommodated in the elongation; its position is adjustable in the tube, several short inner tubes of different lengths being supplied, into the end of any one of which it screws, to enable the interference bands and the silver reference ring to be focussed clearly for different distances of the interference apparatus on its separate pedestal. One of these additional inner tubes is shown at p' .

This second pedestal to the left is similar to the one on the right, being capable of considerable vertical elongation by the strong rack and pinion. It bears at its upper

extremity a counterpoised arm, q in Fig. 706, from which the interference tube p is suspended. This is constructed in part of Berlin porcelain, in order to minimise conduction of heat from the interference chamber c which it carries at its lower extremity; the attachment between porcelain and metal is made by means of flanges s on the porcelain and annular springs u and securing-rings v and t , in such a manner that the difference of expansion is afforded free play. At the upper extremity a pair of refracting prisms are carried, adjustable for azimuth by the mode of attachment of their metal carrier-slide to an inner tube w , provided with a flange above, bearing the rabbeted guides for the carrier-slide of the prisms, this tube w rotating within the fixed tube r without vertical movement. The prisms are also each independently

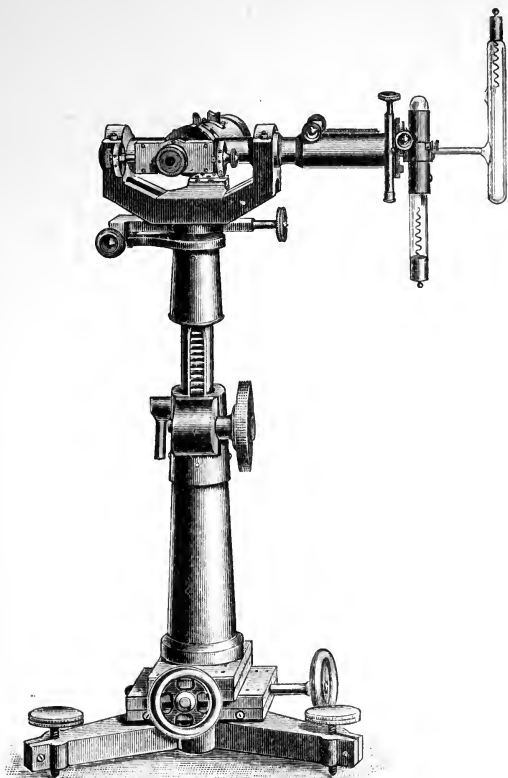


FIG. 707.—The Auto-collimating Telescope.

adjustable to minimum deviation, a pair of silver divided arcs, and indicators on the axes of the prism frames, being provided, best seen in Fig. 704, to enable this setting to be immediately recovered if ever it be disturbed for any other purpose. The refracting prisms are replaceable for adjustment purposes by a single large totally reflecting prism, carried on a similar slider in a detachable upper part x of the flange of the inner tube w .

The angle of the two refracting prisms is such that the total deviation produced is exactly 90° for sodium light. They may be replaced if desired by a single Hilger constant-deviation prism, affording a regular deviation of 90° . This alternative dispersing apparatus has proved very satisfactory in the interferometer devised by the author in connection with a Wave-Length Standard Comparator recently constructed under the author's supervision for the Standards Department of the Board of Trade. Hence, its use can confidently be recommended in this or any other adaptation of

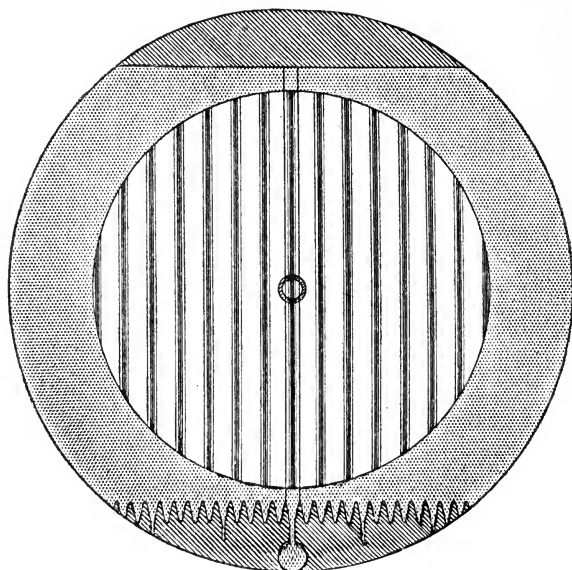


FIG. 708.—The Field of Interference Bands.

the author's "interferometer." For the optical part of the dilatometer constitutes an interferometer of general applicability to the measurement of minute distances or quantities of motion.

The parallel beam of monochromatic light received from the telescope—the objective of which is filled with light from the little rectangular stop at its focus, in front of the small reflection prism—is thus deflected downwards at right angles along the axis of the interference tube p to the interference tripod a . The latter is supported in a detachable chamber c , which is attached by a screw-thread f carried by its perforated roof-plate d to the bottom of the interference tube. The greater part of the walls of this little chamber are cut away so as to leave three wide windows, through any one of which the interference tripod can be introduced. The roof d of the chamber is pierced by a central aperture an inch in diameter, closed by a glass plate e to be referred to directly, as it forms part of the interference apparatus; it is held in a rabbet, through which a pair of small tilting screws are driven, in two positions 120° apart, in order to give the plate a slight inclination from normality to the vertical axis of the whole tube.

The chamber terminates below in an adjustable floor *g*, the level of which can be adjusted by three screws *h*, which push it away from a base *k* rigidly attached to the bottom of the chamber *n* by a screw annulus *o*, a strong spring *l* holding the floor tightly down on the screws. The interference tripod *a* does not rest directly on the gun-metal floor *g*, but on a glass plate *b*, which is separated from *g* by a thick disc of asbestos. Conduction away of heat from the tripod is thus reduced to a minimum.

The interference apparatus, the essential part of the whole instrument, is shown two-thirds its real size at *a*, *b*, and *c* in Fig. 709, which represents three methods of its use. It consists of a tripod of platinum-iridium, the three legs of which are stout screws of fine pitch, with roundly pointed ends, and each furnished near the bottom with a milled head for rotating purposes, the fine adjustment of these screws being the means of adjusting the interference bands. The screws pass through three arms radiating from a thick table, also of solid platinum-iridium (10 per cent of the latter metal being present in the alloy), and three small tightening screws of the same alloy secure the large screws in the adjusted positions at the right height above the table, the arms having each a saw-cut to facilitate the tightening. The upper surface of the table bears three concentric sets of raised points, each set consisting of three points of equal height, but the sets increasing in height as the centre of the table is receded from. These points are intended for the support of the crystal or other object the

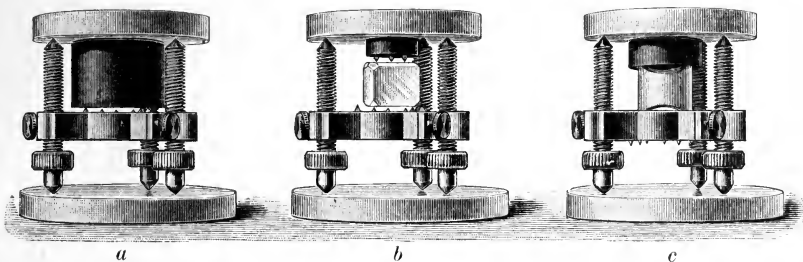


FIG. 709.—The Interference Tripod of Platinum-iridium and three Modes of its Use.

thermal expansion of which is to be determined, which requires to be furnished with two parallel surfaces, as shown at *a* in Fig. 709, one of the sets of points being found suitable to the dimensions of the lower of these two surfaces. The upper surface of the crystal or other object must be ground and polished a true plane. The largest size of object which it is convenient to employ is 25 millimetres (one inch) in diameter, and 12 millimetres in height (thickness). A cylinder of black glass of these dimensions is the object shown as under investigation at *a* in Fig. 709.

The height of the screws projecting above the table top is regulated so that the upper terminations of the screws are just higher than the upper surface of the object when the latter is resting on the table points. A thick circular glass plate is then laid over the screws, a film of air being thus left between the under surface of the plate and the upper surface of the object. These are the two surfaces which are caused to reflect the light waves which interfere. The lower surface of the object, when the latter is transparent, is ground just sufficiently to prevent any reflection, while both surfaces of the glass plate are polished truly plane, for the light has to traverse the glass plate twice in going and returning to the object surface. Hence, there is a disturbing reflection from the upper surface of the glass plate, not required for interference purposes, and this is eliminated by making the two surfaces inclined at a minute angle, 35 minutes being the most convenient amount, instead of employing a parallel-surfaced plate. This small amount of inclination is just

adequate to deflect the undesirable upper reflection to one side of the semicircular aperture *c* in Fig. 705, and behind the iris diaphragm *f*. The device introduces, however, a little dispersion, such a plate being really a prism of small angle, and this in turn is corrected by means of a duplicate wedge-plate, the two in fact being cut from the same carefully prepared plate having truly plane surfaces inclined at 35° . This countervailing wedge-disc is the smaller one, *e* in Fig. 706, of 1-inch diameter already referred to as closing the aperture in the roof *d* of the interference chamber; the two little tilting screws in the rabbet supporting the disc serve to give it a slight inclination at right angles to the direction of the wedge. This direction is clearly marked on both plates by two engraved dots at the thicker end of the diameter perpendicular to the edge of the wedge, and a single dot at the narrower end. Both the reflections from the countervailing wedge-disc are thus deflected out of the field, behind the iris diaphragm. Hence, the only light passing to the eyepiece through that diaphragm is that reflected from the upper surface of the crystal or other object under investigation, and from the lower surface of the large glass wedge-disc resting on the screws of the platinum-iridium tripod. This lower glass surface bears at its centre a miniature silvered ring, which serves as an excellent reference spot, the movement or position of the interference bands being measured with reference to the centre of the ring.

In the not uncommon event, especially in the case of artificial crystals, of the surface of the crystal or other object not taking a good polish, the author places over it a disc of aluminium, the expansion of which is accurately known, an exhaustive series of determinations of the expansion of aluminium having been carried out by the author in the year 1898,¹ by using a block of the metal slightly over 12 mm. thick, as shown at *a* in Fig. 709. Aluminium has three advantages, the first being that its expansion is 2.6 times as great as that of platinum-iridium. Hence, it is easy to select a disc of such thickness as will approximately compensate for the expansion of the platinum-iridium screws, and thus an immediate indication of the real degree of expansion of the crystal or other object is afforded by the alteration of the thickness of the air-film, instead of merely the differential expansion of object and screws being observed. No attempt is made at absolute compensation, and the actual expansions of the aluminium and screws are, of course, taken into account in the calculations; but it is often a convenience to see at once the kind of expansion that is occurring, and in the occasional cases of contraction the unusual fact is at once indicated. In the second place the surface of aluminium just takes that not too brilliant degree of polish as causes the reflection to be of about the same intensity as the partial reflection from the lower surface of the glass cover-wedge. Thirdly, aluminium is particularly light, and does not compress the crystal appreciably. Such a case is that shown at *b* in Fig. 709. A crystal is represented standing on the intermediate set of three points of the tripod table, and an aluminium disc, also furnished below with three points, is seen resting in turn by the latter on the crystal. Still another case of the use of the aluminium disc is shown at *c*, where a specimen of Bayeux porcelain, a piece of tube 12 millimetres high, very inconvenient in itself for the production of interference bands, is seen as arranged for a determination of its thermal expansion; the planely ground upper and lower parallel terminations of the piece of tube were cut away at three places in each case, to enable the porcelain to rest by three-point contact on the platinum-iridium table (the other surface, not furnished with points, of the table being employed in this case), and to permit an aluminium disc to rest in a similar manner on the porcelain.

In making a determination of expansion the interference tube carried by the second pedestal is immersed in the inner chamber of the double air-bath of copper (the outside covering of asbestos of which retards loss of heat by radiation), until about one-third of the porcelain tube is below the level of the top of the bath, the interference chamber then occupying a position slightly below the centre of the bath. The inner bath *y* in

¹ *Phil. Trans. Roy. Soc., A*, 1898, 191, 360.

Fig. 706 is closed by a lid, fitted together in two parts when the interference apparatus is in position. Between this inner air-space and the outer bath \approx two thermometers are inserted, and a Muencke thermostat, which with the aid of a Stott governor and a tap with long lever arm moving over a divided quadrant-arc, seen in front of the right pedestal in Fig. 704, enable an excellent control to be obtained over the temperature, by affording delicate regulation of the gas supply to the Fletcher ring burner arranged beneath the air-bath. The actual temperature of the interference apparatus, however, is measured by a special thermometer suspended in the inner bath alongside the interference tripod, and bent at right angles just above the bulb, in order that the latter may actually lie on the platinum-iridium table quite close to the crystal or other object, where it is secured by tying it to the nearest leg of the tripod with a silk thread.

The Determination of Thermal Expansion.—The interference apparatus is first adjusted to afford a suitable field of bands, of the nature shown in Fig. 708. When the colourless glass 35' wedge-disc is laid over the screws, the crystal or other object under investigation with or without the aluminium disc being already in position, the height of the screws projecting through the table is so regulated that an apparently parallel-surfaced film of air of half a millimetre or more thickness is left between the crystal or aluminium surface and the lower glass surface. The large total reflection prism should then be placed in position at the top of the interference tube instead of the two refracting prisms, and the images of the little rectangular stop in front of the small reflecting prism of the telescope are reviewed in white light, using the common eyepiece, as reflected from the two surfaces in question. Three other images are also seen, derived from the second surface of the glass wedge-disc and the two surfaces of the countervailing wedge-disc, and all four images derived from the wedge-discs are adjusted to occupy the corners of a rhombus; for then the spectra corresponding (produced when the two refracting prisms are reinstated in position) will not overlap and interfere, and will in fact be separated to an extent which will enable the iris diaphragm to shut off all but the one from the surface next to the crystal or aluminium surface, which alone is wanted. This right one may be readily identified, for it is one of the two which move when the glass disc lying on the screws is touched; when the disposition of the wedge is known, as it always will be, never altering, the correct one of the two is at once known. For it will always be either the right or left of these two, according as the thicker part of the wedge is known to be arranged.

The image from the surface of the crystal or other object, or the aluminium disc, may also be identified as a fifth independent image, some distance from the others unless the parallelism of the surfaces has been well attained, by touching the crystal for a moment, which causes the image to tremble; and this fifth image is then made by adjustment of one, or, if necessary, two of the platinum-iridium screws, to approach the one derived from the surface of the wedge-disc which has just been recognised, and eventually partially to overlap and almost cover the latter. Generally it will be found that these two images relevant to the interference should be at precisely the same vertical height in the

field, and that they should overlap horizontally to about four-fifths of their extent, in order to secure the conditions for the production of vertical interference bands of a width equal at least to one hundred drum divisions of the micrometer eyepiece. If now, (1) the large totally reflecting prism be replaced by the two refracting prisms, (2) the common eyepiece be replaced by the special one provided with micrometer, and (3) the goniometer white-light lamp be replaced by the hydrogen Geissler tube excited by the induction coil, the interference bands will themselves be seen conveniently spaced and vertically arranged as expected, as dark rectilinear bands on a brilliantly illuminated ground in the colour corresponding to the wave-length for which the telescope had been adjusted, generally that corresponding to the red line of hydrogen. If the bands, however, do require some slight adjustment for width or parallelism to the spider-lines, it can be readily attained by slight further manipulation of the tripod screws.

The bent thermometer is now attached in its place. The left pedestal, with the whole of the apparatus suspended therefrom, is then moved to a suitable marked position about five feet away from the right pedestal carrying the observing apparatus, in order that no heat may reach the latter, large mica screens being also erected between the two parts of the apparatus, and between the air-bath and the dispersing apparatus, when all is ready, to assist still further in preventing any heating of the optical parts. The left pedestal is mounted on a mahogany base-board, covered below with broadcloth to facilitate sliding, and supplied with two handles to enable it to be drawn gently and steadily into the desired position at the left end of the slate table without any disturbance of the interference apparatus.

The interference bands should now be reviewed after the transference, with the movable lens p of the micrometer eyepiece readjusted in the proper (longer) draw-tube corresponding to the longer-focus position. The suspended tube is then raised by the pedestal rack and pinion, while the air-bath is placed in position on its annular support over the ring of gas burners; the tube is then as delicately lowered into its correct position in the bath, as indicated by a mark on the pedestal lengthening shaft; and the lid, thermometers and thermostat, together with asbestos and mica screens to ward off draughts and the convection currents, all fitted into their proper places. The bands are then finally reviewed in order to be sure that no derangement has occurred, and their position for the ordinary temperature, read on the bent thermometer, is determined after a reasonable interval has elapsed for temperature equilibrium to have been established. That is, if a band does not happen to be placed exactly between the pair of vertical spider-lines, and over the reference centre of the silvered ring carried by the under surface of the glass cover-wedge, as shown in Fig. 708, the fraction of a band, the interval between the reference spot and the centre of the nearest band, requires to be determined. To do this we measure the width of a band in drum divisions (the interval between the centres of the two bands nearest to the reference centre), and the distance of the next

band (the distance of the reference centre from the centre of the nearest band) also in drum divisions, when the quotient of the latter over the former is either the fraction of a band required, or the complementary fraction, according to the direction in which the bands move when the heating commences. The object in producing rectilinear bands, rather than the circular interference rings or curved fringes afforded by the lenticular cover-glass of Fizeau, or broad fields of light and darkness, is to render them suitable for such micrometric measurement, and to obtain a field containing several dark bands as narrow parallel lines separated by broad brilliantly illuminated interspaces, so that the distance between the blackest central part of two adjacent bands can be determined by measuring it between five or six successive pairs of bands, the average of which will give a very accurate quantity for the true "width" of a band. As the drum is divided into a hundred parts, and as matters are so arranged that a convenient width to which to adjust the bands, which still enables about a dozen of them to be visible in the field at once, corresponds to one drum revolution, the interval between two consecutive bands can thus be subdivided into a hundred parts.

Now each pair of consecutive black interference bands correspond to a difference of thickness of the air-film at the two positions where they are produced of half the wave-length of the light which is interfering. If either of the reflecting surfaces be moved in the direction of its normal, keeping it parallel to itself, the bands also move parallel to themselves, and for every black band which has moved past the centre of the reference ring the moving surface has actually travelled through a distance (opposite this spot) equal to the half wave-length of the light employed to produce the bands. The diameter of the silver ring is arranged to be just a trace greater than the separation of the spider-lines, so that a little segment can be arranged to show outside each, when the spider-lines are "adjusted to the centre." The gross unit of the interferometer scale, formed by the black bands themselves, is thus the half wave-length of the monochromatic light employed to produce the bands, and this being divisible into 100 parts by the micrometer, the actual unit is the two-hundredth part of the wave-length. The observer can actually divide the distance between the two consecutive bands readily by the eye and judgment alone into ten parts, even without the micrometer.

The light employed for all ordinary purposes is that of the red hydrogen ray H_{α} , the C of the solar spectrum. The wave-length of this light is 0.0006562 millimetre, and the half wave-length consequently 0.0003281 mm., or $\frac{1}{3048}$ of a millimetre, or $\frac{1}{77419}$ of an inch. The one-hundredth part of this is the unit of measurement of the instrument, namely in round numbers, the three-hundred thousandth part of a millimetre or the eight-millionth part of an inch.

If the air-film between the closely adjacent reflecting surfaces were of equal thickness throughout, that is, if the truly plane surfaces were strictly parallel, alternate fields of the brilliant monochromatic light

and of jet-black darkness would be observed as the distance of separation of the surfaces was varied by moving one of the surfaces normally and parallel to itself. But when the air-film is made slightly wedge-shaped, by tilting one of the discs, then rectilinear interference bands (lines of extinction) are produced, the width of which diminishes as the tilt increases and which rotate if the position of the edge of the air-wedge (that is, of the direction of tilting) be altered.

The thickness of the air-film should theoretically not matter if the light were absolutely monochromatic, that is, if the spectrum line corresponding to it were single and absolutely unresolvable into a double line or a triplet by extreme dispersion. But the only line that has yet resisted resolution is the red line of cadmium vapour, wave-length 0.0006438 mm., and with light corresponding to this wave-length the bands can still be seen when the surfaces are a decimetre apart. With light of the wave-length of the green mercury line, wave-length 0.0005461 mm., the interference bands are still visible for a separation of the surfaces of between one and two centimetres. With red C-hydrogen light, the surfaces must not be further apart than about three millimetres. Between half a millimetre and a millimetre is the best distance of separation for brilliant bands either in the red C-light, or in the greenish-blue F-hydrogen light of wave-length 0.0004861 mm.

The measurement of the position of the bands having been made, and the temperature of the inner bent thermometer read by a small telescope on a stand near the first pedestal, seen in Fig. 704, and which also carries a circular screen to shade off the glare from the Geissler tube, the double ring of Bunsen gas jets is ignited and turned very low, in order to raise the temperature of the bath very slowly. The bands are carefully observed, the Ruhmkorff coil being intermittently actuated every few seconds as the bands begin to move. The passage of each band is noted as it passes between the vertical pair of spider-lines, indeed the author finds it the safest course to note every quarter of a band, being then sure that no band has escaped being counted. When the temperature approaches the neighbourhood of the first higher limit, usually about 60° C., it is carefully controlled by means of the graduated gas tap, and eventually brought to constancy, at which it is maintained for a full hour, the bands having almost simultaneously ceased to move. The temperature of the bent thermometer is then again read, and if the last band does not happen to have stopped in the central adjusted position, but has gone somewhat further, its position with respect to the reference spot and the band width is determined, in order that the last fraction of a band may be calculated. The gas may then be turned higher and the temperature again slowly raised to the higher limit, about 120°, the bands which effect their traverse during the process being counted as before. Constancy is again brought about for an hour at this higher limiting temperature, and the last fraction of a band measured when the bands have settled into perfect repose.

The operations should take a minimum of five hours, but need never exceed seven hours altogether. By raising the temperature

thus slowly the chance of cracking the crystal is minimised, and the temperature of the platinum-iridium tripod and its contents follows closely after that of the inner air-bath. It is much more satisfactory thus to watch the process throughout, than to make observations with several wave-lengths and obtain the number of bands by calculation, a method described by Pulfrich (*loc. cit.*).

After the apparatus has cooled down again the lengths of the platinum-iridium screws above the table, the thickness of the crystal, and that of the aluminium disc when one is used, are to be measured by the thickness measurer *in situ*, under the reference spot.

Determination of Thickness.—A convenient and very accurate thickness measurer is shown in Fig. 710.

It consists of a vertical rod *a* with a silver millimetre scale *b*, suspended from gimbals *c* in a nearly counterpoised manner, with the aid of a pulley *d* and weight *e*. The rod terminates below in a rounded agate point, which is lowered on to the upper surface *f* of the object *g*, the thickness (height) of which, above a thick glass plate *h* intended for its reception, is to be measured. The scale is read by a low-power microscope *k*, provided with a micrometer eyepiece *l*, the drum *m* of which is divided into 100 parts. The scale is divided directly into fifths of a millimetre, and two rotations of the micrometer drum correspond to a scale division, so that one rotation thus occurs for each tenth of a millimetre. Hence, each drum division indicates the thousandth of a millimetre. In the author's varied experience of all modes of fine measurement, the thousandth of a millimetre is the smallest unit attainable with accuracy by mechanical means. For any higher accuracy recourse must be had to a scale of interference bands. The thousandth of a millimetre is just adequate for the determination of thickness. For it is the ten- (or twelve) thousandth part of the ten (or twelve) millimetres of thickness given to the object under determination for expansion, and this fraction of the total expansion, which an error of one drum unit would entail, is smaller than the experimental error of the whole process. It is not necessary that the initial thickness should be determined to the unit of the interference method, and it is mechanically impossible.

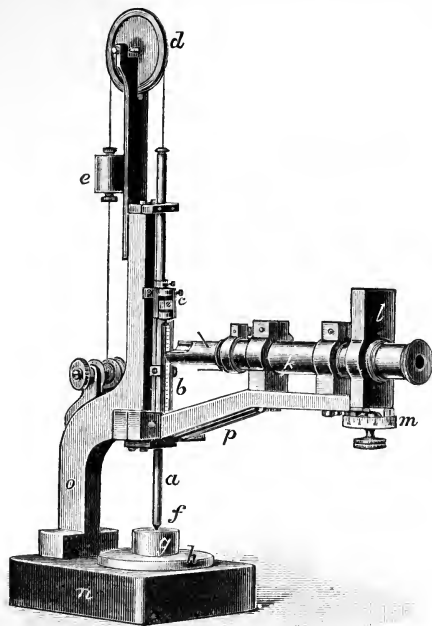


FIG. 710.—The Thickness Measurer.

The whole thickness measurer is supported on a solid base *n*, and a curved pillar *o* supports the scale, and the winch for raising or lowering it by means of the cord, while a forward arm *p* bears the microscope. The instrument was made by Troughton and

Simms, and a similar instrument, but with a different gimbal arrangement and other details, is also constructed in an admirable manner by Zeiss.

On removal of the interference tripod and its contents, except the large glass cover-wedge (which is lifted up while the tripod is removed, and then left behind in the chamber), it is carried without disturbance straight to the thickness measurer, on the glass-plate base h of which it is placed, with the centre of the aluminium disc (if used, or crystal or other object if not) under the agate point of the suspended rod a .

When the height of the centre of the top surface of the aluminium disc, over which the reference ring of the wedge-disc had been situated, and of that of the crystal has been taken the disc or crystal is removed, and finally the height of the upper platinum-iridium surface is taken. If the three points of any of the three sets are used, their mean difference from the surface is known by having determined it for each set once for all. The difference of the first two heights gives the thickness l_a of the aluminium, and that between the second and third heights gives the thickness L_{t_1} of the crystal. The height of the plane of the tops of the platinum-iridium screws is already known from previous measurements; but it may be freshly determined also at the centre with the aid of a glass disc with truly parallel surfaces and of known thickness laid over them. The difference between this and the height of the table top or the mean height of the set of points used is the length l of the screws, while the difference from the height of the aluminium top surface is the thickness d of the air-film. These thicknesses, thus determined all through under the very spot where the reference ring had been, are the original lengths actually expanded, according to the indications of the movement of the interference bands at the reference spot, and are those used in the calculations.

An example taken at random from actual measurements with the alkali sulphates will render the process quite clear. It refers to a crystal of caesium sulphate along the direction of the morphological axis b .

MEASUREMENTS OF THICKNESS.

	Millimetres.	Millimetres.
Height of top of glass disc . . .	40·857	...
Known thickness of glass disc . . .	6·117	...
Height of screws	34·740	$d = 0·145$
„ „ top of compensator . . .	34·595	$l_a = 5·253$
„ „ „ crystal	29·342	$L_{t_1} = 8·379$
„ „ „ tripod table	20·963	$l = 13·777$

By making the determinations for two different intervals we are enabled to calculate not only the mean coefficient of expansion, but also the increment per degree of temperature, as the coefficient varies, usually increasing regularly, with rise of temperature.

The Calculations of Thermal Expansion.—The coefficient of thermal

expansion is signified by a , and it has already been shown that the expression for the actual coefficient at any temperature t , as also for the mean coefficient between any two temperatures the mean of which is t , is :

$$a = a + 2bt.$$

The mean coefficient of expansion between 0° and t° is, however,

$$a + bt.$$

The data afforded by observations of the positions of the interference bands at three adequately separated temperatures, and of the number of bands passing the reference point during the intervals between these temperatures, together with a knowledge of the original thicknesses of the block of crystal and of the aluminium compensator, and the length of the platinum-iridium screws projecting above the tripod table or its raised points, are ample to enable the two constants a and b to be calculated. For it is only necessary to insert respectively in three equations of the form

$$L_t = L_0(1 + at + bt^2)$$

the known values of the three temperatures and the lengths (thicknesses) of the crystal block at those temperatures, and to solve the three equations thus provided, for the three unknown quantities L_0 , a , and b .

The solution of these equations furnishes expressions for the three required quantities of the forms

$$a = \frac{\theta}{L_0}, \quad b = \frac{\phi}{L_0}, \quad \text{and } L_0 = L_{t_1} - \theta t_1 - \phi t_1^2,$$

in which θ and ϕ are terms involving the differences of the lengths, L_{t_1} , L_{t_2} , L_{t_3} , at the three temperatures t_1 , t_2 , and t_3 , and the sums and differences of those temperatures.

The actual expressions for θ and ϕ employed throughout the observations were :

$$\theta = \frac{(t_1 + t_3)}{(t_2 - t_1)} \frac{(L_{t_2} - L_{t_1})}{(t_3 - t_2)} - \frac{(t_1 + t_2)}{(t_3 - t_1)} \frac{(L_{t_3} - L_{t_1})}{(t_3 - t_2)},$$

$$\phi = \frac{L_{t_3} - L_{t_1}}{(t_3 - t_1)(t_3 - t_2)} - \frac{L_{t_2} - L_{t_1}}{(t_2 - t_1)(t_3 - t_2)}.$$

A correction is required for the alteration of the wave-length by the large change of temperature, and for any appreciable change of atmospheric pressure during the observations. The effect of the latter is almost negligibly minute, but the barometer should be read at the time of taking each of the three temperatures, in order to be able to use the correction if necessary ; and as the correction is included in the same formula as that for temperature it is best always to apply it. The effect of change of temperature on the wave-length is, however, such as to require a quite serious correction, amounting in some ordinary cases in the author's investigations to nearly half a band, and when the expansion of the platinum-iridium tripod screws was being determined—when the air-film was the thickness of the whole 12 millimetres of screw-

length employed (green mercury light having been used, hydrogen light not affording bands at such long intervals), the lower reflection being directly that of the polished platinum-iridium table top (the side without points)—the correction amounted to as much as $3\frac{1}{2}$ bands. Hence, this is an important matter requiring careful attention. The most satisfactory formula is due to Pulfrich,¹ and is as follows :

$$f' = f + d(t_2 - t_1) \cdot \frac{b_1}{760} \cdot \frac{1}{1 + at_1} \cdot \frac{1}{1 + at_2} \cdot 2 \frac{(n-1)\alpha}{\lambda} - d(b_2 - b_1) \cdot \frac{1}{1 + at_2} \cdot 2 \frac{n-1}{760\lambda},$$

where f' is the corrected number of bands, f the observed number, d the thickness of the air layer, t_1 and t_2 the limiting temperatures, and b_1 and b_2 the corresponding barometric pressures, α is the coefficient of expansion of air, 0·00367, and n the refractive index of air for the wave-length λ of the light employed. The logarithmic values of $2 \frac{(n-1)\alpha}{\lambda}$ and $2 \frac{n-1}{760\lambda}$ can be found once for all; they are respectively 3·59901 and 3·15353.

The logarithms of the factors $\frac{b_1}{760}$, $\frac{1}{1 + at_1}$, and $\frac{1}{1 + at_2}$ can be extracted directly from Landolt's tables. Hence, the expression, although apparently long and troublesome, lends itself to very easy computation.

With regard to the sign of the correction, the signs given in the above formula for the temperature and pressure portions are the correct ones for use in all cases where the result of the increase of temperature is to effect an increase in the separation of the two reflecting surfaces, as in the determination of the expansion of the tripod screws. For in such cases the effect of change of temperature is to cause the observed number of bands to be less than it would be if such a change did not occur. The contrary is the case when the thickness of the air layer diminishes, as in the determination of expansion of a crystal or other object supported on the tripod table and the expansion of which is greater than that of the screws; and in all cases involving the use of the aluminium disc, except in the few cases of contracting substances. In all these cases of a thinning air-film the signs of the temperature and pressure portions of the above formula should be respectively - and +.

The data and calculated results for the thermal dilatation of a crystal of potassium sulphate may be taken as a concrete example. The crystal employed had two truly plane parallel surfaces ground on it a little over a centimetre apart, and perpendicular to the rhombic crystallographic axis a , so that the dilatation was determined along this direction.

THERMAL DILATATION OF POTASSIUM SULPHATE ALONG AXIS a . EXPERIMENTAL DATA.

L_{r1}	Length (thickness) of crystal block	10·355 mm.
l_a	Thickness of aluminium disc	5·237 mm.
l	Length of platinum-iridium screws above table	15·786 mm.

¹ *Zeitschrift für Instrumentenkunde*, 1893, 456.

d	Thickness of air-film	0·194 mm.
t_1	Initial temperature	17·2°
b_1	„ pressure	766·5 mm.
t_2	Intermediate temperature limit	56·4°
b_2	„ pressure	765·0 mm.
t_3	Final temperature limit	96·6°
b_3	„ pressure	763·5 mm.
$\lambda/2$	Half wave-length of C-hydrogen light used	0·0003281 mm.
$\alpha = \alpha + 2bt$	Mean coefficient of linear expansion of platinum-iridium between any two temperatures of which the mean is t	
	Similar mean coefficient for aluminium	$10^{-9} (8600 + 4·56t)$ $10^{-8} (2204 + 2·12t)$
f_2	Number of interference bands between t_1 and t_2	43·90
f_3	„ „ „ „ t_1 and t_3	90·42
f_2'	Corrected number of int. bands between t_1 and t_2	43·88
f_3'	„ „ „ „ t_1 and t_3	90·38

The two last quantities f_2' and f_3' were obtained by applying the correction referred to on the last page to the observed numbers of bands f_2 and f_3 , which effected their transit during the two respective temperature intervals. The correction in the two cases amounted to -0·02 and -0·04 respectively.

CALCULATED ACTUAL AMOUNTS OF EXPANSION.

$$\text{Diminution of thickness of air-layer} = \begin{cases} f_2' \lambda / 2 = 0·0143970. \\ f_3' \lambda / 2 = 0·0296540. \end{cases}$$

$$\text{Expansion of tripod screws} \left\{ \begin{array}{l} \text{For 1st interval} = l \left[10^{-9} \left(8600 + 4·56 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0·0054256 \\ \text{„ 2nd „} = l \left[10^{-9} \left(8600 + 4·56 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0·0110887 \end{array} \right.$$

$$\text{Expansion of aluminium disc} \left\{ \begin{array}{l} \text{For 1st interval} = l_a \left[10^{-8} \left(2204 + 2·12 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0·0046848 \\ \text{„ 2nd „} = l_a \left[10^{-8} \left(2204 + 2·12 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0·0096523 \end{array} \right.$$

$$\text{Preponderance of expansion of screws over disc} = \begin{cases} 0·0007408 \text{ for 1st interval.} \\ 0·0014364 \text{ for 2nd interval.} \end{cases}$$

As the effect of the expansion of the screws is to increase the thickness of the air-film, while that of the dilatation of the crystal is to thin the film, the full effect due to the crystal is not apparent, so the above two quantities are to be added to $f_2' \lambda / 2$ and $f_3' \lambda / 2$ respectively, in order to arrive at the expansion of the crystal.

$$\text{Expansion of crystal} = \begin{cases} L_{t_2} - L_{t_1} = 0·0151378. \\ L_{t_3} - L_{t_1} = 0·0310904. \end{cases}$$

CALCULATED COEFFICIENT OF LINEAR EXPANSION.

$$\begin{aligned} \theta &= 0·000 \ 375 \ 23 \\ \phi &= 0·000 \ 000 \ 148 \ 5 \\ L_0 &= 10·3485 \\ \alpha &= 0·000 \ 036 \ 26 \\ b &= 0·000 \ 000 \ 014 \ 4 \end{aligned}$$

That is, the mean coefficient of expansion of potassium sulphate along the direction of the axis α , between 0° and t° , is :

$$\alpha + bt = 0·000 \ 036 \ 26 + 0·000 \ 000 \ 014 \ 4t ;$$

and the true coefficient α of linear expansion at t° , or the mean coefficient between any two temperatures the mean of which is t is :

$$\alpha = a + 2bt = 10^{-8} (3626 + 2.88t).$$

The result of this particular determination which has formed our example was confirmed by seven others, four different crystals being employed, all perpendicular to the axis a , and two determinations on different days were carried out with each. The final result was almost identical with that of our example, illustrating the extreme accuracy of the method, namely,

$$\alpha_{K_2SO_4} = 10^{-8} (3616 + 2.88t) = 0.000\ 036\ 16 + 0.000\ 000\ 028\ t.$$

The expansion of the aluminium disc was determined by the ordinary Fizeau method shown at a in Fig. 709, a cylindrical block, with two parallel truly plane-polished normal end-surfaces 12 millimetres apart and 25 mm. in diameter, being prepared from the same casting as all the aluminium discs were cut from. It rested on the outer set of three points of the platinum-iridium table during the determination. The following data concerning the calculation will serve as an example of an ordinary Fizeau determination :

DILATATION OF ALUMINIUM DATA.

Thickness of aluminium block, $L_1 = 12.188$ mm.

Length of platinum-iridium screws above table points, $l = 12.400$ mm.

Thickness of air layer, $d = 0.212$ mm.

Temperatures $f\ t_1 = 10.1^\circ$, $t_2 = 68.9^\circ$, $t_3 = 123.4^\circ$.
 $(t_2 - t_1 = 58.8$, $t_3 - t_1 = 113.3$.

Pressures, $b_1 = 754$, $b_2 = 754.8$, $b_3 = 756$ mm.

Number of transited bands, $f_2 = 30.35$, $f_3 = 60.32$.

Corrections for air refraction, -0.04 , -0.06 .

Corrected number of bands, $f'_2 = 30.31$, $f'_3 = 60.26$.

Half wave-length of C-hydrogen light employed, $\lambda/2 = 0.0003281$.

Diminution of thickness of air layer, $f'_2 \lambda/2 = 0.0099449$,
 $f'_3 \lambda/2 = 0.0197720$.

Elongation of tripod screws = measured length of screws $\times (a + 2bt$ for tripod alloy)
 \times temperature interval,

$$l \left[10^{-9} \left(8600 + 4.56 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0.0064074,$$

$$l \left[10^{-9} \left(8600 + 4.56 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0.0125100.$$

Expansion of aluminium block = diminution of thickness of air layer + elongation of screws,

$$L_2 - L_1 = 0.0099449 + 0.0064074 = 0.0163523.$$

$$L_3 - L_1 = 0.019772 + 0.012510 = 0.032282.$$

$$\theta = 0.000\ 268\ 23.$$

$$\phi = 0.000\ 000\ 125\ 31.$$

$$L_0 = 12.1853.$$

$$a = 0.000\ 022\ 013.$$

$$b = 0.000\ 000\ 010\ 284.$$

In order to form an idea of the kind of concordance afforded by determinations made on different days, the result obtained as above is compared below with the results of similar determinations made on two other days for slightly different temperature intervals :

<i>a</i>	<i>b</i>
0.000 022 013	0.000 000 010 284
22 070	10 617
22 041	10 895

The final mean accepted value for the expansion of aluminium is thus :

Mean coefficient $a + bt = 10^{-8} (2204 + 1.06t)$, or

True coefficient $a + 2bt = 10^{-8} (2204 + 2.12t)$.

It is not always possible to use the three-point method of contact in the case of crystals, for it frequently happens that crystals which are difficult to obtain of considerable size will be adequately long in the direction in which the expansion is to be determined, but narrower in other directions, so that the parallel-ended block will not stand on all three of any set of points. Also the aluminium disc with points below will not always stand on the upper end of the crystal, for a similar reason. In such cases the plane side of the platinum-iridium table is used, and a plane disc of aluminium is employed, and the crystal and disc are placed in position by sliding contact, to get rid of the air film. No appreciable difference has ever been observed by the author between the results in cases when this procedure has been followed and those obtained when three-point contact has occurred. This is probably owing to the perfection of planeness of the crystal surfaces produced by the cutting and grinding goniometer described in Chapter XLII.

Another occasion when the plane side of the tripod table without points is used has already been referred to, namely, when the expansion of the platinum-iridium alloy is itself being determined. This is necessarily performed with the two reflecting surfaces concerned in the production of the interference bands, the table top and the under surface of the glass wedge-disc, separated at a distance of somewhat over a centimetre. Either red cadmium or green mercury light requires to be used, as no other is sufficiently monochromatic to afford bands at so great a distance of separation of the surfaces, that is, no other radiation is sufficiently free from internal secondary interference. Moreover, the polish taken by platinum-iridium is so excellent that the reflection from the table top overpowers that from the glass wedge-disc; but this difficulty is overcome by depositing on the lower relevant surface of the latter a thin film of silver by means of a silvering solution. The reflection from this surface is then enhanced, and at the same time less light reaches the platinum-iridium, so that the two reflections are now adequately equalised to afford black bands. In other respects the determination is carried out as described in the previous pages. The length of screws (mean of the three) employed in an actual determination was 12.369 millimetres, and this length was both L_t and d the thickness of the air layer. The initial temperature was 11.1° , corresponding to this length. This amount of screw produced in green mercury light 20.63 bands by its expansion, on raising the temperature to 68.5° , and 39.40 bands for the whole interval from 11.1° up to the higher limit 117.8° . The correction for alteration of the refraction of air was obviously unusually large, even the barometric pressure part being in this case quite serious; the total corrections for

the two intervals were + 2.17 and + 3.52 bands, bringing up the correct number of bands to 22.80 and 42.92. The particular value for the coefficient of expansion of platinum-iridium obtained from this determination was :

$$\alpha = a + 2bt = 10^{-9} (8588 + 4.52t),$$

and the mean of five such determinations yielded the value which has been used in the calculations of the expansion of aluminium and of potassium sulphate quoted in the previous pages, namely,

$$\alpha = a + 2bt = 10^{-9} (8600 + 4.56t) = 0.000\ 008\ 600 + 0.000\ 000\ 004\ 56t.$$

Cubical Expansion.—The thermal expansion was similarly determined along the axes b and c of a number of excellent large crystals of potassium sulphate, and the final results for the three axes are compared below. Their sum is also taken, for, the directions being rectangular, this sum represents the coefficient of cubical expansion of potassium sulphate. For when the expression for the product of the expansions of the crystal in the three rectangular axial directions, which naturally gives the expansion of the solid, is examined, it is found to consist of a large number of terms of which the only ones that affect the fourth and last trustworthy place of significant figures in the coefficient of expansion for any temperature are the sums of the constants a and b respectively. If the crystal were of cubic symmetry, the expression for the cubical expansion would be three times the linear, that is, obtained by multiplying the constants a and b , as determined from the linear expansion, by three. For a tetragonal, hexagonal, or trigonal crystal, the cubical expansion is obtained by adding to the constants for the direction of the axis twice those for the direction perpendicular to the axis. In the case of a rhombic crystal such as potassium sulphate, in which the expansions along the three rectangular axial directions are unequal, the sum of the three different pairs of constants affords the cubical expansion. The same applies also to monoclinic and triclinic crystals.

CUBICAL EXPANSION OF POTASSIUM SULPHATE.

	a	b
Linear expansion along axis a	0.000 036 16	0.000 000 014 4
„ „ „ b	0.000 032 25	0.000 000 014 1
„ „ „ c	0.000 036 34	0.000 000 041 3
Cubical expansion of K_2SO_4 = sum	0.000 104 75	0.000 000 069 8

Thus the mean coefficient of cubical expansion $\alpha + bt$ of potassium sulphate between 0° and t° is $0.000\ 104\ 75 + 0.000\ 000\ 069\ 8t$.

It may be interesting to quote the analogous results for the two other members of the eutropically isomorphous series, rubidium sulphate and caesium sulphate. For convenience in comparison the values for all three salts are tabulated below.

VALUES OF $\alpha = a + 2bt$, THE CUBICAL COEFFICIENT OF EXPANSION AT ANY TEMPERATURE t , OF ALKALI SULPHATES, OR MEAN COEFFICIENT BETWEEN ANY TWO TEMPERATURES OF WHICH THE MEAN IS t .

K_2SO_4	$0.000\ 104\ 75 + 0.000\ 000\ 139\ 6t,$
Rb_2SO_4	$0.000\ 103\ 14 + 0.000\ 000\ 153\ 4t,$
Cs_2SO_4	$0.000\ 101\ 70 + 0.000\ 000\ 162\ 0t.$

It is interesting to observe that the values of each constant show a regular progression in the order of the atomic weights of the three alkali metals. The values are very close together, showing how similarly, on the whole, analogously constituted chemical substances expand with rise of temperature. There is absolutely no doubt about the values, as the accuracy of the interference method is much greater than the differences between different salts.

As was pointed out on page 530 in Chapter XXXII. on density determinations, in which the cubical expansion of salts in general was required in order to correct density determinations by the Retgers immersion method to the standard temperature of 20° , chemical salts as a rule expand about 0.0001, that is, about one ten-thousandth of their bulk, for each degree centigrade of rise of temperature. For it is scarcely necessary to state that the real meaning of the above figures for the coefficient of expansion is, taking for example the case of potassium sulphate, that when unity represents the volume of the crystal at 0° , its volume at 1° C. would be 1.00010475; and that if, again, say, its volume at 10° were taken as unity, its volume at 11° would be $1.00010475 + 0.00000140 = 1.00010615$. Similarly, if the volume were unity at 50° , at 51° it would be 1.00011173; and if unity at 100° it would be 1.00011871 at 101° .

Hence, even at temperatures as high as 100° , the volume only increases one ten-thousandth on heating through one degree of temperature.

This chapter may fittingly conclude with a short list of coefficients of thermal expansion of some crystalline substances of the various systems of symmetry, as determined by Fizeau, by Benoit, employing Fizeau's apparatus, and by Reimerdes, with the dilatometer of Pulfrich. It must not be taken for granted that expansion is an invariable accompaniment of the rise of temperature of crystals, at any rate along every direction in the crystal. For a remarkable exception was discovered by Fizeau, in the case of hexagonal silver iodide, AgI, the thermal behaviour of that salt having been studied by him between the temperatures of -10° and $+70^{\circ}$ C. A crystal of silver iodide actually contracts along the hexagonal (crystallographic and optic) axis on warming, and this linear contraction increases as the temperature rises; moreover, so predominating is this contraction, as compared with the smaller expansion perpendicular to the axis (see figures in table), that the total volume of the crystal diminishes at an accelerating rate as its temperature rises. This contraction has been shown by other observers to continue until at the temperature of 146° silver iodide suddenly changes with absorption of heat into a cubic modification, and on cooling below this temperature passes back to the hexagonal variety. Hence, the greatest care must always be taken in employing the interference method, to verify that the alteration in length along the direction under observation is actually one of elongation. For the exceptional cases like silver iodide, and those also of calcite and beryl given in the table, will probably prove of the greatest value in the future in elucidating crystal structure.

Another curious phenomenon is that the diamond at $-41^{\circ}7$, cuprite at $-4^{\circ}3$ and beryl at $-4^{\circ}2$, each reach a maximum density, at which the crystals begin to expand with further cooling. Hence, if the crystals were warmed from an original temperature lower than this limit they would resemble silver iodide in contracting with rise of temperature, until the limit was reached, after which they would commence to expand. It is noteworthy that the coefficients are very small in all three cases, that of cuprite being remarkably low.

VALUES OF α , COEFFICIENT OF LINEAR EXPANSION AT t° .

$$\alpha = \alpha + 2bt.$$

System.	Substance.	Observer.	Direction.	α .	$2b$.
Cubic	Rock Salt, NaCl	Fizeau	Any direction	0.000 038 59	0.000 000 044 9
"	Fluorspar, CaF ₂	"	"	017 96	028 8
"	Pyrites, FeS ₂	"	"	008 42	017 8
"	Diamond, C	"	"	000 60	014 4
"	Cuprite, Cu ₂ O	"	"	000 09	021 0
Tetragonal	Anatase, TiO ₂	Fizeau	Par. axis	006 95	031 1
		"	Perp. axis	003 50	029 5
Hexagonal	Silver iodide, AgI	Fizeau	Par. axis	-002 26	-042 7
		"	Perp. axis	+000 10	+013 8
"	Beryl, Al ₂ Be ₃ (SiO ₃) ₆	Fizeau	Par. axis	-001 52	011 4
		Benoit ¹	"	-001 340	008 06
		Fizeau	Perp. axis	+000 84	013 3
		Benoit ¹	"	000 994	009 30
Trigonal	Quartz, SiO ₂	Fizeau	Par. axis	006 99	020 5
		Benoit ¹	"	007 111	017 12
		Reimerdes ²	"	006 925	016 89
		Fizeau	Perp. axis	013 24	023 8
		Benoit ¹	"	013 163	025 26
"	Calcite, CaCO ₃	Fizeau	Par. axis	025 57	016 0
		Benoit ¹	"	024 963	027 34
		Fizeau	Perp. axis	-005 75	008 7
		Benoit ¹	"	-005 541	001 94
"	Sapphire, Al ₂ O ₃	Fizeau	Par. axis	005 37	020 5
		"	Perp. axis	004 53	022 5
Rhombic	Topaz (AlF) ₂ SiO ₄	Fizeau	Par. axis a	004 23	015 3
		"	" b	003 47	016 8
		"	" c	005 19	018 3
"	Potassium sulphate, K ₂ SO ₄	Tutton	Par. axis a	036 16	028 8
		"	" b	032 25	028 2
		"	" c	036 34	082 6

¹ Employing the apparatus of Fizeau.

² Employing the apparatus of Pulfrich.

Only one axis is identical with a crystallographic axis, the symmetry axis b , in the case of a monoclinic crystal, the other two axes of the thermal ellipsoid lying rectangularly in the symmetry plane. For

gypsum, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$, the expansion along the symmetry axis was found by Fizeau to be

$$a = 0.000\ 037\ 89, \ 2b = 0.000\ 000\ 093\ 6;$$

and for the directions of the other two rectangular axes lying in the symmetry plane :

$$a = 0.000\ 001\ 13, \ 2b = 0.000\ 000\ 010\ 9, \text{ and} \\ a = 0.000\ 027\ 96, \ 2b = 0.000\ 000\ 034\ 3.$$

In order to obtain these values Fizeau¹ first prepared a cube of gypsum, of which one pair of faces was parallel to the symmetry plane, the expansion afforded along the direction perpendicular to this pair being obviously that for the direction of the symmetry axis b ; the other two pairs of parallel faces possessed known orientations with respect to the crystallographic axes a and c . A fourth pair of parallel faces were then ground at two opposite corners of the cube. The expansion along the direction of the normal to each of the four pairs of faces then existing on the block was determined by this interference method. The results were subsequently combined in four equations, from which it was possible to calculate the four unknown quantities, namely, the three linear expansions above given for the three axial directions of the thermal ellipsoid, and the angle of inclination of one of the two thermal axes lying in the symmetry plane with the inclined crystallographic axis a . This angle in the case of gypsum proved to be $+47^\circ\ 44'$. The method is of general application for the determination of the coefficients of expansion of monoclinic crystals.

¹ *Comptes rendus*, 1868, 66, 1005 and 1072; *Pogg. Ann.* 1868, 135, 372.

CHAPTER LIV

ELASTICITY OF CRYSTALS AND ITS MEASUREMENT.

CRYSTALS exhibit their orientated structure very clearly in the different elasticity, or resistance to temporary deformation, which they show in different directions. The attempted deformation may obviously be of two kinds: (1) of compression such as that by a weight placed on (above) the crystal, which may conveniently be an elongated prism resting by one of its plane ends (cut perpendicularly to its upright length) on a relatively immovable support, and (2) of extension such as that effected by means of a weight suspended from the lower end of (below) the crystal, while the latter is fixed at its upper end. The amount of deformation, whether compression or dilatation, will clearly depend on the force employed and on the dimensions of the crystal. But a third factor also enters into the case, namely, the specific power of the crystal to resist such deformation, a power which in turn depends both on the nature of the substance and on the symmetry of its structure. This specific resistance to change of form is known in mechanics for non-crystallised substances, or substances which, if crystalline, are composed of a mass of interlacing anyhow-orientated crystals, and so produce an average effect, as the "**Modulus of Elasticity**," and is distinguished by the letter E .

Let us suppose we are dealing simply with a vertical wire, of length L and sectional area Q , attached at the top to a rigid support, and stretched by a weight just adequate to keep it taut; if we then add another weight P at the lower end, the wire will stretch further by an amount l , and the modulus of elasticity E corresponding to the stretching is, according to Kohlrausch:

$$E = \frac{L}{l} \cdot \frac{P}{Q}.$$

We can define the modulus of elasticity as that weight in kilograms which must be hung on a wire of one square millimetre section in order to double its length, supposing the elongation to be proportional to the load. For if $Q = 1$ and $l = L$ we have $E = P$.

The modulus of elasticity E , the coefficient of resistance to dilatation or bending, is usually determined by observations of the bending

of a rod when loaded with a given weight. The simplest manner is to clamp the rod, horizontally arranged, and of free length L , at one end, the free end pointing to a scale; to load this free end with a weight of P grammes, and to observe the deflection s produced. If the rod be of rectangular section with vertical side or thickness t and horizontal side or breadth b , then E is afforded by the formula of Kohlrausch:

$$E = 4 \frac{P}{s} \cdot \frac{L^3}{t^3 b}.$$

A much better method of procedure, however, and the one currently employed, which avoids all the uncertainty introduced by the clamping of the rod, is to lay it with both ends loose upon the parallel edges of two wedge-shaped supports, as shown in Fig. 711. Here L represents the distance between the two supports, that is, the free length of the rod, and the weight P is suspended from the middle of the rod, generally from a wedge-stirrup, producing a deflection there of s . The Kohlrausch formula then becomes:

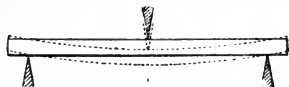


FIG. 711.

$$E = \frac{1}{4} \cdot \frac{P}{s} \cdot \frac{L^3}{t^3 b}.$$

All lengths are conveniently expressed in millimetres and the weights in grammes, so that the result shall be expressed in terms of the currently accepted unit of the modulus of elasticity.

There are three obvious assumptions made, in order that the formula may correctly express the facts, namely, that the deflection s produced by P is small in comparison with L , that the load used is far from the breaking strain, and that the substance is a truly elastic one, not suffering any appreciable permanent deflection as the result of the operations.

In the year 1878 Warburg and Koch¹ showed that the above formula of Kohlrausch affords only a first approximation, and that the experimental facts are better interpreted when a second term is introduced, namely, the product of the Kohlrausch formula with $3\left(\frac{t}{L}\right)^2$, the whole formula being then:

$$E = \frac{1}{4} \cdot \frac{P}{s} \cdot \frac{L^3}{t^3 b} \left\{ 1 + 3\left(\frac{t}{L}\right)^2 \right\}.$$

For most substances the second term proves to be far from negligible, when the determinations are carried out with good crystals or a homogeneous rod of whatever material is under observation, and with the accuracy possible to the interference method introduced by Koch in the same memoir. The method formerly used was the mechanical one of Neumann, as improved by Voigt, to whom we owe most of the work on the elasticity of crystals which has yet been carried out. It is simply the ordinary mechanical laboratory method above referred to

¹ *Ann. der. Phys. und Chem.*, 1878, 5, 253.

carried to extreme refinement, with crystal plates or rods as large as are procurable. The interference method of Koch was based on the same principle as that underlying the Fizeau dilatometer explained in the last chapter, as regards the mode of measuring the bending, sodium light being used. The plate was laid on the edges of two sloping blocks, and a bending weight applied above its centre, either through a point at the end of a hook, or by a stirrup knife-edge. The lower surface of the plate, and the upper surface of a totally reflecting prism supported very closely beneath it, with only a thin film of air intervening, were the two surfaces affording the interfering reflections producing the interference fringes. A complicated and somewhat cumbersome method was adopted for suspending the weight from the hook or stirrup, and for regulating and graduating the application of the load. The excellent results obtained for the thermal expansion of crystals by the author's interferometer-dilatometer described in the last chapter, and the desirability of improving Koch's apparatus as regards the mode of application and control of the weight, caused the author¹ to devise an interference elasticity apparatus involving the use of the interferometer part of the dilatometer, combined with a much more refined and accurate arrangement for applying and controlling the load. The whole apparatus, as arranged for the purpose of the determination of bending, has been termed the "elasmometer," and is shown in Fig. 712.

The Measurement of s , the Bending at the Centre of an elongated Plate supported near both Ends, by the Elasmometer.—The optical portion of the apparatus consists of the author's interferometer, exactly as described in the last chapter in connection with the dilatometer. It is shown in position in Fig. 712, and the elasmometer proper by itself in Fig. 713.

The auto-collimating telescope pedestal stands on a small plinth attachable in front to the main plinth of the elasmometer proper. The second pedestal of the interferometer, carrying the refracting prisms and the partially porcelain suspended tube, is mounted on three toe-plates, ready fixed for the reception of its levelling screws, on the back part of the metal base of the elasmometer proper. The lower end of the suspended tube carries a simple cap, with a circular aperture of one inch diameter, in which is mounted slightly obliquely the countervailing wedge-disc (e in Fig. 706, p. 890) of the interference apparatus referred to on page 894. This cap replaces the interference chamber of the dilatometer, the rest of the interference apparatus being carried by the elasmometer proper, immediately under the cap when the suspended tube is swung into position.

The **Elasmometer** proper consists of seven essential parts, which will be briefly described in turn.

(1) **A pair of platinum-iridium wedges** are arranged parallel to each other, and with their knife-edges downwards (the method of Fig. 711 being inverted), up against which the plate (usually an elongated one) of the crystal or other

¹ *Phil. Trans.*, 1904, A, 202, 143.

substance under investigation is to be bent, by a weight applied under its centre by means of an upright agate point carried at the end of a balance beam, as shown in

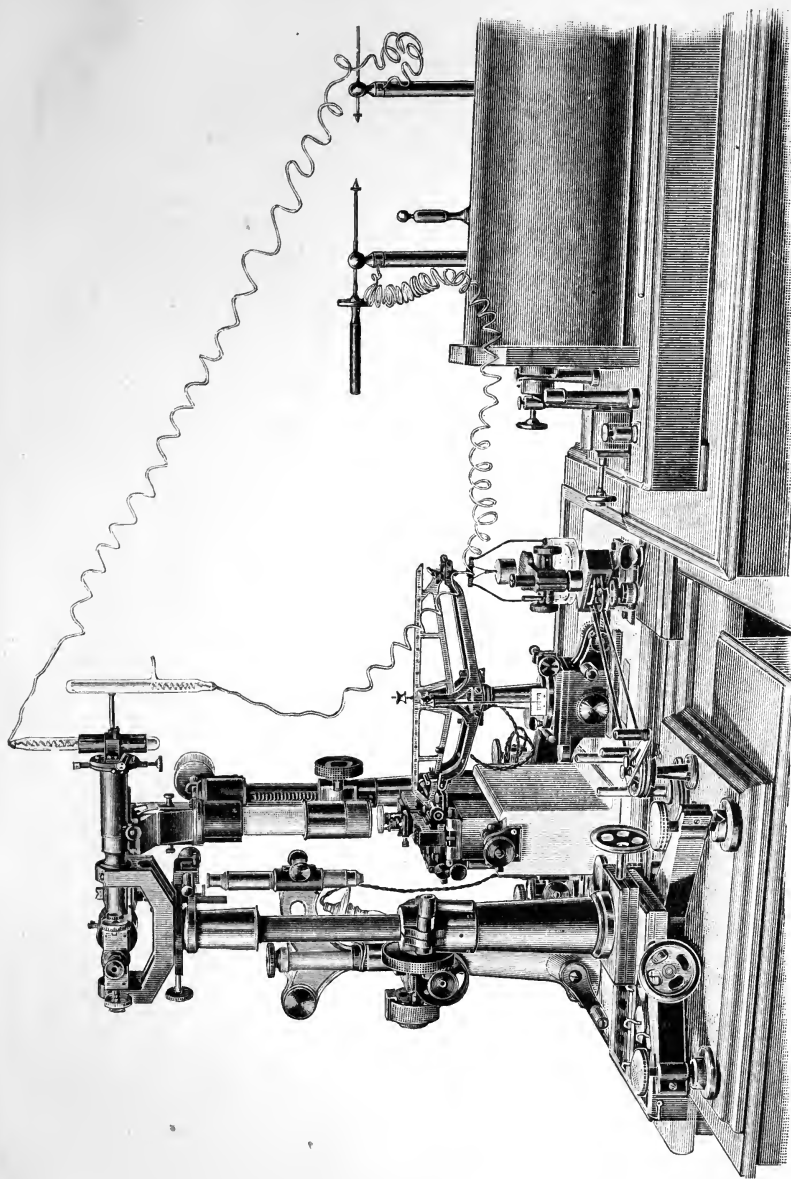


FIG. 712.—The Elasmometer, with Interferometer in Action, employing Red Hydrogen Light.

Fig. 714. They are carried by a pair of gun-metal blocks arranged in the same straight line, and underneath overhanging portions of the blocks; the latter approach

each other, thus forming a commodious recess, and the blocks and therefore the wedges are adjustable for their distance apart. This recess formed underneath these over-

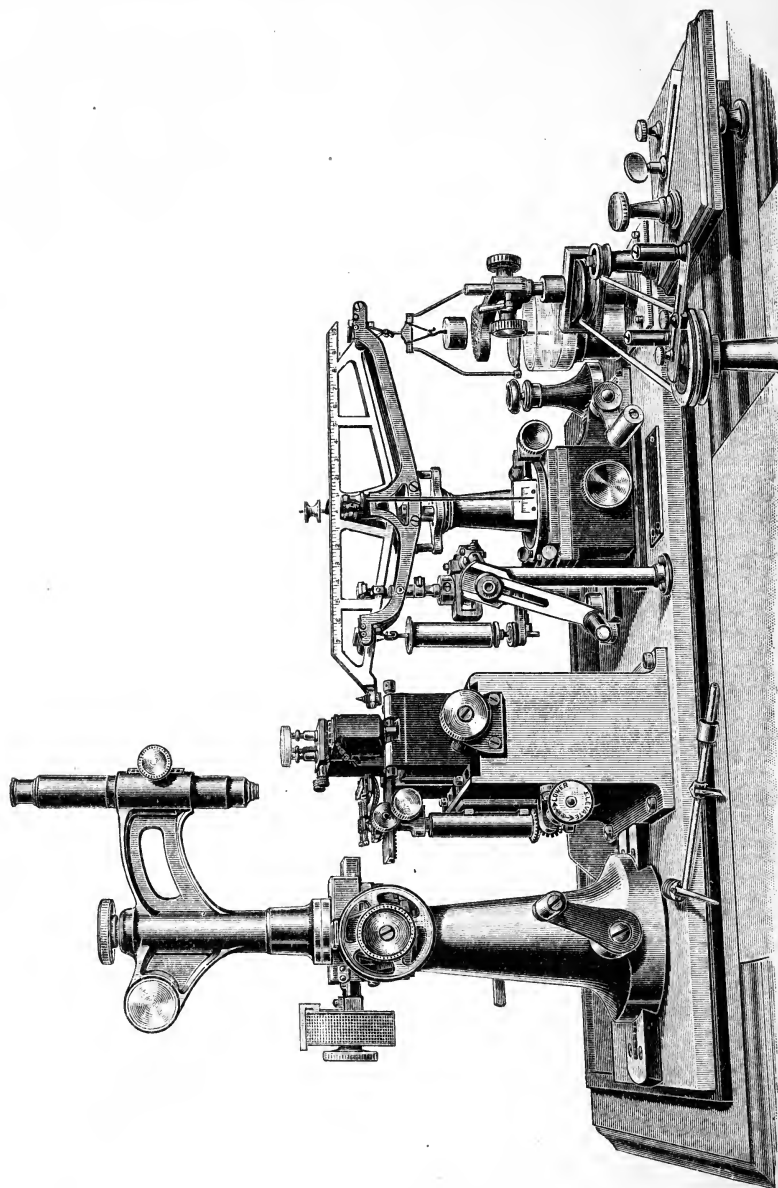


FIG. 713.—The Elasmometer Proper.

hanging parts of the blocks accommodates the plate-supporting apparatus and the weight-applying end of the balance, both of which can be independently moved either

into or out of the recess at will. These gun-metal blocks slide on a large rigid block of steel, mounted to the front and left of the centre of the iron base, and the front block is provided with fine adjustments for altitude and azimuth, to enable the parallelism of the wedges to be accurately attained.

(2) **The interference tripod**, consisting of three adjustable capstan screws, for supporting the large colourless glass wedge-disc, the under surface of which is the upper one of the two surfaces which are to reflect the interfering light, is carried by the gun-metal blocks, two of its screws being supported on the front block and the third on the one behind. The pair of screws and the single screw are in each case directly mounted on a small but thick plate with wide dovetailed groove, sliding over a corresponding dovetail fixed to the block, in order that the separation of the two front screws from the back screw may be adjusted to suit the size of glass wedge-disc employed.

(3) **A transmitter of the bending motion** of the centre of the plate to the inter-

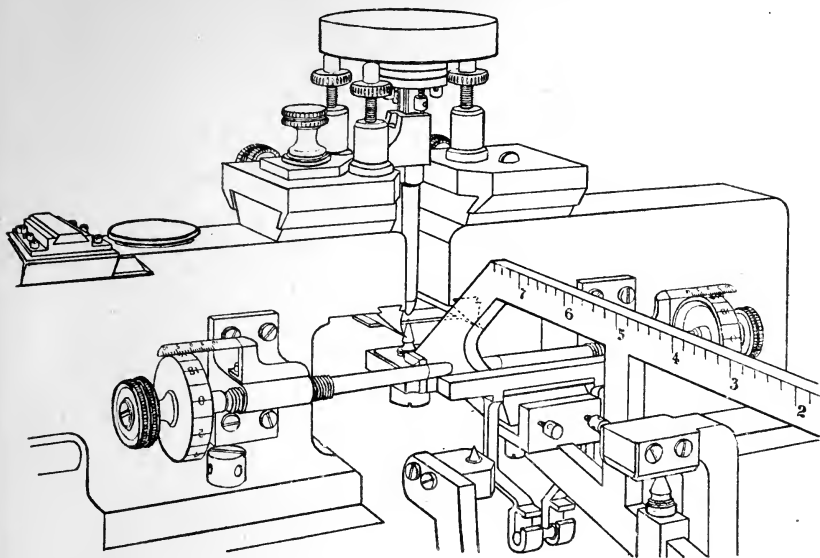


FIG. 714.—The Platinum-iridium Wedges, Crystal Plate, Balance and its Pressure Point, and Interference Apparatus of the Elasmometer.

ference apparatus, and itself furnishing the second (lower) reflecting surface of the latter, is fitted between the two inner closely approached ends of the blocks. It is best shown in Fig. 714, and consists of a vertical rod of aluminium, resting by its roundly pointed lower end on the centre of the crystal-plate, and carrying a head-piece above terminating in an adjustable horizontal black-glass disc, the truly plane polished upper surface of which is the reflecting surface just alluded to, relevant to the interference. The upper portion of the rod is of fluted square section, in order to slide without friction in a square vertical boring in an arm screwed with slot adjustment to the front block, between the two tripod screws.

(4) **A pair of mechanical fingers**, shown in Fig. 715, are provided for preliminarily adjusting the crystal-plate and supporting it up against the platinum-iridium wedges until the pressure point of the balance is brought into action under the centre of the plate. They are carried to the left of the steel block in such an adjustable manner that they may be racked into or out of the recess by means of the large milled head

shown below in Fig. 715. The two "fingers" terminate in spring tables, each having a knife-edge as shown separately to the right in Fig. 715, and which are capable of adjustment for their distance apart parallel to the plate by the second large milled-headed screw, which is right- and left-handed in its two parts moving respectively the two knife-edges. The plate is laid over the latter, and its position thereon may be delicately adjusted by two pairs of screws parallel to and perpendicular to its length, in a manner which will be clear from the figure.

(5) **A balance for the delivery of the load**, specially constructed by Oertling, is arranged to the right of the steel block. It carries at the left end instead of a pan the "pressure point," an upright agate cone, which is covered with a soft kid-leather cone during the experiment in order to protect the crystal from scratching. The bending weight is applied through this cone, and is equal to the weight placed in the pan at the right end. The whole balance is movable in the direction of its length, like the mechanical fingers on the other side of the steel block,

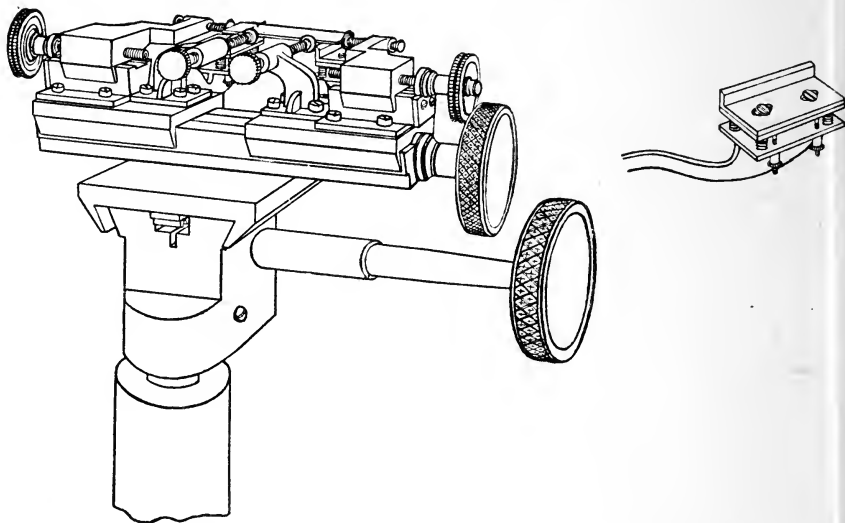


FIG. 715.—Mechanical Fingers for adjusting the Crystal-plate.

so as to be able to bring the pressure point in or out of the recess, by means of a basal rack and a pinion provided with handle. A fine adjustment for azimuth is also provided, by means of the screw seen approaching the observer on the right at the top of the basal box on which the balance stands. These two movements thus enable the pressure point to be accurately centred under the crystal plate. In order to avoid frequent repetition of this centring, during a series of observations with the same plate, a pair of fine horizontal adjusting screws with ivory tips and divided drum-heads are arranged, one on each side of the beam near the pressure point. They are carried by the blocks on their right side, and are well shown in Fig. 714. The ivory tips are arranged just to touch the beam when the centring has been accomplished, and the readings of the drums are noted for these adjusted positions; they are then withdrawn for a noted number of turns, say one or two, of the screws during an observation, and reinstated before the next observation in order to ensure the continued adjustment of the pressure point, and withdrawn again before the observation is carried out.

The aluminium pan at the right end is connected by a central vertical rod with a second lower pan, a plane disc of aluminium, which is immersed during the observations in a dish of cedar oil, and the function of which is so to steady the balance during the application of the weight as to prevent absolutely all flickering of the interference bands, which is otherwise a serious difficulty. A cylindrical counterpoise for the pan and aluminium steadier is suspended on the left side by the usual stirrup and agate wedge and plate. Weights up to 500 grammes may be employed as the load, and the sensibility of the balance up to this load is scarcely affected by the cedar oil immersion of the aluminium disc.

(6) **A control apparatus** is provided for the application of the load. If the weight were laid directly on the pan—when the pressure point has been adjusted under the centre of the crystal-plate—the bending would occur at once as soon as the balance were set in motion, by lowering the usual rests with the aid of the large milled head in front of the basal box, and the interference bands would flash past far too rapidly to enable them to be counted. Each of the platinised and carefully standardised weights, therefore, is provided with a hook instead of a knob, for suspension in a loop of soft but strong string carried by and below the suspension bar of the pan fitting. The weight is laid on a stout little circular table adjustable for height just below the string loop, covered with broadcloth and forming the flat expanded end of an arm proceeding from a slider about a stout column. This coarse vertical motion is by rack and pinion, manipulated by either of the two milled heads seen to right and left of the column; fixation at any height can be accomplished by a clamping screw with lever head. The column itself is cylindrical and is susceptible of a very fine slow vertical motion, without rotation, in a correspondingly cylindrical keyed boring in a basal casting, which provides at the same time a rectangular box-like expansion, open at two opposite sides, for the driving-nut wheel of the screw tapped on the lower part of the column. The screw and nut are of quarter-millimetre pitch, so that a very fine motion is achieved, either by manipulation of the driving-nut itself, or more conveniently by means of the pulley wheel seen in front in Figs. 712 and 713, which is provided with a radial arm and handle for greater radius of rotation. The band is kept taut by a third pulley pressed in from the right by the little attachment seen to the extreme right in Figs. 712 and 713.

With the aid of the coarse adjustment the weight table can be lowered until the hook of the weight just touches and all but rests in the string loop. The fine movement of the table is then resorted to and the weight brought extremely slowly into action, the retardation being assisted by the regularly growing tension on the string, until the full weight is eventually operative. So perfect is this control, that the interference bands may be made to pass the reference spot in the centre of the field with any desired degree of slowness, or each band may even be held at the centre for an indefinite length of time. The counting of the bands is, therefore, a perfectly easy matter.

(7) **A measuring microscope** is also provided, wherewith to determine *in situ* the dimensions of the crystal-plate and to find its centre. It stands to the left of the apparatus, and is best shown in Fig. 713. Each of its two rectangular measuring movements reads directly on the single screw itself to the thousandth of a millimetre, the head being provided with a large silvered drum divided into 100 parts directly; the tenths of these are obtained by engraving ten other circles on the cylinder of the drum, parallel to the divided edge-circle and equidistant from each other, and oblique lines from every division of the edge circle (the first) to the next division on the other edge-circle (the eleventh). The tenth-part of a division of the edge-circle is then read off by means of a glass plate carried in a frame above the drum, the centre of the glass being engraved with a single line coloured red parallel to the screw axis, the reading being afforded by the coincidence of the red line with the nearest

intersection of an oblique line with a circle. Each screw is provided with a special device to prevent backlash, so that the readings afforded in this manner by a single screw, in the case of each of the two rectangular movements, are most accurate and very easy to take.

The conical column supporting the microscope is mounted on the top of the upper measuring movement, and the outer cone, from an arm of which the microscope is suspended, is capable of rotation about this inner cone, so that the microscope may be turned out of the way after it has served its purpose, and replaced over the interference apparatus by the interferometer suspended tube. A fixing screw is provided at the summit of the cone, and the microscope is counterbalanced on the other side of the column. The lower measuring movement is mounted in an adjustable manner, and made rigid after the adjustment, on a stout pedestal movable, in the direction of the length of the elasmometer, over a bevelled guiding bed by rack and pinion, the latter provided with the handle seen in front. The two-thirds of a rotation possible to the handle are adequate to bring the microscope over the interference apparatus or remove it well out of the way, when more room is required than is obtained by merely rotating the upper part carrying the microscope. A clamping lever is just seen emerging behind the pedestal, for fixing the microscope rigidly during measuring operations.

The Determination of Bending is carried out in the following manner by means of the elasmometer. The plate of the crystal or other substance the elasticity of which is to be determined, and the thickness of which has been measured on a thickness measurer such as that shown in Fig. 710 (page 899), is laid temporarily on the gun-metal knife-edges of the mechanical fingers, and its length and breadth measured with the aid of the measuring microscope. It is then adjusted to the platinum-iridium knife-edges by means of the adjusting screws of the mechanical fingers. The pressure point is then adjusted below the plate centre, while the balance is out of action in its rests. The transmitter is next placed in position above the centre. A small preponderance of weight is then given to the pan side of the balance, in order that when the latter is released the pressure-point may press up against the centre, with just enough weight to keep the plate in full contact with the platinum-iridium knife-edges, when the gun-metal ones are lowered and withdrawn.

The operation of bending the plate is thus always started with a small constant weight in action, and the plate very slightly preliminarily bent. It is the effect of the additional weights subsequently added from the control apparatus, in producing movement of the interference bands corresponding to further bending of the plate, that is to be measured. This small preliminary weight is effectively given to the pan end of the balance by merely removing the cylindrical counterpoise from the left side; the weight of this is about 83 grammes, which, when the relative positions on the two arms of the balance are taken into account, is equivalent to a weight of 60 grammes on the pan, an amount of the right order to ensure effective contact of the plate with the platinum-iridium knife-edges.

The interference apparatus is next adjusted to give an excellent field of vertical bands (as shown in Fig. 708, p. 892) as seen through the telescope. The position of the bands is then measured in order to determine the

initial fraction of a band, in the event of a band not being already centred on the silver ring and between the vertical pair of spider-lines.

The weight, the bending effect of which is to be measured, is then placed on the control table, and the latter lowered by the quick movement until its hook nearly touches the string loop. An eye must then be given to the interference bands, and the right hand used to continue the lowering of the weight with the utmost gentleness by one of the milled heads of the coarse movement, until the hook touches the string; this can easily be done so gently that only a single band moves parallel to itself very slowly past the reference centre of the silvered ring, as the tension on the string gradually increases. Recourse must then be had to the fine movement, and the driving wheel of the latter slowly rotated by means of the long lever handle, in the direction which is known by experience to lower the weight. The bands will immediately recommence their movement and pass one by one as the wheel is rotated, so steadily in fact, that each band can be held over the reference centre as long as may be desired, and the counting is a perfectly easy matter. Their movement is very slow at first as the tension on the string is increasing, but they go somewhat more quickly as the loop tightens, slackening again, however, towards the finish, until eventually the weight is left altogether suspended in the loop and quite clear of the table, when the bands also come completely to rest. No trembling of the bands is perceived, owing to the resistance of the aluminium disc immersed in the dish of cedar oil. Moreover, the bands can be recounted as the weight is again taken up by the table and as slowly removed from the loop by the reverse motion of the driving wheel. The bands repass equally steadily and deliberately.

When the weight is in full operation, and the bands are at rest, the last fraction is measured with the micrometer just like the first fraction, and the two fractions are added to the whole number of traversed bands, in order to arrive at the total corresponding to the bending. Similarly, on arriving back at the starting-point, with the weight entirely removed, the position of the bands is again measured for any fraction, which should be the same as at first. The operations can be repeated as often as desired, the lateral screws for the fine adjustment of the pressure-point being employed between each pair of determinations, and then again withdrawn. After the completion of the determinations for the first load, another greater weight is used, and a set of observations taken for it, and so on for as many different loads as one wishes to employ. The loads generally employed by the author are 100, 200, 300, and 400 grammes, and occasionally, if the plate will stand it, 500 grammes, the limit of the instrument.

The most convenient dimensions of the plate of the crystal or other substance investigated are about a couple of centimetres in length, a thickness of a millimetre or less, and a width not greater than half the length.

This description may conclude with an example taken from a series of observations made with four plates of an excellent cubic crystal of

iron pyrites, FeS_2 . Each plate was cut by Messrs. Hilger truly parallel to a face of the cube, and with its edge-faces parallel to the two other rectangular facial planes of the cube. The length of the plate serving as our example was 17.261 mm., its breadth was 8.833 mm., and its thickness 0.875 mm. The number of interference bands in red hydrogen light which were caused to effect their transit for successive loads of 100, 200, 300, and 400 grammes are given below, together with the differences for each increment of 100 grammes, and their mean. Plate No. 1 broke under a load of 500 grammes, so that 400 grammes is the safe limit of weight the plates will withstand. The observations were made on different days, with at least four days' interval, in order to permit recovery from any slight strain. Red C-hydrogen light was employed as source of light, the half wave-length of which is 0.0003281 mm.

PLATE 2 OF PYRITES.

Corrected No. of Bands for load of				Difference in No. of Bands for loads of			Mean difference for 100 grammes additional load.
100 gr.	200 gr.	300 gr.	400 gr.	100 and 200.	200 and 300.	300 and 400.	
2.14	4.44	6.66	8.99	2.30	2.22	2.33	2.28
2.30	4.54	6.76	9.02	2.24	2.22	2.26	2.24
2.16	4.45	6.69	8.98	2.29	2.24	2.29	2.27
2.27	4.52	6.82	9.17	2.25	2.30	2.35	2.30

Final mean corrected number of bands passing for increment of 100 grammes 2.27.

The actual distance s through which the centre of the plate was bent for a load increment P of 100 grammes was :

$$s = 2.27 \times \lambda/2 = 2.27 \times 0.0003281 = 0.0007448 \text{ millimetre.}$$

The above results are observed to be satisfactorily concordant, notwithstanding the fact that the number of bands passing per 100 grammes of load is so small with pyrites, on account of the extreme resistance to bending exhibited exceptionally by this mineral. With most other substances very many more bands would pass for the same amount of load and size of plate. Perhaps the best indication of trustworthiness of the method is the closeness of the numbers of bands in the four experiments for the highest load of 400 grammes.

It will be observed that the number of bands recorded in the table is referred to as corrected. For before commencing the experiments it is essential, as Warburg and Koch pointed out, to determine whether there is any bending of the parts of the apparatus, recorded by the movement of the interference bands; a preliminary experiment is made with the same loads as are to be used in the determinations, but with a thick relatively unbendable block of glass instead of a thin plate. With the author's apparatus this correction is reduced to very minute dimensions, only just attaining the equivalent of a single interference band for the maximum load of 500 grammes. The actual values of

the correction are: for 500 grammes 1.04 band, for 400 gr. 0.86 band, for 300 gr. 0.69 band, for 200 gr. 0.51 band, and for 100 gr. 0.26 band. Special provision for lowering the balance by the thickness of the block of glass one centimetre thick is made, in order that this correction can be determined, a metal packing plate also one centimetre thick being inserted during the construction between the balance pedestal and the rectangular box on which it stands; this plate is temporarily removed during the determination of the correction, lowering the pressure-point to just the right amount for it to come under the glass block when the latter is in contact with the platinum-iridium knife-edges.

We may now work out for the example of pyrites the actual value of E , the modulus of elasticity, that is, the coefficient of resistance to bending or dilatation, by use of the formula of Warburg (see page 911),

$$E = \frac{PL^3}{4st^3b} \left\{ 1 + 3 \left(\frac{t}{L} \right)^2 \right\}.$$

The experimental data are :

$P = 100$ grammes.

$L = 17.261$ millimetres.

$b = 8.833$.

$t = 0.875$.

$n = 2.27$ bands.

$s = n \frac{\lambda}{2} = 0.0007448$ millimetre.

The result for E is found to be, for the direction normal to the cube face, that is, along any cubic axis :

$$E = 29\ 172\ 000 + 225\ 000 = 29\ 397\ 000.$$

An approximate determination by Voigt, by the older method, with a similarly orientated plate of pyrites from Cornwall gave the value 35 500 000. Other results of Voigt's, however, were much lower, and very similar to that derived from the author's experiments. Pyrites crystals would, indeed, appear to show considerable differences in elasticity, possibly due to cavities and enclosures. In any case the coefficient E of resistance to bending or dilatation is exceptionally high, which accounts for the very small number of interference bands which effect their transit for 100 grammes of load, compared with the considerable number usually observed with a plate of the same dimensions of other substances.

Variation of Elasticity with Symmetry.—When the elasticity E , the resistance to deformation, is thus determined along different directions in a non-crystalline substance such as glass, or in a crystalline substance composed of small individual crystals interlaced in a mass with every possible orientation, such as cast iron, the values found for the various directions are all equal within the limits of the possible experimental error. Hence for such a substance E has only one value, its modulus of elasticity.

In the case of a crystal, however, the value varies in general for different directions, and in a manner which is compatible with the degree of symmetry exhibited by the crystal. Moreover, this variation of the modulus of elasticity affects all regular vibratory movements transmitted through a crystal, such as sound waves, the velocity of which is proportional to the square root of the modulus of elasticity and inversely proportional to the square root of the density, that is, proportional to $\sqrt{\frac{E}{d}}$. This has been proved by Savart for sound travelling through quartz, plates of which set in vibration give different dust figures according to their orientation in the crystal, and by von Groth for rock salt by direct measurement of the velocity of sound in the crystal.

In order to be able to specify the modulus of elasticity E for *any* direction in a crystal, and thus to arrive at the surface of elasticity, the envelope on which lie the ends of all the radiating straight lines drawn from a given point to represent the elasticity, it is not sufficient, in the present state of our knowledge, to determine only the resistance to bending with a number of plates cut in different directions out of the crystal, but in addition we have to determine another factor, the resistance to torsion, T . The experimental determination of the torsion coefficient of a solid is carried out by supporting it firmly at one end while a twist is given to it at the other end, the force acting in the plane of the cross-section. The experiment, however, is clearly not one that can be easily carried out with a rod of an object so small as a crystal—even the maximum one that can be grown—often is, and the methods at present used are all mechanical and require a large-sized crystal. That of Voigt¹ is the best of these, which is comparable to the mechanical method of Neumann, as also improved by Voigt, for the determination of the bending.

The author hopes eventually, however, to extend the interference method to the determination of the torsion coefficients of crystals, for the accuracy of the mechanical method can never hope to approach anywhere near that of the determination of bending by the interference method. An investigation which the author is conducting, in collaboration as regards the mathematical portion with Professor A. E. H. Love, whose original contributions to the subject of elasticity are well known, has already shown, however, that torsion determinations are not imperatively necessary, but that a formula can be derived which will enable all the elastic constants of crystals to be calculated from the results of bending determinations with the elasmometer alone.

The work of Neumann and especially of Voigt has shown that for the complete expression of the elasticity relations in different directions of a triclinic crystal, the general case, no less than 21 elasticity constants are required, which reduce in stages as symmetry is introduced. For a monoclinic crystal there are 13, for a rhombic crystal 9, for a

¹ *Ann. der Phys.* N.F., 1886, 29, 604; 1887, 31, 721; 1888, 35, 642. See also Liebisch's *Physikalische Kristallographie*, Fig. 290, page 562.

hexagonal crystal these reduce to 7, and still further to 6 for a tetragonal crystal, while finally for a cubic crystal there are only 3 elasticity constants. When both bending and torsion determinations are available for several differently orientated plates these constants can be more or less readily calculated. The surface of elasticity is a sphere for substances which are either not crystalline, or are composed of interlacing crystals having every possible orientation, so that the substances behave as if amorphous. For a crystal the surface is no longer spherical, but is centro-symmetrical at least, the two parts of every diameter passing through the central point chosen (from which lines representative of the elasticity are drawn) being equal. The shape of this surface is not an ellipsoid, like the surface of optical vibration velocity, of refractive index, or of thermal deformation, but is merely conditioned by the fact that every direction of equal value in regard to the morphological symmetry has an elasticity diameter of equal length.

In the case of a cubic crystal the surface is one characterised by the three constants above referred to; the three rectangular cubic (tetragonal) axes are equal axes of maximum or minimum elasticity, the four trigonal axes joining the corners of the cube and equally inclined to the cubic axes are the converse, the directions either of the minimum or maximum elasticity, while the six digonal axes of symmetry bisecting the angles of the cubic axes and proceeding from the middle of the edges are axes of intermediate elasticity. Thus Fig. 716 represents the elasticity surface of iron pyrites. Here the cubic axes c are the minima, while the trigonal axes o normal to the octahedron faces are the maxima of elasticity of the whole crystal. The digonal axes d normal to the rhombic dodecahedron faces are for this crystal greater than c but less than o . Sections of the elasticity surface parallel to the faces of the octahedron are circular. Fluorspar and rock salt exhibit similar surfaces, while the alums, the cubic double sulphates of the series $R_2SO_4 \cdot M_2(SO_4)_3 \cdot 24H_2O$, are characterised by elasticity surfaces which have protuberances instead of depressions and *vice versa*.

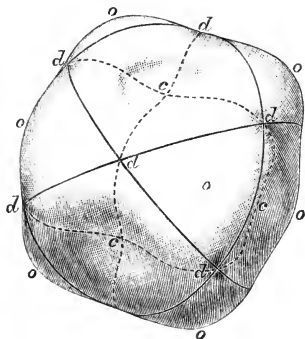


FIG. 716.—Elasticity Surface of Iron Pyrites.

Crystals belonging to the other systems of symmetry exhibit analogous surfaces of elasticity, characterised by the symmetry of the particular system, that is, they possess the same planes of symmetry as the crystal itself.

Thus the property of elasticity, like the optical and thermal properties and that of hardness to be referred to in the next chapter, is a function of the symmetry, and we have in this fact the final expression of the fundamental importance of symmetry of internal structure, as the dominant fact of crystallography, causing and governing not only symmetry of exterior form, the most obvious of all the properties of crystals, but also the symmetrical display of every physical property.

CHAPTER LV

HARDNESS OF CRYSTALS, AND LIQUID CRYSTALS.

THE **Hardness of a Crystal** is measured by its capability of being just abraded or scratched by contact with a sharp fragment of another substance of slightly greater hardness, and of which the degree of hardness is known with reference to a conventional scale. Hardness, as thus measured, is a highly characteristic crystallographic property. It has been defined by Dana as "the resistance offered by a smooth surface to abrasion." What occurs during the abrasion is that the particles of the softer substance are torn away by the harder substance, in the form of powder, their cohesion being overcome. Hence, hardness is intimately connected with cohesion, and therefore with the structure of crystals.

Most artificially prepared crystals, such as those of metallic salts or other chemical preparations, are low in the scale of hardness, and the determination of their hardness is not of much value. But in the case of naturally occurring mineral crystals, usually higher in the scale, the degree of hardness displayed is frequently a most valuable help towards identification. Thus the diamond is uniquely distinguished by remaining unabraded by any other substance, a sharp fragment of another diamond being alone capable of scratching it. The hardness of the diamond is thus obviously the maximum known. The diamond is, therefore, placed at the head of the scale of hardness. This scale has ten units, corresponding to the ten distinctly marked degrees of hardness of certain specific minerals. These substances are stated in the following table in ascending order of their hardness. It is usually known as Mohs' scale of hardness, the order having first been definitely arranged by Mohs.

MOHS' SCALE OF HARDNESS.

- | | |
|---------------|------------------------|
| 1. Talc. | 6. Orthoclase felspar. |
| 2. Gypsum. | 7. Quartz. |
| 3. Calcite. | 8. Topaz. |
| 4. Fluorspar. | 9. Corundum. |
| 5. Apatite. | 10. Diamond. |

The degree of hardness is experimentally determined by observing that the crystal under investigation is just able to be scratched by a particular one of these minerals forming the scale of hardness, and itself

scratches the mineral next below in the scale. For instance, a crystal which is scratched with some little difficulty by quartz, but which is itself capable of scratching orthoclase felspar, is possessed of a hardness between 6 and 7. Talc and gypsum are so soft as to be scratched by the finger nail. A penknife is a most useful article for testing hardness; for good steel has a hardness of about 6, and so will not scratch quartz, but readily scratches any mineral below orthoclase in the scale. The three varieties of corundum, aluminium oxide Al_2O_3 ,—the ruby, sapphire, and emery—are only slightly less hard than the diamond, and are absolutely impervious to the penknife, the point of which they will readily blunt; but glass imitations of these gem-stones have a hardness about 6, and are just scratched by the knife. In the absence of a diamond, glass can often be scratched by a sharp fragment or pyramidal point of a quartz crystal, sufficiently to break readily along the line of scratch.

It must not be overlooked, however, that the hardness varies slightly with the cleavage direction, being always lower along the direction of cleavage than in the direction perpendicular thereto. High specific gravity, that is, great density, is generally accompanied by great hardness, as it is also as a rule by a high refractive index.

The powder scratched off a coloured or opaque crystal by a harder substance frequently possesses a characteristic colour, known as the "Streak." The streak is best recognised by spreading and smearing the powder on white paper. The colour of the streak is often more distinctive of the particular mineral than the colour of the mineral itself.

Exact Measurement of Hardness by the Sclerometer.—The most precise measurements of hardness yet possible are obtained by the method of Pfaff with the instrument known as the "Sclerometer." The instrument is seen in Fig. 717 as constructed by Steeg and Reuter.

It consists of a horizontal goniometer-circle divided directly into single degrees, with the crystal-adjusting apparatus carried immediately above it and terminating in a little tabular support on which the crystal can be cemented. The non-rotatable basal table and central axial cone for the independently rotatable circle and crystal-holder is mounted on three wheels rolling on three corresponding guiding rails. The traverse is brought about by laying a convenient weight in a pan connected by a cord with the goniometer table, and passing over a pulley. The fixed tripod table which carries the rails is also fitted with a spirit level, and a vertical column rises from it near the edge, which carries the balance beam, to the end of one arm of which the scratching point is attached. Two alternative points are given, one of which is shown in position and the other lying on the tripod table; one is of hardened steel and the other is of diamond. It is carried at that end of the balance beam which comes over the centre, a little pan for the reception of weights being arranged immediately above it, in order that the scratching may occur with a given pressure. The other arm of the balance carries an adjustable counterpoise for the weight of the pan and scratching point. An adjustable stop is arranged to prevent the point from cutting too deeply into the crystal. In order to provide for crystals of different sizes the supporting column of the balance is adjustable for height, by sliding in an outer pedestal column.

A similar instrument is also supplied by Fuess, the centring and adjusting movements for the crystal being the same as those on

the No. 2a goniometer, but which is otherwise constructed on lines resembling those of the instrument shown in Fig. 717.

The crystal is fixed upon the adjusting apparatus with the face to be tested uppermost, and the load on the little pan above the diamond increased until the point of the diamond resting on the crystal surface just scratches it as the crystal is drawn underneath it by the traversing of the goniometer, under the influence of a weight in the pan suspended from the cord. After raising the diamond again the goniometer circle is rotated for some definite angle, and another scratch made, the process being repeated until the scratching effect for all kinds of azimuths of the crystal surface has been tested, the minimum load required to produce a scratch in each direction having been recorded. A drawing

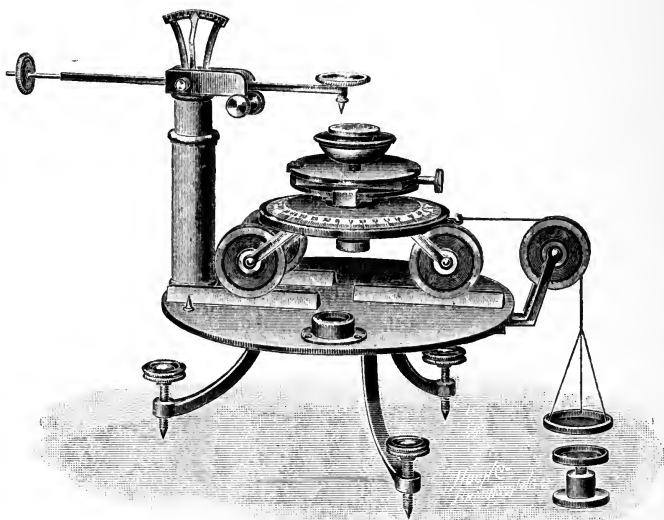


FIG. 717.—The Sclerometer.

of the crystal face is then made, and lines drawn from the centre of it along each experimental direction, of a length proportional to the observed hardness, their terminations being arranged so as not quite to reach the boundary lines representing the crystal face; these terminations of the lines are then joined by a curve, the **Curve of Hardness**. Such a curve will exhibit the symmetry of the system. For instance, the curve of hardness on the face (111) of calcite perpendicular to the optic axis is a three-leaved rosette, with maxima 120° apart, and minima at the half-way positions; that for a cube face is a square with rounded corners or maxima, and with depressions or minima in the middle of each side.

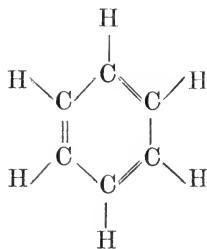
A somewhat more delicate and accurate modification of this method has also been introduced by Pfaff, according to which the load on the diamond point is kept constant at a weight adequate to effect a con-

venient depth of scratch, and the traverse is performed for a definite number of times, a hundred or more, the diamond being only allowed to come into operation during the movement in one and the same direction; the amount of material scratched off is then determined by the loss of weight of the crystal, which with its attached crystal-holder is carefully weighed before and after the experiment. The hardness is then considered to be inversely proportional to the weight of crystal dust thus removed.

A delicate form of sclerometer for use with the crystallographic microscope, such as the Dick microscope or the No. 1 Fuess instrument, has been described by Jagger,¹ which he terms a "microsclerometer." It is supplied with a rigid standard and with apparatus for adjusting the instrument to the microscope; also with rotating, lifting, and fixing adjustments, a balance beam with a rotatory diamond at one end, and a recording apparatus for the number of revolutions of the diamond and the depth of its penetration. It is capable of measuring variations of the rate, depth, or duration of the scratching, as well as of very accurately recording the weight above the diamond point. The operations are watched under a low-power objective of the microscope. Almost all the varieties of modes of procedure which have from time to time been described may be followed with this instrument, involving abrasion, penetration, friction, and fracture.

"LIQUID CRYSTALS."

Reference has several times been made during the course of this book to the "liquid crystals" discovered and so ably worked out by Lehmann,² and although the theory concerning their nature is still a matter of discussion and more or less of controversy, there can be no question about the fact of the existence of substances certainly liquid, which exhibit many of the attributes of crystals, especially double refraction and dichroism. The substances are chiefly compounds of carbon, and mostly derivatives of benzene C_6H_6 containing a large number of atoms, and often derivatives of that hydrocarbon in which the replacement of two hydrogen atoms of the benzene ring has occurred in the "para" position, that is, at two opposite points in the hexagonal ring, thus conferring on them an elongated and more or less rectilinear configuration. One of the most prominent of the substances not aromatic in their nature is the ammonium salt of oleic acid, $C_{18}H_{34}O_2$, a very long and straight chain compound belonging to the unsaturated series of fatty acids. Vorländer,³ who discovered several of these substances, which were afterwards studied further by Lehmann, has shown that this extended



¹ *Amer. Journ. of Science*, 1897, 4, 399.

² *Flüssige Krystalle*, Leipzig (Engelmann), 1904; *Zeitschrift für physikalische Chemie*, 1910, 71, 355.

³ *Ber. der deutsch. Chem. Ges.*, 1907, 40, 1970.

chain-like character is one of the most favourable conditions for the production of liquid crystals. Also they are generally substances which exist in solid crystalline condition at the ordinary temperature, and after fusion above their melting point form, on partial cooling, at a specific temperature, a second modification consisting of "liquid crystals."

The discovery of the fact that these substances produce crystal-like bodies, when examined under the microscope at certain particular temperatures, came about as a side issue from the work of Lehmann on chemical reactions under the microscope at higher temperatures. His heating microscope is now a well-known instrument, and one form of it, constructed by the firm of Steeg and Reuter (formerly Voigt and Hochgesang) has already been described in Chapter LI. (Fig. 690, page 848).

An excellent form of this microscope, as constructed with the most recent improvements (1910) by Zeiss, is shown in Fig. 718. It can be employed equally well for ordinary eye observation and for screen demonstration, the eyepiece, shown separately at A in Fig. 718, being removed when the instrument is required for the latter purpose, and the mirror fitting shown at the top of the instrument placed in position instead. Adjustment of the mirror for azimuth is afforded by rotation of the inner tube (to which its collar is attached) within the optical tube. Its altitude tilt may be regulated by means

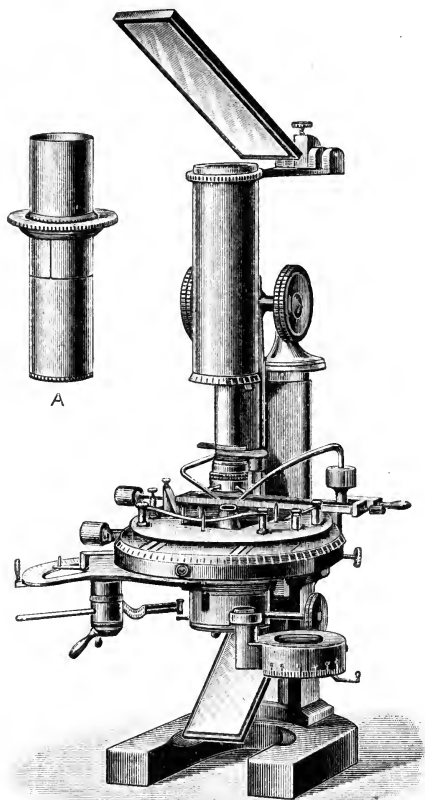


FIG. 718.—Lehmann's Heating Microscope as constructed in its latest form by Zeiss.

after adjustment of the projected screen picture, by means of a tightening screw. The best Zeiss microscope is at once adapted for the purpose by making certain additions to its ordinary equipment. Firstly, the microscope-slip, which is somewhat wider than the usual 3 by 1 inch slip, is raised from the ordinary stage by means of eight little columnar supports. Two other pairs of little columns, insulated in this case, are arranged with holes and capstan screws to form binding screws for the attachment of electric wires, in the event of the action of the electric current on the

substance on the slide being desired to be studied. Secondly, an adjustable miniature Bunsen burner is provided (seen on the left in Fig. 718), which can be swung under the centre of the stage, where it is open, and its flame may be regulated from the minutest size to that of a jet nearly an inch high; the flame can, moreover, be converted into a blowpipe one if desired, as provision is also made of an air blast attachment, both gas and compressed air from a weighted gas-holder being delivered by caoutchouc tubing to two little tubulures carried by a cylindrical mixing chamber. Both gas and air blast can be delicately controlled by means of a long lever gas-tap moving over a graduated arc, and by use of this it is possible to attain a constant temperature at any height up to 700°C . for quite a considerable time.

Thirdly, a means of rapidly cooling the substance on the slide, after it has been heated by the Bunsen flame, is provided, in the form of a couple of air-blasts, delivered from above by two equal tubes adjustable about ball-and-socket joints, one on each side of the stage, and directed down on to the slide. The compressed air is evenly supplied to both, so that the cooling effect on each side of the centre may be arranged to be quite equal. These are the three main attachments. Subsidiary fittings are: a water jacket for the protection of the objective and of the analysing Nicol prism, fitted with tubulures for the delivery of a stream of cold water and its removal after circulation; the mirror-fitting already alluded to for demonstration purposes, to direct the rays proceeding from the objective (used without eyepiece for screen demonstration) to the screen, and certain further protective arrangements such as a convex screen which may be fitted if desired round the optical tube just above the objective.

The most useful objective is the Zeiss 8-millimetre, magnifying without eyepiece about 30 diameters, and with the lowest-power eyepiece about 200 diameters. This objective serves admirably for the screen projections, the light from the electric lantern, in the form of a somewhat narrow parallel beam, being first directed into the microscope by the ordinary sub-stage mirror, and then concentrated on the object by the sub-stage condenser, and subsequently directed to the screen by the upper mirror attached for the purpose.

One of the most interesting of the many substances now known to produce "liquid crystals" is ammonium oleate, $\text{C}_{18}\text{H}_{33}(\text{NH}_4)\text{O}_2$, which has already been referred to as a very long chain-like compound belonging to the fatty series of organic substances, exhibiting the elongated character of molecule like the para-benzene compounds also already alluded to. The "liquid crystals" of ammonium oleate are obtained at a temperature only slightly above the ordinary, by simply allowing a warm solution in alcohol to cool. They take the form of steep double pyramids seen with crossed Nicols and as shown at *e* in Fig. 719. That they are liquid is at once seen by merely touching the cover-glass which is laid over the crystallising drop of solution on the slide, for they immediately become distorted, but regain their shape on removal of the pressure. The crystals polarise strongly between crossed Nicols, and are obviously doubly refractive. When two such crystals approach each other they become blended at

the point of contact, as shown at *a* and *b* in Fig. 719, and are gradually drawn into each other, as represented in the successive figures *c* and *d*, until they are merged into a single individual *e* exactly like each separately before their union, but of double the volume. When an individual is bent above a certain amount it either breaks, the two parts then becoming perfect smaller individuals, or it forms a knee-shaped twin.

The chief other substances which afford the phenomena of liquid crystals are para-azoxy-anisol, the esters of para-azoxy-benzoic acid, para-azoxy-phenetol, cholesteryl benzoate, and cholesteryl acetate.

Some of these, and of the numerous kindred substances now known to produce liquid crystals at specific temperatures, show formations which are obvious attempts at the production of prismatic crystals, more or less rounded at the edges, while others exhibit the liquid crystals as spherical drops, showing double refraction in an unmistakable manner. Generally, the substances are solids at the ordinary

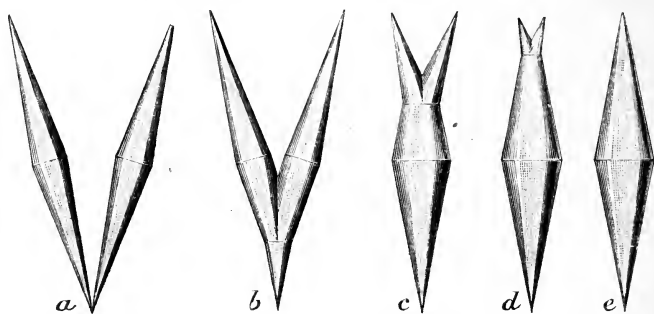


FIG. 719.—Liquid Crystals of Ammonium Oleate.

temperature. On melting under the heat of the little Bunsen flame they produce a non-polarising liquid, but on allowing this to cool, accelerated by gentle and carefully regulated use of the pair of cold air blasts, at some particular temperature, corresponding to the formation of a specific modification of the substance, the liquid crystals form, and if the temperature be then maintained constant they can be studied and photographed at leisure. On cooling further, however, the ordinary solid modification is again reproduced.

A particularly beautiful example is afforded by the substance cholesteryl acetate. A slide is taken on which a very small quantity of the solid has been preliminarily carefully melted, and the drop of liquid is then covered with a cover-glass. After resolidification it shows under the microscope a confused mass of radiating crystals of the ordinary solid variety. While still in position on the stage the little Bunsen flame is now brought underneath it until it again melts. The Nicols being crossed the field is practically dark, either to the eye, using the microscope in the ordinary way with an eyepiece, or on the screen when the eyepiece is removed and the mirror substituted and the

parallel beam from the electric lantern replaces the ordinary table microscope lamp. The Bunsen flame is then at once removed, and the two cooling air jets above the stage allowed to play gently on each side of the centre of the slide. After two or three minutes, the Nicols being crossed for the production of the dark field, colour becomes visible, due to the double refraction of the liquid crystalline modification which now forms. Presently, however, spherulites commence to grow out of this liquid-crystal phase; small spots of light appear, probably in several places at once over the field, and rapidly develop into relatively large circular discs, of a pinkish to a yellowish colour, beautifully shaded both concentrically and radially, and strongly marked by a black cross parallel to the planes of vibration of the crossed Nicols. The field is soon almost covered with these beautiful apparitions, as if it were strewn with bright cruciferous flowers on a dark background, each one quite isolated if the cooling has been gradual, and the size never exceeding a moderate one, such as is represented in the typical field more or less faithfully reproduced, but with a lighter background, in Fig. 720. When the growth of these spherulites, somewhat flattened by the cover-glass, has become arrested, the remaining portion of the field is traversed by numerous radiating acicular crystals of the ordinary solid modification of cholesteryl acetate, polarising in brilliant colours and eventually completely filling the field.

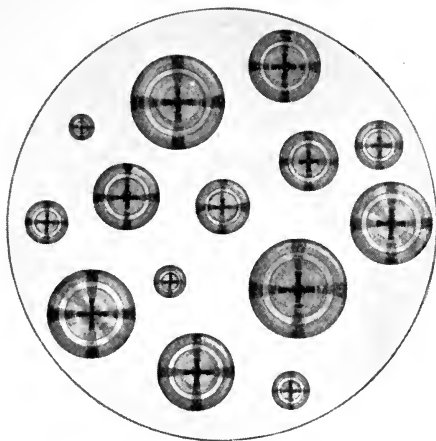


FIG. 720.—Spherulites crystallising out of the "Liquid Crystal" Variety of Cholesteryl Acetate.

Whatever be the truth concerning the much-discussed theories of Lehmann there can be no doubt of the reality of the phenomena, and that homogeneous structures in the liquid condition are built up by these long-chain molecules of the substances exhibiting the phenomena. There can be no getting away from the fact thus revealed that it is possible to these molecules, while forming a substance which is liquid, in some cases as mobile as water, to arrange themselves in an organised manner, akin to that in a crystal, and that when the "liquid crystals" are disturbed, or broken, they at once endeavour, usually with complete success, to recover their form and to repair the damage. It thus appears that the molecules themselves, which are more or less free agents in the liquid condition, must be endowed with a directive force, a configuration-producing force, the "Gestaltungskraft" of Lehmann. In the case of the

bipyramids of ammonium oleate and of the prismatic forms of other substances producing "liquid crystals," the molecules would appear to be arranged just as if the crystal were solid. In the spherical drops, however, they appear to take up a concentric arrangement.

Although many of Lehmann's theoretical conclusions have not been accepted, having probably been carried too far, yet one solid fact is obvious, namely, that the molecule possesses definite and even prominent individuality in these liquid bodies approximating to the nature of crystals. Indeed, the inevitable outcome of Lehmann's work is that the molecular directive force concerned in crystallisation is re-established as a fact, after having been more or less discarded for a time as the effect of the results of the geometricians, who have pushed their magnificent work beyond its true limits. It had been assumed much too readily that no molecular directive force was concerned in crystallisation, because such a force was not absolutely necessary to account for the production of a homogeneous structure; and that the mere mechanical fitting together, along the lines of greatest facility, of the structural units to form a particular kind of crystal structure was sufficient to account for that structure. The essential part of the splendid work of the geometricians is not at all affected by Lehmann's discoveries, the establishment of the 230 types of homogeneous structures possible to crystals standing as the great truth it undoubtedly represents. But the mode of production of the particular type of structure in each individual case, and our ideas as to the *modus operandi* of the act of crystallisation, are profoundly affected by this restoration to the molecule of its orientating directive force.

The importance of the space-lattice, as the expression of the molecular arrangement, is at the same time even further enhanced. For there can be no question but that a space-lattice arrangement of the molecules is being constantly striven after and more or less attained, in the cases of the bipyramidal and prismatic "liquid crystals" of such substances as ammonium oleate and the esters of para-azoxy-benzoic acid, although some of the substances exhibiting such forms are nearly as mobile as water. The directive molecular force may possibly and even probably be eventually traced to the electronic corpuscles composing the atoms present in the molecule, as ultimate source. Much of the force is doubtless used up in interatomic attraction and repulsion, resulting in the stereometric arrangement of the atoms in the molecule, which gives rise on crystallisation to the particular Sohncke-Barlow point-system determinative of the crystal-class; Lehmann also suggests that astatic combinations are produced, which account for the negative results of crystallisation experiments in the magnetic field. But as Lehmann shows, such astatic molecules would endeavour to arrange themselves in space-lattices by reason of the considerable residual force (involving polarity in certain positions or directions) left after the maximum possible inner expenditure of energy had occurred. Whatever be the correct theoretical explanation, however, there can be no possible doubt about the fact that a directive force is at work endeavouring to pro-

duce and maintain a space-lattice arrangement of the chemical molecules of these remarkable liquids.

The resuscitation of the molecular directive force renders many phenomena described by workers in this domain of crystallography—notably Wulff and Miers, both of whom as well as Lord Kelvin have emphasised the prime importance of the molecular space-lattice—and which have been referred to in several chapters of this book, at once comprehensible, and it is in harmony with many facts also described by the author. In the character here ascribed to the force, as ultimately derived from the electronic corpuscles composing the atoms, it further supplies just the intermolecular and interatomic force required by the theory of Pope and Barlow, connecting crystalline form with chemical valency, which has lately been brought forward, an account of which has been given in Chapter XXXIII., and according to which the sizes of the atoms present in the molecule of any one substance are roughly proportional to the valencies of their respective chemical elements. This hypothesis, together with the author's experimental elucidation of the true meaning of the specific gravity of crystals, and of molecular volume in the crystalline state and its directional expression in molecular distance ratios—correlating the similarly constructed substances belonging to the same isomorphous series by the law of progression with the atomic weight of the interchangeable elements—would appear to offer a comprehensive explanation of the intimate relationship between crystalline form and chemical constitution. For the theory of Pope and Barlow concerns the horizontal progression, and the author's results express the vertical progression, of the periodic law of Mendeléeff, as it affects crystals. It is of considerable importance that the work on liquid crystals should thus lend some support to the basis of this interesting theory.

The definite experimental proof of the individuality of the molecule, in the borderland between the truly liquid and crystalline states, affords strong presumptive evidence that such individuality persists in the truly crystalline state. After thus emphasising the solid fact which has been revealed as the main result of the surprising and highly interesting work of Lehmann and his fellow-workers on "liquid crystals,"¹ the subject may now be well left for further experimental results to accumulate.

That the near future may see many more keen and highly trained investigators taking up this fascinating branch of study, of such superlative importance to chemistry, and thus rapidly adding to the number of experimentally well grounded and closely related facts, is the author's hope and main thought in concluding this attempt to set the existing store of crystallographic knowledge concisely forth within the limits of a single volume.

¹ Some further facts concerning "liquid crystals," illustrated by a number of reproductions of photographs of striking examples, will be found in the author's book on *Crystals*, in the International Scientific Series recently published (1911) by Messrs. Kegan Paul, Trench, Trübner, and Co.



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